Let (X, Y) be a continuous bivariate random variable with density $f_{XY} : S \to \mathbb{R}^2$ given by

$$f_{XY}(x, y) = \begin{cases} c \cdot (x^2 + y^2) & \text{for } x^2 + y^2 \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

where $c \in \mathbb{R}$ is a suitable constant.

1. Determine the constant c.
2. Find $\mathrm{E}[X]$ and $\mathrm{E}[Y]$.
3. Find $\mathrm{Var}[X]$ and $\mathrm{Var}[Y]$.
4. Find the correlation coefficient ρ_{XY}.

We substitute (x, y) with (r, θ)

$x = r\cos\theta$ $\quad r \in [0, 1]$

$y = r\sin\theta$ $\quad \theta \in [0, 2\pi]$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

i) We must have $\int_{\mathbb{R}^2} f(x, y)\, dx\, dy = 1$, so

$$\begin{aligned} \int_{\mathbb{R}^2} f(x, y)\, dx\, dy &= \iint_{x^2+y^2 \le 1} c \cdot (x^2 + y^2)\, dx\, dy \\ &= \int_0^{2\pi} \int_0^1 c \cdot r^2 \cdot r\, dr\, d\theta \\ &= c \cdot 2\pi \cdot \frac{1}{4} \end{aligned}$$

Jacobian

implies that $c = 2/\pi$.

Give 2 Marks if the calculation and the answer are correct. Give 1 Mark if the calculation is present but the answer is wrong. Give no marks if there is no calculation.

ii) We calculate

$$\mathrm{E}[X] = \int_{\mathbb{R}^2} x f(x, y)\, dx\, dy = \frac{2}{\pi} \int_0^{2\pi} \int_0^1 r\cos(\theta) \cdot r^3\, dr\, d\theta = 0$$

by symmetry of the cosine function. The expectation of Y is the same, since X and Y enter into the density the same way.

It's symmetric for X and Y.

iii) We calculate

$$\begin{aligned} \mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2 &= \mathrm{E}[X^2] \\ &= \int_{\mathbb{R}^2} x^2 f(x, y)\, dx\, dy \\ &= \frac{2}{\pi} \int_0^{2\pi} \int_0^1 r^2 \cos^2(\theta) \cdot r^3\, dr\, d\theta \\ &= \frac{2}{\pi} \underbrace{\int_0^{2\pi} \cos^2(\theta)\, d\theta}_{=\pi} \int_0^1 r^5\, dr \\ &= \frac{1}{3}. \end{aligned}$$

The variance of Y is the same, since X and Y enter into the density the same way.

$$\int_0^{2\pi} \cos^2\theta\, d\theta = \int_0^{2\pi} (1 - \sin^2\theta)\, d\theta$$

$$\int_0^{2\pi} \cos^2\theta\, d\theta + \int_0^{2\pi} \sin^2\theta\, d\theta = 2\pi$$

$$\int_0^{2\pi} \cos^2\theta\, d\theta = \int_0^{2\pi} \sin^2\theta\, d\theta$$

iv) We calculate

$$\begin{aligned} \mathrm{Cov}[X, Y] = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] &= \mathrm{E}[XY] \\ &= \int_{\mathbb{R}^2} xy f(x, y)\, dx\, dy \\ &= \frac{2}{\pi} \int_0^{2\pi} \int_0^1 r^2 \cos(\theta)\sin(\theta) \cdot r^3\, dr\, d\theta \\ &= 0 \end{aligned}$$

by orthogonality of the sine and cosine functions. It follows that the correlation coefficient is zero.

$$\int_0^{2\pi} \cos\theta \sin\theta\, d\theta = 0.$$

(*Multivariate* R.V.) Let (X, Y) be a continuous bivariate random variable with density $f_{XY} : S \to \mathbb{R}^2$ given by

$$f_{XY}(x,y) = \begin{cases} \frac{2}{\pi} \cdot (x^2+y^2) & \text{for } x^2+y^2 \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the density of $U = X/Y$.

$U = X/Y \quad V = Y$

$X = UV \quad Y = V$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = |v|$$

$u, v \in \mathbb{R}$.

v) Formally, we have, for suitable $u \in \mathbb{R}$

$$f_U(u) = \int_{-\infty}^{\infty} f_{XY}(uv, v) \cdot |v| \, dv = \frac{2}{\pi} \int_I (u^2v^2 + v^2) \cdot |v| \, dv$$

For multivariate, do the marginal to eliminate the other variable.

Here the integral is taken over all v such that $u^2v^2 + v^2 \leq 1$. We note that this is possible for all $u \in \mathbb{R}$ and that

$$I = \left[-\frac{1}{\sqrt{1+u^2}}, \frac{1}{\sqrt{1+u^2}} \right].$$

It follows that, using the fact that the integrand is even

$$\begin{aligned} f_U(u) &= \frac{2}{\pi}(1+u^2) \int_{-\frac{1}{\sqrt{1+u^2}}}^{\frac{1}{\sqrt{1+u^2}}} v^2 \cdot |v| \, dv \\ &= \frac{4}{\pi}(1+u^2) \int_0^{\frac{1}{\sqrt{1+u^2}}} v^3 \, dv \\ &= \frac{4}{\pi}(1+u^2) \frac{1}{4(\sqrt{1+u^2})^4} \\ &= \frac{1}{\pi} \frac{1}{1+u^2} \end{aligned}$$

Jacobian

Symmetric.

(*Univariate* R.V.) Let X be a continuous *uniformly distributed* random variable on $[-1, 1]$. Does X^2 also follow uniform distribution?

