Motivation

Matrix Fisher Distribution

Matrix Fisher–Gaussian Distribution

Attitude Estimation With MFG

Attitude Observability with Single Direction Measurements

Attitude Estimation Based on MFG

6D Pose Estimation With MFG

Loosely Coupled IMU-GNSS Integration

Visual-Inertial Navigation

Summary

Geometric Formulation of Uncertainties and Estimation for Three-Dimensional Rotations

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Overview

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Attitude Estimation in Engineering

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- 3D Rigid Body Attitude
  - The orientation of a reference frame fixed to the rigid body.
  - The space of 3D attitude: three dimensional special orthogonal group.

$$\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \,|\, RR^T = I_{3\times 3},\ \det(R) = 1\}.$$

  - Rotation matrix $R \in \mathrm{SO}(3)$: transform the coordinates of a vector from the body-fixed frame x^B to inertial frame x^I.

$$x^I = Rx^B$$

- Applications in Engineering
  - Alignment of two satellites: laser communication.
  - Attitude control for UAVs.
  - Inertial navigation.

Uncertainty for 3D Attitude

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- State Estimation of a Dynamical System
  - Uncertainty Propagation: propagete the mean and covariance matrix of the state through the kinematic equations.
  - Measurement Update: use new measurements to correct propagation errors.
  - Covariance matrices are used to weigh the propagation and measurement noises.
- What is Covariance Matrix for 3D Rotation?
  - Covariance for lower dimensional parameterizations:
    - Three dimensional: Euler angles, (modified) Rodrigues parameters, etc.
    - Four dimensional: Unit quaternions.
  - Problems with three dimensional parameterizations:
    - Singularities: such as gimbal lock for Euler angles.
  - Problems with quaternions:
    - The unit norm constraint makes the covariance matrix singular.

Multiplicative Extended Kalman Filter

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- Covariance Matrix for Attitude Error

$$q_t = q \otimes q\{\delta v\}.$$

  - $q_t, q \in \mathbb{S}^3$: true and estimated attitude represented by unit quaternions.
  - $q\{\delta v\} \in \mathbb{S}^3$: error attitude expressed as a rotation vector $\delta v \in \mathbb{R}^3$.
  - $\delta v \sim \mathcal{N}(0, \Sigma)$, where $\Sigma \in \mathbb{R}^{3\times 3}$ is the covariance matrix, representing the uncertainty of the attitude.
    - δv is three dimensional.
    - δv is usually small, so singularity is avoided.
- Multiplicative Extended Kalman Filter (MEKF)
  - Using Gaussian distribution to model the uncertainty of attitude error.
  - Problems: not suitable for large uncertainty
    - Linearization error.
    - Wrapping error.

Multiplicative Extended Kalman Filter

- Covariance Matrix for Attitude Error

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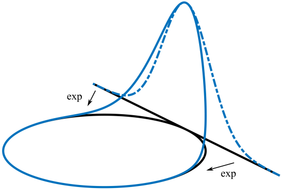

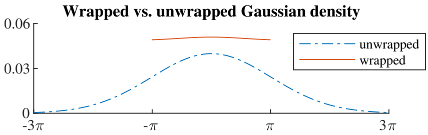

Large Attitude Uncertainty

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- Large Attitude Uncertainty in Estimation Problems
  - Unknown initial conditions, e.g., unknown heading direction in a building.
  - Not properly observed degree of freedom, e.g., single reference direction that is slowly varying in the inertial frame.
  - Sensor failure, e.g., no GPS signal in a tunnel.
- Goal of This Dissertation
  - Deal with large attitude uncertainty using probability distribution defined intrinsically on $\mathrm{SO}(3)$ for attitude.
    - Matrix Fisher distribution on $\mathrm{SO}(3)$.
    - Bingham distribution on $\mathbb{S}^3$ for unit quaternions.
    - No singularities, no geometric constraints, able to model arbitrarily large attitude uncertainty.
  - Design Bayesian filters for classical estimation problems involving the attitude/pose of a rigid body.

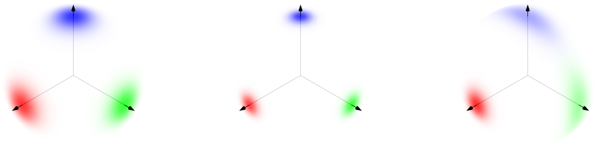
Matrix Fisher Distribution

- Matrix Fisher Distribution: Defined on $\mathrm{SO}(3)$
  - Construction: condition a Gaussian distribution from $\mathbb{R}^9$ to $\mathrm{SO}(3)$.
  - Density function for $R \sim \mathcal{M}(F)$:

$$p(R) = \frac{1}{c(F)} \exp\left\{\mathrm{tr}\left(FR^T\right)\right\}.$$

  - $F \in \mathbb{R}^{3\times 3}$ is the parameter, $c(F) \in \mathbb{R}$ is the normalizing constant.

- Bingham Distribution: Defined on $\mathbb{S}^3$
  - Equivalent to the matrix Fisher distribution under the homomorphism from $\mathrm{SO}(3)$ to $\mathbb{S}^3$.



Properties

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- Shape of the Density Function
  - proper singular value decomposition (pSVD) of the parameter: $F = USV^T$.
  - Mean attitude: $M = UV^T$ (uni-modal).
  - Principal axes:
    - Columns of U in the inertial frame.
    - Columns of V in the body-fixed frame of the mean attitude M.
  - Dispersion: $s_i + s_j$ specifies the dispersion along the k-th principal axis, for $i \neq j \neq k$.
  - Analogous to a Gaussian distribution.

- Maximum Likelihood Estimation (MLE) for Parameters
  - Moments

$$\mathrm{E}[R] = UDV^T, \qquad d_i = \frac{1}{c(S)} \frac{\partial c(S)}{\partial s_i}.$$

  - MLE for $U, V \in \mathrm{SO}(3)$: the pSVD of $\mathrm{E}[R] = USV^T$.
  - MLE for S: solving $d_i = \frac{1}{c(S)} \frac{\partial c(S)}{\partial s_i}$ from D.

Central Moments

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- Motivation
  - Designing Bayesian filters using matrix Fisher distribution sometimes needs to evaluate its higher order moments.
- Central Moments for Matrix Fisher Distribution: $Q = U^T R V$

$$\mathrm{E}\left[Q_{i_1 j_1} \cdots Q_{i_n j_n}\right] = \frac{1}{c(S)} \left.\frac{\partial c(S+T)}{\partial T_{i_1 j_1} \cdots \partial T_{i_n j_n}}\right|_{T=0}.$$

  - $\mathrm{E}\left[Q_{i_1 j_1} \cdots Q_{i_n j_n}\right] = 0$ if $\{i_k, j_k\}_{k=1}^{n}$ has odd number of 1, 2, or 3.
  - $\mathrm{E}\left[Q_{i_1 j_1} \cdots Q_{i_n j_n}\right] = \mathrm{E}\left[Q_{j_1 i_1} \cdots Q_{j_n i_n}\right]$.
- Computation
  - $\mathrm{E}\left[Q_{i_1 j_1} \cdots Q_{i_n j_n}\right]$ is a linear combination of the derivatives of $c(S)$ up to n-th order.
  - The coefficients and derivatives of $c(S)$ can be calculated recursively.
  - The recursion starts from $c(S)$ and its first order derivatives $\frac{\partial c(S)}{\partial s_i}$.

