VE401 RC Week2

Wang Yangyang

UM-SJTU JI

2022 Spring

Outline

1. **Elementary Probability**
   - Counting Principles
   - Probability Measurements
2. **Conditional Probability**
   - Condition and Independence
   - Bayes's Theorem
3. **Exercise and Discussion**
   - Banach Matchbox problem
   - Two Children Paradox Revisit
   - Penney's Game

Outline

Cardano's Principle

Principle

Let A be a random outcome of an experiment that may proceed in various ways. Assume each of these ways is equally likely. Then the probability $P[A]$ of the outcome A is

$$P[A] = \frac{\text{number of ways leading to outcome } A}{\text{number of ways the experiment can proceed.}}$$

When applying Cardano's principle, it is crucial that all outcomes are equally likely! —— D'Alembert's Error

Basic Principles of Counting

Suppose a set A of n objects is given.

- ***Permutation*** of k objects: $\frac{n!}{(n-k)!}$ ways of choosing an ordered tuple of k objects from A.
- ***Combination*** of k objects: $\frac{n!}{k!(n-k)!}$ ways of choosing an unordered set of k objects from A.
- ***Permutation*** of k ***indistinguishable*** objects: $\frac{n!}{n_1!n_2!\dots n_k!}$ ways of partitioning A into k disjoint subsets $A_1, \dots, A_k$ whose union is A, where each A_i has n_i elements.

Basic Principles of Counting

Example

If the letters $s, s, s, t, t, t, i, i, a, c$ are arranged in a random order, what is the probability that they will spell the word "statistics"?

Basic Principles of Counting

Example

If the letters $s, s, s, t, t, t, i, i, a, c$ are arranged in a random order, what is the probability that they will spell the word "statistics"?

Solution

$$p = 1/\frac{10!}{3! \times 3! \times 2! \times 1 \times 1} = \frac{1}{50400}$$

Binomial Coefficient

We define binomial coefficients by for $\alpha \in \mathbb{R}$

$$\begin{pmatrix} \alpha \\ 0 \end{pmatrix} := 1,$$

and, for $n \in \mathbb{N}\setminus\{0\}$ and $\alpha \in \mathbb{R}$,

$$\begin{pmatrix} \alpha \\ n \end{pmatrix} := \frac{\alpha \cdot (\alpha - 1) \cdot (\alpha - 2) \cdots (\alpha - n + 1)}{n!}.$$

If $\alpha \in \mathbb{N}$, this may be expressed as the perhaps more familiar

$$\begin{pmatrix} \alpha \\ n \end{pmatrix} = \frac{\alpha!}{(\alpha - n)!n!}.$$

It also implies that $\begin{pmatrix} m \\ n \end{pmatrix} = 0$ whenever $n > m$ and $m, n \in \mathbb{N}$.

Sample Space and Events

Definition

Sample Space: a space containing all possible outcomes of an experiment.
Event: a subset of sample space, containing possible outcomes of the experiment.
σ-field: $\mathscr{F}$ on S is a family of subsets of S such that

1. $\varnothing \in \mathscr{F}$;
2. if $A \in \mathscr{F}$, then $S \backslash A \in \mathscr{F}$;
3. if $A_1, A_2, \ldots \in \mathscr{F}$ is a finite or countable sequence of subsets, then the union $\bigcup_k A_k \in \mathscr{F}$.

In probability, we consider families of events that are σ-fields.
For any set S, the smallest possible σ-field is $\mathscr{F} = \{\varnothing, S\}$.

Outline

1. **Elementary Probability**
   - Counting Principles
   - Probability Measurements
2. **Conditional Probability**
   - Condition and Independence
   - Bayes's Theorem
3. **Exercise and Discussion**
   - Banach Matchbox problem
   - Two Children Paradox Revisit
   - Penney's Game

Sample Space and Events

Example

Suppose a coin is tossed three times. Then the sample space S contains the following possible outcomes:

$$s_1 = HHH, s_2 = THH, s_3 = HTH, s_4 = HHT,$$

$$s_5 = HTT, s_6 = THT, s_7 = TTH, s_8 = TTT,$$

where H denotes head and T denotes tail.

