Introduction and Solving Equations

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Empirical Studies in Economics

- Credibility revolution aiming to establish causality, $X \rightarrow Y$, from their correlation
- Gold standard: Quasi-experiments
- Limitations: *What you achieve may not be what you want!*
  - X and/or Y are not observed
  - Difficult to clarify the underlying mechanisms
  - Causality $X \rightarrow Y$ is not sufficiently useful in policy evaluation
- Example: Tariffs $\rightarrow$ Trade and welfare

Gravity Equation

- The "naive" gravity equation proposed by Tinbergen (1962)

$$\log X_{in} = -\epsilon \log (1 + t_{in}) + D_{in}'\gamma + \alpha X_i + \beta X_n + u_{in}, \tag{1}$$

  where X_{in} is the total export from country i to n, t_{in} is the tariff rates of country n on imports from country i, D_{in} is a vector of distance measures such as physical distance, common border, common language, and so on. X_i is the total expenditure of country i

- The fixed-effect gravity equation:

$$\log X_{in} = -\epsilon \log (1 + t_{in}) + D_{in}'\gamma + fe_i + fe_n + u_{in}, \tag{2}$$

  where the fixed effects fe_i and fe_n do not only absorb the total expenditure but also all factors that are exporter- or importer-specific

- Tariff $\Rightarrow$ Trade flows
  - What does ϵ measure?
  - Tariff $\Rightarrow$ Wage $\Rightarrow$ Expenditure $\Rightarrow$ Trade flows?

Structural gravity equation: Armington Model

- N countries. Each country i produces a distinctive variety of goods. Consumers in country n has a CES preference over all varieties:

$$U_n = \left[\sum_{i=1}^{N} C_{in}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \quad \sigma > 1, \tag{3}$$

  where C_{in} is the quantity of goods from country i consumed in country n

- Country i is endowed with L_i workers. The variety i is produced using labor under perfect competition. The unit cost of producing good i is

$$c_i = \frac{w_i}{A_i}, \tag{4}$$

  where w_i is the wage and A_i is the productivity in country i

- Exporting from country i to n incurs
  - an iceberg trade cost, $\tau_{in} \geq 1$, with $\tau_{ii} = 1$
  - an import tariff, $t_{in} \geq 0$, with $t_{ii} = 0$

Armington Model: Equilibrium

- Let X_n be the total expenditure in country n and X_{in} be the value of exports from i to n. Then bilateral trade share can be expressed as

$$\lambda_{in} \equiv \frac{X_{in}}{X_n} = \left[\frac{w_i \kappa_{in}}{A_i}\right]^{1-\sigma} P_n^{\sigma-1}, \quad \kappa_{in} \equiv \tau_{in}\left(1+t_{in}\right), \tag{5}$$

where $P_n = \left[\sum_{i=1}^{N}\left[\frac{w_i \kappa_{in}}{A_i}\right]^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$

- Labor market clearing:

$$w_i L_i = \sum_{n=1}^{N} \frac{1}{1+t_{in}} \lambda_{in} X_n. \tag{6}$$

- Total expenditure equates total income:

$$X_n = w_n L_n + \sum_{i=1}^{N} \frac{t_{in}}{1+t_{in}} \lambda_{in} X_n. \tag{7}$$

- Welfare measure: real income

$$U_i = \frac{X_i}{P_i}. \tag{8}$$

Armington Model: Gravity

- Equation (5) can be re-written into

$$\log\left(X_{in}\right) = -\left(\sigma - 1\right)\log\left(1 + t_{in}\right) - \left(\sigma - 1\right)\log\tau_{in} + \left(\sigma - 1\right)\log\left(\frac{w_i}{A_i}\right) + \log\left(P_n^{\sigma-1}X_n\right) \tag{9}$$

- Suppose that $\tau_{in} = D_{in}'\tilde{\gamma} + \tilde{u}_{in}$. Then we have the fixed-effect gravity equation:

$$\log X_{in} = -\left(\sigma - 1\right)\log\left(1 + t_{in}\right) + D_{in}'\gamma + fe_i + fe_n + u_{in}. \tag{10}$$

- The coefficient of $\log\left(1 + t_{in}\right)$ has a structural interpretation: $-\left(\sigma - 1\right)$

Armington Model: Counterfactuals

- How do changes in (t_{in}) affect trade share λ_{in}, considering direct and indirect effects?

