VE401 RC Week8

Wang Yangyang

UM-SJTU JI

2022 Spring

Outline

1. **Single Sample Tests**
   - Single Sample Tests for the Mean and Variance (Done)
   - Non-Parametric Single Sample Tests for Median
   - Inferences on Proportions
2. **Comparison Tests**
   - Proportion Comparison, Pooled Test
   - Comparison of Two Variances
   - Comparison of Two Means
   - Non-Parametric Methods
   - Paired Test, Correlation
   - Categorical Data
3. **Supplementary Materials**
   - Prepare MMA File

Outline

1. **Single Sample Tests**
   - Single Sample Tests for the Mean and Variance (Done)
   - Non-Parametric Single Sample Tests for Median
   - Inferences on Proportions
2. **Comparison Tests**
   - Proportion Comparison, Pooled Test
   - Comparison of Two Variances
   - Comparison of Two Means
   - Non-Parametric Methods
   - Paired Test, Correlation
   - Categorical Data
3. **Supplementary Materials**
   - Prepare MMA File

Test for Mean (Variance Known)

Let $X_1, \ldots, X_n$ be a random sample of size n from a ***normal*** distribution with ***unknown*** mean μ and ***known*** variance σ^2. Let μ_0 be a null value of the mean. Then the test statistic is given by

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

We reject at significance level α

- $H_0 : \mu = \mu_0$ if $|Z| > z_{\alpha/2}$,
- $H_0 : \mu \leqslant \mu_0$ if $Z > z_\alpha$,
- $H_0 : \mu \geqslant \mu_0$ if $Z < -z_\alpha$.

OC curve. The abscissa is defined by

$$d = \frac{|\mu - \mu_0|}{\sigma}.$$

Test for Mean (Variance Unknown)

Let $X_1, \ldots, X_n$ be a random sample of size n from a ***normal*** distribution with ***unknown*** mean μ and ***unknown*** variance σ^2. Let μ_0 be a null value of the mean. Then the test statistic is given by

$$T_{n-1} = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

We reject at significance level α

- $H_0 : \mu = \mu_0$ if $|T_{n-1}| > t_{\alpha/2,n-1}$,
- $H_0 : \mu \leqslant \mu_0$ if $T_{n-1} > t_{\alpha,n-1}$,
- $H_0 : \mu \geqslant \mu_0$ if $T_{n-1} < -t_{\alpha,n-1}$.

OC curve. The abscissa is defined by

$$d = \frac{|\mu - \mu_0|}{\sigma}$$

Comments on T-test

1. The T-distribution may be used for $\frac{\bar{X}-\mu_0}{S/\sqrt{n}}$ when a sample is obtained from a normal population.
2. If a sample is obtained from a non-normal population, then for large to medium sample sizes ($n \geqslant 25$) it can be shown that violating the normality assumption does not significantly change α and β.
3. For small sample sizes, a T-test cannot be used and an alternative (non-parametric) test must be employed.

Comments on Abscissa

Abscissa of OC Curves.

$$d = \frac{|\mu - \mu_0|}{\sigma}$$

where σ is the unknown standard deviation of the random variable. We have three options:

1. If available, we can use prior experiments to insert a rough estimate for σ
2. We can express the difference $\delta = |\mu - \mu_0|$ relative to σ, e.g., prescribing $d = \delta/\sigma < 1$ for a small difference in the mean or $d = \delta/\sigma < 2$ for a moderately large difference.
3. We substitute the sample standard deviation s for σ.

Test for Variance

Let $X_1, \ldots, X_n$ be a random sample of size n from a ***normal*** distribution with ***unknown*** variance σ^2. Let σ_0^2 be a null value of the variance. Then the test statistic is given by

$$\chi_{n-1}^2 = \frac{(n-1)S^2}{\sigma_0^2}.$$

We reject at significance level α

- $H_0 : \sigma = \sigma_0$ if $\chi_{n-1}^2 \in \left(0, \chi_{1-\alpha/2,n-1}^2\right) \cup \left(\chi_{\alpha/2,n-1}^2, \infty\right)$,
- $H_0 : \sigma \leqslant \sigma_0$ if $\chi_{n-1}^2 > \chi_{\alpha,n-1}^2$,
- $H_0 : \sigma \geqslant \sigma_0$ if $\chi_{n-1}^2 < \chi_{1-\alpha,n-1}^2$.

