

# Introduction and Solving Equations

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# Empirical Studies in Economics

- Credibility revolution aiming to establish causality,  $X \rightarrow Y$ , from their correlation
- Gold standard: Quasi-experiments
- Limitations: *What you achieve may not be what you want!*
  - $X$  and/or  $Y$  are not observed
  - Difficult to clarify the underlying mechanisms
  - Causality  $X \rightarrow Y$  is not sufficiently useful in policy evaluation
- Example: Tariffs  $\rightarrow$  Trade and welfare

# Gravity Equation

- The “naive” gravity equation proposed by [Tinbergen \(1962\)](#)

$$\log X_{in} = -\epsilon \log (1 + t_{in}) + D'_{in}\gamma + \alpha X_i + \beta X_n + u_{in}, \quad (1)$$

where  $X_{in}$  is the total export from country  $i$  to  $n$ ,  $t_{in}$  is the tariff rates of country  $n$  on imports from country  $i$ ,  $D_{in}$  is a vector of distance measures such as physical distance, common border, common language, and so on.  $X_i$  is the total expenditure of country  $i$

- The fixed-effect gravity equation:

$$\log X_{in} = -\epsilon \log (1 + t_{in}) + D'_{in}\gamma + fe_i + fe_n + u_{in}, \quad (2)$$

where the fixed effects  $fe_i$  and  $fe_n$  do not only absorb the total expenditure but also all factors that are exporter- or importer-specific

- Tariff  $\Rightarrow$  Trade flows
  - What does  $\epsilon$  measure?
  - Tariff  $\Rightarrow$  Wage  $\Rightarrow$  Expenditure  $\Rightarrow$  Trade flows?

## Structural gravity equation: Armington Model

- $N$  countries. Each country  $i$  produces a distinctive variety of goods. Consumers in country  $n$  has a CES preference over all varieties:

$$U_n = \left[ \sum_{i=1}^N C_{in}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \quad \sigma > 1, \quad (3)$$

where  $C_{in}$  is the quantity of goods from country  $i$  consumed in country  $n$

- Country  $i$  is endowed with  $L_i$  workers. The variety  $i$  is produced using labor under perfect competition. The unit cost of producing good  $i$  is

$$c_i = \frac{w_i}{A_i}, \quad (4)$$

where  $w_i$  is the wage and  $A_i$  is the productivity in country  $i$

- Exporting from country  $i$  to  $n$  incurs
  - an iceberg trade cost,  $\tau_{in} \geq 1$ , with  $\tau_{ii} = 1$
  - an import tariff,  $t_{in} \geq 0$ , with  $t_{ii} = 0$

## Armington Model: Equilibrium

- Let  $X_n$  be the total expenditure in country  $n$  and  $X_{in}$  be the value of exports from  $i$  to  $n$ . Then bilateral trade share can be expressed as

$$\lambda_{in} \equiv \frac{X_{in}}{X_n} = \left[ \frac{w_i \kappa_{in}}{A_i} \right]^{1-\sigma} P_n^{\sigma-1}, \quad \kappa_{in} \equiv \tau_{in} (1 + t_{in}), \quad (5)$$

where  $P_n = \left[ \sum_{i=1}^N \left[ \frac{w_i \kappa_{in}}{A_i} \right]^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$

- Labor market clearing:

$$w_i L_i = \sum_{n=1}^N \frac{1}{1 + t_{in}} \lambda_{in} X_n. \quad (6)$$

- Total expenditure equates total income:

$$X_n = w_n L_n + \sum_{i=1}^N \frac{t_{in}}{1 + t_{in}} \lambda_{in} X_n. \quad (7)$$

- Welfare measure: real income

$$U_i = \frac{X_i}{P_i}. \quad (8)$$

## Armington Model: Gravity

- Equation (5) can be re-written into

$$\log(X_{in}) = -(\sigma - 1) \log(1 + t_{in}) - (\sigma - 1) \log \tau_{in} + (\sigma - 1) \log \left( \frac{w_i}{A_i} \right) + \log(P_n^{\sigma-1} X_n) \quad (9)$$

- Suppose that  $\tau_{in} = D'_{in} \tilde{\gamma} + \tilde{u}_{in}$ . Then we have the fixed-effect gravity equation:

$$\log X_{in} = -(\sigma - 1) \log(1 + t_{in}) + D'_{in} \gamma + fe_i + fe_n + u_{in}. \quad (10)$$