- Parameters: $(A_i, L_i, \tau_{in}, t_{in}; \sigma)$

- Equilibrium outcomes (w_i, X_i) such that
    1. $$w_i L_i = \sum_{n=1}^{N} \frac{1}{1+t_{in}} \lambda_{in} X_n, \quad \lambda_{in} = \frac{\left(\frac{w_i \kappa_{in}}{A_i}\right)^{1-\sigma}}{\sum_{k=1}^{N} \left(\frac{w_k \kappa_{kn}}{A_k}\right)^{1-\sigma}}, \quad \kappa_{in} = \tau_{in}\left(1+t_{in}\right). \tag{11}$$
    2. $$X_n = w_n L_n + \sum_{i=1}^{N} \frac{t_{in}}{1+t_{in}} \lambda_{in} X_n. \tag{12}$$

- Counterfactual:
    - Solve (w_i, X_i) and (λ_{in}) under baseline (t_{in}).
    - Do it again under alternative (t_{in}).

Reduced-Form vs. Structural Trade Elasticities

- Reduced-form trade elasticity:
  - Fixed-effect gravity equation:

$$\log \lambda_{in} = -(\sigma - 1)\log(1 + t_{in}) + D'_{in}\gamma + fe_i + fe_n + u_{in} \tag{13}$$

  - Ceteris paribus: capture, by design, the **direct** effect

- Structural trade elasticity:
  - *Structural interpretation* of reduced-form trade elasticity: Elasticity of substitution $\Rightarrow$ Deep parameters
  - GE counterfactual elasticity for *policy evaluation*: $\Delta(1 + t_{in}) \Rightarrow_{\textbf{Overall effects}} \Delta X_{in}$?

Reductions-Form vs. Structural Trade Elasticities

- Log-linearizing the equilibrium system: for any $Z > 0$, denote $\tilde{Z} = d\log Z$

$$w_i L_i = \sum_{n=1}^{N} \frac{1}{1+t_{in}} \lambda_{in} X_n, \quad \lambda_{in} = \frac{\left(\frac{w_i \kappa_{in}}{A_i}\right)^{1-\sigma}}{\sum_{k=1}^{N} \left(\frac{w_k \kappa_{kn}}{A_k}\right)^{1-\sigma}}, \quad \kappa_{in} = \tau_{in}\left(1+t_{in}\right)$$
$$X_n = w_n L_n + \sum_{i=1}^{N} \frac{t_{in}}{1+t_{in}} \lambda_{in} X_n \tag{14}$$

- Linearized system: Deriving $\frac{\partial \log \lambda_{in}}{\partial \log(1+t_{in})}$ from a GE model

$$\tilde{w}_i + \tilde{L}_i = \sum_{n=1}^{N} \chi_{in} \left(\tilde{\lambda}_{in} + \tilde{X}_n - \widetilde{1+t_{in}}\right), \quad \chi_{in} \equiv \frac{\frac{1}{1+t_{in}} \lambda_{in} X_n}{w_i L_i}$$
$$\tilde{\lambda}_{in} = (1-\sigma)\left(\tilde{w}_i + \tilde{\kappa}_{in} - \tilde{A}_i\right) - (1-\sigma) \sum_{k=1}^{N} \lambda_{kn} \left(\tilde{w}_k + \tilde{\kappa}_{kn} - \tilde{A}_k\right), \quad \tilde{\kappa}_{in} \equiv \tilde{\tau}_{in} + \widetilde{1+t_{in}} \tag{15}$$
$$\tilde{X}_n = \frac{w_n L_n}{X_n}\left(\tilde{w}_n + \tilde{L}_n\right) + \sum_{i=1}^{N} \left[\frac{t_{in}\lambda_{in}}{1+t_{in}} \left(\tilde{\lambda}_{in} + \tilde{X}_n\right) + \frac{\lambda_{in}}{1+t_{in}} \widetilde{1+t_{in}}\right]$$

