# Numerical Differentiation and Integration

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# Numerical Differentiation

- Forward differencing:
  - $f'(x) \simeq \frac{f(x+h)-f(x)}{h}$
  - $\frac{\partial f(x)}{\partial x_i} \simeq \frac{f(x_1,\ldots,x_i+h_i,\ldots,x_n)-f(x_1,\ldots,x_i,\ldots,x_n)}{h_i}$
- Backward differencing:
  - $f'(x) \simeq \frac{f(x)-f(x-h)}{h}$
  - $\frac{\partial f(x)}{\partial x_i} \simeq \frac{f(x_1,\ldots,x_i,\ldots,x_n)-f(x_1,\ldots,x_i-h_i,\ldots,x_n)}{h_i}$
- Note:
  - In practice, one uses forward or backward differences depending on whether we care more about left or right derivative
  - $h = \max(|x|, 1)\sqrt{\epsilon}$ where $\epsilon$ is the machine precision (about $10^{-15}$ in the Matlab)

# Numerical Differentiation

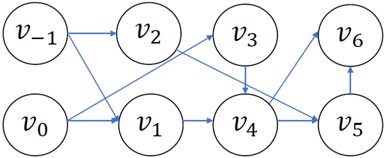
- Centered differencing:
  - $f'(x) \simeq \frac{f(x+h)-f(x-h)}{2h}$
  - $\frac{\partial f(x)}{\partial x_i} \simeq \frac{f(x_1,\ldots,x_i+h_i,\ldots,x_n)-f(x_1,\ldots,x_i-h_i,\ldots,x_n)}{2h_i}$
- Richardson's extrapolation (a fourth-order approximation):

$$f'(x) \simeq \frac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h} \tag{1}$$

- Example: $y = f(x_1, x_2) = \left[x_1^2 + x_1/x_2 - \exp(x_2)\right][x_1/x_2 - \exp(x_2)]$ at $(1, 1)$

# Automatic Differentiation

- Example: $y = \left[x_1^2 + x_1/x_2 - \exp(x_2)\right] \left[x_1/x_2 - \exp(x_2)\right]$
  - Intermediate variables: $v_{-1} = x_1$, $v_0 = x_2$, $v_1 = v_{-1}/v_0$, $v_2 = v_{-1}^2$, $v_3 = \exp(v_0)$, $v_4 = v_1 - v_3$, $v_5 = v_2 + v_4$, and $v_6 = v_4 \cdot v_5 = y$
  - Computational graph:




  - By the chain rule

$$
\begin{aligned}
\frac{\partial y}{\partial x_1} &= \frac{\partial y}{\partial v_6}\left(\frac{\partial v_6}{\partial v_4}\frac{\partial v_4}{\partial v_1}\frac{\partial v_1}{\partial v_{-1}} + \frac{\partial v_6}{\partial v_5}\left(\frac{\partial v_5}{\partial v_4}\frac{\partial v_4}{\partial v_1}\frac{\partial v_1}{\partial v_{-1}} + \frac{\partial v_5}{\partial v_2}\frac{\partial v_2}{\partial v_{-1}}\right)\right)\frac{\partial v_{-1}}{\partial x_1} \\
\frac{\partial y}{\partial x_2} &= \frac{\partial y}{\partial v_6}\left(\frac{\partial v_6}{\partial v_4}\left(\frac{\partial v_4}{\partial v_1}\frac{\partial v_1}{\partial v_0} + \frac{\partial v_4}{\partial v_3}\frac{\partial v_3}{\partial v_0}\right) + \frac{\partial v_6}{\partial v_5}\frac{\partial v_5}{\partial v_4}\frac{\partial v_4}{\partial v_1}\frac{\partial v_1}{\partial v_0}\right)\frac{\partial v_0}{\partial x_2}
\end{aligned}
\tag{2}
$$

# Numerical Quadrature Methods

- Integration:

$$S = \int_I f(x)dx \tag{3}$$

- Quadrature:

$$\int_I f(x)dx \simeq \sum_{i=1}^{n} w_i f\left(x_i\right) \tag{4}$$

- Quadrature methods differ only in how the quadrature weights $w_i$ and the quadrature nodes $x_i$ are chosen