Highly Concentrated Approximations

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- Motivation
  - Evaluating the normalizing constant $c(S)$ and its derivatives is hard.
  - The MLE $d_i = \frac{1}{c(S)}\frac{\partial c(S)}{\partial s_i}$ is very hard to solve.
- Highly Concentrated in Three Degrees of Freedom [1]

Theorem

Let $R \sim \mathcal{M}(F)$, where $F = USV^T$ is the pSVD of F. Suppose $s_2 + s_3 \gg 0$. Let $Q = U^T RV = \exp(\hat{\eta})$, then $\eta \overset{\sim}{\approx} \mathcal{N}\left(0, \left(\mathrm{tr}(S) I_{3\times 3} - S\right)^{-1}\right)$.

  - The matrix Fisher distribution is approximated by a Gaussian distribution in $\mathbb{R}^3$.

[1] T. Lee, "Bayesian attitude estimation with approximate matrix Fisher distributions on SO(3)," in *Conference on Decision and Control*, 2018, pp. 5319–5325.

Highly Concentrated Approximations

- Motivation
  - Evaluating the normalizing constant $c(S)$ and its derivatives is hard.
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- Highly Concentrated in Three Degrees of Freedom
  - Approximations of $c(S)$ and its derivatives ($i \neq j \neq k$):

$$c(S) \approx \frac{\operatorname{etr}(S)}{\sqrt{8\pi(s_1+s_2)(s_1+s_3)(s_2+s_3)}}$$

$$\frac{1}{c(S)}\frac{\partial c(S)}{\partial s_i} \approx 1 - \frac{1}{2}\left(\frac{1}{s_i+s_j}+\frac{1}{s_i+s_k}\right),$$

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  - The MLE $d_i = \frac{1}{c(S)} \frac{\partial c(S)}{\partial s_i}$ is very hard to solve.
- Highly Concentrated in Two Degrees of Freedom

Theorem

Let $R \sim \mathcal{M}(F)$, where $F = USV^T$ is the pSVD of F. Suppose $s_1 + s_3 \gg s_2 + s_3 \geq 0$. Let $Q = U^T R V = \exp(\hat{\eta}) \exp(\hat{\eta}')$, where $\eta = [0, \eta_2, \eta_3]^T$, and $\eta' = [\eta_1, 0, 0]^T$. Then $\eta_3 \stackrel{\sim}{\approx} \mathcal{VM}(0, s_2 + s_3)$, and $[\eta_2, \eta_3]^T \stackrel{\sim}{\approx} \mathcal{N}\left(0, \mathrm{diag}\left(\frac{1}{s_1+s_3}, \frac{1}{s_1+s_2}\right)\right)$, and they are approximately independent.

  - The matrix Fisher distribution is approximated by a combination of Gaussian distribution in $\mathbb{R}^2$, and a von Mises distribution on $\mathbb{S}^1$.

Highly Concentrated Approximations

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  - The MLE $d_i = \frac{1}{c(S)} \frac{\partial c(S)}{\partial s_i}$ is very hard to solve.
- Highly Concentrated in Two Degrees of Freedom
  - Approximations of $c(S)$ and its derivatives ($j \in \{2, 3\}$):

$$c(S) \approx \frac{\exp(s_1) I_0(s_2 + s_3)}{2\sqrt{(s_1 + s_2)(s_1 + s_3)}},$$

$$\frac{1}{c(S)} \frac{\partial c(S)}{\partial s_1} \approx 1 - \frac{1}{2}\left(\frac{1}{s_1 + s_2} + \frac{1}{s_1 + s_3}\right),$$

$$\frac{1}{c(S)} \frac{\partial c(S)}{\partial s_j} \approx \frac{I_1(s_2 + s_3)}{I_0(s_2 + s_3)} - \frac{1}{2} \frac{1}{s_1 + s_j},$$

Correlation Between $SO(3)$ and $\mathbb{R}^n$

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- Matrix Fisher Distribution
  - My contributions:
    - An algorithm to compute higher order central moments.
    - A approximation when it is highly concentrated in two degrees of freedom.
  - Cannot model the correlation between $SO(3)$ and $\mathbb{R}^n$.
  - Correlation: transfer information from observed state to unobserved state.
- Examples
  - Attitude estimation: correlation between attitude and gyroscope bias.
  - Inertial navigation: correlation between attitude and position.
  - SLAM: correlation between attitude and landmark locations.

Correlation Between $\mathrm{SO}(3)$ and $\mathbb{R}^n$

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- Existing Models
  - In Markley 2006[1], the matrix Fisher distribution on $\mathrm{SO}(3)$ is combined with Gaussian distribution in $\mathbb{R}^n$.
  - In Darling 2016[2], the Bingham distribution on $\mathbb{S}^3$ is combined with Gaussian distribution in $\mathbb{R}^n$.
  - Problems:
    - These models do not have a generic geometric construction.
    - Their MLEs are complicated and need numerical optimizations, so they are not suitable for real time implementations.
- Objective: a New Model on $\mathrm{SO}(3) \times \mathbb{R}^n$
  - Generic construction from directional statistics.
  - Closed form maximum likelihood estimation.

[1] F. L. Markley, "Attitude filtering on $\mathrm{SO}(3)$," *The Journal of the Astronautical Sciences*, vol. 54, no. 3-4, pp. 391–413, 2006.

[2] J. E. Darling and K. J. DeMars, "Uncertainty propagation of correlated quaternion and Euclidean states using the Gauss-Bingham density," *Journal of Advances in Information Fusion*, vol. 11, no. 2, pp. 1–20, 2016

Matrix Fisher–Gaussian Distribution

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- Matrix Fisher–Gaussian Distribution (MFG)
  - Density: $\mathrm{SO}(3) \times \mathbb{R}^n \ni (R, x) \sim \mathcal{MG}(\mu, \Sigma, P, U, S, V)$ if

$$f(R, x) = \frac{1}{c(S)\sqrt{(2\pi)^n \det(\Sigma_c)}} \exp\left\{-\frac{1}{2}(x-\mu_c)^T \Sigma_c^{-1}(x-\mu_c)\right\} \exp\left\{\mathrm{tr}\left(FR^T\right)\right\}$$

    - $F = USV^T$;
    - $\Sigma_c = \Sigma - P(\mathrm{tr}(S)I - S)P^T$;
    - Two definitions: $\mu_c = \mu + P(QS - SQ^T)^\vee$ (MFGI), or $\mu_c = \mu + P(SQ - Q^T S)^\vee$ (MFGB), where $Q = U^T RV$.
  - The correlation between $\mathrm{SO}(3)$ and $\mathbb{R}^n$ is quantified by P.
- Bingham-Gaussian Distribution
  - Density: $\mathbb{S}^3 \times \mathbb{R}^n \ni (q, x) \sim \mathcal{MG}(\mu, \Sigma, P, M, Z)$ if

$$f(q, x) = \frac{1}{c(Z)\sqrt{(2\pi)^n \det(\Sigma_c)}} \exp\left\{-\frac{1}{2}(x-\mu_c)^T \Sigma_c^{-1}(x-\mu_c)\right\} \exp\left\{q^T MZM^T q\right\}$$

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Construction

- Conditioning from a $(9+n)$-variate Gaussian distribution (MFGI).

$$\begin{bmatrix} x \\ \mathrm{vec}(R^T) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_R \\ \mu \end{bmatrix}, \begin{bmatrix} \Sigma_R & P_R^T \\ P_R & \Sigma \end{bmatrix} \right)$$

  - $\mu_R = \mathrm{vec}(M^T) = \mathrm{VU}^{\mathrm{T}} \in \mathbb{R}^9$.
  - $\Sigma_R^{-1} = I_{3\times3} \otimes K \in \mathbb{R}^{9\times9}$, with $K = VSV^T$.
  - MFGI uses row vectorization $\mathrm{vec}(R^T)$, MFGB uses column vectorization $\mathrm{vec}(R)$.