Structural Modeling

- Theoretical model: $F(Y, X, \theta) = 0$: Y: equilibrium outcomes; X: policies; θ: deep parameters
- Model characterization: properties of the equilibrium; elasticities $\nabla_X Y$, given θ
- Empirical model: $G(Y, X, \theta, \varepsilon) = 0$: ε: structural errors to fit the data
- Solve the model: Given θ and ε, solve (X, Y) from $G(Y, X, \theta, \varepsilon) = 0$
- Bring the model to the data: $G(Y, X, \theta, \varepsilon) = 0$ + Data on (X, Y) + Assumptions $\Rightarrow \theta$
- Counterfactuals: Given θ, changes in $X \Rightarrow$ changes in Y

Model Meets Data: A "Typical" JMP Today

- Motivational facts: may include reduced-form results
- Model:
  - Setup and equilibrium
  - Theoretical results: key mechanisms
- Bring model to data
  - Estimate key parameters
  - Calibrate other parameters
- Counterfactuals
  - Quantify existing or proposed policies/shocks
  - Optimal policies

Trade-offs

- Structural modeling:
  - Answering more questions in interest
  - Useful in policy evaluation (both actual and hypothetical policies)
  - With stronger assumptions (some may not be easily justifiable)
- Structural modeling excels in the topics with:
  - Well-established and uncontroversial theories
  - Rich data that can characterize agents' entire behaviors
  - Relevant to important policy questions

Solving Non-Linear Equations

- General problem:

$$f(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^N, \quad f(\mathbf{x}) \in \mathbb{R}^N, \forall \mathbf{x} \in \mathcal{X} \tag{16}$$

- Example from Burden and Faires

$$\begin{aligned} 3x_1 - \cos(x_2 x_3) - \frac{1}{2} &= 0 \\ x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 &= 0 \\ \exp[-x_1 x_2] + 20x_3 + \frac{10\pi - 3}{3} &= 0 \end{aligned} \tag{17}$$

Solving Non-Linear Equations

- Jacobian matrix:

$$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_N} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x})}{\partial x_N} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_N(\mathbf{x})}{\partial x_1} & \frac{\partial f_N(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_N(\mathbf{x})}{\partial x_N} \end{bmatrix} \tag{18}$$

- Jacobian matrix for the example from Burden and Faires

$$J(\mathbf{x}) = \begin{bmatrix} 3 & x_3 \sin(x_2 x_3) & x_2 \sin(x_2 x_3) \\ 2x_1 & -162(x_2 + 0.1) & \cos x_3 \\ -x_2 \exp[-x_1 x_2] & -x_1 \exp[-x_1 x_2] & 20 \end{bmatrix} \tag{19}$$

Simple Iteration

1. Initial guess $\mathbf{x}^{(0)}$
2. Compute $f\left(\mathbf{x}^{(0)}\right)$
3. Update $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - f\left(\mathbf{x}^{(0)}\right)$
4. Iterate until $\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)}$ where $k = 1, 2, \ldots$

Matlab Programming

- Matrix manipulation
- Function
- Iteration

Newton's Method

- First-order Taylor expansion at $\mathbf{x} = \mathbf{x}^{(0)}$:

$$f(\mathbf{x}) = f(\mathbf{x}_0) + J\left(\mathbf{x}^{(0)}\right)\left[\mathbf{x} - \mathbf{x}^{(0)}\right] \tag{20}$$