# Newton-Cotes Methods

- Integration:

$$\int_a^b f(x)dx \tag{5}$$

- Trapezoid rule:
  - $x_i = a + (i-1)\,h$ where $h = (b-a)\,/n$
  - $\int_a^b f(x)dx \simeq \sum_{i=1}^n w_i f(x_i)$ where $w_1 = w_n = h/2$ and $w_i = h$, otherwise
- Simpson's rule: piece-wise quadratic
  - $x_i = a + (i-1)\,h$ where $h = (b-a)\,/(n-1)$ and $n$ is odd
  - $\int_{x_{2j-1}}^{x_{2j+1}} f(x)dx \simeq \frac{h}{3}\left[f(x_{2j-1}) + 4f(x_{2j}) + f(x_{2j+1})\right]$
  - $\int_a^b f(x)dx \simeq \sum_{i=1}^n w_i f(x_i)$ where $w_1 = w_n = h/3$ and, otherwise, $w_i = 4h/3$ if $i$ is odd and $w_i = 2h/3$ if $i$ is even

# Newton-Cotes Methods: Higher dimensional Integration

- Integration:

$$\int_{x_1 \in [a_1, b_1]} \int_{x_2 \in [a_2, b_2]} f(x_1, x_2)\, dx_1 dx_2 \tag{6}$$

- Newton-Cotes nodes and weights:
  - $\{(x_{1i}, w_{1i}) \,|i = 1, 2, \ldots, n_1\}$
  - $\{(x_{2j}, w_{2j}) \,|j = 1, 2, \ldots, n_2\}$
  - Nodes: $\{(x_{1i}, x_{2j}) \,|i = 1, 2, \ldots, n_1; j = 1, 2, \ldots, n_2\}$
  - Weights: $\{w_{ij} = w_{1i} w_{2j} | i = 1, 2, \ldots, n_1; j = 1, 2, \ldots, n_2\}$

# Gaussian Quadrature

- Integration:

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i) \tag{7}$$

- Nodes: $x_i$ are the roots of a Legendre polynomial of degree $n$, $P_n(x)$

- Weights: $w_i = \frac{2}{\left(1-x_i^2\right)\left[P_n'(x_i)\right]^2}$

- Legendre polynomials:
  - Generating function: $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(t)t^n$, where $P_0(x) = 1$ and $P_1(x) = x$
  - Recursive: $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$
  - Rodrigues' formula: $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$

# Monte Carlo Integration

- Integration (multi-dimensional): $S = \int_I f(x)dx$
- Let $V = \int_I dx$
- $n$ uniform samples: $x_1, \ldots, x_n \in I$
- Then $S \simeq \frac{V}{n} \sum_{i=1}^{n} f(x_i)$

# Numerical Integration: Example

- $H(x, y) = \begin{cases} 1 & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{else} \end{cases}$

- Let $I = [-1, 1] \times [-1, 1]$. Compute

$$S = \int_I H(x, y) dxdy \tag{8}$$

# Example: Variable Markups in the Quantitative Trade Model

- $i = 1, \ldots, N$ countries with labor $(L_i)_{i=1}^{N}$
- The *Kimball*'s preference over a continuum of varieties:

$$\int_{\omega \in \Omega_n} H\left(\frac{q_n(\omega)}{Q_n}\right) d\omega = 1, \tag{9}$$

  where $Q_n$ is the aggregate consumption and the function $H(.)$ is strictly increasing, strictly concave, and satisfies $H(1) = 1$

- CES as a special case: $H(q) = q^{\frac{\sigma-1}{\sigma}}$ for $\sigma > 1$

# Example: Variable Markups in the Quantitative Trade Model

- The inverse demand function of variety $\omega$ in country $n$ can be expressed as

$$\frac{p_n(\omega)}{P_n} = H'\left(\frac{q_n(\omega)}{Q_n}\right) D_n, \tag{10}$$

where the *demand index*

$$D_n = \left[\int_{\omega \in \Omega_n} H'\left(\frac{q_n(\omega)}{Q_n}\right) \frac{q_n(\omega)}{Q_n} d\omega\right]^{-1}, \tag{11}$$

and the price index for final goods

$$P_n = \int_{\omega \in \Omega_n} p_n(\omega) \frac{q_n(\omega)}{Q_n} d\omega = \int_{\omega \in \Omega_n} p_n(\omega) H'^{-1}\left(\frac{p_n(\omega)}{P_n} \frac{1}{D_n}\right) d\omega \tag{12}$$

