Hint_Quantitative Trade Model with Variable Markups

Preference

- Consider N countries, indexed by $i=1,\ldots,N$ countries, with labor $(L_i)_{i=1}^N$
- The representative consumer in country n has the Kimball's preference over a continuum of varieties:

$$\int_{\omega\in\Omega_n} H\left(\frac{q_n(\omega)}{Q_n}\right)d\omega=1$$

  - Q_n is the aggregate quantity consumed
  - Function $H(.)$ is strictly increasing, strictly concave, and satisfies $H(1)=1$.
  - CES: $H(x)=x^{\frac{\sigma-1}{\sigma}}$ for $\sigma>1\Rightarrow Q_n=\left(\int_{\omega\in\Omega_n} q_n(\omega)^{\frac{\sigma-1}{\sigma}}d\omega\right)^{\frac{\sigma}{\sigma-1}}$
- The inverse demand function of variety ω in country n:

$$\frac{p_n(\omega)}{P_n}=H'\left(\frac{q_n(\omega)}{Q_n}\right)D_n$$

  - Demand index: $D_n\equiv\left[\int_{\omega\in\Omega_n} H'\left(\frac{q_n(\omega)}{Q_n}\right)\frac{q_n(\omega)}{Q_n}d\omega\right]^{-1}$
  - Price index: $P_n=\int_{\omega\in\Omega_n} p_n(\omega)\frac{q_n(\omega)}{Q_n}d\omega=\int_{\omega\in\Omega_n} p_n(\omega)H'^{-1}\left(\frac{p_n(\omega)}{P_n}\frac{1}{D_n}\right)d\omega$
  - Please derive D_n and P_n under CES
- Klenow and Willis (2016):

$$H(x)=1+(\sigma-1)\exp\left(\frac{1}{\varepsilon}\right)\varepsilon^{\frac{\sigma}{\varepsilon}-1}\left[\Gamma\left(\frac{\sigma}{\varepsilon},\frac{1}{\varepsilon}\right)-\Gamma\left(\frac{\sigma}{\varepsilon},\frac{x^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right)\right]$$

  - $\Gamma(s,x)$ denotes the upper incomplete Gamma function: $\Gamma(s,x):=\int_x^\infty t^{s-1}e^{-t}dt$
  - $\sigma>1;\ \varepsilon\geq 0$
  - F.O.C.: $H'(x)=\frac{\sigma-1}{\sigma}\exp\left(\frac{1-x^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right)$
  - CES: $\varepsilon=0\Rightarrow H(x)=x^{\frac{\sigma-1}{\sigma}}$

Technology

- Each variety ω is produced by a firm using labor under monopolistic competition
- To serve destination market n, firm needs to pay a fixed marketing cost $F_n>0$ in units of labor in n
- Exporting from country i to n incurs an iceberg trade cost $\tau_{in}\geq 1$ with $\tau_{ii}=1$
- Before entry, each potential firm in country i pays a fixed entry cost f^e in units of labor in i

- After paying the fixed entry cost f^e, firm ω in country i draws its productivity $\varphi_i(\omega)$ from a Pareto distribution:

$$\Pr\left(\varphi_i(\omega) \leq \varphi\right) = 1 - T_i \varphi^{-\theta}$$

with support $\varphi \geq T_i^{\frac{1}{\theta}}$

Intenstive Margin of International Trade

- Transform productivity distribution into cost distribution:
  - The effective cost of firm ω from country i serving in country n: $c_{in}(\omega) \equiv \frac{w_i \tau_{in}}{\varphi_i(\omega)}$
  - The CDF of $c_{in}(\omega)$:

$$G_{in}(c) \equiv \Pr\left(c_{in}(\omega) \leq c\right) = \bar{T}_{in}^{\theta} c^{\theta}, \quad c \leq \frac{1}{\bar{T}_{in}}, \quad \bar{T}_{in} \equiv T_i^{\frac{1}{\theta}} \left(w_i \tau_{in}\right)^{-1}$$

  where $1/\bar{T}_{in}$ is the fundamental cost upper bound of entry of international trade from i to n

