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Exercise Report

S-114.4202 Special Course in Computational Engineering II

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Round 1

Exercise 1.1

a)

The problem can be written in matrix form as follows:

$$
\begin{aligned}
\mathbf{y} &= \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T \\
\mathbf{a} &= \begin{bmatrix} a_1 & a_2 \end{bmatrix}^T \\
\mathbf{X} &= \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \\
\mathbf{y} &= \mathbf{X}\mathbf{a}
\end{aligned}
$$

b)

$$
\mathrm{E}(a_1, a_2) = (\mathbf{y} - \mathbf{X}\mathbf{a})^T(\mathbf{y} - \mathbf{X}\mathbf{a})
$$

c)

To compute the LS-estimate $\hat{\mathbf{a}}$, we need to find the global minimum of E, which can be found by setting its gradient to zero (it's a quadratic form):

$$
\begin{aligned}
\nabla \mathrm{E} &= -2\mathbf{X}^T(\mathbf{y} - \mathbf{X}\mathbf{a}) \\
&= 2\mathbf{X}^T\mathbf{X}\mathbf{a} - 2\mathbf{X}^T\mathbf{y} \\
\Rightarrow \hat{\mathbf{a}} &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}
\end{aligned}
$$

Exercise 1.2

a)

The second order differential equation can be written as a first order vector valued differential equation as follows:

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} &= \begin{bmatrix} 0 & -c^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x'(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t) \\
\Leftrightarrow \mathbf{x}'(t) &= \mathbf{F}\mathbf{x}(t) + \mathbf{L}w(t).
\end{aligned}
$$

Let us proceed to solve this equation:

$$
\begin{aligned}
\mathbf{x}'(t) - \mathbf{F}\mathbf{x}(t) &= \mathbf{L}w(t) \\
e^{-\mathbf{F}t}\mathbf{x}'(t) - e^{-\mathbf{F}t}\mathbf{F}\mathbf{x}(t) &= e^{-\mathbf{F}t}\mathbf{L}w(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\mathbf{F}t}\mathbf{x}(t)\right) &= e^{-\mathbf{F}t}\mathbf{L}w(t)
\end{aligned}
$$

so that

$$
\mathbf{x}(t) = e^{\mathbf{F}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{F}(t-s)}\mathbf{L}w(s)\ \mathrm{d}s \tag{1}
$$

where

$$\mathbf{x}(t_0) = \begin{bmatrix} v_0 \\ x_0 \end{bmatrix}$$

B)

After discretizing the time into $n+1$ instants as $\{t_0 = 0, t_1 = \Delta t, \dots, t_n = n\Delta t\}$ and assuming $w(s) = w(t_{k-1})\ \forall s \in [t_{k-1}, t_k)$ we can write (1) as

$$\begin{aligned}
\mathbf{x}(t_k) &= e^{\mathbf{F}(t_k - t_0)}\mathbf{x}(t_0) + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} e^{\mathbf{F}(t_j - s)}\mathbf{L}w(s)\ \mathrm{d}s \\
&= e^{k\mathbf{F}\Delta t}\mathbf{x}(t_0) + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} e^{\mathbf{F}(t_j - s)}\ \mathrm{d}s\mathbf{L}w(t_{j-1}) \\
&= e^{k\mathbf{F}\Delta t}\mathbf{x}(t_0) + \sum_{j=1}^{k} (-\mathbf{F}^{-1})\left(1 - e^{\mathbf{F}\Delta t}\right)\mathbf{L}w(t_{j-1}) \\
&= e^{k\mathbf{F}\Delta t}\mathbf{x}(t_0) + \mathbf{F}^{-1}(e^{\mathbf{F}\Delta t} - 1)\mathbf{L}\sum_{j=1}^{k} w(t_{j-1}) \\
&= e^{\mathbf{F}\Delta t}\mathbf{x}(t_{k-1}) + \mathbf{F}^{-1}(e^{\mathbf{F}\Delta t} - 1)\mathbf{L}w(t_{k-1}) \\
&= \mathbf{A}\mathbf{x}(t_{k-1}) + \mathbf{B}w(t_{k-1})
\end{aligned}$$

C)