The event that at most two tails are obtained is given by

$$a = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}.$$

Probability Measures and Spaces

Definition

Let S be a sample space and $\mathscr{F}$ a σ-field on S. Then a function

$$P : \mathscr{F} \to [0, 1], \quad A \mapsto P[A],$$

is called a *probability measure* (or *probability function* or just *probability*) on S if

1. $P[S] = 1$,
2. For any set of events $\{A_k\} \subset \mathscr{F}$ such that $A_j \cap A_k = \varnothing$ for $j \neq k$,

$$P\left[\bigcup_k A_k\right] = \sum_k P\left[A_k\right]$$

The triple $(S, \mathscr{F}, P)$ is called a *probability space*.

Probability Measures and Spaces

Properties

For a probability sample space $(S, \mathscr{F}, P)$:

$$P[S] = 1$$

$$P[\varnothing] = 0$$

$$P[S \backslash A] = 1 - P[A]$$

$$P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]$$

where $A, A_1, A_2 \in S$ are any events.

Outline

Condition and Independence

Definition

- *Conditional probability* of "B occurs given A has occurred":

$$P[B \mid A] = \frac{P[B \cap A]}{P[A]}$$

- *Total probability* for $P[B]$ on a sample space S, given events $A_1, \ldots, A_n \in S$ are mutually exclusive and $A_1 \cup \cdots \cup A_n = S$:

$$P[B] = \sum_{k=1}^{n} P\left[B \mid A_k\right] \cdot P\left[A_k\right]$$

- *Independence* of events A and $B : P[A \cap B] = P[A]P[B]$, which is equivalent to

$$\begin{aligned} P[A \mid B] &= P[A] \quad \text{if } P[B] \neq 0 \\ P[B \mid A] &= P[B] \quad \text{if } P[A] \neq 0 \end{aligned}$$

Conditional Probability

Example

You meet a mother with two children. What is the probability of the event A that...

1. One of her children is a girl, the other child is a girl?
2. One of her children is born on Monday, the other child is not born on Monday?
3. The older child is a girl, the younger child is also a girl?

Conditional Probability

Example

You meet a mother with two children. What is the probability of the event A that...

1. One of her children is a girl, the other child is a girl?
2. One of her children is born on Monday, the other child is not born on Monday?
3. The older child is a girl, the younger child is also a girl?

Solution

1. $P[A] = \frac{P[GG]}{P[GG]+P[GB]+P[BG]} = \frac{1/4}{3/4} = \frac{1}{3}$
2. $P[A] = \frac{12/49}{13/49} = \frac{12}{13}$
3. $P[A] = \frac{P[GG]}{P[GG]+P[GB]} = \frac{1/4}{2/4} = \frac{1}{2}$

Total Probability

Example

Suppose Keven plays a game where his score must be $1, 2, \ldots, 50$, and each of these 50 scores is equally likely. Suppose he get X for his first try. He then continue to play the game until he obtains another score Y such that $Y \geqslant X$. What is the probability of the event A that $Y = 50$?

Total Probability

Example

Suppose Keven plays a game where his score must be $1, 2, \ldots, 50$, and each of these 50 scores is equally likely. Suppose he get X for his first try. He then continue to play the game until he obtains another score Y such that $Y \geqslant X$. What is the probability of the event A that $Y = 50$?

Solution

For each $i = 1, \ldots, 50$, let B_i be the event that $X = i$. Then conditional on B_i, the value of Y is equally likely to be any one of the numbers $i, i+1, \ldots, 50$. Therefore, the probability is given by

$$P[A] = \sum_{i=1}^{50} P\left[B_i\right] \cdot P\left[A \mid B_i\right] = \sum_{i=1}^{50} \frac{1}{50} \cdot \frac{1}{51 - i} \approx 0.09$$

Independence

Example

Given events A and B, what happens to $P[A \cap B]$, $P[A \cup B]$, $P[A \mid B]$, $P[A \mid \neg B]$, and $P[\neg A \mid B]$? You are encouraged to fill out this table by yourself first.