OC curve. The abscissa is defined by

$$\lambda = \frac{\sigma}{\sigma_0}$$

Comments on Chi-squared Test

- If the distribution is non-normal, we cannot use Chi-squared test.
- ** Normality of the data must first be tested if we do not know the distribution.

Abscissa of OC Curves.

$$\lambda = \frac{\sigma}{\sigma_0}$$

Note that the OC curves for the left- and right-tailed chi-squared distributions are distinct.

Outline

1. **Single Sample Tests**
   - Single Sample Tests for the Mean and Variance (Done)
   - Non-Parametric Single Sample Tests for Median
   - Inferences on Proportions
2. **Comparison Tests**
   - Proportion Comparison, Pooled Test
   - Comparison of Two Variances
   - Comparison of Two Means
   - Non-Parametric Methods
   - Paired Test, Correlation
   - Categorical Data
3. **Supplementary Materials**
   - Prepare MMA File

Sign Test for Median

Let $X_1, \ldots, X_n$ be a random sample of size n from an arbitrary continuous distribution and let

$$Q_+ = \#\left\{X_k : X_k - M_0 > 0\right\}, \quad Q_- = \#\left\{X_k : X_k - M_0 < 0\right\}.$$

We reject at a significance level α

- $H_0 : M \leqslant M_0$ if $P\left[Y \leqslant q_- \mid M = M_0\right] < \alpha$,
- $H_0 : M \geqslant M_0$ if $P\left[Y \leqslant q_+ \mid M = M_0\right] < \alpha$,
- $H_0 : M = M_0$ if $P\left[Y \leqslant \min\left(q_-, q_+\right) \mid M = M_0\right] < \alpha/2$,

where $q_-, q_+$are values of Q_-, Q_+, and Y follows a binomial distri-bution with parameters n' and $1/2$, i.e.,

$$P\left[Y \leqslant k \mid M = M_0\right] = \sum_{y=0}^{k} \binom{n'}{y} \frac{1}{2^{n'}}, \quad n' = q_+ + q_-.$$

Wilcoxon Signed Rank Test for Median

Let $X_1, \ldots, X_n$ be a random sample of size n from a symmetric distribution. Order the n absolute differences $|X_i - M_0|$ according to the magnitude, so that $X_{R_i} - M_0$ is the R_i th smallest difference by modulus. If ties in the rank occur, the mean of the ranks is assigned to all equal values. Let

$$W_+ = \sum_{R_i > 0} R_i, \quad |W_-| = \sum_{R_i < 0} |R_i| .$$

We reject at significance level α

- $H_0 : M \leqslant M_0$ if $|W_-|$is smaller than the critical value for α,
- $H_0 : M \geqslant M_0$ if $W_+$is smaller than the critical value for α,
- $H_0 : M = M_0$ if $W = \min(W_+, |W_-|)$is smaller than the critical value for $\alpha/2$.

As is in the sign test, we use n' after discarding data with $X_i = M_0$.

Wilcoxon Signed Rank Test for Median

For non-small sample sizes $(n \geqslant 10)$ a ***normal distribution*** with parameters

$$\mathrm{E}[W] = \frac{n(n+1)}{4}, \quad \mathrm{Var}[W] = \frac{n(n+1)(2n+1)}{24}$$

may be used as an approximation.
However, the variance needs to be reduced if there are ***ties***: for each group of t ties, the variance is reduced by $\left(t^3 - t\right)/48$.

Proof(Brief)

Let I_i be a ***bernoulli*** random variable with $p = 1/2$ and $I_i = 1$ if $X_i < M_0$. Then $|W_-| = \sum_{i=1}^{n} |R_i| I_i$

$$\mathrm{E}\left[|W_-|\right] = \mathrm{E}\left[\sum_{i=1}^{n} |R_i| I_i\right] = \sum_{i=1}^{n} \frac{|R_i|}{2} = \frac{n(n+1)}{4},$$

$$\mathrm{Var}[|W_-|] = \sum_{i=1}^{n} |R_i|^2 \,\mathrm{Var}\, I_i = \sum_{i=1}^{n} \frac{|R_i|^2}{4} = \frac{n(n+1)(2n+1)}{24}.$$

Remarks

Sign Test

- Not very powerful, as the magnitude of $X_i - M_0$ is not needed.
- If $X_i - M_0 = 0$, then the data excluded from the analysis.