- The coefficient of  $\log(1 + t_{in})$  has a structural interpretation:  $-(\sigma - 1)$

## Armington Model: Counterfactuals

- How do changes in  $(t_{in})$  affect trade share  $\lambda_{in}$ , considering direct and indirect effects?
- Parameters:  $(A_i, L_i, \tau_{in}, t_{in}; \sigma)$
- Equilibrium outcomes  $(w_i, X_i)$  such that

1.

$$w_i L_i = \sum_{n=1}^N \frac{1}{1+t_{in}} \lambda_{in} X_n, \quad \lambda_{in} = \frac{\left(\frac{w_i \kappa_{in}}{A_i}\right)^{1-\sigma}}{\sum_{k=1}^N \left(\frac{w_k \kappa_{kn}}{A_k}\right)^{1-\sigma}}, \quad \kappa_{in} = \tau_{in} (1 + t_{in}). \quad (11)$$

2.

$$X_n = w_n L_n + \sum_{i=1}^N \frac{t_{in}}{1+t_{in}} \lambda_{in} X_n. \quad (12)$$

- Counterfactual:
  - Solve  $(w_i, X_i)$  and  $(\lambda_{in})$  under baseline  $(t_{in})$ .
  - Do it again under alternative  $(t_{in})$ .

# Reduced-Form vs. Structural Trade Elasticities

- Reduced-form trade elasticity:
  - Fixed-effect gravity equation:

$$\log \lambda_{in} = -(\sigma - 1) \log (1 + t_{in}) + D'_{in} \gamma + fe_i + fe_n + u_{in} \quad (13)$$

- Ceteris paribus: capture, by design, the **direct effect**
- Structural trade elasticity:
  - *Structural interpretation* of reduced-form trade elasticity: Elasticity of substitution  $\Rightarrow$  Deep parameters
  - GE counterfactual elasticity for *policy evaluation*:  $\Delta (1 + t_{in}) \Rightarrow_{\text{Overall effects}} \Delta X_{in}$ ?

## Reduced-Form vs. Structural Trade Elasticities

- Log-linearizing the equilibrium system: for any  $Z > 0$ , denote  $\tilde{Z} = d \log Z$

$$w_i L_i = \sum_{n=1}^N \frac{1}{1+t_{in}} \lambda_{in} X_n, \quad \lambda_{in} = \frac{\left(\frac{w_i \kappa_{in}}{A_i}\right)^{1-\sigma}}{\sum_{k=1}^N \left(\frac{w_k \kappa_{kn}}{A_k}\right)^{1-\sigma}}, \quad \kappa_{in} = \tau_{in} (1 + t_{in})$$

$$X_n = w_n L_n + \sum_{i=1}^N \frac{t_{in}}{1+t_{in}} \lambda_{in} X_n$$
(14)

- Linearized system: Deriving  $\frac{\partial \log \lambda_{in}}{\partial \log(1+t_{in})}$  from a GE model

$$\tilde{w}_i + \tilde{L}_i = \sum_{n=1}^N \chi_{in} \left( \tilde{\lambda}_{in} + \tilde{X}_n - \widetilde{1+t_{in}} \right), \quad \chi_{in} \equiv \frac{\frac{1}{1+t_{in}} \lambda_{in} X_n}{w_i L_i}$$

$$\tilde{\lambda}_{in} = (1 - \sigma) \left( \tilde{w}_i + \tilde{\kappa}_{in} - \tilde{A}_i \right) - (1 - \sigma) \sum_{k=1}^N \lambda_{kn} \left( \tilde{w}_k + \tilde{\kappa}_{kn} - \tilde{A}_k \right), \quad \tilde{\kappa}_{in} \equiv \tilde{\tau}_{in} + \widetilde{1+t_{in}}$$
(15)

$$\tilde{X}_n = \frac{w_n L_n}{X_n} \left( \tilde{w}_n + \tilde{L}_n \right) + \sum_{i=1}^N \left[ \frac{t_{in} \lambda_{in}}{1+t_{in}} \left( \tilde{\lambda}_{in} + \tilde{X}_n \right) + \frac{\lambda_{in}}{1+t_{in}} \widetilde{1+t_{in}} \right]$$