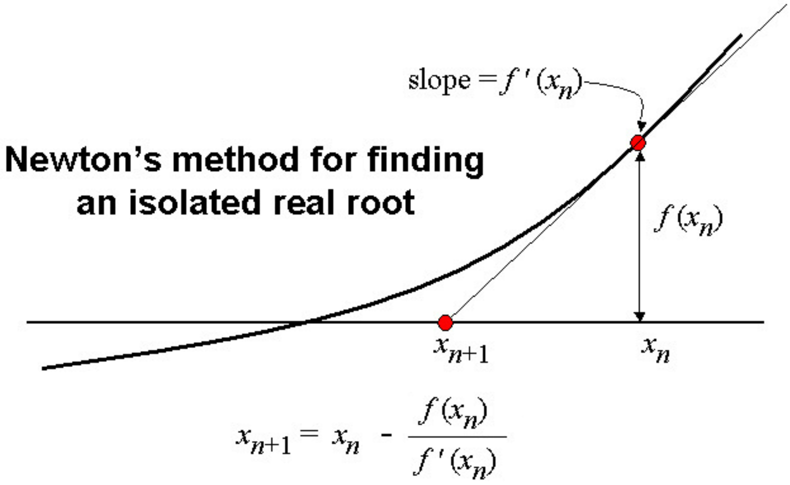
- Since $f(\mathbf{x}) = 0$, we approximate $\mathbf{x}$ by

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - J\left(\mathbf{x}^{(0)}\right)^{-1} f\left(\mathbf{x}^{(0)}\right) \tag{21}$$

- Newton's iterative method:

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} - J\left(\mathbf{x}^{(k-1)}\right)^{-1} f\left(\mathbf{x}^{(k-1)}\right), \quad k = 1, 2, \dots \tag{22}$$

Newton's Method



Newton's Method

1. Initial guess $\mathbf{x}^{(0)}$
2. Compute $f\left(\mathbf{x}^{(0)}\right)$ and $J\left(\mathbf{x}^{(0)}\right)$
3. Update $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - J\left(\mathbf{x}^{(0)}\right)^{-1} f\left(\mathbf{x}^{(0)}\right)$
4. Iterate until $\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)}$ where $k = 1, 2, \ldots$

Broyden's Method (Quansi-Newton)

1. Approximate the inverse of the Jacobian matrix: Initial guess $\mathbf{x}^{(0)}$ and $B_0 = \mathbf{I}$
2. Update $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - B_0 f\left(\mathbf{x}^{(0)}\right)$
3. Let $\mathbf{h} = \mathbf{x}^{(1)} - \mathbf{x}^{(0)}$ and $\mathbf{y} = f\left(\mathbf{x}^{(1)}\right) - f\left(\mathbf{x}^{(0)}\right)$. By the definition of the Jacobian matrix, we want to obtain B_1 such that $B_1\mathbf{y} = \mathbf{h}$
4. Update $B_1 = B_0 + \frac{1}{\mathbf{h}' B_0 \mathbf{y}} \left(\mathbf{h} - B_0\mathbf{y}\right) \mathbf{h}' B_0$. It is straightforward to verify that $B_1\mathbf{y} = \mathbf{h}$
5. Iterate until $\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)}$ where $k = 1, 2, \ldots$

Armington Model: Counterfactuals

- Equilibrium outcomes (w_i, X_i) such that

  1. $$w_i L_i = \sum_{n=1}^{N} \frac{1}{1+t_{in}} \lambda_{in} X_n, \quad \lambda_{in} = \frac{\left(\frac{w_i \kappa_{in}}{A_i}\right)^{1-\sigma}}{\sum_{k=1}^{N} \left(\frac{w_k \kappa_{kn}}{A_k}\right)^{1-\sigma}}, \quad \kappa_{in} = \tau_{in}\left(1+t_{in}\right). \tag{23}$$

  2. $$X_n = w_n L_n + \sum_{i=1}^{N} \frac{t_{in}}{1+t_{in}} \lambda_{in} X_n. \tag{24}$$

