CSE250A Homework1 Answer

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1.1 Conditioning on background evidence

(a)

$$
\begin{aligned}
\text{Right} &= P(X|Y,E)P(Y|E) \\
&= \frac{P(X|Y,E)P(Y|E)P(E)}{P(E)} \\
&= \frac{P(X|Y,E)P(Y,E)}{P(E)} && \text{(Product Rule)} \\
&= \frac{P(X,Y,E)}{P(E)} && \text{(Product Rule)} \\
&= P(X,Y|E) && \text{(Product Rule)} \\
&= \text{Left}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\text{Right} &= \frac{P(Y|X,E)P(X|E)}{P(Y|E)} \\
&= \frac{P(Y|X,E)P(X|E)P(E)}{P(Y|E)P(E)} \\
&= \frac{P(Y|X,E)P(X,E)}{P(Y,E)} && \text{(Product Rule)} \\
&= \frac{P(X,Y,E)}{P(Y,E)} && \text{(Product Rule)} \\
&= P(X|Y,E) \\
&= \text{Left}
\end{aligned}
$$

1.2 Conditional independence

$$P(X, Y|E) = P(X|E)P(Y|E) \tag{1}$$
$$P(X|Y, E) = P(X|E) \tag{2}$$
$$P(Y|X, E) = P(Y|E) \tag{3}$$

The thing we need to do is to show (1)→(2)(3), (2)→(1)(3), (3)→(1)(2)

With (1):

$$\begin{aligned} P(X, Y|E) &= P(X|E)P(Y|E) \\ P(X, Y|E)P(E) &= P(X|E)P(Y|E)P(E) \\ P(X, Y, E) &= P(X|E)P(Y, E) \\ P(X|Y, E) &= P(X|E) \end{aligned}$$

(1)→(2)

$$\begin{aligned} P(X, Y|E) &= P(X|E)P(Y|E) \\ P(X, Y|E)P(E) &= P(X|E)P(E)P(Y|E) \\ P(X, Y, E) &= P(X, E)P(Y|E) \\ P(Y|X, E) &= P(Y|E) \end{aligned}$$

(1)→(3)

With (2):

$$\begin{aligned} P(X|Y, E) &= P(X|E) \\ P(X|Y, E) * P(Y|E) &= P(X|E)P(Y|E) \\ P(X, Y|E) &= P(X|E)P(Y|E) \end{aligned}$$

(2)→(1)

$$
\begin{aligned}
P(X|Y,E) &= P(X|E) \\
\frac{P(Y|X,E)P(X|E)}{P(Y|E)} &= P(X|E) \\
P(Y|X,E) &= P(Y|E)
\end{aligned}
$$

(2)→(3)

With (3):

$$
\begin{aligned}
P(Y|X,E) &= P(Y|E) \\
P(Y|X,E)P(X|E) &= P(Y|E)P(X|E) \\
P(X,Y|E) &= P(X|E)P(Y|E)
\end{aligned}
$$

(3)→(1)

$$
\begin{aligned}
P(Y|X,E) &= P(Y|E) \\
\frac{P(X|Y,E)P(Y|E)}{P(X|E)} &= P(Y|E) \\
P(X|Y,E) &= P(X|E)
\end{aligned}
$$

(3)→(2)

One equation can be derived from others, so the three equations are equivalent.

1.3 Creative writing

(a)

Y = Bell completed the first assignment
Z = Bell completed the second assignment
X = Bell passed the course

$P(X = 1) < P(X = 1|Y = 1) < P(X = 1|Y = 1, Z = 1)$

Using our common sense, we have more confidence in a student passing a course when we know he completed two assignments than that when we know he completed one assignment, which is more than that when we don't know whether he did assignment or not.

(b)

X = Bell completed the first assignment
Z = Bell completed the second assignment
Y = Bell passed the course

$$P(X = 1|Y = 1) > P(X = 1)$$

$$P(X = 1|Y = 1, Z = 1) < P(X = 1|Y = 1)$$

Because we have more confidence in that Bell completed the first assignment when we know he passed the course. And Bell completing the second assignment "explains away" the Bell passing the course, thus decreasing our belief in Bell completing the first assignment.