# Structural Modeling

- Theoretical model:  $F(Y, X, \theta) = 0$ :  $Y$ : equilibrium outcomes;  $X$ : policies;  $\theta$ : deep parameters
- Model characterization: properties of the equilibrium; elasticities  $\nabla_X Y$ , given  $\theta$
- Empirical model:  $G(Y, X, \theta, \varepsilon) = 0$ :  $\varepsilon$ : structural errors to fit the data
- Solve the model: Given  $\theta$  and  $\varepsilon$ , solve  $(X, Y)$  from  $G(Y, X, \theta, \varepsilon) = 0$
- Bring the model to the data:  $G(Y, X, \theta, \varepsilon) = 0 + \text{Data on } (X, Y) + \text{Assumptions} \Rightarrow \theta$
- Counterfactuals: Given  $\theta$ , changes in  $X \Rightarrow$  changes in  $Y$

## Model Meets Data: A “Typical” JMP Today

- Motivational facts: may include reduced-form results
- Model:
  - Setup and equilibrium
  - Theoretical results: key mechanisms
- Bring model to data
  - Estimate key parameters
  - Calibrate other parameters
- Counterfactuals
  - Quantify existing or proposed policies/shocks
  - Optimal policies

## Trade-offs

- Structural modeling:
  - Answering more questions in interest
  - Useful in policy evaluation (both actual and hypothetical policies)
  - With stronger assumptions (some may not be easily justifiable)
- Structural modeling excels in the topics with:
  - Well-established and uncontroversial theories
  - Rich data that can characterize agents' entire behaviors
  - Relevant to important policy questions

# Solving Non-Linear Equations

- General problem:

$$f(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^N, \quad f(\mathbf{x}) \in \mathbb{R}^N, \forall \mathbf{x} \in \mathcal{X} \quad (16)$$

- Example from Burden and Faires

$$\begin{aligned} 3x_1 - \cos(x_2 x_3) - \frac{1}{2} &= 0 \\ x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 &= 0 \\ \exp[-x_1 x_2] + 20x_3 + \frac{10\pi - 3}{3} &= 0 \end{aligned} \quad (17)$$

# Solving Non-Linear Equations

- Jacobian matrix:

$$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_N} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x})}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_N(\mathbf{x})}{\partial x_1} & \frac{\partial f_N(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_N(\mathbf{x})}{\partial x_N} \end{bmatrix} \quad (18)$$

- Jacobian matrix for the example from Burden and Faires

$$J(\mathbf{x}) = \begin{bmatrix} 3 & x_3 \sin(x_2 x_3) & x_2 \sin(x_2 x_3) \\ 2x_1 & -162(x_2 + 0.1) & \cos x_3 \\ -x_2 \exp[-x_1 x_2] & -x_1 \exp[-x_1 x_2] & 20 \end{bmatrix} \quad (19)$$

## Simple Iteration

1. Initial guess  $\mathbf{x}^{(0)}$
2. Compute  $f(\mathbf{x}^{(0)})$
3. Update  $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - f(\mathbf{x}^{(0)})$
4. Iterate until  $\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)}$  where  $k = 1, 2, \dots$

# Matlab Programming

- Matrix manipulation
- Function
- Iteration

## Newton's Method

- First-order Taylor expansion at  $\mathbf{x} = \mathbf{x}^{(0)}$ :

$$f(\mathbf{x}) = f(\mathbf{x}_0) + J(\mathbf{x}^{(0)}) [\mathbf{x} - \mathbf{x}^{(0)}] \quad (20)$$