Wilcoxon Signed Rank Test

- Assumes a ***symmetric*** distribution around the median.
- Fairly ***powerful***; may even be used as an alternative to the T-test without much loss of power for data following normal distributions or with large sample size.
- As in the sign test, observations where $X_i - M_0 = 0$ are discarded.

Outline

Estimating Proportions

Let $X_1, \ldots, X_n$ be a random sample of X with sample space $\{0, 1\}$, an unbiased estimator for proportion is given by

$$\widehat{p} = \frac{1}{n} \sum_{i=1}^{n} X_i .$$

- Statistic and distribution (by central limit theorem).

$$Z = \frac{\widehat{p} - p}{\sqrt{p(1-p)/n}} \sim \mathrm{Normal}(0, 1)$$

- $100(1-\alpha)\%$ two-sided confidence interval for p.

$$\widehat{p} \pm z_{\alpha/2} \sqrt{\widehat{p}(1-\widehat{p})/n}$$

Estimating Proportions

Let $X_1, \ldots, X_n$ be a random sample of X with sample space $\{0, 1\}$, an unbiased estimator for proportion is given by

$$\widehat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Choose ***sample size***.
$\widehat{p}$ differs from p by at most d with $100(1-\alpha)\%$ confidence.

$$d = z_{\alpha/2}\sqrt{\widehat{p}(1-\widehat{p})/n} \Rightarrow n = \frac{z_{\alpha/2}^2\widehat{p}(1-\widehat{p})}{d^2}.$$

When no estimate for p is available, we use

$$n = \frac{z_{\alpha/2}^2}{4d^2}.$$

Hypothesis Testing on Proportion

Let $X_1, \ldots, X_n$ be a random sample of size n from a Bernoulli distribution with parameter p and let $\widehat{p} = \bar{X}$ denote the sample mean. The test statistic is

$$Z = \frac{\widehat{p} - p_0}{\sqrt{p_0\left(1 - p_0\right)/n}}.$$

We reject at significance level α

- $H_0 : p = p_0$ if $|Z| > z_{\alpha/2}$,
- $H_0 : p \leqslant p_0$ if $Z > z_{\alpha}$,
- $H_0 : p \geqslant p_0$ if $Z < -z_{\alpha}$.

Outline

Comparing Two Proportions

Suppose we have random samples of sizes n_1, n_2 of $X^{(1)}$ and $X^{(2)}$, respectively.

- Statistic and distribution. For large sample sizes,

$$Z = \frac{\widehat{p}_1 - \widehat{p}_2 - (p_1 - p_2)}{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}} \sim \text{Normal}(0, 1).$$

- $100(1-\alpha)\%$ two-sided confidence interval for $p_1 - p_2$.

$$\widehat{p}_1 - \widehat{p}_2 \pm z_{\alpha/2}\sqrt{\frac{\widehat{p}_1\left(1-\widehat{p}_2\right)}{n_1} + \frac{\widehat{p}_2\left(1-\widehat{p}_2\right)}{n_2}}$$

Hypothesis Testing on Difference of Proportions

Let $X_1^{(i)}, \ldots, X_{n_i}^{(i)}, i = 1, 2$ be random samples of sizes n_i from two Bernoulli distributions with parameters p_i and let $\widehat{p}_i = \bar{X}_i$ denote the corresponding sample means. The test statistic is given by

$$Z = \frac{\widehat{p}_1 - \widehat{p}_2 - (p_1 - p_2)_0}{\sqrt{\frac{\widehat{p}_1(1-\widehat{p}_1)}{n_1} + \frac{\widehat{p}_2(1-\widehat{p}_2)}{n_2}}}$$

We reject at significance level α

- $H_0 : p_1 - p_2 = (p_1 - p_2)_0$ if $|Z| > z_{\alpha/2}$,
- $H_0 : p_1 - p_2 \leqslant (p_1 - p_2)_0$ if $Z > z_\alpha$,
- $H_0 : p_1 - p_2 \geqslant (p_1 - p_2)_0$ if $Z < -z_\alpha$.