Assuming $w_{k-1} \sim N(0, \dfrac{q_c}{\Delta t})$ and that at each timestep t_k we measure the value of $x(t_k)$ and additive zero mean gaussian noise with variance R_k and designating $\mathbf{x}_k = \mathbf{x}(t_k)$ we get the following linear Gaussian state space model:

$$\begin{aligned}
\mathbf{x}_k &= \mathbf{A}\mathbf{x}_{k-1} + q_{k-1} \\
y_k &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x_k} + r_k
\end{aligned}$$

so that

$$\begin{aligned}
\mathbf{x}_k | \mathbf{x}_{k-1} &\sim N\left(\mathbf{A}\mathbf{x}_{k-1},\ \mathbf{B}\frac{q_c}{\Delta t}\mathbf{B}^T\right) \\
\mathbf{y}_k | \mathbf{x}_k &\sim N\left(x(k),\ R_k\right)
\end{aligned}$$

D)

Solving the co

Round 2

Exercise 2.1

A)

The posterior is of the form

$$p(\mathbf{a}|y_{1:n}) = Ze^{f(\mathbf{a})}$$

where the exponent $f(\mathbf{a}) : \mathbb{R}^2 \to \mathbb{R}$ can be written as

$$\begin{aligned} f(\mathbf{a}) &= -\frac{1}{2}\left((\mathbf{X}\mathbf{a}-\mathbf{y})^T(\mathbf{X}\mathbf{a}-\mathbf{y}) + \frac{1}{\sigma^2}\mathbf{a}^T\mathbf{a}\right) \\ &= -\frac{1}{2}\left((\mathbf{a}-\mathbf{m})^T\mathbf{P}^{-1}(\mathbf{a}-\mathbf{m})\right) \end{aligned}$$

with suitably defined mean $\mathbf{m}$ and covariance matrix $\mathbf{P}$. Here $\mathbf{a}$, $\mathbf{X}$ and $\mathbf{y}$ are defined as in Round 1 exercise 1.

B)

The maximum of the posterior, which also is its mean in this case, is at the maximum of $f(\mathbf{a})$, which can be found at the point where its gradient vanishes:

$$\frac{\partial f(\mathbf{a})}{\partial \mathbf{a}} = -\mathbf{X}^T(\mathbf{X}\mathbf{a}-\mathbf{y}) - \frac{1}{\sigma^2}^T\mathbf{a} = -\left(\mathbf{X}^T\mathbf{X} + \frac{1}{\sigma^2}\mathbf{I}\right)\mathbf{a} + \mathbf{X}^T\mathbf{y}$$

so that at the maximum $\mathbf{m}$

$$\begin{aligned} \mathbf{X}^T\mathbf{X}\mathbf{m} + \frac{1}{\sigma^2}\mathbf{m} &= \mathbf{X}^T\mathbf{y} \\ \mathbf{m} &= \left(\mathbf{X}^T\mathbf{X} + \frac{1}{\sigma^2}\mathbf{I}\right)^{-1}\mathbf{X}^T\mathbf{y} \end{aligned}$$

C)

The Hessian matrix of f is

$$\frac{\partial^2 f(\mathbf{a})}{\partial \mathbf{a}^2} = -\left(\mathbf{X}^T\mathbf{X} + \frac{1}{\sigma^2}\mathbf{I}\right).$$

In order to relate this to $\mathbf{P}$, we can calculate that

$$\frac{\partial^2}{\partial \mathbf{a}^2}\left[-\frac{1}{2}\left((\mathbf{a}-\mathbf{m})^T\mathbf{P}^{-1}(\mathbf{a}-\mathbf{m})\right)\right] = -\mathbf{P}^{-1} \tag{2}$$

so that

$$\mathbf{P} = \left(\mathbf{X}^T\mathbf{X} + \frac{1}{\sigma^2}\mathbf{I}\right)^{-1}$$

D)

The resulting posterior distribution is then

$$\begin{aligned} p(\mathbf{a}|y_{1:n}) &= \mathrm{N}(\mathbf{a}|\mathbf{m}, \mathbf{P}) \\ \mathbf{m} &= \mathbf{P}\mathbf{X}^T\mathbf{y} \\ \mathbf{P} &= \left(\mathbf{X}^T\mathbf{X} + \frac{1}{\sigma^2}\mathbf{I}\right)^{-1} \end{aligned}$$

Comparing $\mathbf{m}$ to the LS estimate $\mathbf{m}_{\mathrm{LS}}$ in Round 1 Exercise 1 we can see that if the prior variance σ^2 approaches infinity, then $\mathbf{m} \to \mathbf{m}_{\mathrm{LS}}$.