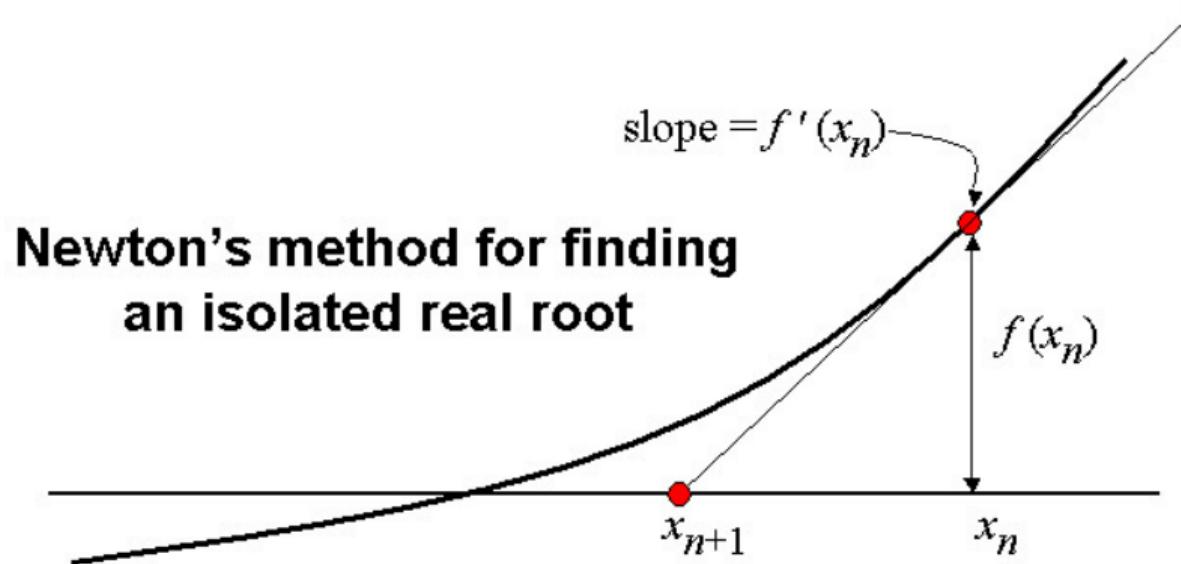
- Since  $f(\mathbf{x}) = 0$ , we approximate  $\mathbf{x}$  by

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - J(\mathbf{x}^{(0)})^{-1} f(\mathbf{x}^{(0)}) \quad (21)$$

- Newton's iterative method:

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} - J(\mathbf{x}^{(k-1)})^{-1} f(\mathbf{x}^{(k-1)}), \quad k = 1, 2, \dots \quad (22)$$

## Newton's Method



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

## Newton's Method

1. Initial guess  $\mathbf{x}^{(0)}$
2. Compute  $f(\mathbf{x}^{(0)})$  and  $J(\mathbf{x}^{(0)})$
3. Update  $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - J(\mathbf{x}^{(0)})^{-1} f(\mathbf{x}^{(0)})$
4. Iterate until  $\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)}$  where  $k = 1, 2, \dots$

## Broyden's Method (Quansi-Newton)

1. Approximate the inverse of the Jacobian matrix: Initial guess  $\mathbf{x}^{(0)}$  and  $B_0 = \mathbf{I}$
2. Update  $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - B_0 f(\mathbf{x}^{(0)})$
3. Let  $\mathbf{h} = \mathbf{x}^{(1)} - \mathbf{x}^{(0)}$  and  $\mathbf{y} = f(\mathbf{x}^{(1)}) - f(\mathbf{x}^{(0)})$ . By the definition of the Jacobian matrix, we want to obtain  $B_1$  such that  $B_1 \mathbf{y} = \mathbf{h}$
4. Update  $B_1 = B_0 + \frac{1}{\mathbf{h}' B_0 \mathbf{y}} (\mathbf{h} - B_0 \mathbf{y}) \mathbf{h}' B_0$ . It is straightforward to verify that  $B_1 \mathbf{y} = \mathbf{h}$
5. Iterate until  $\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)}$  where  $k = 1, 2, \dots$

## Armington Model: Counterfactuals

- Equilibrium outcomes  $(w_i, X_i)$  such that

1.

$$w_i L_i = \sum_{n=1}^N \frac{1}{1+t_{in}} \lambda_{in} X_n, \quad \lambda_{in} = \frac{\left(\frac{w_i \kappa_{in}}{A_i}\right)^{1-\sigma}}{\sum_{k=1}^N \left(\frac{w_k \kappa_{kn}}{A_k}\right)^{1-\sigma}}, \quad \kappa_{in} = \tau_{in} (1 + t_{in}). \quad (23)$$

2.

$$X_n = w_n L_n + \sum_{i=1}^N \frac{t_{in}}{1+t_{in}} \lambda_{in} X_n. \quad (24)$$