Pooled Test

Let $X_1^{(i)}, \ldots, X_{n_i}^{(i)}$, $i = 1, 2$ be random samples of sizes n_i from two Bernoulli distributions with parameters p_i and let $\widehat{p}_i = \overline{X}_i$ denote the corresponding sample means. The test statistic is given by

$$Z = \frac{\widehat{p}_1 - \widehat{p}_2}{\sqrt{\widehat{p}(1-\widehat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \quad \widehat{p} = \frac{n_1 \widehat{p}_1 + n_2 \widehat{p}_2}{n_1 + n_2}$$

We reject at significance level α

- $H_0 : p_1 = p_2$ if $|Z| > z_{\alpha/2}$
- $H_0 : p_1 \leqslant p_2$ if $Z > z_\alpha$
- $H_0 : p_1 \geqslant p_2$ if $Z < -z_\alpha$.

Outline

F-Distribution

Let $\chi^2_{\gamma_1}$ and $\chi^2_{\gamma_2}$ be independent chi-squared random variables with γ_1 and γ_2 degrees of freedom, respectively. Then the random variable

$$F_{\gamma_1,\gamma_2} = \frac{\chi^2_{\gamma_1}/\gamma_1}{\chi_{\gamma_2}/\gamma_2}$$

follows a ***F-distribution*** with γ_1 and γ_2 degrees of freedom, with density function

$$f_{\gamma_1,\gamma_2}(x) = \gamma_1^{\gamma_1/2}\gamma_2^{\gamma_2/2}\frac{\Gamma\left(\frac{\gamma_1+\gamma_2}{2}\right)}{\Gamma\left(\frac{\gamma_1}{2}\right)\Gamma\left(\frac{\gamma_2}{2}\right)}\frac{x^{\gamma_1/2-1}}{(\gamma_1 x+\gamma_2)^{(\gamma_1+\gamma_2)/2}}$$

for $x \geqslant 0$ and $f_{\gamma_1,\gamma_2}(x) = 0$ for $x < 0$. Furthermore,

$$P\left[F_{\gamma_1,\gamma_2} < x\right] = P\left[\frac{1}{F_{\gamma_1,\gamma_2}} > \frac{1}{x}\right] = P\left[F_{\gamma_2,\gamma_1} > \frac{1}{x}\right]$$

Comparing Variances

Let S_1^2 and S_2^2 be sample variances based on independent random samples of sizes n_1 and n_2 drawn from ***normal*** populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. The test statistic is given by

$$F_{n_1-1,n_2-1} = \frac{S_1^2}{S_2^2}.$$

We reject at significance level α

- $H_0 : \sigma_1 \leqslant \sigma_2$ if $S_1^2/S_2^2 > f_{\alpha,n_1-1,n_2-1}$,
- $H_0 : \sigma_1 \geqslant \sigma_2$ if $S_2^2/S_1^2 > f_{\alpha,n_2-1,n_1-1}$,
- $H_0 : \sigma_1 = \sigma_2$ if $S_1^2/S_2^2 > f_{\alpha/2,n_1-1,n_2-1}$ or $S_2^2/S_1^2 > f_{\alpha/2,n_2-1,n_1-1}$.

OC curve. The abscissa is defined by

$$\lambda = \frac{\sigma_1}{\sigma_2}.$$

Outline

Basic Cases

For two ***Normally Distributed*** Populations:

- $X^{(1)} \sim N\left(\mu_1, \sigma_1^2\right)$
- $X^{(2)} \sim N\left(\mu_2, \sigma_2^2\right)$

Goal: compare μ_1 and μ_2.
Three Basic Cases:

- σ_1^2 and σ_2^2 are known
- σ_1^2 and σ_2^2 are unknown but $\sigma_1^2 = \sigma_2^2$
- σ_1^2 and σ_2^2 are unknown and not necessarily equal