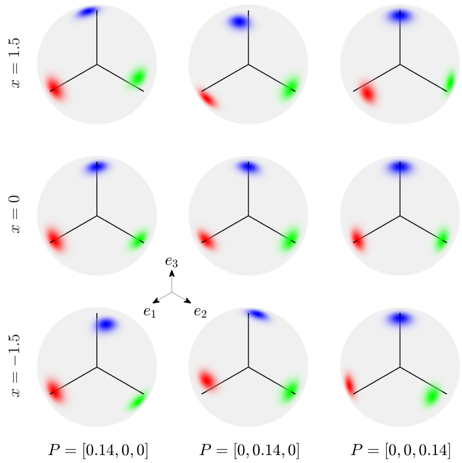
- Correlation: only in the tangent space of the mean attitude $M = UV^T$.
  - Basis of the tangent space at M: $t_i = \mathrm{vec}[(M\widehat{Ve_i})^T]$ for $i \in \{1,2,3\}$.
  - Construct correlation: $P_R = P \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix}^T \in \mathbb{R}^{n\times9}$.
  - Avoids over-parameterization.

Theorem

The conditional distribution $(R,x) \mid RR^T = I_{3\times3},\ \det(R) = 1$ is a matrix Fisher–Gaussian distribution $\mathcal{MG}(\mu,\Sigma,P,U,S,V)$.

Geometric Interpretation of the Correlation

Conditional distribution of attitude in MFG



Difference Between MFGI and MFGB

- Difference between MFGI and MFGB
  - MFGI: x is correlated with rotations expressed in the inertial frame.
    - $R|x \mathrel{\dot\sim} \mathcal{MG}\left(\exp(\widehat{Uv(x)})USV^T\right)$.
  - MFGB: x is correlated with rotations expressed in the body-fixed frame.
    - $R|x \mathrel{\dot\sim} \mathcal{MG}\left(USV^T\exp(\widehat{Vv(x)})\right)$.

Difference Between MFGI and MFGB

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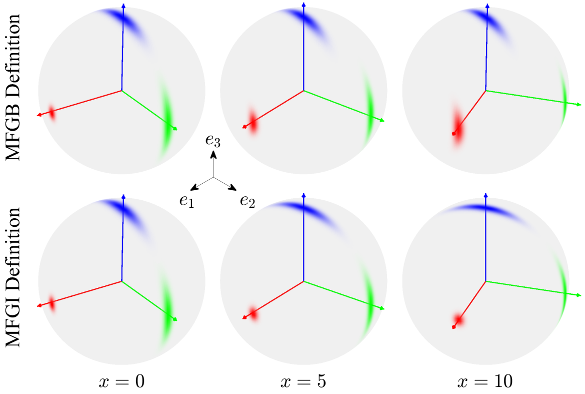
- Difference between MFGI and MFGB



Properties

- Moments
  - Attitude: $\mathrm{E}[R] = UDV^T$, $\mathrm{E}[\nu_R] = 0$,
    $\mathrm{E}\left[\nu_R\nu_R^T\right]_{ii} = s_j\mathrm{E}[Q_{jj}] + s_k\mathrm{E}[Q_{kk}]$ for $i \neq j \neq k$.
  - Linear: $\mathrm{E}[x] = \mu$,
    $\mathrm{E}\left[xx^T\right] = \mu\mu^T + \Sigma - P(\mathrm{tr}(S)I - S)P^T + P\mathrm{E}\left[\nu_R\nu_R^T\right]P^T$.
  - Correlation: $\mathrm{E}\left[x\nu_R^T\right] = P\mathrm{E}\left[\nu_R\nu_R^T\right]$.
- Maximum Likelihood Estimation for Parameters
  - Marginal Likelihood for R: U, S, V.
    - $UDV^T = \mathrm{E}[R]$ is the proper SVD.
    - Solve S from $d_i = \frac{1}{c(S)}\frac{\partial c(S)}{\partial s_i}$ for $i = 1, 2, 3$.
  - Conditional Likelihood for $x|R$: μ, Σ, P
    - $P = \mathrm{cov}(x, \nu_R)\mathrm{cov}(\nu_R, \nu_R)^{-1}$.
    - $\mu = \mathrm{E}[x] - P\mathrm{E}[\nu_R]$.
    - $\Sigma = \mathrm{cov}(x, x) - P\mathrm{cov}(x, \nu_R)^T + P(\mathrm{tr}(S)I_{3\times 3} - S)P^T$.
  - The marginal-conditional MLE is an approximation to the full MLE.
  - This closed form solution makes the calculation efficient, thus suitable for real time implementations.

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- Matrix Fisher Gaussian Distribution
  - Defined on $\mathrm{SO}(3) \times \mathbb{R}^n$.
  - Extends the matrix Fisher distribution to deal with the correlation between $\mathrm{SO}(3)$ and $\mathbb{R}^n$.
  - Applications: simultaneous estimation of attitude and other Euclidean quantities.
    - Sensor biases.
    - Velocity, position.
    - Landmark locations.
- Application of MFG in Attitude Estimation
  - Attitude Observability with Single Direction Measurements
  - Attitude Estimation Based on MFG

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- Attitude kinematics

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = R(t)\hat{\Omega}(t) \tag{1a}$$

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = \hat{\omega}(t)R(t) \tag{1b}$$

  - $\Omega(t)$: angular velocity measured in body-fixed frame.
  - $\omega(t)$: angular velocity measured in inertial frame.

- Single direction measurements

$$x(t) = R(t)^T a \tag{2a}$$

$$y(t) = R(t)b \tag{2b}$$

  - a: reference vector in inertial frame, e.g., the direction of magnetic field, or towards a star. $x(t)$: measurement in body-fixed frame.
  - b: reference vector in body-fixed frame, e.g., the direction towards solar panel from the center. $y(t)$: measurement in inertial frame.

Observable Conditions

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$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = R(t)\hat{\Omega}(t) \tag{1a}$$

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = \hat{\omega}(t)R(t) \tag{1b}$$

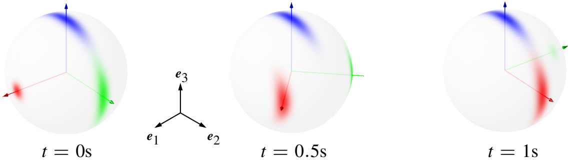
$$x(t) = R(t)^T a \tag{2a}$$

$$y(t) = R(t)b \tag{2b}$$

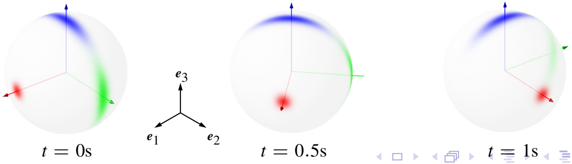
ang. vel. / ref. vec.	body-fixed frame (1a)	inertial frame (1b)
body-fixed frame (2b)	observable	unobservable
inertial frame (2a)	unobservable	observable

Uncertainty Propagation

- Angular velocity in body-fixed frame: $\frac{\mathrm{d}R(t)}{\mathrm{d}t} = R(t)\hat{\Omega}(t)$
  - The uncertainty is rotated in the body-fixed frame, but is unchanged in the inertial frame.



- Angular velocity in inertial frame: $\frac{\mathrm{d}R(t)}{\mathrm{d}t} = \hat{\omega}(t)R(t)$
  - The uncertainty is unchanged in the body-fixed frame, but is rotated in the inertial frame.



