Ex. 1.

Exercise

For $0 < p < 1$ and $n = 2, 3, \ldots$, determine the value of

$$\sum_{x=2}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} (x^2 - x) \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} - \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

We note that for a random variable $X \sim Bin(n, p)$,

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Therefore, it turns out to be $E[X^2] - E[X]$

$$E[X^2] = Var[X] + E[X]^2 = np(1-p) + (np)^2$$

$$E[X] = np.$$

So the result is $(n^2 - n)p^2$.

Ex 2.

Discussion

Prove that we can approximate the *Binomial distribution* with *Poisson distribution* when n is large (Have Fun!).

For $n \in \mathbb{N} \setminus \{0\}$, $0<p<1$

$f(x;n,p)$ denotes P.D.F of Binomial Dist. with parameters n, p.

$f(x;k)$ denotes P.D.F of Poisson Dist. with parameter k.

Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of numbers between 0 and 1 such that $\lim_{n\to\infty} np_n = k$. We need to prove $\lim_{n\to\infty} f(x;n,p_n) = f(x;k)$

For binomial, $f(x;n,p_n) = \frac{n(n-1)\cdots(n-x+1)}{x!} p_n^x (1-p_n)^{n-x}$.

Suppose $k_n = np_n$ so that $\lim_{n\to\infty} k_n = k$. The expression can be written as $f(x;n,p_n) = \frac{k_n^x}{x!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdot \cdots \cdot \frac{n-x+1}{n} \left(1-\frac{k_n}{n}\right)^n \left(1-\frac{k_n}{n}\right)^{-x}$

for each $x \geq 0$, $\lim_{n\to\infty} \frac{n}{n} \cdot \frac{n-1}{n} \cdot \cdots \cdot \frac{n-x+1}{n} \left(1-\frac{k_n}{n}\right)^{-x} \to 1 \cdot 1 \cdot \cdots 1 \cdot 1 = 1$

$$\lim_{n\to\infty} \left(1-\frac{k_n}{n}\right)^n = e^{-k}$$

Therefore, $\lim_{n\to\infty} f(x;n,p_n) = \frac{k^x e^{-k}}{x!} = f(x;k)$

For theorem $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$, it can be proved that

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} e^{n \ln(1+\frac{x}{n})} = \lim_{n\to\infty} e^{x\left(\frac{\ln(1+x/n)}{x/n}\right)} = \lim_{h\to 0} e^{x\left(\frac{\ln(1+h)}{h}\right)}$$

Since $\lim_{h\to 0} \frac{\ln(1+h)}{h} \to 1$, it finally goes to $\lim_{h\to 0} e^x$

Ex. 2.

Exercise

A factory produces drill bits. It is known that 2% of the drill bits are defective. The drill bits are shipped in packages of 100 . Use the Poisson approximation to the binomial distribution to answer the following questions.

1. What is the probability that there are no more than three defective drill bits in a package?
2. How many drill bits must a package contain so that with probability greater than 0.9 there are at least 100 non-defective drill bits in the package?

We use the Poisson approximation with $\lambda = np = 2$

i) $P[X \le 3 | \lambda = 2] = \frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!} + \frac{2^3 e^{-2}}{3!} = 0.857$

ii) $P[X \le 4 | \lambda = 2] = 0.947 > 0.9 > P[X \le 3 | \lambda = 2] = 0.857$

So we choose $4 + 100 = 104$ bills λ changed!

For $n' = 104$ bills, $\lambda = n'p = 2.08$. check $P[X \le 4 | \lambda = 2.08]$

For $n' = 103$ bills, $\lambda = n'p = 2.06$. check $P[X \le 3 | \lambda = 2.06]$

When λ increases, $P[X \le 4]$, $P[X \le 3]$ decreases.

We only need to make sure that $P[X \le 4 | \lambda = 2.08] > 0.9$.

Ex 3.

Discussion

Prove that the time between successive arrivals of a *Poisson-distributed* random variable is *exponentially distributed* with parameter $\beta = \lambda$.