Variance Known

Let $X_1^{(i)}, \ldots, X_{n_i}^{(i)}$ with $i = 1, 2$ be samples of sizes n_1 and n_2 from normal distributions with unknown means μ_1, μ_2 and ***known*** variances σ_1^2, σ_2^2. Then the test statistic is given by

$$Z = \frac{\bar{X}^{(1)} - \bar{X}^{(2)} - (\mu_1 - \mu_2)_0}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

We reject at significance level α

- $H_0 : \mu_1 - \mu_2 = (\mu_1 - \mu_2)_0$ if $|Z| > z_{\alpha/2}$,
- $H_0 : \mu_1 - \mu_2 \leqslant (\mu_1 - \mu_2)_0$ if $Z > z_\alpha$,
- $H_0 : \mu_1 - \mu_2 \geqslant (\mu_1 - \mu_2)_0$ if $Z < -z_\alpha$.

Variance Known

When testing equality of means $H_0 : \mu_1 = \mu_2$, we have $(\mu_1 - \mu_2)_0 = 0$. We can use the OC curves for normal distributions with

$$d = \frac{|\mu_1 - \mu_2|}{\sqrt{\sigma_1^2 + \sigma_2^2}}$$

with $n = n_1 = n_2$. When $n_1 \neq n_2$, we use the equivalent sample size

$$n = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2/n_1 + \sigma_2^2/n_2}.$$

Variance Equal but Unknown

Variances equal but unknown. Let $X_1^{(i)}, \ldots, X_{n_i}^{(i)}$ with $i = 1, 2$ be samples of sizes n_1 and n_2 from normal distributions with unknown means μ_1, μ_2 and ***equal but unknown*** variances $\sigma^2 = \sigma_1^2 = \sigma_2^2$. Then the test statistic is given by

$$T_{n_1+n_2-2} = \frac{\bar{X}^{(1)} - \bar{X}^{(2)} - (\mu_1 - \mu_2)_0}{\sqrt{S_p^2\left(1/n_1 + 1/n_2\right)}},$$

with pooled estimator for variance

$$S_p^2 = \frac{(n_1 - 1)\, S_1^2 + (n_2 - 1)\, S_2^2}{n_1 + n_2 - 2}.$$

We reject at significance level α

- $H_0 : \mu_1 - \mu_2 = (\mu_1 - \mu_2)_0$ if $|T_{n_1+n_2-2}| > t_{\alpha/2, n_1+n_2-2}$,
- $H_0 : \mu_1 - \mu_2 \leqslant (\mu_1 - \mu_2)_0$ if $T_{n_1+n_2-2} > t_{\alpha, n_1+n_2-2}$,
- $H_0 : \mu_1 - \mu_2 \geqslant (\mu_1 - \mu_2)_0$ if $T_{n_1+n_2-2} < -t_{\alpha, n_1+n_2-2}$.

Variance Equal but Unknown

OC curve. When testing equality of means $H_0 : \mu_1 = \mu_2$, we have $(\mu_1 - \mu_2)_0 = 0$. We can use the OC curves for the T-test in case of ***equal*** sample sizes $n = n_1 = n_2$

$$d = \frac{|\mu_1 - \mu_2|}{2\sigma}.$$

When reading the charts, we must use the ***modified sample size*** $n^* = 2n - 1$.

Variance Not Necessarily Equal and Unknown

Let $X_1^{(i)}, \dots, X_{n_i}^{(i)}$ with $i = 1, 2$ be samples of sizes n_1 and n_2 from normal distributions with unknown means μ_1, μ_2 and ***not necessarily equal and unknown*** variances σ_1^2, σ_2^2. The test statistic is given by

$$T_\gamma = \frac{\bar{X}^{(1)} - \bar{X}^{(2)} - (\mu_1 - \mu_2)_0}{\sqrt{S_1^2/n_1 + S_2^2/n_2}}, \quad \gamma = \frac{\left(S_1^2/n_1 + S_2^2/n_2\right)^2}{\frac{\left(S_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(S_2^2/n_2\right)^2}{n_2 - 1}}$$

We reject at significance level α

- $H_0 : \mu_1 - \mu_2 = (\mu_1 - \mu_2)_0$ if $T_\gamma > t_{\alpha/2,\gamma}$,
- $H_0 : \mu_1 - \mu_2 \leqslant (\mu_1 - \mu_2)_0$ if $T_\gamma > t_{\alpha,\gamma}$,
- $H_0 : \mu_1 - \mu_2 \geqslant (\mu_1 - \mu_2)_0$ if $T_\gamma < -t_{\alpha,\gamma}$.