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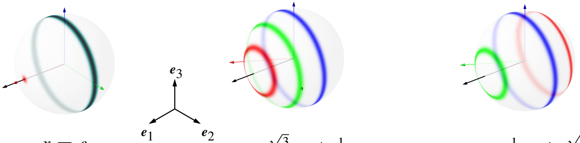
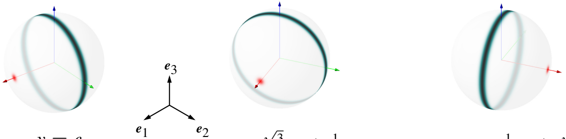
Correction From Direction Measurements

- Reference direction in inertial frame: $x(t) = R(t)^T a$
  - Non-informative along direction a; a is fixed in inertial frame.



$x = e_1$ $\quad x = \frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2$ $\quad x = -\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$

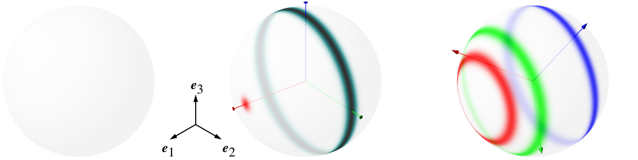
- Reference direction in body-fixed frame: $y(t) = R(t)b$
  - Non-informative along direction b; b is fixed in body-fixed frame.



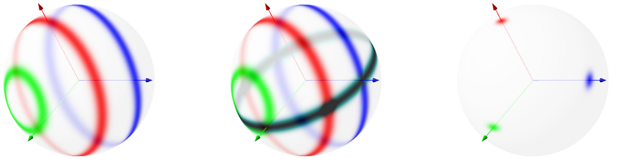
$y = e_1$ $\quad y = \frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2$ $\quad y = -\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$

Observability

- Observable:
  - Angular velocity in body-fixed frame: $\frac{\mathrm{d}R(t)}{\mathrm{d}t} = R(t)\hat{\Omega}(t)$
  - Reference direction in body-fixed frame: $y(t) = R(t)b$



Prior dist. at $t = 0$ | Posterior dist. at $t = 0$ | Propagated dist. at $t = 0.5$



Propagated dist. at $t = 1$ | Propagated dist. overlapped with measured dist. at $t = 1$ | Posterior dist. at $t = 1$

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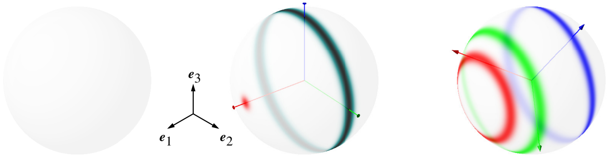
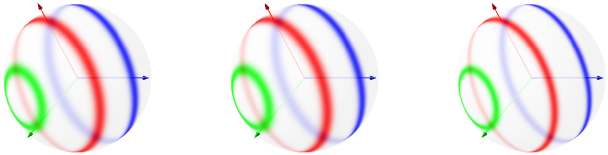
Unobservability

- Unobservable:
  - Angular velocity in body-fixed frame: $\frac{\mathrm{d}R(t)}{\mathrm{d}t} = R(t)\hat{\Omega}(t)$
  - Reference direction in inertial frame: $x(t) = R(t)^T a$



Prior dist. at $t = 0$ | Posterior dist. at $t = 0$ | Propagated dist. at $t = 0.5$

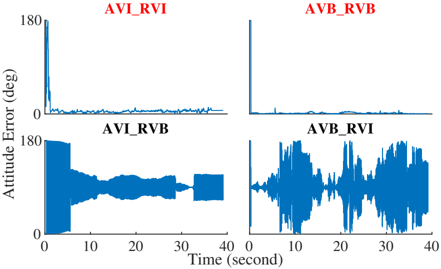


Propagated dist. at $t = 1$ | Propagated dist. overlapped with measured dist. at $t = 1$ | Posterior dist. at $t = 1$

Experimental Verification

ref. vec. \ ang. vel.	body-fixed frame (1a)	inertial frame (1b)
body-fixed frame (2a)	AVB_RVB	AVI_RVB
inertial frame (2b)	AVB_RVI	AVI_RVI

- Attitude error [3]



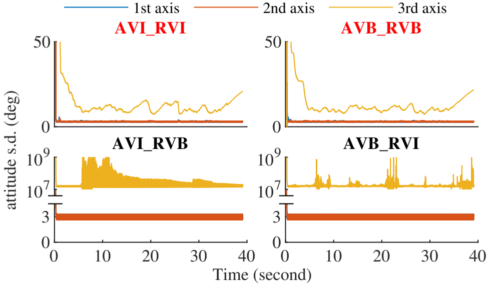
- AVB_RVB and AVI_RVI : attitude error converges to around zero.
- AVB_RVI and AVI_RVB: attitude error does not converge.

[3] I would like to thank Kanishke Gamagedara for conducting the experiment.

Experimental Verification

ref. vec. \ ang. vel.	body-fixed frame (1a)	inertial frame (1b)
body-fixed frame (2a)	AVB_RVB	AVI_RVB
inertial frame (2b)	AVB_RVI	AVI_RVI

- Attitude uncertainty



- AVB_RVB and AVI_RVI : standard deviations for all axes are small.
- AVB_RVI and AVI_RVB: standard deviation for one axis is large.

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- Gyroscope Model

$$R_{k+1} = R_k \exp\left\{h(\hat{\Omega}_k + \hat{x}_k) + (H_u \Delta W_u)^\wedge\right\},$$
$$x_{k+1} = x_k + H_v \Delta W_v.$$

  - Angle random walk noise: $H_u \Delta W_u$.
  - Bias random walk noise: $H_v \Delta W_v$.

- Measurement Model
  - Attitude measurement: $Z_i = R_t \delta R$, $\delta R \sim \mathcal{M}(F_{Z_i})$, $i = 1, \ldots, N_a$.
  - Vector measurement: $z_j \in \mathbb{S}^2 \sim \mathcal{VM}(R^T B_j a_j, \kappa_j)$, $j = 1, \ldots, N_v$.
    - $\mathcal{VM}$: von Mises–Fisher distribution on $\mathbb{S}^2$.
    - a_j: a reference vector in the inertial frame.

- Bayesian Assumed Density Filter
  - Uncertainty propagation: $(R, x)_k^+ \sim \mathcal{MG}(\mu, \Sigma, P, U, S, V)_k^+$ $\Rightarrow (R, x)_{k+1}^- \sim \mathcal{MG}(\mu, \Sigma, P, U, S, V)_{k+1}^-$.
  - Measurement update: $(R, x)_{k+1}^- \sim \mathcal{MG}(\mu, \Sigma, P, U, S, V)_{k+1}^-$ $\Rightarrow (R, x)_{k+1}^+ \sim \mathcal{MG}(\mu, \Sigma, P, U, S, V)_{k+1}^+$.