Exercise 2.2

The linear regression model in the last exercise can be written in the following state space form

$$\begin{aligned}\mathbf{a}_k &= \mathbf{a}_{k-1} = \mathbf{a} = \begin{bmatrix} a_1 & a_2 \end{bmatrix}^T \sim N(\mathbf{a}|\mathbf{o}, \sigma^2 \mathbf{I}) \\ y_k &= \mathbf{H}_k \mathbf{a} + \varepsilon_k, \;\; k = 1, \ldots, n \\ \mathbf{H}_k &= \begin{bmatrix} x_k & 1 \end{bmatrix} \\ \varepsilon_k &\sim N(\varepsilon_k|0, 1)\end{aligned}$$

From these definitions we can deduce that the measurement distribution is

$$p(y_k|x_k, \mathbf{a}) = N(y_k|\mathbf{H}_k \mathbf{a}, 1)$$

As before, we are interested in the posterior distribution of $\mathbf{a}$. The "states" x_k are fixed and have no associated uncertainty.

A)

After the first observation we have

$$p(\mathbf{a}|y_1, x_1) = \frac{p(y_1|x_1, \mathbf{a})p(\mathbf{a})}{p(y_1|x_1)}$$

Now noting that $p(\mathbf{a}) = N(a_1|0, \sigma^2)N(a_2|0, \sigma^2)$ and designating $Z = p(y_1|x_1)^{-1}$, we find that this distribution is exactly the same as the posterior in the previous exercise with $n = 1$. Similarly after $k \leq n$ measurements we get (the measurements are considered i.i.d)

$$p(\mathbf{a}|y_{1:k}, x_{1:k}) = \frac{\prod_{j=1}^{k} p(y_j|x_j, \mathbf{a})p(\mathbf{a})}{p(y_{1:k}|x_{1:k})}.$$

that is again the same as the posterior in the previous exercise with $n = k$.

We can then use the results from the last exercise by replacing n with k in the equations. If we designate by $\mathbf{X}_k$ and $\mathbf{y}_k$ the $\mathbf{X}$ and $\mathbf{y}$ of the previous exercise but with k instead of n elements and with $\mathbf{m}_k$ and $\mathbf{P}_k$ the mean and covariance matrix of the posterior after k measurements, we get

$$\begin{aligned}\mathbf{m}_k &= \mathbf{P}_k \mathbf{X}_k^T \mathbf{y}_k \\ \mathbf{P}_k &= \left(\mathbf{X}_k^T \mathbf{X}_k + \frac{1}{\sigma^2} \mathbf{I} \right)^{-1}\end{aligned} \tag{3}$$

B)

For the covariance matrix we can write

$$\begin{aligned}\mathbf{X}_k^T \mathbf{X}_k &= \begin{bmatrix} \sum_i^k x_i^2 & \sum_i^k x_i \\ \sum_i^k x_i & k \end{bmatrix} \\ \mathbf{H}_k^T \mathbf{H}_k &= \begin{bmatrix} x_k^2 & x_k \\ x_k & 1 \end{bmatrix} \\ \Leftrightarrow \mathbf{X}_k^T \mathbf{X}_k &= \mathbf{X}_{k-1}^T \mathbf{X}_{k-1} + \mathbf{H}_k^T \mathbf{H}_k\end{aligned}$$

giving

$$\mathbf{P}_k = \left(\mathbf{X}_{k-1}^T\mathbf{X}_{k-1} + \mathbf{H}_k^T\mathbf{H}_k + \frac{1}{\sigma^2}\mathbf{I}\right)^{-1}.$$

Similarly for the mean we get

$$\mathbf{X}_k^T\mathbf{y}_k = \begin{bmatrix} \sum_i^k x_i y_i \\ \sum_i^k y_i \end{bmatrix}$$
$$\mathbf{H}_k^T y_k = \begin{bmatrix} x_k y_k & \\ y_k & 1 \end{bmatrix}$$
$$\Leftrightarrow \mathbf{X}_k^T\mathbf{y}_k = \mathbf{X}_{k-1}^T\mathbf{y}_{k-1} + \mathbf{H}_k^T y_k$$

giving

$$\mathbf{m}_k = \mathbf{P}_k\mathbf{X}_{k-1}^T\mathbf{y}_{k-1} + \mathbf{P}_k\mathbf{H}_k^T y_k \tag{4}$$

c)