Variance Not Necessarily Equal and Unknown

Remarks:

- Round γ down to the nearest integer.
- No simple OC curves for Welch's test.
- ***!!!*** It is not a good idea to pre-test for equal variances and then make a decision whether to use Student's or Welch's test. ***!!!***
  It is fine to test for normality, equality of variances or other properties and then to gather ***new data*** for a comparison of means test. But using the ***same data*** creates serious problems.
- When variances are unknown, current recommendations are to always use Welch's test.

Outline

Wilcoxon Rank-Sum Test

Let X and Y be two random samples following some continuous distributions. Decide whether to reject the null hypothesis

$$H_0 : P[X > Y] = \frac{1}{2} \quad \text{or} \quad H_0 : P[X > Y] \leqslant \frac{1}{2}$$

Procedures:

1. Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$, $m \leqslant n$, be random samples from X and Y and associate the rank R_i, $i = 1, \dots, m + n$, to the R_i th smallest among the $m + n$ total observations. If ties in the rank occur, the mean of the ranks is assigned to all equal values.
2. Sum up the ranks of smaller samples. Then the test based on the statistic
$$W_m := \text{ sum of the ranks of } X_1, \dots, X_m$$
is called the Wilcoxon rank-sum test.

Wilcoxon Rank-Sum Test

We reject $H_0 : P[X > Y] = 1/2$ at significance level α if

- for small m : W_m falls into the corresponding critical region, or
- for large $m (m \geqslant 20)$: perform a Z-test, since W_m is approximately normally distributed with

$$\mathrm{E}\left[W_m\right] = \frac{m(m+n+1)}{2}, \quad \mathrm{Var}\left[W_m\right] = \frac{mn(m+n+1)}{12}$$

If there are many ties, the variance may be corrected by taking

$$\mathrm{Var}\left[W_m\right] = \frac{mn(m+n+1)}{12 - \sum_{\text{groups}} \frac{t^3+t}{12}}$$

where the sum is taken over all groups of t ties (not always a good way).

Outline

Paired Tests for Mean

Comparing means (or the location) of two related populations X and Y. Method: Pair the samples as $D = X - Y$.

- Set the hypothesis as, i.e.,

$$H_0 : \mu_D = \mu_X - \mu_Y = (\mu_X - \mu_Y)_0 = \mu_{D0}$$

- Then use a ***T-test*** for D is called a paired T-test for X and Y

$$T_{n-1} = \frac{\bar{D} - \mu_{D_0}}{\sqrt{S_D^2/n}}$$

Suppose the two independent random variables X and Y do not follow a normal distribution, but the same distribution (differing in their location). Assumption: $D = X - Y$ is symmetric about $M_0 \in \mathbb{R}$.

- Set the hypothesis as, i.e.,

$$H_0 : X - Y \geqslant M_0$$

- Apply the ***Wilcoxon signed rank test***. Rank $|D_i - M_0|$ and calculate W_+or $W_-$to test the hypothesis.

Paired vs. Pooled T-Tests

Assume that we have two populations of normally distributed random variables X and Y with equal variances σ^2. We want to test

$$H_0 : \mu_X - \mu_Y = (\mu_X - \mu_Y)_0$$

Then we could either perform a paired test or a pooled test. Which is more powerful? Let us compare the test statistics:

$$T_{\text{pooled}} = \frac{\bar{X} - \bar{Y} - (\mu_X - \mu_Y)_0}{\sqrt{2S_p^2/n}}, \qquad \text{critical value} = t_{\alpha/2,2n-2}$$

$$T_{\text{paired}} = \frac{\bar{X} - \bar{Y} - (\mu_X - \mu_Y)_0}{\sqrt{S_D^2/n}}, \qquad \text{critical value} = t_{\alpha/2,n-1}$$