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- Analytical Propagation
  - Calculate analytical approximations $O(h^2)$ of moments for MLE.
  - Marginal MLE: $\mathrm{E}\left[R_{k+1}^-\right] \to U_{k+1}^-, S_{k+1}^-, V_{k+1}^-$.
  - Conditional MLE: $\mathrm{E}\left[x_{k+1}^-\right], \mathrm{E}\left[\nu_{R_{k+1}}^-\right], \mathrm{E}\left[xx^T\right]_{k+1}^-, \mathrm{E}\left[x\nu_R^T\right]_{k+1}^-$, $\mathrm{E}\left[\nu_R\nu_R^T\right]_{k+1}^- \Rightarrow \mu_{k+1}^-, \Sigma_{k+1}^-, P_{k+1}^-$.
- Unscented Propagation
  - Select sigma points $(R_i, x_i, w_i)_{i=1}^{13}$ from $\mathcal{MG}(\mu, \Sigma, P, U, S, V)_k^+$.
  - Select sigma points $(\eta_j, w_j)_{j=1}^{7}$ from noise $H_u \Delta W_u$.
  - Propagate sigma points through the kinematic equations.

$$R_{ij} = R_i \exp(h(\hat{\Omega}_k + \hat{x}_k) + \hat{\eta}_j), \qquad x_{ij} = x_i$$

  - Update weights: $w_{ij} = w_i w_j$.
  - Recover $\mathcal{MG}_{k+1}^-$ from the sigma points $(R_{ij}, x_{ij}, w_{ij})_{i=1,j=1}^{i=13,j=7}$ using MLE.
  - Set $\Sigma_{k+1}^- = \Sigma_{k+1}^- + hH_vH_v^T$.

Measurement Update

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- Measurement Model
  - Attitude measurement: $Z_i = R_t \delta R$, $\delta R \sim \mathcal{M}(F_{Z_i})$, $i = 1, \ldots, N_a$.
  - Vector measurement: $z_j \in \mathbb{S}^2 \sim \mathcal{VM}(R_j^T B_j a_j, \kappa_j)$, $j = 1, \ldots, N_v$.
- Bayes's formula

$$
\begin{aligned}
&p(x, R \mid Z_{1:N_a}, z_{1:N_v}) \propto p(x, R) \cdot p(Z_{1:N_a}, z_{1:N_v} \mid x, R) \\
&= \operatorname{etr}\left\{ \left(F + \sum_{i=1}^{N_a} Z_i F_i^T + \sum_{j=1}^{N_v} \kappa_j B_j a_j z_j^T \right) R^T \right\} \cdot \exp\left\{ -\tfrac{1}{2} (x - \mu_c)^T \Sigma_c^{-1} (x - \mu_c) \right\}
\end{aligned}
$$

- Match to a New MFG
  - Marginal MLE:
    - $U_{k+1}^+ S_{k+1}^+ (V_{k+1}^+)^T = F + \sum_{i=1}^{N_a} Z_i F_i^T + \sum_{j=1}^{N_v} \kappa_j B_j a_j z_j^T$.
  - Conditional MLE:
    - Calculate $\mathrm{E}\left[x_{k+1}^+\right]$, $\mathrm{E}\left[xx^T\right]_{k+1}^+$, $\mathrm{E}\left[x\nu_R^T\right]_{k+1}^+$, $\mathrm{E}\left[\nu_R \nu_R^T\right]_{k+1}^+$.
    - $\Rightarrow \mu_{k+1}^+, \Sigma_{k+1}^+, P_{k+1}^+$.

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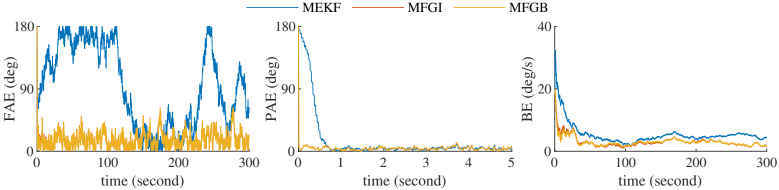
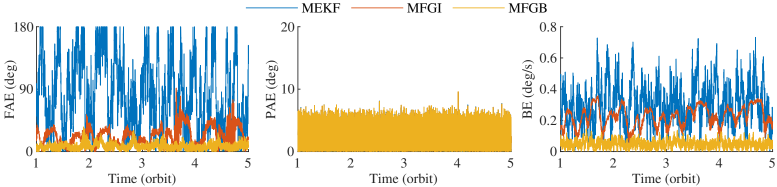
Simulations with Two Direction Measurements

- Settings
  - False initial conditions.
  - Two direction measurements: accurate and inaccurate.
    - Uncertainty around the first reference direction is very large.
    - Uncertainties in the two other axes are small.
- Estimation Errors
  - Compared with MEKF: the state of art method.
  - MFG has better accuracy around the first reference direction.
  - MFG has faster convergence with false initial values.



Simulations with Single Direction Measurements

- Settings
  - A spacecraft orbiting around the Earth.
  - False initial conditions.
  - Time-varying magnetic field direction measurements.
    - Uncertainty around the magnetic field is very large.
    - Uncertainties in the two other axes are small.
- Estimation Errors
  - MFG has better accuracy around the magnetic field.
  - MFG has better accuracy of gyroscope bias.
  - MFGB is more accurate than MFGI.



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Summary

- Application of MFG in Rigid Body Pose Estimation
  - Loosely Coupled IMU-GNSS Integration
  - Visual-Inertial Navigation

Loosely Coupled IMU-GNSS Integration

Problem Formulation

- IMU Kinematics

$$
\begin{aligned}
R_{k+1} &= R_k \exp\left\{ h(\hat{\Omega}_k + \hat{b}_{g,k}) + H_{gu}\Delta W_{gu} \right\}, \\
b_{g,k+1} &= b_{g,k} + H_{gv}\Delta W_{gv}, \\
p_{k+1} &= p_k + h v_k, \\
v_{k+1} &= v_k - gh + R_k \left(h(a_k + b_{a,k}) + H_{au}\Delta W_{au} \right), \\
b_{a,k+1} &= b_{a,k} + H_{av}\Delta W_{av}.
\end{aligned}
$$

  - $x = [b_g, p, v, b_a] \in \mathbb{R}^{12}$, where b_g, b_a are gyroscope and accelerometer biases, and v, p are velocity and position.

- Measurements
  - Attitude and direction measurements (optional).
  - GNSS receiver: position measurements.

$$
y_k | p_k = p_k + \mathcal{N}(0, \Sigma_y).
$$

Bayesian Assumed Density Filter

Motivation

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Summary

- Uncertainty Propagation
  - Calculate analytical approximations $O(h^2)$ of moments.
  - Use MLE to match the propagated moments to an MFG.
- Measurement Update
  - Posterior probability density

$$
\begin{aligned}
p(R,x|\mathcal{Z}) \propto \mathrm{etr}&\left\{\left(F+\sum_{i=1}^{N_a} Z_i F_i^T + \sum_{j=1}^{N_v} \kappa_j B_j a_j z_j^T\right) R^T\right\} \\
&\cdot \exp\left\{-\tfrac{1}{2}(x-\mu_c)^T \Sigma_c^{-1}(x-\mu_c)\right\} \cdot \exp\left\{-\tfrac{1}{2}(Hx-y)^T \Sigma_y^{-1}(Hx-y)\right\}.
\end{aligned}
$$

  - Factorization of posterior probability density

$$
\begin{aligned}
p(R,x|\mathcal{Z}) &\propto p_R(R)\cdot p_{x|R}(x|R), \\
p_R(R) &= \mathrm{etr}\left(\tilde{F}R^T\right)\exp\left\{-\tfrac{1}{2}(H\mu_c-y)^T(\Sigma_y+H\Sigma_c H^T)^{-1}(H\mu_c-y)\right\}, \\
p_{x|R}(x|R) &= \exp\left\{-\tfrac{1}{2}\left(x-K_p\mu_c-K_m y\right)^T\left((I-K_m H)\Sigma_c\right)^{-1}\left(x-K_p\mu_c-K_m y\right)\right\}.
\end{aligned}
$$

  - Marginal likelihood for $p_R(R)$, conditional likelihood for $p_{x|R}(x|R)$.