# Example: Variable Markups in the Quantitative Trade Model

- Klenow and Willis (2016):

$$H(q) = 1 + (\sigma - 1) \exp\left(\frac{1}{\varepsilon}\right) \varepsilon^{\frac{\sigma}{\varepsilon}-1} \left[\Gamma\left(\frac{\sigma}{\varepsilon}, \frac{1}{\varepsilon}\right) - \Gamma\left(\frac{\sigma}{\varepsilon}, \frac{q^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right)\right], \tag{13}$$

  with $\sigma > 1$ and $\varepsilon \geq 0$ and where $\Gamma(s, x)$ denotes the upper incomplete Gamma function

$$\Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t} dt \tag{14}$$

- $\varepsilon = 0$: the CES case $H(q) = q^{\frac{\sigma-1}{\sigma}}$

- We have

$$H'(q) = \frac{\sigma - 1}{\sigma} \exp\left(\frac{1 - q^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right) \tag{15}$$

# Example: Variable Markups in the Quantitative Trade Model

- Each variety $\omega$ is produced by a firm using labor under monopolistic competition
- Fixed marketing cost of firms to serve market $n$: $F_n$ in units of $n$'s production labor
- Iceberg trade cost from country $\ell$ to $n$: $\tau_{\ell n} \geq 1$ with $\tau_{\ell\ell} = 1$
- After paying a fixed entry cost $f^e$ in terms of country $i$'s labor, firm $\omega$ draws a productivity $\varphi_i(\omega)$ from

$$\Pr\left(\varphi_i(\omega) \leq \varphi\right) = 1 - T_i \varphi^{-\theta}, \tag{16}$$

  with support $\varphi \geq T_i^{\frac{1}{\theta}}$

# Example: Variable Markups in the Quantitative Trade Model

- Conditional on firm $\omega$ from country $i$ serving country $n$, its effective cost $c_{in}(\omega) = \frac{w_i \tau_{in}}{\varphi_i(\omega)}$ satisfies

$$\Pr\left(c_{in}(\omega) \leq c\right) = \bar{T}_{in}^{\theta} c^{\theta}, \quad c \leq \frac{1}{\bar{T}_{in}}, \quad \bar{T}_{in} \equiv T_i^{\frac{1}{\theta}} \left(w_i \tau_{in}\right)^{-1} \tag{17}$$

- The operating profit of firm $\omega$ from country $i$ serving market $n$ is

$$\tilde{\pi}_{in}(\omega) = \max_{q_{in}(\omega) \geq 0} \left[ H'\left(\frac{q_{in}(\omega)}{Q_n}\right) D_n P_n - c_{in}(\omega) \right] q_{in}(\omega). \tag{18}$$

- Let $s_{in}(\omega) := \frac{q_{in}(\omega)}{Q_n}$ be the relative output. The optimal price can be expressed as a markup over the unit cost:

$$p_{in}(\omega) = \mu\left(s_{in}(\omega)\right) c_{in}(\omega), \quad \mu\left(s_{in}(\omega)\right) := \frac{\sigma s_{in}(\omega)^{-\frac{\varepsilon}{\sigma}}}{\sigma s_{in}(\omega)^{-\frac{\varepsilon}{\sigma}} - 1}. \tag{19}$$

- Let $X_n := P_n Q_n$. Then the operating profit can be expressed as

$$\tilde{\pi}_{in}(\omega) = \frac{s_{in}(\omega)^{\frac{\varepsilon}{\sigma}}}{\sigma} \frac{\sigma - 1}{\sigma} \exp\left(\frac{1 - s_{in}(\omega)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right) s_{in}(\omega) D_n X_n. \tag{20}$$

# Example: Variable Markups in the Quantitative Trade Model

- Combining Equation (10) and (19), we can can express $s_{in}(\omega)$ in terms of $c_{in}(\omega)$:

$$\frac{\sigma}{\sigma - s_{in}(\omega)^{\frac{\varepsilon}{\sigma}}} c_{in}(\omega) = \frac{\sigma - 1}{\sigma} \exp\left(\frac{1 - s_{in}(\omega)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right) D_n P_n \Rightarrow s_{in}(\omega) = s_n(c_{in}(\omega)) \tag{21}$$