$Y = X^2 \qquad X = \pm\sqrt{Y} \qquad y \in [0, 1]$ Correct ?

$F_{X^2}(\ ) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = \sqrt{y}$

$$f_{X^2}(y) = \frac{dF_{X^2}(y)}{dy} = \begin{cases} \frac{1}{2\sqrt{y}} & y \in (0, 1), \\ 0 & \text{otherwise} \end{cases}$$

For both cases, $y = 0$ seems to be undefined.

For continuous R.V., $P[Y=a]$ is invalid. $P[a<Y<b] = P[a \leq Y \leq b]$.

Exercise

Let X, Y be independent random variables such that $X, Y \sim N(\mu, \sigma^2)$, show that $X+Y$ and $X-Y$ are independent.

Independence $\Rightarrow$ $Cov = 0$
$\not\Leftarrow$

To prove independence, recall that

$P(X=x, Y=y) = P(X=x)P(Y=y)$ (discrete)

$f_{X,Y}(x,y) = f_X(x) f_Y(y)$ (continuous)

So we need to prove

$f_{X+Y, X-Y}(x+y, x-y) = f_{X+Y}(x+y) f_{X-Y}(x-y)$

Use M.G.F. because it's uniquely defined.

Recall that for an $\mathcal{N}(\mu, \sigma^2)$ random variable, the moment generating function of it is

$$M(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right). \tag{1}$$

By condition, $X+Y \sim \mathcal{N}(2\mu, 2\sigma^2)$ and $X-Y \sim \mathcal{N}(0, 2\sigma^2)$. Therefore by (1), we have:

$$M_{X+Y}(t) = \exp(2\mu t + \sigma^2 t^2),\ M_{X-Y}(t) = \exp(\sigma^2 t^2).$$

On the other hand, as a bivariate random vector $(X+Y, X-Y)$, its MGF can be computed by definition as follows:

$$\begin{aligned}
&M_{(X+Y,X-Y)}(t_1, t_2)\\
=&E[\exp(t_1(X+Y) + t_2(X-Y))]\\
=&E\{\exp[(t_1+t_2)X] \times \exp[(t_1-t_2)Y]\}\\
=&E\{\exp[(t_1+t_2)X]\} \times E\{\exp[(t_1-t_2)Y]\} \quad \text{by independence of } X \text{ and } Y.\\
=&M_X(t_1+t_2)M_Y(t_1-t_2)\\
=&\exp\left(\mu(t_1+t_2) + \frac{1}{2}\sigma^2(t_1+t_2)^2\right)\exp\left(\mu(t_1-t_2) + \frac{1}{2}\sigma^2(t_1-t_2)^2\right)\\
=&\exp(2\mu t_1 + \sigma^2 t_1^2)\exp(\sigma^2 t_2^2)\\
=&M_{X+Y}(t_1)M_{X-Y}(t_2).
\end{aligned}$$

Hence $X+Y$ and $X-Y$ are independent.

Recall that $E[e^{t_1 X_1 + t_2 X_2}]$ for bivariate M.G.F.

Exercise

We can also derive *Chi-Squared* following *Uniform-Exponential-Gamma* thread.

1. Gamma distribution with parameters $\alpha = r$, $\beta = \lambda$ has M.G.F. $m_X(t) = \frac{1}{(1-t/\beta)^\alpha}$. What is the M.G.F. for Chi-squared distribution with γ degrees of freedom?
2. If the random variables $X_1, X_2, \ldots, X_n$ are independent and follow the uniform distribution $U(0,1)$. Find the distribution of the random variable $Z = \sum_{i=1}^{n} Y_i$, where $Y = -2\ln X$.

1. Chi-squared: $\alpha = \frac{\gamma}{2}$ $\beta = \frac{1}{2}$. $m_{\chi^2_\gamma}(t) = \frac{1}{(1-2t)^{\frac{\gamma}{2}}}$

2. First, $x = e^{-\frac{y}{2}}$ $\left|\frac{d\phi^{-1}}{dy}\right| = \frac{e^{-\frac{y}{2}}}{2}$

$f_Y(y) = \begin{cases} \frac{1}{2}e^{-\frac{y}{2}} & (y>0) \\ 0 & \text{otherwise} \end{cases}$ follows Exp. Distribution $(\lambda = \frac{1}{2})$

Sum of exponential: $\chi^2_\gamma(\beta = \lambda = \frac{1}{2}, \alpha = n, \gamma = 2n.)$

Next to find the distribution for $Z = \sum_{i=1}^{n} Y_i$, we use moment generating functions.

To, this end, the moment generating function for Y is given by

$$\begin{aligned} m_Y(t) &= E[e^{tY}] \\ &= \int_0^\infty e^{ty} \cdot \frac{1}{2}e^{-y/2}\,dy \\ &= \frac{1}{2}\int_0^\infty e^{-(\frac{1}{2}-t)y}\,dy \\ &= \frac{1}{1-2t}, \text{ assuming } t < \frac{1}{2}. \end{aligned}$$

Next, $\{Y_i\}$ is independent, because $\{X_i\}$ is independent. Hence, we can find the moment generating function for $Z = \sum_{i=1}^{n} Y_i$ as follows:

$$m_Z(t) = \prod_{i=1}^{n} m_{Y_i}(t) = \frac{1}{(1-2t)^n}.$$

However, since the moment generating function for the chi-squared distribution with k degrees of freedom is $m_{\chi^2_k}(t) = (1-2t)^{-k/2}$, we conclude by the uniqueness of moment generating functions that Z **is a chi-squared distribution with $2n$ degrees of freedom**.