A and B are ...	mutually exclusive	independent
$P[A \cap B]$		
$P[A \cup B]$		
$P[A \mid B]$		
$P[A \mid \neg B]$		
$P[\neg A \mid B]$		

Independence

Example

Given events A and B, what happens to $P[A \cap B], P[A \cup B]$, $P[A \mid B], P[A \mid \neg B]$, and $P[\neg A \mid B]$? You are encouraged to fill out this table by yourself first.

A and B are ...	mutually exclusive	independent
$P[A \cap B]$	0	$P[A]P[B]$
$P[A \cup B]$	$P[A] + P[B]$	$P[A] + P[B] - P[A]P[B]$
$P[A \mid B]$	0	$P[A]$
$P[A \mid \neg B]$	$\frac{P[A]}{1-P[B]}$	$P[A]$
$P[\neg A \mid B]$	1	$1 - P[A]$

Outline

Bayes's Theorem

Theorem

Let $A_1, \ldots, A_n \subset S$ be a set of pairwise mutually exclusive events whose union is S and who each have non-zero probability of occurring. Let $B \subset S$ be any event such that $P[B] \neq 0$. Then for any $A_k, k = 1, \ldots, n$

$$P\left[A_k \mid B\right] = \frac{P\left[B \cap A_k\right]}{P[B]} = \frac{P\left[B \mid A_k\right] \cdot P\left[A_k\right]}{\sum_{j=1}^{n} P\left[B \mid A_j\right] \cdot P\left[A_j\right]}.$$

Intuition

Continue to expand Conditional Theorem in order to substitute $P[A \mid B]$ with $P[B \mid A]$. But this is only for calculation. You can learn more in the course VE414.

Bayes's Theorem

Example

A box contains one fair coin and one coin with heads on both sides. Suppose one coin is selected at random and when it is tossed twice, two heads are obtained. What is the probability that the coin is the fair coin?

Bayes's Theorem

Example

A box contains one fair coin and one coin with heads on both sides. Suppose one coin is selected at random and when it is tossed twice, two heads are obtained. What is the probability that the coin is the fair coin?

Solution

Let E_1 be the event that the selected coin is fair, and E_2 be the event that the selected coin have two heads. Using Bayes's theorem, we have

$$\begin{aligned} P[E_1 \mid HH] &= \frac{P[E_1]\,P[HH \mid E_1]}{P[HH \mid E_2]\,P[E_2] + P[HH \mid E_1]\,P[E_1]} \\ &= \frac{\frac{1}{2}\cdot\frac{1}{4}}{1\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{2}} = \frac{1}{5} \end{aligned}$$

Outline

Banach Matchbox problem

Exercise

Suppose a mathematician carries two matchboxes at all times: one in his left pocket and one in his right. Each time he needs a match, he is equally likely to take it from either pocket. Suppose he reaches into his pocket and discovers for the first time that the box picked is empty. If it is assumed that each of the matchboxes originally contained n matches, what is the probability that there are exactly k matches in the other box?

Outline

Two Children Paradox Revisit

Exercise

Suppose we were told not only that Mr. Smith has two children, and one of them is a boy, but also that the boy was born on a Sunday. What is the probability that the other child is a boy? Is it still $1/3$?

Outline

Penney's Game (Have Fun)

Exercise

The game is played by two players, A and B, who each select a sequence of three flips. For example, assume that Player A selected "heads-heads-heads" (HHH) and Player B has selected "tails-heads-heads" (THH). Then the coin is flipped repeatedly, resulting in a sequence like the following:

HTHTHHHHTHHHTTTTHTHH...

The player whose sequence showed up first (HHH for Player A or THH for Player B) is declared the winner.
Suppose you are player B. Player A has selected a pattern and now it's your turn to select. Can you maximize your winning probability?

End

Credit to Zhanpeng Zhou (TA of SP21)
Credit to Fan Zhang (TA of SU21)
Credit to Jiawen Fan (TA of SP21)
Credit to Zhenghao Gu (TA of SP20)