Compare the two denominators, which estimate

$$\frac{2\sigma^2}{n} \quad \text{with} \quad \frac{\sigma_D^2}{n}$$

Paired vs. Pooled T-Tests

A direct calculation yields

$$\frac{\sigma_D^2}{n} = \frac{\mathrm{Var}[D]}{n} = \frac{\mathrm{Var}[X]}{n} + \frac{\mathrm{Var}[Y]}{n} - \frac{2}{n}\mathrm{Cov}[X, Y]$$
$$= \frac{\sigma^2}{n} + \frac{\sigma^2}{n} - \frac{2\sigma^2}{n}\frac{\mathrm{Cov}[X, Y]}{\sqrt{\mathrm{Var}[X]}\sqrt{\mathrm{Var}[Y]}} = \frac{2\sigma^2}{n}(1 - \rho_{XY})$$

Conclusion: From $\frac{\sigma_D^2}{n} = \frac{2\sigma^2}{n}(1 - \rho_{XY})$ we see

- If $\rho_{XY} > 0$, paired T-test is more powerful. The denominator of the paired statistic will be smaller than that of the pooled statistic, leading to a larger value of the statistic.
- If ρ_{XY} is zero (or even negative), pairing is unnecessary and pooled T-test is more powerful. The reason is that it is easier to reject H_0 when comparing with $t_{\alpha/2,2n-2}$ than with $t_{\alpha/2,n-1}$.

$\Rightarrow$ *Positive correlation* makes a paired T-test more powerful.

Test for Correlation Coefficient

First, find the estimation of ρ. Since

$$\widehat{\mathrm{Var}[X]]} = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$

$$\widehat{\mathrm{Cov}[X, Y]} = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)$$

The natural choice for an estimator for the correlation coefficient is then

$$R := \widehat{\rho} = \frac{\sum\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)}{\sqrt{\sum\left(X_i - \bar{X}\right)^2}\sqrt{\sum\left(Y_i - \bar{Y}\right)^2}}$$

Fisher Transformation of R

Next, suppose that (X, Y) follows a bivariate normal distribution, i.e., they have the joint density

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}$$

with $\mu_X, \mu_Y \in \mathbb{R}$, $\sigma_X, \sigma_Y > 0$ and correlation coefficient $\rho \in (-1, 1)$. For large n the ***Fisher transformation*** of R,

$$\frac{1}{2}\ln\left(\frac{1+R}{1-R}\right) = \operatorname{Artanh}(R)$$

is approximately ***normally distributed*** with

$$\mu = \frac{1}{2}\ln\left(\frac{1+\rho}{1-\rho}\right) = \operatorname{Artanh}(\rho), \quad \sigma^2 = \frac{1}{n-3}$$

Test for Correlation Coefficient

- Hypothesis test: We can test $H_0 : \rho = \rho_0$, by ***Z-test***, using the test statistic

$$Z = \frac{\sqrt{n-3}}{2}\left(\ln\left(\frac{1+R}{1-R}\right) - \ln\left(\frac{1+\rho_0}{1-\rho_0}\right)\right)$$
$$= \sqrt{n-3}\left(\operatorname{Artanh}(R) - \operatorname{Artanh}(\rho_0)\right)$$

- Confidence interval: A $100(1-\alpha)\%$ confidence interval for ρ,

$$\left[\frac{1+R-(1-R)e^{2z_{\alpha/2}/\sqrt{n-3}}}{1+R+(1-R)e^{2Z_{\alpha/2}/\sqrt{n-3}}}, \frac{1+R-(1-R)e^{-2z_{\alpha/2}/\sqrt{n-3}}}{1+R+(1-R)e^{-2z_{\alpha/2}/\sqrt{n-3}}}\right]$$

or

$$\tanh\left(\operatorname{Artanh}(R) \pm \frac{z_{\alpha/2}}{\sqrt{n-3}}\right)$$

Outline

Outline

It is suggested that you solve problems in assignments using Mathematica. It's the best way to prepare for Final Exam.
This notebook file, for your reference, is credited to previous TA Zhang Xingjian and Joy Dong.
It would be better to write your own notebook file.

End

Credit to Zhanpeng Zhou (TA of SP21)
Credit to Fan Zhang (TA of SU21)
Credit to Liying Han (TA of SP21)
Credit to Zhenghao Gu (TA of SP20)