Let's start by substituting first $\mathbf{K}_k$ and then $\mathbf{S}_k$ into $\mathbf{P}_k$

$$\begin{aligned}\mathbf{P}_k &= \mathbf{P}_{k-1} - \mathbf{P}_{k-1}\mathbf{H}_k^T\mathbf{S}_k^{-1}\mathbf{S}_k\mathbf{S}_k^{-1}\mathbf{H}_k\mathbf{P}_{k-1} \\ &= \mathbf{P}_{k-1} - \mathbf{P}_{k-1}\mathbf{H}_k^T\left(\mathbf{H}_k\mathbf{P}_{k-1}\mathbf{H}_k^T + 1\right)^{-1}\mathbf{H}_k\mathbf{P}_{k-1} \end{aligned}\tag{5}$$

apply the matrix inversion lemma

$$\begin{aligned} &= \left(\mathbf{P}_{k-1}^{-1} + \mathbf{H}_k^T\mathbf{H}_k\right)^{-1} \\ &= \left(\mathbf{X}_{k-1}^T\mathbf{X}_{k-1} + \frac{1}{\sigma^2}\mathbf{I} + \mathbf{H}_k^T\mathbf{H}_k\right)^{-1} \\ &= \left(\mathbf{X}_k^T\mathbf{X}_k + \frac{1}{\sigma^2}\mathbf{I}\right)^{-1}\end{aligned}$$

d)

In part b) it was proved that the result in part a) can be written as in (4). By using the identities $\mathbf{K}_k = \mathbf{P}_k\mathbf{H}_k^T = \mathbf{P}_{k-1}\mathbf{H}_k^T\left(\mathbf{H}_k\mathbf{P}_{k-1}\mathbf{H}_k^T + 1\right)^{-1}$ this can be derived from the Kalman filter equations:

$$\begin{aligned}\mathbf{m}_k &= \mathbf{m}_{k-1} + \mathbf{K}_k(\mathbf{y}_k - \mathbf{H}_k\mathbf{m}_{k-1}) \\ &= (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\,\mathbf{m}_{k-1} + \mathbf{K}_k\mathbf{y}_k\end{aligned}$$

apply equation (3)

$$\begin{aligned} &= (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\,\mathbf{P}_{k-1}\mathbf{X}_{k-1}^T\mathbf{y}_{k-1} + \mathbf{K}_k\mathbf{y}_k \\ &= (\mathbf{P}_{k-1} - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_{k-1})\,\mathbf{X}_{k-1}^T\mathbf{y}_{k-1} + \mathbf{K}_k\mathbf{y}_k\end{aligned}$$

apply the identities

$$= \left(\mathbf{P}_{k-1} - \mathbf{P}_{k-1}\mathbf{H}_k^T\left(\mathbf{H}_k\mathbf{P}_{k-1}\mathbf{H}_k^T + 1\right)^{-1}\mathbf{H}_k\mathbf{P}_{k-1}\right)\mathbf{X}_{k-1}^T\mathbf{y}_{k-1} + \mathbf{P}_k\mathbf{H}_k^T\mathbf{y}_k$$

apply equation (5)

$$= \mathbf{P}_k\mathbf{X}_{k-1}^T\mathbf{y}_{k-1} + \mathbf{P}_k\mathbf{H}_k^T y_k$$

E)

- step 1
  - mean

$$\begin{aligned} \mathbf{m}_0 &= 0 \\ \mathbf{P}_0 &= \sigma^2 \mathbf{I} \\ \Leftrightarrow \mathbf{m}_1 &= \mathbf{m}_0 + \mathbf{P}_1 \mathbf{H}_1^T (\mathbf{y}_1 - \mathbf{H}_1 \mathbf{m}_0) \\ &= \mathbf{P}_1 \mathbf{X}_1^T \mathbf{y}_1 \end{aligned}$$

  - variance

$$\mathbf{P}_1 = \left(\mathbf{X}_1^T \mathbf{X}_1 + \mathbf{P}_0^{-1} \right)^{-1}$$

- step k
  - mean (assume $\mathbf{m}_{k-1} = \mathbf{P}_{k-1} \mathbf{X}_{k-1}^T \mathbf{y}_{k-1}$)

$$\begin{aligned} \mathbf{m}_k &= \mathbf{m}_{k-1} + \mathbf{K}_k (\mathbf{y}_k - \mathbf{H}_k \mathbf{m}_{k-1}) \\ &= \mathbf{P}_k \mathbf{X}_{k-1}^T \mathbf{y}_{k-1} + \mathbf{P}_k \mathbf{H}_k^T y_k \\ &= \mathbf{P}_k \mathbf{X}_k^T \mathbf{y}_k \end{aligned}$$