- Suppose that firm ω from country i serves market n. Then it decides quantity $q_{in}(\omega)$ to maximize its operating profit at market n:

$$\tilde{\pi}_{in}(\omega) = \max_{q_{in}(\omega) \geq 0} \left[H'\left(\frac{q_{in}(\omega)}{Q_n}\right) D_n P_n - c_{in}(\omega)\right] q_{in}(\omega)$$

- **Solution under Klenow and Willis:** Let $s_{in}(\omega) \equiv \frac{q_{in}(\omega)}{Q_n}$ be the relative output. Then the optimal pricing:

$$p_{in}(\omega) = \mu\left(s_{in}(\omega)\right) c_{in}(\omega), \quad \mu\left(s_{in}(\omega)\right) \equiv \frac{\sigma s_{in}(\omega)^{-\frac{\varepsilon}{\sigma}}}{\sigma s_{in}(\omega)^{-\frac{\varepsilon}{\sigma}} - 1}$$

where markup $\mu\left(s_{in}(\omega)\right)$ is increasing with $s_{in}(\omega)$ if $\varepsilon > 0$

Extensive Margin of International Trade

- Let $X_n \equiv P_n Q_n$ be the aggregate expenditure in country n. Then the operating profit of firm ω from country i in country n:

$$\tilde{\pi}_{in}(\omega) = \frac{s_{in}(\omega)^{\frac{\varepsilon}{\sigma}}}{\sigma} \frac{\sigma - 1}{\sigma} \exp\left(\frac{1 - s_{in}(\omega)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right) s_{in}(\omega) D_n X_n$$

- $s_{in}(\omega)$ summarizes firm ω's performance in market n: connect it with $c_{in}(\omega)$
  - Inverse demand function: $p_{in}(\omega) = H'\left(s_{in}(\omega)\right) D_n P_n = \frac{\sigma-1}{\sigma} \exp\left(\frac{1-s_{in}(\omega)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right) D_n P_n$
  - Optimal pricing: $p_{in}(\omega) = \frac{\sigma s_{in}(\omega)^{-\frac{\varepsilon}{\sigma}}}{\sigma s_{in}(\omega)^{-\frac{\varepsilon}{\sigma}} - 1} c_{in}(\omega)$
  - Combining these two equations leads to

$$\frac{\sigma}{\sigma - s_{in}(\omega)^{\frac{\varepsilon}{\sigma}}} c_{in}(\omega) = \frac{\sigma - 1}{\sigma} \exp\left(\frac{1 - s_{in}(\omega)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right) D_n P_n$$

    - Therefore, $s_{in}(\omega)$ can be expressed as a function of $c_{in}(\omega)$: $s_{in}(\omega) = s_n(c_{in}(\omega))$
- Firm ω from country i will serve market n if and only if $\tilde{\pi}(\omega) \geq w_n F_n$, or equivalently

$$\frac{s_n(c_{in}(\omega))^{\frac{\varepsilon}{\sigma}}}{\sigma} \frac{\sigma-1}{\sigma} \exp\left(\frac{1-s_n(c_{in}(\omega))^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right) s_n(c_{in}(\omega)) D_n X_n \geq w_n F_n$$

- The cost cut-off of entering into trade from i to n satisfies:

$$\frac{s_n(c_n^*)^{\frac{\varepsilon}{\sigma}}}{\sigma} \frac{\sigma-1}{\sigma} \exp\left(\frac{1-s_n(c_n^*)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right) s_n(c_n^*) D_n X_n = w_n F_n$$

- To ensure that for any pair (i, n) there are firms that do not operate, we assume that F_n is sufficiently large so that

$$c_n^* < \min\left\{\frac{1}{\bar{T}_{in}}, \frac{\sigma-1}{\sigma} \exp\left(\frac{1}{\varepsilon}\right) D_n P_n\right\}, \quad \forall (i, n)$$

Aggregation

- Let M_i be the mass of firms in country i. Let X_{in} be the value of exports from i to n. Then

$$\lambda_{in} \equiv \frac{X_{in}}{X_n} = \frac{M_i \bar{T}_{in}^{\theta}}{\sum_{k=1}^{N} M_k \bar{T}_{kn}^{\theta}}$$