Progressive Unscented Measurement Update

- Marginal Likelihood for $p_R(R)$.

$$p_R(R) = \mathrm{etr}\left(\tilde{F}R^T\right) \exp\left\{-\tfrac{1}{2}(H\mu_c - y)^T(\Sigma_y + H\Sigma_c H^T)^{-1}(H\mu_c - y)\right\}.$$

  - Unscented update
    - Select sigma points $\{R_i, w_i\}_{i=1}^7$ from $\mathcal{M}(\tilde{F})$.
    - Reweigh the weights by $w_i^+ = w_i f_m(R_i)$, where $f_m(R)$ is the second term on the right hand side.
    - Calculate $\mathrm{E}[R]^+$: $w_i^+ = w_i^+ / \sum_{j=1}^7 w_j^+$, $\mathrm{E}[R]^+ = \sum_{i=1} w_i^+ R_i$.
    - Use the marginal MLE: $U^+ D^+ (V^+)^T = \mathrm{E}[R]^+$, $d_i^+ = \frac{1}{c(S^+)} \frac{\partial c(S^+)}{\partial s_i^+}$.
    - Sample degeneration: w_i^+ can be close to zero.
  - Progressive unscented update [3]

$$f_m(R) = f_m(R)^{\lambda_1} f_m(R)^{\lambda_2} \cdots f_m(R)^{\lambda_l}, \qquad \lambda_1 + \cdots + \lambda_l = 1$$

    - Update the weights progressively: $w_i^k = w_i f_m(R_i)^{\lambda_k}$.
    - Avoid sample degeneration: $\frac{\min_i\{w_i^k\}}{\max_i\{w_i^k\}} < \tau$, where $0 < \tau < 1$.

[3] U. D. Hanebeck, "PGF 42: Progressive Gaussian filtering with a twist," in *International Conference on Information Fusion*, 2013, pp. 1103–1110.

Conditional MLE

Motivation

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Summary

$$p(R,x|\mathcal{Z}) \propto \operatorname{etr}\left(F^{+}R^{T}\right)\cdot p_{x|R}(x|R),$$

$$p_{x|R}(x|R) = \exp\left\{-\frac{1}{2}\left(x-K_p\mu_c-K_my\right)^T\left((I-K_mH)\Sigma_c\right)^{-1}\left(x-K_p\mu_c-K_my\right)\right\}.$$

- Conditional MLE for $p_{x|R}(x|R)$.
  - Calculate the moments $\mathrm{E}[x]$, $\mathrm{E}\left[xx^T\right]$, $\mathrm{E}\left[x\nu_R^T\right]$, $\mathrm{E}[\nu_R]$, $\mathrm{E}\left[\nu_R\nu_R^T\right]$, with respect to the above density function.
  - $\Rightarrow \mu^+, \Sigma^+, P^+$.

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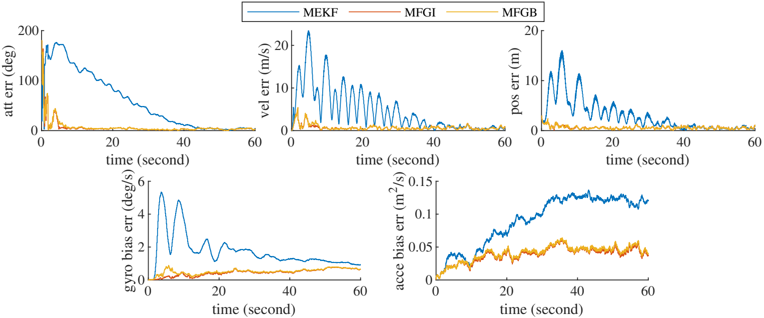
Summary

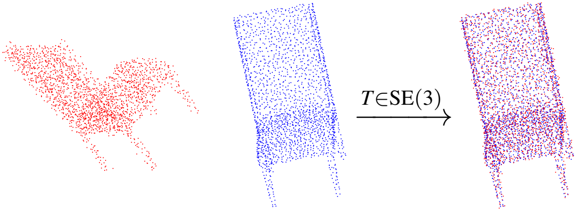
- Initial conditions: highly accurate pitch and roll, reversed yaw.
  - For example from indoors to outdoors.
- Estimation Errors
  - MFG has Faster convergence of yaw.
  - MFG has Better accuracy of position and velocity.



Visual-Inertial Navigation

- 3D Feature Location Measurements
  - Camera types: RGB-D camera, stereo camera.
  - Measures the coordinates of landmark or feature locations in the body-fixed frame.

- Two Point Sets Alignment



$\{p_i \in \mathbb{R}^3\}$ $\{p_i' \in \mathbb{R}^3\}$ Align $\{p_i\}$ and $\{p_i'\}$ using $T \in \mathrm{SE}(3)$

Visual-Inertial Navigation

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Visual-Inertial Navigation

Summary

- 3D Feature Location Measurements
  - Camera types: RGB-D camera, stereo camera.
  - Measures the coordinates of landmark or feature locations in the body-fixed frame.
- Two Point Sets Alignment
  - map point set: landmark/feature coordinates $\{p_i\}_{i=1}^N$ in inertial frame.
    - Without noise: $\{p_i\}_{i=1}^N$ are deterministic.
    - With noise: $p_i \sim \mathcal{N}(\bar{p}_i, B_i)$.
  - measurement point set: coordinates in body-fixed frame.

$$p_i' = R^T(p_i - t) + \mathcal{N}(0, A_i).$$

  - The correspondences between two point sets are known.
  - Objective: find the distribution of $(R, t) \in \mathrm{SE}(3)$, and match it to an MFG.

Deterministic Map Points

Motivation

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Summary

- Likelihood for (R, t)

$$p(p_1', \dots, p_N' | R, t) = \exp\left\{ -\frac{1}{2} \sum_{i=1}^{N} \left(R^T(p_i - t) - p_i' \right)^T A_i^{-1} \left(R^T(p_i - t) - p_i' \right) \right\}.$$

- Factorization of Likelihood

$$\begin{aligned} p(p_1', \dots, p_N' | R, t) &\propto p_R(R) p_{t|R}(t|R), \\ p_R(R) &= \exp\left\{ -\tfrac{1}{2} (\mathbf{T}\boldsymbol{p}' - \mathbf{T}\boldsymbol{p}_m)^T \left(\mathbf{T}\mathbf{A}\mathbf{T}^T \right)^{-1} (\mathbf{T}\boldsymbol{p}' - \mathbf{T}\boldsymbol{p}_m) \right\}, \\ p_{t|R}(t|R) &= \exp\left\{ -\tfrac{1}{2} (t - \mu_{t|R})^T \Sigma_{t|R}^{-1} (t - \mu_{t|R}) \right\}. \end{aligned}$$

  - Motivated by the conventional SVD method.
  - $\mathbf{T}\boldsymbol{p}_m$ is independent of t.
  - If $A_i = \sigma_i^2 I_{3\times 3}$, $\Sigma_{t|R}$ is independent of R, $\mu_{t|R}$ is linear in R.
  - Use marginal MLE to $p_R(R)$.
  - Use conditional MLE to $p_{t|R}(t|R)$.