(c)

X = Mary calls
Y = Bell calls
Z = Alarm goes off

$$P(X = 1, Y = 1) \neq P(X = 1)P(Y = 1)$$

$$P(X = 1, Y = 1|Z = 1) = P(X = 1|Z = 1)P(Y = 1|Z = 1)$$

These events are from the lecture. Because Mary calling and Bell calling both depend on whether alarm going off, which gives us the first inequation. But when we know the alarm goes off, Mary calling and Bell calling are independent.

1.4 Bayes Rule

From the question, we know two events, which are represented in the below way:
X = Cyclists using performance-enhancing drugs
Y = The drug test is positive
$P(X = 1) = 0.01$ $\quad$ $P(Y = 1|X = 0) = 0.05$ $\quad$ $P(Y = 0|X = 1) = 0.1$
From the above, we can derive:
$P(X = 0) = 0.99$ $\quad$ $P(Y = 0|X = 0) = 0.95$ $\quad$ $P(Y = 1|X = 1) = 0.9$
$P(Y = 0) = P(Y = 0, X = 0) + P(Y = 0, X = 1) = P(Y = 0|X = 0) * P(X = 0) + P(Y = 0|X = 1) * P(X = 1) = 0.9415$
$P(Y = 1) = 0.0585$

(a)

This question is to compute $P(X=0|Y=0)$
$P(X=0|Y=0) = \frac{P(Y=0|X=0)P(X=0)}{P(Y=0)} = \frac{0.95*0.99}{0.9415} \approx 0.9989$

(b)

This question is to compute $P(X=1|Y=1)$
$P(X=1|Y=1) = \frac{P(Y=1|X=1)P(X=1)}{P(Y=1)} = \frac{0.9*0.01}{0.0585} \approx 0.1538$

1.5 Entropy

(a)

Let $f(x) = H(X) + \lambda(\sum_i p_i - 1)$
If we want to get maximum of H(X) under the constraint that $\sum_i p_i = 1$,
we just need to get the maximum of f(x)
$\frac{\partial f}{\partial p_i} = -1 - \log p_i + \lambda$, for all i
$\frac{\partial f}{\partial \lambda} = \sum_i p_i - 1$
Let $\frac{\partial f}{\partial p_i} = 0, \frac{\partial f}{\partial \lambda} = 0$
We got $p_i = e^{\lambda - 1}$ for all i, $and \sum_i p_i = 1$
Finally we got $p_i = \frac{1}{n}$ and $\lambda = \log\frac{1}{n} + 1$, for all i
And after testing, we find$(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$ is the maximum point.

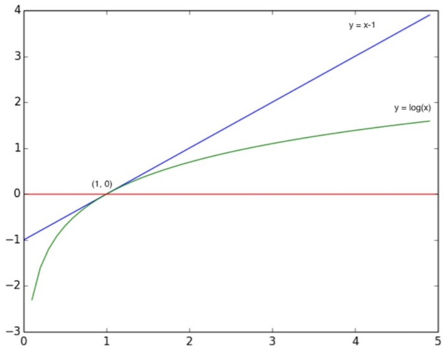
(b)