- Then firm $\omega$ from country $i$ will serve destination market $n$ if and only if

$$\frac{s_n(c_{in}(\omega))^{\frac{\varepsilon}{\sigma}}}{\sigma} \frac{\sigma - 1}{\sigma} \exp\left(\frac{1 - s_n(c_{in}(\omega))^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right) s_n(c_{in}(\omega)) D_n X_n \geq w_n F_n \tag{22}$$

- We assume that the cost cut-off $c_n^*$ above which firms from country $i$ will not serve market $n$ satisfies:

$$c_n^* < \min\left\{\frac{1}{\bar{T}_{in}}, \frac{\sigma - 1}{\sigma} \exp\left(\frac{1}{\varepsilon}\right) D_n P_n\right\}, \quad \forall (i, n) \tag{23}$$

# Example: Variable Markups in the Quantitative Trade Model

- Let $X_{in}$ be the total sales of firms originated from country $i$ in destination market $n$ and $\Pi_{in}$ be the associated profit. Let $M_i$ be the mass of firms in country $i$
  - $$\lambda_{in} \equiv \frac{X_{in}}{X_n} = \frac{M_i \bar{T}_{in}^{\theta}}{\sum_{k=1}^{N} M_k \bar{T}_{kn}^{\theta}} \tag{24}$$
  - $$\frac{\Pi_{in}}{X_{in}} = \eta_n = \frac{\int_0^{c_n^*} \frac{s_n(c)^{\frac{\varepsilon}{\sigma}}}{\sigma} \frac{\sigma-1}{\sigma} \exp\left[\frac{1-s_n(c)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right] s_n(c) c^{\theta-1} dc}{\int_0^{c_n^*} \frac{\sigma-1}{\sigma} \exp\left[\frac{1-s_n(c)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right] s_n(c) c^{\theta-1} dc} \tag{25}$$
- The price index can be given by

$$P_n = \sum_{i=1}^{N} \nu_{in}^{P} M_i \bar{T}_{in}^{\theta}, \quad \nu_{in}^{P} \equiv \left[\theta \int_0^{c_n^*} \frac{\sigma}{\sigma - s_n(c)^{\frac{\varepsilon}{\sigma}}} c s_n(c) c^{\theta-1} dc\right] \tag{26}$$

- By the definition of $H(.)$, we have

$$\sum_{i=1}^{N} M_i \bar{T}_{in}^{\theta} \theta \int_0^{c_n^*} H\left(s(c)\right) c^{\theta-1} dc = 1 \tag{27}$$

# Example: Variable Markups in the Quantitative Trade Model

- Equilibrium consists of $(w_i, M_i, P_i, D_i, c_i^*)_{i=1}^N$ such that
  1. $(w_i)$ is determined by labor market clearing

$$w_i L_i = \sum_{n=1}^{N} (1-\eta_n) \lambda_{in} X_n + w_i F_i (c_i^*)^\theta \sum_{k=1}^{N} M_k \bar{T}_{ki}^\theta + \sum_{n=1}^{N} \left[\eta_n \lambda_{in} X_n - w_n F_n (c_n^*)^\theta M_i \bar{T}_{in}^\theta\right] \tag{28}$$

  2. Firm mass $M_i$ is determined by the free-entry condition:

$$M_i w_i f^e = \sum_{n=1}^{N} \left[\eta_n \lambda_{in} X_n - w_n F_n (c_n^*)^\theta M_i \bar{T}_{in}^\theta\right] \tag{29}$$

  3. Total absorption: $X_i = w_i L_i$
  4. The price index is determined by Equation (26)
  5. $(c_n^*, D_n)$ are jointly determined by Equation (22) and (27)

# Example: Variable Markups in the Quantitative Trade Model

- Draw $J$ numbers from the uniform distribution $U[0,1]$, sorting them as $u_1 < u_2 < \ldots < u_J$
- Initial guess $(w_i, M_i, P_i, D_i)_{i=1}^N$
- Compute the relative quantity cutoff below which firms will not serve the market $n$ by solving:

$$\frac{(s_n^*)^{\frac{\varepsilon}{\sigma}}}{\sigma} \frac{\sigma - 1}{\sigma} \exp\left(\frac{1 - (s_n^*)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right) s_n^* D_n X_n = w_n F_n \tag{30}$$