Marginal MLE

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Summary

- Marginal MLE for $p_R(R)$

$$p_R(R) = \exp\left\{-\tfrac{1}{2}(\mathbf{T}\boldsymbol{p}' - \mathbf{T}\boldsymbol{p}_m)^T\left(\mathbf{T}\mathbf{A}\mathbf{T}^T\right)^{-1}(\mathbf{T}\boldsymbol{p}' - \mathbf{T}\boldsymbol{p}_m)\right\}$$

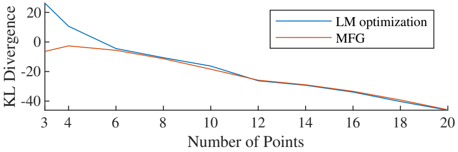
- $\mathbf{T}$ is used to center $\{p_i\}$ and $\{p_i'\}$, $i = 1, \dots, N-1$.

$$q_i = p_i - \frac{1}{N}\sum_{i=1}^{N} p_i, \qquad q_i' = p_i' - \frac{1}{N}\sum_{i=1}^{N} p_i' = R^T q_i + \text{noise},$$

- Wahba's problem:
  - $\{q_i\}_{i=1}^{N-1}$ are reference vectors in the inertial frame.
  - $\{q_i'\}_{i=1}^{N-1}$ are measurements in the body-fixed frame.
  - However, $\{q_i'\}$ are correlated.
- Importance sampling
  - Find a matrix Fisher distribution $\mathcal{M}(\tilde{F})$ through Wahba's problem.
  - Select sigma points $\{R_i, w_i\}_{i=1}^{7}$ from $\mathcal{M}(\tilde{F})$.
  - Reweigh the weights as $w_i^+ = w_i p_R(R_i)/p_{\text{prop}}(R_i)$.
  - Use MLE to the reweighed sigma points $\{R_i, w_i^+\}_{i=1}^{7}$.
  - Progressive update can also be applied to improve accuracy.

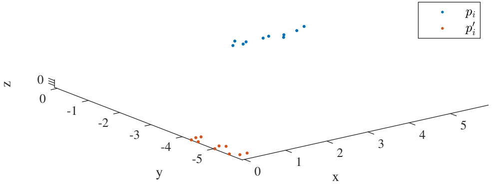
Simulation: KL-Divergence

- Comparison with LM-Optimization
  - Find the best $(R, t) \in \mathrm{SE}(3)$ that maximizes the likelihood $p(p'_i, \ldots, p'_N | R, t)$.
  - Gradient based Gauss-Newton method for nonlinear least square.
  - The Jacobian $(J_r^T J_r)^{-1} \in \mathbb{R}^{6\times 6}$ represents the covariance matrix.
- KL-Divergence
  - A metric for the difference between two probability distributions.
  - Simulation results with varying number of map points.



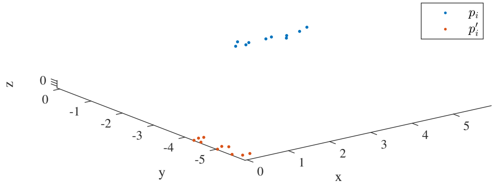
  - The MFG is a much better approximation of the true likelihood when the number of map points is small, i.e., the uncertainty is large.

Simulation: Pose Accuracy



- Two Point Sets Alignment
  - Ten map points scattered around x-axis.
  - The uncertainty in x-axis in inertial frame is large.
  - The uncertainties in two other axes are small.
  - Concentration of map points around x-axis, $a \in \{1, 0.1, 0.01, 0\}$.

Simulation: Pose Accuracy



- Estimation Errors

estimator	MFG			
a	1	0.1	0.01	0
att err (deg)	3.93±1.79	24.0±21.7	82.4±51.4	90.1±0.2
pos err	0.28±0.15	0.36±0.20	0.14±0.06	0.11±0.08

estimator	LM optimization			
a	1	0.1	0.01	0
att err (deg)	3.93±1.79	24.0±21.8	80.7±51.2	90.1±0.1
pos err	0.28±0.15	0.39±0.22	0.39±0.22	0.38±0.20

- MFG has better accuracy in translation when the uncertainty about x-axis is large.

Simulation: Visual-Inertial Navigation

Motivation

Matrix Fisher Distribution

Matrix Fisher–Gaussian Distribution

Attitude Estimation With MFG

Attitude Observability with Single Direction Measurements

Attitude Estimation Based on MFG

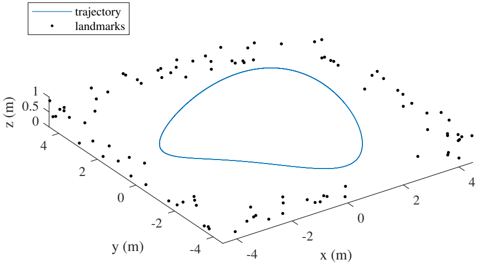
6D Pose Estimation With MFG

Loosely Coupled IMU-GNSS Integration

Visual-Inertial Navigation

Summary

- Bayesian Assumed Density Filter
  - Uncertainty propagation: IMU kinematics.
  - Measurement update: landmark coordinate measurements combined with pose prior.
- Settings
  - A circular trajectory with landmarks scattered around.



  - Initial conditions: reversed yaw direction.

Simulation: Visual-Inertial Navigation

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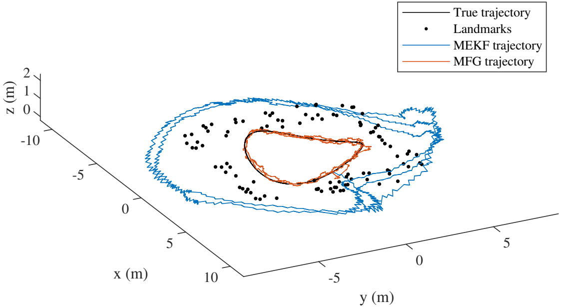
- Bayesian Assumed Density Filter
  - Uncertainty propagation: IMU kinematics.
  - Measurement update: landmark coordinate measurements combined with pose prior.
- Estimation Errors



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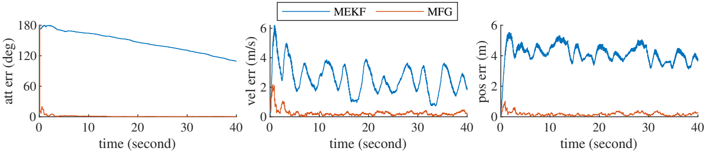
Summary

Simulation: Visual-Inertial Navigation

- Bayesian Assumed Density Filter
  - Uncertainty propagation: IMU kinematics.
  - Measurement update: landmark coordinate measurements combined with pose prior.
- Estimation Errors
  - MFG has Faster convergence of yaw.
  - MFG has Better accuracy of position and velocity.



Noisy Map Points

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Visual-Inertial Navigation
Summary

- Posterior Density for Pose (R, t) and Map Points $\{p_i\}$

$$p\left(R, t, \{p_i\} | \{p_i'\}\right) \propto \exp\left\{-\frac{1}{2}\sum_{i=1}^{N}\left(R^T(p_i - t) - p_i'\right)^T A_i^{-1}\left(R^T(p_i - t) - p_i'\right)\right\} \cdot \exp\left\{-\frac{1}{2}\sum_{i=1}^{N}(p_i - \bar{p}_i)^T B_i^{-1}(p_i - \bar{p}_i)\right\}.$$

- Factorization of Posterior Density

$$p\left(R, t, \{p_i\} | \{p_i'\}\right) \propto p_R(R|\{p_i'\}) \cdot p_{x|R}(x|R),$$

$$p_R(R) = \exp\left\{-\tfrac{1}{2}(\mathbf{T}\boldsymbol{p}' - \mathbf{T}\boldsymbol{p}_m)^T\left(\mathbf{TAT}^T\right)^{-1}(\mathbf{T}\boldsymbol{p}' - \mathbf{T}\boldsymbol{p}_m)\right\},$$

$$p_{x|R}(x|R) = \exp\left\{-\tfrac{1}{2}(x - \mu_{x|R})^T \Sigma_{x|R}^{-1}(x - \mu_{x|R})\right\}.$$

  - $x = [t, p_1, \ldots, p_N]$ is the translation and all map points.
  - Use marginal MLE to $p_R(R)$.
  - Use conditional MLE to $p_{x|R}(x|R)$ with $\{p_i\}$ marginalized.
  - Calculate $\mathrm{E}[p_i]$, $\mathrm{E}\left[p_i p_i^T\right]$, and match them to Gaussian distributions.