If we know $X_1, X_2, ..., X_n$ are independent, then
$H(X_1, X_2, ..., X_n) = -\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} P(x_1)P(x_2)\cdots P(x_n)\log P(x_1)P(x_2)\cdots P(x_n)$
$H(X_1, X_2, ..., X_n) = -\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} P(x_1)P(x_2)\cdots P(x_n)(\log P(x_1)+\log P(x_2)+\cdots+\log P(x_n))$
$H(X_1, X_2, ..., X_n) = -\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} P(x_1)P(x_2)\cdots P(x_n)\log P(x_1)$
$-\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} P(x_1)P(x_2)\cdots P(x_n)\log P(x_2)$
$-\cdots-\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} P(x_1)P(x_2)\cdots P(x_n)\log P(x_n)$
Because the sum calculation is commutative in this case, then
$H(X_1, X_2, ..., X_n) = -\sum_{x_2}\cdots\sum_{x_n} P(x_2)\cdots P(x_n)\sum_{x_1} P(x_1)\log P(x_1)$
$-\sum_{x_1}\cdots\sum_{x_n} P(x_1)\cdots P(x_n)\sum_{x_2} P(x_2)\log P(x_2)$
$-\cdots-\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n-1} P(x_1)P(x_2)\cdots P(x_n-1)\sum_{x_n} P(x_n)\log P(x_n)$
$H(X_1, X_2, ..., X_n) = \sum_{x_2}\cdots\sum_{x_n} P(x_2)\cdots P(x_n)H(X_1)$
$+\sum_{x_1}\cdots\sum_{x_n} P(x_1)\cdots P(x_n)H(X_2)$
$+\cdots+\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n-1} P(x_1)P(x_2)\cdots P(x_n-1)H(X_n)$
$H(X_1, X_2, ..., X_n) = \sum_{x_2}\cdots\sum_{x_n-1} P(x_2)\cdots P(x_n-1)\sum_{x_n} P(x_n)H(X_1)$
$+\sum_{x_1}\sum_{x_3}\cdots\sum_{x_n-1} P(x_1)P(x_3)\cdots P(x_n-1)\sum_{x_n} P(x_n)H(X_2)$

$+\cdots+\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n-2}P(x_1)P(x_2)\cdots P(x_n-2)\sum_{x_n-1}P(x_n-1)H(X_n)$
$H(X_1,X_2,...,X_n)=\sum_{x_2}\cdots\sum_{x_n-1}P(x_2)\cdots P(x_n-1)H(X_1)$
$+\sum_{x_1}\sum_{x_3}\cdots\sum_{x_n-1}P(x_1)P(x_3)\cdots P(x_n-1)H(X_2)$
$+\cdots+\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n-2}P(x_1)P(x_2)\cdots P(x_n-2)H(X_n)$
$\cdots$
$\cdots$
$H(X_1,X_2,...,X_n)=H(X_1)+H(X_2)+\cdots+H(X_n)=\sum_{i=1}^{n}H(X_i)$

1.6 Kullback-Leibler distance

(a)



Let $f(x)=\log(x)-(x-1) \qquad (x>0)$
$\frac{df(x)}{dx}=\frac{1}{x}-1 \qquad (x>0)$
When $0<x<1, \frac{df(x)}{dx}>0$.When $x=1, \frac{df(x)}{dx}=0$.When $x>1, \frac{df(x)}{dx}=<0$.
So when $0<x<1, f(x)$ increases, when $x>1, f(x)$ decreases and when $x=1, f(x)$ is the maximum value. $f(0)=0$
So $\log(x)<=x-1$ with equality if and only if $x=1$.

(b)

We know $\log(x)<=x-1$ with equality if and only if $x=1$
Let $x=\frac{q_i}{p_i}$, for all i, we have:
$\log(\frac{q_i}{p_i})<=\frac{q_i}{p_i}-1$, for all i
$p_i\log(\frac{q_i}{p_i})<=q_i-p_i$, for all i $\qquad$ (We know $p_i>=0$)
$p_i-q_i<=p_i\log(\frac{p_i}{q_i})$, for all i
$\sum(p_i-q_i)<=\sum_i p_i\log(\frac{p_i}{q_i})$
$\sum p_i-\sum q_i<=\sum_i p_i\log(\frac{p_i}{q_i})$
$1-1<=\sum_i p_i\log(\frac{p_i}{q_i})$
$KL(p,q)>=0$, and when the two distributions are equal, $\log(1)=0$,

so every term in the KL is 0 when two distributions are equal
$KL(p,q) >= 0$, with equality if and only if the two distributions are equal