- A toy example:

- $N = 3$  and  $\sigma = 4$
- $A = [2, 1, 1]$ ,  $L = [1, 2, 4]$
- $\tau_{in} = 2$  for all  $i \neq n$
- Initially  $t_{in} = 0$  for all  $i, n$

# Bringing the Armington Model to Data

- Challenges:
  - High-dimensional ( $A_i, L_i, \tau_{in}$ )
  - How to get  $\sigma$ ?
- Data:
  - Bilateral trade flows:  $X_{in}$  for all  $i, n$
  - Bilateral tariffs:  $t_{in}$  for all  $i, n$
  - Bilateral distance measures:  $D_{in}$  for all  $i \neq n$

## Exact-Hat Algebra

- Starting from the observed economy, how would changes in tariffs affect trade flows, wages, and welfare?
  - Observed economy: a set of parameters, e.g.  $(A_i, L_i, \tau_{in}) \Leftarrow \text{Data}$ , e.g.  $(X_{in}, t_{in})$
  - Shortcut: computing counterfactuals based on **Data**, without explicitly obtaining parameter values
- Notation: For any variable  $Z > 0$ 
  - Let  $Z'$  be the value of  $Z$  after shocks in interest
  - Let  $\hat{Z} = Z'/Z$
  - WTH:  $(\hat{w}_i, \hat{X}_i, \hat{P}_i)$  under  $\widehat{(1 + t_{in})}$

# Exact-Hat Algebra

- Equilibrium in relative changes:  $(\hat{w}_i, \hat{X}_i)$  satisfy  
1.

$$\hat{w}_i \hat{L}_i w_i L_i = \sum_{n=1}^N \frac{1}{1 + t'_{in}} \hat{\lambda}_{in} \hat{X}_n \lambda_{in} X_n, \quad \hat{\lambda}_{in} = \frac{\left( \frac{\hat{w}_i \hat{\kappa}_{in}}{\hat{A}_i} \right)^{1-\sigma}}{\sum_{k=1}^N \lambda_{kn} \left( \frac{\hat{w}_k \hat{\kappa}_{kn}}{\hat{A}_k} \right)^{1-\sigma}}, \quad \hat{\kappa}_{in} = \hat{\tau}_{in} \widehat{1 + t_{in}}. \quad (25)$$

2.

$$\hat{X}_n X_n = \hat{w}_n \hat{L}_n w_n L_n + \sum_{i=1}^N \frac{t'_{in}}{1 + t'_{in}} \hat{\lambda}_{in} \hat{X}_n \lambda_{in} X_n. \quad (26)$$

- Welfare: measured by real income  $\hat{U}_i \equiv \frac{\hat{X}_i}{\hat{P}_i}$  where  $\hat{P}_i = \left[ \sum_{k=1}^N \lambda_{ki} \left( \frac{\hat{w}_k \hat{\kappa}_{ki}}{\hat{A}_k} \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$
- Exogenous shocks:  $(\widehat{1 + t_{in}}; \hat{\tau}_{in}, \hat{A}_i, \hat{L}_i)$

## Exact-Hat Algebra

- Data:  $(X_{in}, t_{in}, D_{in})$
- Exact-hat algebra requires
  - $\lambda_{in} \equiv \frac{X_{in}}{X_n}$  where  $X_n = \sum_{k=1}^N X_{kn}$
  - $w_i L_i = \sum_{n=1}^N \frac{1}{1+t_{in}} X_{in}$
  - $\sigma$ : estimated by the gravity equation
- Trade imbalances in the data?
  - Exogenous trade imbalances
  - Eliminate trade imbalances in the data, starting from a balanced world

## Summary of Structural Gravity Model

- Question: Tariff changes  $\Rightarrow$  Equilibrium outcomes: trade, wages, welfare
- Armington model
- Model characterization: equilibrium existence and uniqueness; equilibrium in relative changes; stylized version of the model
- Estimation and calibration: data; estimation of  $\sigma$
- Counterfactual experiments
- Sensitivity analysis: e.g. the value of  $\sigma$

# Summary

- Reduced-form vs. Structural modeling
- Example: Armington model
  - Build an equilibrium system
  - Numerically solve the equilibrium system
  - Bring the model to the data
- Structural modeling: sketch