- A toy example:
  - $N = 3$ and $\sigma = 4$
  - $A = [2, 1, 1]$, $L = [1, 2, 4]$
  - $\tau_{in} = 2$ for all $i \neq n$
  - Initially $t_{in} = 0$ for all i, n

Bringing the Armington Model to Data

- Challenges:
  - High-dimensional (A_i, L_i, τ_{in})
  - How to get σ?
- Data:
  - Bilateral trade flows: X_{in} for all i, n
  - Bilateral tariffs: t_{in} for all i, n
  - Bilateral distance measures: D_{in} for all $i \neq n$

Exact-Hat Algebra

- Starting from the observed economy, how would changes in tariffs affect trade flows, wages, and welfare?
  - Observed economy: a set of parameters, *e.g.* $(A_i, L_i, \tau_{in}) \Leftarrow$ Data, *e.g.* (X_{in}, t_{in})
  - Shortcut: computing counterfactuals based on Data, without explicitly obtaining parameter values
- Notation: For any variable $Z > 0$
  - Let Z' be the value of Z after shocks in interest
  - Let $\hat{Z} = Z'/Z$
  - WTH: $\left(\hat{w}_i, \hat{X}_i, \hat{P}_i\right)$ under $\left(\widehat{1 + t_{in}}\right)$

Exact-Hat Algebra

- Equilibrium in relative changes: $\left(\hat{w}_i, \hat{X}_i\right)$ satisfy
  1. $$\hat{w}_i \hat{L}_i w_i L_i = \sum_{n=1}^{N} \frac{1}{1+t'_{in}} \hat{\lambda}_{in} \hat{X}_n \lambda_{in} X_n, \quad \hat{\lambda}_{in} = \frac{\left(\frac{\hat{w}_i \hat{\kappa}_{in}}{\hat{A}_i}\right)^{1-\sigma}}{\sum_{k=1}^{N} \lambda_{kn} \left(\frac{\hat{w}_k \hat{\kappa}_{kn}}{\hat{A}_k}\right)^{1-\sigma}}, \quad \hat{\kappa}_{in} = \hat{\tau}_{in} \widehat{1+t_{in}}. \tag{25}$$
  2. $$\hat{X}_n X_n = \hat{w}_n \hat{L}_n w_n L_n + \sum_{i=1}^{N} \frac{t'_{in}}{1+t'_{in}} \hat{\lambda}_{in} \hat{X}_n \lambda_{in} X_n. \tag{26}$$
- Welfare: measured by real income $\hat{U}_i \equiv \frac{\hat{X}_i}{\hat{P}_i}$ where $\hat{P}_i = \left[\sum_{k=1}^{N} \lambda_{ki} \left(\frac{\hat{w}_k \hat{\kappa}_{ki}}{\hat{A}_k}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$
- Exogenous shocks: $\left(\widehat{1+t_{in}}; \hat{\tau}_{in}, \hat{A}_i, \hat{L}_i\right)$

Exact-Hat Algebra

- Data: (X_{in}, t_{in}, D_{in})
- Exact-hat algebra requires
  - $\lambda_{in} \equiv \frac{X_{in}}{X_n}$ where $X_n = \sum_{k=1}^{N} X_{kn}$
  - $w_i L_i = \sum_{n=1}^{N} \frac{1}{1+t_{in}} X_{in}$
  - σ: estimated by the gravity equation
- Trade imbalances in the data?
  - Exogenous trade imbalances
  - Eliminate trade imbalances in the data, starting from a balanced world

Summary of Structural Gravity Model

- Question: Tariff changes $\Rightarrow$ Equilibrium outcomes: trade, wages, welfare
- Armington model
- Model characterization: equilibrium existence and uniqueness; equilibrium in relative changes; stylized version of the model
- Estimation and calibration: data; estimation of σ
- Counterfactual experiments
- Sensitivity analysis: *e.g.* the value of σ

Summary

- Reduced-form vs. Structural modeling
- Example: Armington model
  - Build an equilibrium system
  - Numerically solve the equilibrium system
  - Bring the model to the data
- Structural modeling: sketch