(c)

We know $\log(x) <= x - 1$ with equality if and only if $x = 1$
Let $x = \frac{\sqrt{q_i}}{\sqrt{p_i}}$, for all i, we have:
$\log(\frac{\sqrt{q_i}}{\sqrt{p_i}}) <= \frac{\sqrt{q_i}}{\sqrt{p_i}} - 1$, for all i
$p_i \log(\frac{\sqrt{q_i}}{\sqrt{p_i}}) <= p_i(\frac{\sqrt{q_i}}{\sqrt{p_i}} - 1)$, for all i
$2p_i \log(\frac{\sqrt{q_i}}{\sqrt{p_i}}) <= 2p_i(\frac{\sqrt{q_i}}{\sqrt{p_i}} - 1)$, for all i
$p_i \log(\frac{q_i}{p_i}) <= 2\sqrt{p_i q_i} - 2p_i$, for all i
$\sum_i p_i \log(\frac{p_i}{q_i}) >= \sum_i (2p_i - 2\sqrt{p_i q_i})$
$\sum_i p_i \log(\frac{p_i}{q_i}) >= \sum_i (p_i - 2\sqrt{p_i q_i} + p_i)$ Because $\sum_i p_i = \sum_{q_i} = 1$
$\sum_i p_i \log(\frac{p_i}{q_i}) >= \sum_i (\sqrt{p_i} - \sqrt{p_i})^2$
$KL(p,q) >= \sum_i (\sqrt{p_i} - \sqrt{p_i})^2$

(d)

Let p = Tomorrow will rain.
Let q = Earthquake occurs.
$P(p = 1) = 0.3$
$P(p = 0) = 0.7$
$P(q = 1) = 0.1$
$P(q = 0) = 0.9$
$KL(p,q) \approx 0.1537$
$KL(q,p) \approx 0.1163$
$KL(p,q) \neq KL(q,p)$

1.7 Hangman

(a)

The eight most frequent 5-leter words (with their probabilities) are:
('THREE', 0.03562714868653127), ('SEVEN', 0.023332724928853858),
('EIGHT', 0.021626496097709325), ('WOULD', 0.02085818430793947),
('ABOUT', 0.020541544349751077), ('THEIR', 0.018974130893766185),
('WHICH', 0.018545160072784138), ('AFTER', 0.01436452108630337)
The fourteen(There are 9 words whose frequence are 7 with the same probability) least frequent 5-leter words (with their probabilities) are:

('TROUP', 7.827934689453437e-07), ('MAPCO', 7.827934689453437e-07), ('CAIXA', 7.827934689453437e-07), ('BOSAK', 7.827934689453437e-07), ('OTTIS', 7.827934689453437e-07), ('NIAID', 9.13259047102901e-07), ('SERNA', 9.13259047102901e-07), ('CLEFT', 9.13259047102901e-07), ('CCAIR', 9.13259047102901e-07), ('FABRI', 9.13259047102901e-07), ('FOAMY', 9.13259047102901e-07), ('PAXON', 9.13259047102901e-07), ('TOCOR', 9.13259047102901e-07), ('YALOM', 9.13259047102901e-07)

(b)

correctly guessed	incorrectly guessed	best next guess l	$P(L_i = l$ for some i $\in \{1,2,3,4,5\}\|E)$
- - - - -	{}	E	0.5394
- - - - -	{E, O}	I	0.6366
D - -I -	{}	A	0.8207
D - -I -	{A}	E	0.7521
-U - - -	{A, I, E, O, S}	Y	0.6270