- Let Π_{in} be the aggregate operating profit from trade value X_{in}:

$$\frac{\Pi_{in}}{X_{in}} = \eta_n \equiv \frac{\int_0^{c_n^*} \frac{s_n(c)^{\frac{\varepsilon}{\sigma}}}{\sigma} \frac{\sigma-1}{\sigma} \exp\left[\frac{1-s_n(c)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right] s_n(c) c^{\theta-1} dc}{\int_0^{c_n^*} \frac{\sigma-1}{\sigma} \exp\left[\frac{1-s_n(c)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right] s_n(c) c^{\theta-1} dc}$$

- Price index: $P_n = \sum_{i=1}^{N} M_i \int_0^{c_n^*} p_{in}(c) s_{in}(c) dG_{in}(c)$. Therefore,

$$P_n = \sum_{i=1}^{N} \nu_{in}^P M_i \bar{T}_{in}^{\theta}, \quad \nu_{in}^P \equiv \left[\theta \int_0^{c_n^*} \frac{\sigma}{\sigma - s_n(c)^{\frac{\varepsilon}{\sigma}}} c s_n(c) c^{\theta-1} dc\right]$$

- Finally, by the definition of $H(.)$, we have

$$\sum_{i=1}^{N} M_i \bar{T}_{in}^{\theta} \theta \int_0^{c_n^*} H\left(s(c)\right) c^{\theta-1} dc = 1$$

Equilibrium

Equilibrium consists of $(w_i, M_i, P_i, D_i, c_i^*)_{i=1}^{N}$ such that:

- (w_i) is determined by labor market clearing:

$$w_i L_i = \underbrace{\sum_{n=1}^{N}(1-\eta_n)\lambda_{in}X_n}_{\text{production wage income}} + \underbrace{w_i F_i (c_i^*)^{\theta} \sum_{k=1}^{N} M_k \bar{T}_{ki}^{\theta}}_{\text{fixed marketing cost}} + \underbrace{\sum_{n=1}^{N}\left[\eta_n \lambda_{in} X_n - w_n F_n (c_n^*)^{\theta} M_i \bar{T}_{in}^{\theta}\right]}_{\text{net profit}}$$

- Firm mass M_i is determined by the free-entry condition:

$$M_i w_i f^e = \sum_{n=1}^{N}\left[\eta_n \lambda_{in} X_n - w_n F_n (c_n^*)^{\theta} M_i \bar{T}_{in}^{\theta}\right]$$

- Total expenditure in country i: $X_i = w_i L_i$
- Price index:

$$P_n = \sum_{i=1}^{N} \nu_{in}^{P} M_i \bar{T}_{in}^{\theta}, \quad \nu_{in}^{P} \equiv \left[\theta \int_0^{c_n^*} \frac{\sigma}{\sigma - s_n(c)^{\frac{\varepsilon}{\sigma}}} c s_n(c) c^{\theta-1} dc\right]$$

- (c_n^*, D_n) are jointly determined by:

$$\frac{s_n(c_n^*)^{\frac{\varepsilon}{\sigma}}}{\sigma} \frac{\sigma-1}{\sigma} \exp\left(\frac{1 - s_n(c_n^*)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right) s_n(c_n^*) D_n X_n = w_n F_n$$

and

$$\sum_{i=1}^{N} M_i \bar{T}_{in}^{\theta} \theta \int_0^{c_n^*} H\left(s(c)\right) c^{\theta-1} dc = 1$$

Algorithm

- Draw J numbers from the uniform distribution $U[0,1]$, sorting them as $u_1 < u_2 < \ldots < u_J$
- Initial guess $(w_i, M_i, P_i, D_i)_{i=1}^{N}$
- Compute the cutoff of $s_{in}(\omega)$ below which firms will not serve the market n by solving the nonlinear equation:

$$\frac{(s_n^*)^{\frac{\varepsilon}{\sigma}}}{\sigma} \frac{\sigma-1}{\sigma} \exp\left(\frac{1 - (s_n^*)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right) s_n^* D_n X_n = w_n F_n$$

- Obtain the cost cut-off:

$$c_n^* = \frac{\sigma - (s_n^*)^{\frac{\varepsilon}{\sigma}}}{\sigma} \frac{\sigma-1}{\sigma} \exp\left(\frac{1 - (s_n^*)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right) D_n P_n$$