- The cost cutoff can then by computed by

$$c_n^* = \frac{\sigma - (s_n^*)^{\frac{\varepsilon}{\sigma}}}{\sigma} \frac{\sigma - 1}{\sigma} \exp\left(\frac{1 - (s_n^*)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right) D_n P_n \tag{31}$$

- Compute $\bar{T}_{in}$ by its definition and $\lambda_{in}$ by Equation (24)

# Example: Variable Markups in the Quantitative Trade Model

- Let $c_n^j = u_j c_n^*$. The corresponding relative quantity $s_n^j$ can be solved by Equation (21)

- Compute

$$\eta_n = \frac{\sum_{j=1}^{J} \frac{\left(s_n^j\right)^{\frac{\varepsilon}{\sigma}}}{\sigma} \exp\left[\frac{1-\left(s_n^j\right)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right] s_n^j \left(c_n^j\right)^{\theta-1}}{\sum_{j=1}^{J} \exp\left[\frac{1-\left(s_n^j\right)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right] s_n^j \left(c_n^j\right)^{\theta-1}} \tag{32}$$

- Compute

$$\nu_n^P = \frac{c_n^*}{J} \sum_{j=1}^{J} \frac{\theta\sigma}{\sigma - \left(s_n^j\right)^{\frac{\varepsilon}{\sigma}}} s_n^j \left(c_n^j\right)^{\theta} \tag{33}$$

- Update $D_n$ by:

$$D_n = D_n \times \left[\sum_{i=1}^{N} M_i \bar{T}_{in}^{\theta} \frac{c_n^*}{J} \theta \sum_{j=1}^{J} H\left(s_n^j\right) \left(c_n^j\right)^{\theta-1}\right]^{\frac{1}{1+\theta}} \tag{34}$$

# Example: Variable Markups in the Quantitative Trade Model

- $(w_i, M_i, P_i)$ are updated, respectively, by Equation (28), (29), and (26). Repeat until convergence. Notice that we update $P_n$ by the following equation:

$$P_n^{(t+1)} = \left[ \left( P_n^{(t)} \right)^{-\theta-1} \nu_n^P \sum_{i=1}^{N} M_i \bar{T}_{in}^{\theta} \right]^{-\frac{1}{\theta}} \tag{35}$$

# Example: Variable Markups in the Quantitative Trade Model

- Welfare: $U_i = \frac{w_i}{P_i}$

- Aggregate markup: as suggested by Edmond et al. (2019), we compute a sales-weighted harmonic average:

$$\begin{aligned}
\bar{\mu}_n^D &:= \left[\theta\left(\sum_{i=1}^{N} M_i \bar{T}_{in}^{\theta}\right)\int_0^{c_n^*}\left(\frac{\sigma}{\sigma - s_n(c)^{\frac{\varepsilon}{\sigma}}}\right)^{-1}\frac{p_n(c)}{P_n}s_n(c)c^{\theta-1}dc\right]^{-1} \\
&= \left[\theta\left(\sum_{i=1}^{N} M_i \bar{T}_{in}^{\theta}\right)\frac{1}{P_n}\int_0^{c_n^*} s_n(c)c^{\theta}dc\right]^{-1} \\
&= \left[\frac{\theta}{\nu_n^P}\int_0^{c_n^*} s_n(c)c^{\theta}dc\right]^{-1}
\end{aligned} \tag{36}$$

# Example: Variable Markups in the Quantitative Trade Model

- Quantification:
  - How do changes in trade costs, $\tau_{in}$, affect markups in exporting country $i$ as well as importing country $n$?
  - How do welfare gains from trade rely on the markup variations? e.g. Comparing with the constant-markup models?

# Summary

- Numerical differentiation:
  - Crucial for nonlinear solvers and nonlinear optimization
  - Trade-off between accuracy and efficiency of computation
  - Promising direction: automatic differentiation and deep learning
- Numerical integration:
  - Heterogeneous-agent models
  - Random variables $\Rightarrow$ Moments