(c)

```
import string
import sys
#Usage instruction:
#Python hangman.py [word length] [correctly guessed(don't leave out the space)] [incorrectly guessed]

class Hangman():

	#This function is to load the corresponding corpora and load the correctly guessed word
	#and incorrectly guessed word.
	#And calculating word frequence.

	def __init__(self, fileNum):
		wordDict = {}
		totalCount = 0
		if int(fileNum) < 10:
			fileNum = '0' + str(fileNum)
		filename = "hw1_word_counts_" + fileNum + ".txt"
		with open(filename, 'r') as file:
		    for line in file:
			line = line.upper().strip().split(' ')
			wordDict[line[0]] = float(line[1])
			totalCount += int(line[1])
		wordDict = dict(map(lambda x: (x[0], x[1]/totalCount), wordDict.items()))
		self.wordDict = wordDict
		self.alphabeta = string.uppercase

	#This function is to output the most frequent words and the least frequent words.

	def outputFreq(self):
		wordDict = self.wordDict
		mostFreq = sorted(wordDict.items(), key = lambda x: x[1])[-1:-9:-1]
		leastFreq = sorted(wordDict.items(), key = lambda x: x[1])[0:8]
		print mostFreq
		print leastFreq

	#This function is to calculate the probability of evidence E given the word W.

	def conditionalProb(self, word):
		corrGuessed = self.corrGuessed
		inCorrGuessed = self.inCorrGuessed
		alphabeta = self.alphabeta
		cond = {}
		for i in range(26):
			c = alphabeta[i]
```

```
                #If the c is the next right letter in the word.
                if c not in corrGuessed and c not in inCorrGuessed and c in word:
                        cond[c] = 1
                else:
                        cond[c] = 0
        return cond

    #This function is to test whether the word and the evidence is compatible.

    def compatible(self, word):
        corrGuessed = self.corrGuessed
        inCorrGuessed = self.inCorrGuessed
        for i in range(len(corrGuessed)):
            if corrGuessed[i] != ' ':
                if corrGuessed[i] != word[i]:
                    return False
            else:
                if word[i] in corrGuessed or word[i] in inCorrGuessed:
                    return False
        return True

    #This function is to compute the posterior probability of a word given evidence E.

    def posteriorProb(self):
        corrGuessed = self.corrGuessed
        inCorrGuessed = self.inCorrGuessed
        wordDict = self.wordDict
        postProbDict = {}
        denominator = 0.0
        for key in wordDict:
            singProb = wordDict[key] * (1 if self.compatible(key) else 0)
            denominator += singProb
            postProbDict[key] = singProb

        postProbDict = dict(map(lambda x: (x[0], x[1]/denominator), postProbDict.items()))
        return postProbDict

    #Calculate and then sumarize all the predictive probability of a leter through all words.

    def pred4i(self, postProbDict, condProbDict, c):
        prob = 0.0
        for key in postProbDict.keys():
            prob += postProbDict[key]*condProbDict[key][c]
        return prob

    #This function is to give the predictive probabilities of 26 leters.

    def predictiveProb(self, corrGuessed, inCorrGuessed):
        self.corrGuessed = corrGuessed
        self.inCorrGuessed = inCorrGuessed
        wordDict = self.wordDict
        alphabeta = self.alphabeta
        prob = {}
        #Calculate the posterior probability
        postProbDict = self.posteriorProb()
        #Calculate the conditional probability
        condProbDict = dict(map(lambda x: (x, self.conditionalProb(x)), wordDict.keys()))
        #Use product rule and marginiztion to get the final predictive probability.
        prob = dict(map(lambda x: (x, self.pred4i(postProbDict, condProbDict, x)), alphabeta))
        return prob


def main():

    wordLen = sys.argv[1]
    corrGuessed = sys.argv[2]
    inCorrGuessed = sys.argv[3]

    h = Hangman(wordLen)

    h.outputFreq()

    prob = h.predictiveProb(corrGuessed, inCorrGuessed)
    print sorted(prob.items(), key=lambda x: x[1])

if __name__ == '__main__':
    main()
```

Program output.