  - variance (assume $\mathbf{P}_{k-1} = \left(\mathbf{X}_{k-1}^T \mathbf{X}_{k-1} + \mathbf{P}_0^{-1} \right)^{-1}$)

$$\begin{aligned} \mathbf{P}_k &= \left(\mathbf{P}_{k-1}^{-1} + \mathbf{H}_k^T \mathbf{H}_k \right)^{-1} \\ &= \left(\mathbf{X}_{k-1}^T \mathbf{X}_{k-1} + \mathbf{P}_0^{-1} + \mathbf{H}_k^T \mathbf{H}_k \right)^{-1} \\ &= \left(\mathbf{X}_k^T \mathbf{X}_k + \mathbf{P}_0^{-1} \right)^{-1} \end{aligned}$$

Exercise 2.3

A)

$$\begin{aligned} p(\mathbf{x}) &= N(\mathbf{x}|\mathbf{m}, \mathbf{P}),\ \mathbf{x} \in \mathbb{R}^n \\ p(\mathbf{y}|\mathbf{x}) &= N(\mathbf{y}|\mathbf{H}\mathbf{x}, \mathbf{R}),\ \mathbf{y} \in \mathbb{R}^m \\ \Leftrightarrow p(\mathbf{x}, \mathbf{y}) &= p(\mathbf{y}|\mathbf{x}) p(\mathbf{x}) \\ &= C \exp\left(-\frac{1}{2} (\mathbf{x} - \mathbf{m})^T \mathbf{P}^{-1} (\mathbf{x} - \mathbf{m}) - \frac{1}{2} (\mathbf{y} - \mathbf{H}\mathbf{x})^T \mathbf{R}^{-1} (\mathbf{y} - \mathbf{H}\mathbf{x}) \right) \\ &= C \exp\left(-\frac{1}{2} \begin{bmatrix} \mathbf{x} - \mathbf{m} \\ \mathbf{y} - \mathbf{H}\mathbf{m} \end{bmatrix}^T \begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{H}^T \\ \mathbf{H}\mathbf{P} & \mathbf{H}^T \mathbf{P} \mathbf{H} + \mathbf{R} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x} - \mathbf{m} \\ \mathbf{y} - \mathbf{H}\mathbf{m} \end{bmatrix} \right) \end{aligned}$$

Now from the last form we can see that the joint distribution is clearly normal, which means that all the marginal distributions must be normal too. To get the mean and variance of a variable from its conditional distribution, we can use the following well known identities (easily provable by writing the expectations as integrals):

$$\begin{aligned} \mathrm{E}\left[\mathbf{y}\right] &= \mathrm{E}\left[\mathrm{E}\left[\mathbf{y}|\mathbf{x}\right] \right] \\ \mathrm{var}\left(\mathbf{y}\right) &= \mathrm{E}\left[\mathrm{var}\left(\mathbf{y}|\mathbf{x}\right) \right] + \mathrm{var}\left(\mathrm{E}\left[\mathbf{y}|\mathbf{x}\right] \right) \end{aligned}$$

Now it's easy to see that

$$\begin{aligned}\mathrm{E}\left[\mathbf{y}\right] &= \mathrm{E}\left[\mathbf{Hx}\right] = \mathbf{H}\mathrm{E}\left[\mathbf{x}\right] = \mathbf{Hm} \\ \mathrm{var}\left(\mathbf{y}\right) &= \mathrm{E}\left[\mathbf{R}\right] + \mathrm{var}\left(\mathbf{Hx}\right) = \mathbf{R} + \mathbf{H}\mathrm{var}\left(\mathbf{x}\right)\mathbf{H}^T = \mathbf{R} + \mathbf{HPH}^T\end{aligned}$$

$$\Leftrightarrow \mathbf{y} \sim N(\mathbf{Hm}, \mathbf{HPH}^T + \mathbf{R})$$

B)