- Compute $\bar{T}_{in} \equiv T_i^{\frac{1}{\theta}} (w_i \tau_{in})^{-1}$ and $\lambda_{in} = \frac{M_i \bar{T}_{in}^{\theta}}{\sum_{k=1}^{N} M_k \bar{T}_{kn}^{\theta}}$
- Let $c_n^j = u_j c_n^*$ be the simulated cost for firm j. Then compute the corresponding s_n^j by solving the following nonlinear equation:

$$\frac{\sigma}{\sigma-\left(s_n^j\right)^{\frac{\varepsilon}{\sigma}}}c_n^j=\frac{\sigma-1}{\sigma}\exp\left(\frac{1-\left(s_n^j\right)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right)D_nP_n$$

- Monte Carlo integration: Compute

$$\eta_n=\frac{\sum_{j=1}^{J}\frac{\left(s_n^j\right)^{\frac{\varepsilon}{\sigma}}}{\sigma}\exp\left[\frac{1-\left(s_n^j\right)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right]s_n^j\left(c_n^j\right)^{\theta-1}}{\sum_{j=1}^{J}\exp\left[\frac{1-\left(s_n^j\right)^{\frac{\varepsilon}{\sigma}}}{\varepsilon}\right]s_n^j\left(c_n^j\right)^{\theta-1}}$$

- Monte Carlo integration: Compute

$$\nu_n^P=\frac{c_n^*}{J}\sum_{j=1}^{J}\frac{\theta\sigma}{\sigma-\left(s_n^j\right)^{\frac{\varepsilon}{\sigma}}}s_n^j\left(c_n^j\right)^{\theta}$$

- Update D_n by

$$D_n=D_n\times\left[\sum_{i=1}^{N}M_i\bar{T}_{in}^{\theta}\frac{c_n^*}{J}\theta\sum_{j=1}^{J}H\left(s_n^j\right)\left(c_n^j\right)^{\theta-1}\right]^{\frac{1}{1+\theta}}$$

- Update w_i by

$$w_iL_i=\sum_{n=1}^{N}(1-\eta_n)\lambda_{in}X_n+w_iF_i(c_i^*)^{\theta}\sum_{k=1}^{N}M_k\bar{T}_{ki}^{\theta}+\sum_{n=1}^{N}\left[\eta_n\lambda_{in}X_n-w_nF_n(c_n^*)^{\theta}M_i\bar{T}_{in}^{\theta}\right]$$

- Update M_i by

$$M_iw_if^e=\sum_{n=1}^{N}\left[\eta_n\lambda_{in}X_n-w_nF_n(c_n^*)^{\theta}M_i\bar{T}_{in}^{\theta}\right]$$

- Update P_n by

$$P_n=\left[(P_n)^{-\theta-1}\nu_n^P\sum_{i=1}^{N}M_i\bar{T}_{in}^{\theta}\right]^{-\frac{1}{\theta}}$$

- Iterate until convergence
- Verify whether $c_n^*<\min\left\{\frac{1}{\bar{T}_{in}},\frac{\sigma-1}{\sigma}\exp\left(\frac{1}{\varepsilon}\right)D_nP_n\right\},\quad\forall(i,n)$

Welfare and Aggregate Markup

- Welfare: $U_i=\frac{w_i}{P_i}$
- Aggregate markup: as suggested by Edmond et al. (2019), we compute a sales-weighted harmonic average:

$$\bar{\mu}_n^D \equiv \left[\theta\left(\sum_{i=1}^{N} M_i \bar{T}_{in}^{\theta}\right)\int_0^{c_n^*}\left(\frac{\sigma}{\sigma - s_n(c)^{\frac{\varepsilon}{\sigma}}}\right)^{-1}\frac{p_n(c)}{P_n}s_n(c)c^{\theta-1}dc\right]^{-1} = \left[\frac{\theta}{\nu_n^P}\int_0^{c_n^*}s_n(c)c^{\theta}dc\right]^{-1}$$

CES case

- D_n will be constant
- Aggregate fixed marketing cost, $w_i F_i (c_i^*)^{\theta} M_k \bar{T}_{ki}^{\theta}$, will be a constant share of trade value X_{ki} (you need to derive that share)
- Solve for the CES equilibrium (w_i, M_i, P_i) and utilize it as the initial guess for the non-CES case