For Poiss-dist. events, probability of x arrival in time $[0, t)$

is $p_x(t) = \frac{(\lambda t)^x}{x!} e^{-\lambda t}$, $x \in \mathbb{N}$.

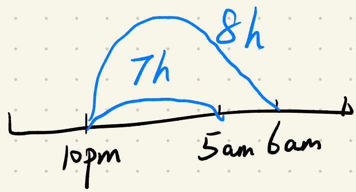
Then $p_0(t) = e^{-\lambda t}$ is the probability of no arrival in $[0, t]$

Also interpeted as: first arrival occurs at a time $T > t$.

$P[T > t] = e^{-\lambda t}$, $t \geq 0$ $\quad F_T(t) = P[T \leq t] = 1 - e^{-\lambda t}$, $t \geq 0$

$$f_T(t) = F'_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \qquad \beta = \lambda$$

A certain widget has a mean time between failures of 24 hours, i.e., failures occur at a constant rate of one failure every 24 hours.
One evening, the widget was observed to be working at 10pm and then left unobserved for the night. The next morning at 6am, it was observed to have failed earlier. What is the probability that it was still working at 5 am that morning?



Failures of Widget $\sim$ Poiss$(\lambda = \frac{1}{24})$

Time to failure $\sim$ Exp$(\beta = \lambda = \frac{1}{24})$

density $f_\beta(t) = \beta e^{-\beta t}$. $\quad P[\text{fail between } T_1 \text{ and } T_2] = \int_{T_1}^{T_2} \beta e^{-\beta t} dt$

$P[\text{fail between } T_1 \text{ and } T_2 \mid \text{faild not after } T_2]$

$$= \frac{\int_{T_1}^{T_2} \beta e^{-\beta t} dt}{\int_0^{T_2} \beta e^{-\beta t} dt} \qquad \begin{matrix} T_1 = 7 \\ T_2 = 8 \\ \beta = \frac{1}{24} \end{matrix} \qquad \text{result is } \frac{e^{-\frac{7}{24}} - e^{-\frac{1}{3}}}{1 - e^{-\frac{1}{3}}} = 0.108$$

Ex. 4.

We shall consider the two problems just described in the case where:

- The demand is random in the sense that the number of demands r during a fixed interval of time t has a Poisson distribution
$$\frac{e^{-\lambda t}(\lambda t)^r}{\Gamma(r+1)}$$
- The lead time t has a gamma type probability density function
$$\frac{\mu e^{-\mu t}(\mu t)^{k-1}}{\Gamma(k)}$$
With mean k/μ and variance k/μ^2.

Calculate the probability P_{rk} of exactly r demands during the lead time when the parameter value is k.

$$P_{rk} = \int_0^\infty f(r|t) f(t)\, dt$$
$$= \int_0^\infty \frac{e^{-\lambda t}(rt)^r}{\Gamma(r+1)} \frac{\mu^{-\lambda t}(\mu t)^{k-1}}{\Gamma(k)}\, dt$$
$$= \frac{\lambda^r \mu^k}{\Gamma(r+1)\Gamma(k)} \int_0^\infty t^{r+k-1} e^{-(\lambda+\mu)t}\, dt$$

Replace $(\lambda+\mu)t = z$. $\left|\frac{d\phi^{-1}}{dz}\right| = \frac{1}{\lambda+\mu}$

$$P_{rk} = \frac{\lambda^r \mu^k}{\Gamma(r+1)\Gamma(k)} \cdot \frac{1}{(\lambda+\mu)^{r+k}} \int_0^\infty z^{r+k-1} e^{-z}\, dz. = \frac{\lambda^r \mu^k}{(\lambda+\mu)^{r+k}} \frac{\Gamma(r+k)}{\Gamma(r+1)\Gamma(k)}$$
$$= \binom{r+k-1}{k-1} \left(\frac{\lambda/\mu}{1+\lambda/\mu}\right)^r \left(\frac{1}{1+\lambda/\mu}\right)^k$$

$$r \sim NG\left(k, \frac{\lambda/\mu}{1+\lambda/\mu}\right)$$

Obtaining k success $p = \frac{\lambda/\mu}{1+\lambda/\mu}$