1

('Q', 0.0042710516321439245), ('Z', 0.004292056590227289), ('J', 0.01563055905674421), ('X', 0.020749506481334088), ('K', 0.05958258581993074), ('V', 0.06369355618767535), ('P', 0.0772493211549799), ('B', 0.08609971458045411), ('M', 0.09001668263347834), ('F', 0.104002201215234), ('Y', 0.10544123654231159) ('W', 0.10968606459324606), ('G', 0.11007759179329682), ('C', 0.1490947580626736), ('U', 0.156309765465942 ('D', 0.17146229864431772), ('L', 0.20103153914025876), ('N', 0.23201711395267857), ('H', 0.259732569048579 ('I', 0.3025530678524059), ('O', 0.3146699279582104), ('A', 0.34689936159278384), ('S', 0.3498943294049688), ('R', 0.3831019939445695), ('T', 0.43069322754488293), ('E', 0.539417238964795)

2

('E', 0.0), ('O', 0.0), ('Z', 0.004600009630695304), ('Q', 0.008808839896183281), ('J', 0.01151625558566803), ('V', 0.028985687596476063), ('X', 0.04363029600320741), ('B', 0.0780064677152583), ('K', 0.090374336469441 ('M', 0.10066457208061445), ('W', 0.10232343229323645), ('P', 0.10274977992779151), ('G', 0.11908030106202 ('D', 0.1265246121075156), ('U', 0.1443002785867984), ('Y', 0.18198713707022385), ('F', 0.18460744366178544), ('C', 0.20778549736319163), ('L', 0.2463845666484698), ('N', 0.2588422497520642), ('H', 0.27253569173551095), ('R', 0.3098751959278396), ('S', 0.42672258241142885), ('A', 0.4425520909700327), ('T', 0.4857500877854112), ('I', 0.6365554141009607)

3

('C', 0.0), ('D', 0.0), ('G', 0.0), ('F', 0.0), ('I', 0.0), ('H', 0.0), ('Q', 0.0), ('W', 0.0), ('Y', 0.0), ('X', 0.0), ('Z', 0.0), ('U', 0.001860119047619047 7), ('K', 0.003162202380952381), ('J', 0.006882440476190476), ('P', 0.006882440476190476), ('T', 0.01525297619047619), ('B', 0.017113095238095236) ('M', 0.017857142857142856), ('O', 0.04538690476190476), ('L', 0.07384672619047619), ('E', 0.1408110119047619), ('R', 0.18098958333333331), ('N', 0.18601190476190474), ('S', 0.7395833333333334), ('V', 0.7436755952380953), ('A', 0.8206845238095238)

4

('A', 0.0), ('C', 0.0), ('D', 0.0), ('G', 0.0), ('F', 0.0), ('I', 0.0), ('H', 0.0), ('K', 0.0), ('Q', 0.0), ('W', 0.0), ('Y', 0.0), ('X', 0.0), ('Z', 0.0), ('U', 0.010373443983402 49), ('J', 0.03838174273858921), ('P', 0.03838174273858921), ('T', 0.0850622406639004), ('B', 0.0954356846473029), ('M', 0.0995850622406639), ('O', 0.1991701244813278), ('R', 0.237551867219917), ('N', 0.30186721991701243), ('S', 0.37033195020746884), ('L', 0.37551867219917007), ('V', 0.39626556016597503), ('E', 0.7520746887966804)

5

('A', 0.0), ('E', 0.0), ('I', 0.0), ('O', 0.0), ('Q', 0.0), ('S', 0.0), ('U', 0.0), ('W', 0.0), ('X', 0.0), ('J', 0.001577932324235871), ('V', 0.0028052130208637708), ('Z', 0.018759862077026467), ('G', 0.020104026649523692), ('P', 0.05908479925194317), ('M', 0.06574718017649463), ('R', 0.0993512944889252), ('K', 0.10291625270293957), ('B', 0.15644906785108986), ('T', 0.19320904681199222), ('D', 0.274326456665303), ('F', 0.3075214774121909), ('N', 0.3091578516743614), ('H', 0.3877622581964817), ('L', 0.4471392671380981), ('C', 0.4625094968149143), ('Y', 0.6269651101630527)