Simulation: Pose and Map Accuracy

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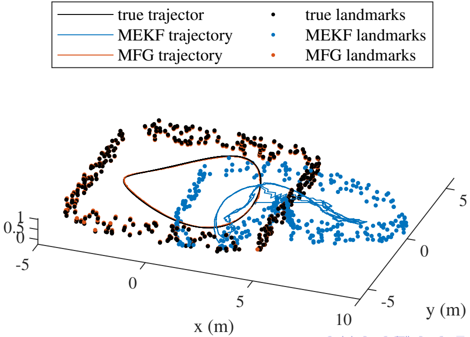
Summary

- Two Point Sets Alignment
  - Ten map points:
    - N_1 of them have small uncertainty.
    - The rest of $10 - N_1$ have large uncertainty.
- Estimation Accuracy

estimator	MFG		
N_1	4	2	0
att err (deg)	16.5±7.9	50.6±30.9	77.1±41.1
pos err	0.305±0.147	0.755±0.372	1.05±0.38
map err (N_1)	0.138±0.030	0.149±0.046	-
map err ($10 - N_1$)	0.225±0.056	0.501±0.189	0.661±0.173
estimator	**LM optimization**		
N_1	4	2	0
att err (deg)	16.0±7.5	48.6±29.5	76.4±40.8
pos err	0.294±0.149	0.772±0.455	1.17±0.55
map err (N_1)	0.138±0.030	0.153±0.048	-
map err ($10 - N_1$)	0.230±0.056	0.518±0.216	0.708±0.196

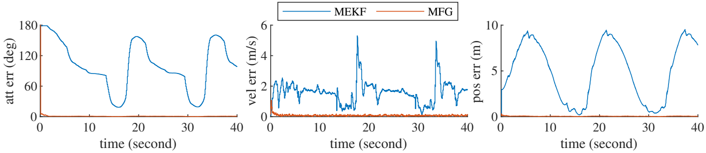
Simulation: Visual-Inertial Navigation

- Bayesian Assumed Density Filter
  - Uncertainty propagation: IMU kinematics, static map points.
  - Measurement update: landmark coordinate measurements combined with pose and map priors.
- Initial conditions: reversed yaw direction.
- Estimation Errors



Simulation: Visual-Inertial Navigation

- Bayesian Assumed Density Filter
  - Uncertainty propagation: IMU kinematics, static map points.
  - Measurement update: landmark coordinate measurements combined with pose and map priors.
- Initial conditions: reversed yaw direction.
- Estimation Errors
  - MFG converges very quickly from wrong initial attitude
  - MEKF does not converge.



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Summary

Visual-Inertial Odometry

- No Prior Information for the Map
  - The features captured by cameras are usually geometric corners.
  - The map must be built incrementally from previously captured features.

- Build Map From Feature Coordinate Measurements

$$p_{p_i}(p_i) = \int_{R\in \mathrm{SO}(3)} \int_{t\in\mathbb{R}^3} p_{R,t}(R,t) p_{p_i'}\left(R^T(p_i - t)\right) \mathrm{d}t\mathrm{d}R.$$

  - $p_{R,t}(R,t)$ is the probability density for pose.
  - $p_{p_i'}(p_i')$ is the probability density for measurements.
  - Calculate $\mathrm{E}\left[p_i\right]$, $\mathrm{E}\left[p_i p_i^T\right]$, and match them to Gaussian distributions.
  - When R has large uncertainty, $p_{p_i}(p_i)$ is far from a Gaussian distribution, so this method is not suitable for large uncertainty.

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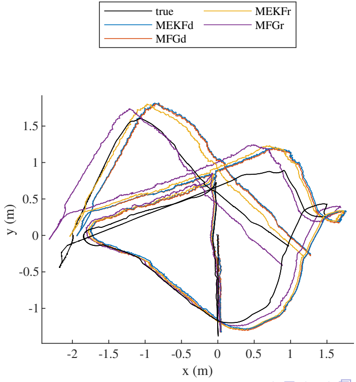
Summary

- VCU-RVI Dataset: Visual-Inertial Odometry
  - Inertial and RGB-D camera measurements.
  - Ground truth provided by a motion capture system.
- Preliminary Results (Proof of Concept)



Summary of Contributions

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Summary

- Objective: Deal With Large Attitude Uncertainty
  - Unknown initial conditions.
  - Not properly observed degree of freedom for attitude.
  - Sensor failure.
- Contributions
  - Use probability distribution intrinsically defined on $\mathrm{SO}(3)$.
  - Matrix Fisher distribution
    - Higher order moments and highly concentrated approximations.
  - Matrix Fisher–Gaussian distribution (MFG)
    - Correlation between attitude and Euclidean quantities.
  - MFG in classical estimation problems
    - Attitude estimation with gyroscope and direction sensors.
    - Loosely coupled IMU-GNSS navigation.
    - Visual-inertial navigation/odometry.
    - Improved accuracy and convergence speed when the attitude uncertainty is large.

List of Publications

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- W. Wang and T. Lee, "Matrix Fisher-Gaussian distribution on $\mathrm{SO}(3) \times \mathbb{R}^n$ for attitude estimation with a gyrobias," in *American Control Conference*, 2020, pp. 4429–4434.
- W. Wang and T. Lee, "Spectral uncertainty propagation for generalized stochastic hybrid systems with applications to a bouncing ball," in *American Control Conference*, 2020, pp. 1803–1808.
- W. Wang and T. Lee "Spectral Bayesian estimation for general stochastic hybrid systems," *Automatica*, vol. 117, p. 108989, 2020.
- W. Wang and T. Lee, "Higher-order central moments of matrix Fisher distribution on $\mathrm{SO}(3)$," *Statistics & Probability Letters*, vol. 169, p. 108983, 2021.
- W. Wang and T. Lee, "Matrix Fisher–Gaussian distribution on $\mathrm{SO}(3) \times \mathbb{R}^n$ and Bayesian attitude estimation," *IEEE Transactions on Automatic Control*, vol. 67, no. 5, pp. 2175–2191, 2021.
- W. Wang and T. Lee, "Bingham-Gaussian distribution on $\mathbb{S}^3 \times \mathbb{R}^n$ for unscented attitude estimation," in *International Conference on Multisensor Fusion and Integration for Intelligent Systems*, 2021, pp. 1–7.
- W. Wang and T. Lee, "Spacecraft attitude and gyro-bias estimation with a single magnetometer on $\mathrm{SO}(3) \times \mathbb{R}^n$," in *AIAA Scitech Forum*, 2022.
- W. Wang, K. Gamagedara, and T. Lee, "On the observability of attitude with single direction measurements," *IEEE Transactions on Automatic Control*, 2022, accepted.
- W. Wang and T. Lee, "Uncertainty propagation for general stochastic hybrid systems on compact Lie groups," *SIAM Journal on Applied Dynamical Systems*, 2022, accepted.