$$\begin{aligned} p(\mathbf{x}) &= N(\mathbf{x}|\mathbf{m}, \mathbf{P}) \\ p(\mathbf{y}|\mathbf{x}) &= N(\mathbf{y}|\mathbf{Hx}, \mathbf{R}) \\ \Leftrightarrow p(\mathbf{x}, \mathbf{y}) &= p(\mathbf{y}|\mathbf{x})p(\mathbf{x}) \\ &= \frac{1}{(2\pi)^{\frac{n+m}{2}}\sqrt{|\mathbf{P}|\,|\mathbf{R}|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T\mathbf{P}^{-1}(\mathbf{x}-\mathbf{m}) - \frac{1}{2}(\mathbf{y}-\mathbf{Hx})^T\mathbf{R}^{-1}(\mathbf{y}-\mathbf{Hx})\right) \end{aligned}$$

$$\begin{aligned} \Rightarrow p(\mathbf{y}) &= \int p(\mathbf{y}|\mathbf{x})p(\mathbf{x}) \\ &= \frac{1}{(2\pi)^{\frac{m}{2}}\sqrt{\left|\mathbf{HPH}^T + \mathbf{R}\right|}} \exp\left(-\frac{1}{2}(\mathbf{y}-\mathbf{Hm})^T(\mathbf{HPH}^T+\mathbf{R})^{-1}(\mathbf{y}-\mathbf{Hm})\right) \end{aligned}$$

C)

Here we prove the following result: let

$$\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}^T \sim N\left(\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}^{-1}, \begin{bmatrix}\mathbf{A} & \mathbf{C}\\ \mathbf{C}^T & \mathbf{B}\end{bmatrix}\right)$$

then

$$\mathbf{x}|\mathbf{y} \sim N\left(\mathbf{a} + \mathbf{CB}^{-1}(\mathbf{y}-\mathbf{b}), \mathbf{A} - \mathbf{CB}^{-1}\mathbf{C}^T\right)$$

Let's denote the inverse of the joint covariance matrix as

$$\begin{bmatrix}\mathbf{A} & \mathbf{C}\\ \mathbf{C}^T & \mathbf{B}\end{bmatrix}^{-1} = \begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{12}^T & \mathbf{D}_{22}\end{bmatrix}$$

and then expand the quadratic form in the exponent:

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}) &= -\frac{1}{2}\begin{bmatrix}\mathbf{x}-\mathbf{a}\\ \mathbf{y}-\mathbf{b}\end{bmatrix}^T \begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{12}^T & \mathbf{D}_{22}\end{bmatrix} \begin{bmatrix}\mathbf{x}-\mathbf{a}\\ \mathbf{y}-\mathbf{b}\end{bmatrix} \\ &= -\frac{1}{2}\begin{bmatrix}(\mathbf{x}-\mathbf{a})^T & (\mathbf{y}-\mathbf{b})^T\end{bmatrix} \begin{bmatrix}\mathbf{D}_{11}(\mathbf{x}-\mathbf{a}) & \mathbf{D}_{12}(\mathbf{y}-\mathbf{b})\\ \mathbf{D}_{12}^T(\mathbf{x}-\mathbf{a}) & \mathbf{D}_{22}(\mathbf{y}-\mathbf{b})\end{bmatrix} \\ &= -\frac{1}{2}\left((\mathbf{x}-\mathbf{a})^T\mathbf{D}_{11}(\mathbf{x}-\mathbf{a}) + 2(\mathbf{x}-\mathbf{a})^T\mathbf{D}_{12}(\mathbf{y}-\mathbf{b}) + (\mathbf{y}-\mathbf{b})^T\mathbf{D}_{22}^T(\mathbf{y}-\mathbf{b})\right) \end{aligned}$$

In this Gaussian case the mean is also the maximum and the maximum of the exponential function can be found at the maximum of the exponent. Thus if we take the partial derivative of the exponent with respect to $\mathbf{x}$ (meaning that $\mathbf{y}$ is held fixed) and see where it vanishes, we have found the mean $\mathbf{m}$ of $\mathbf{x}|\mathbf{y}$:

$$\begin{aligned}\frac{\partial}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{y}) &= 0 \\ \Leftrightarrow -\mathbf{D}_{11}(\mathbf{m} - \mathbf{a}) - \mathbf{D}_{12}(\mathbf{y} - \mathbf{b}) &= 0 \\ \Leftrightarrow \mathbf{m} &= \mathbf{D}_{11}^{-1}\left(\mathbf{D}_{11}\mathbf{a} - \mathbf{D}_{12}(\mathbf{y} - \mathbf{b})\right)\end{aligned}$$

apply the identity $\mathbf{D}_{12} = -\mathbf{D}_{11}\mathbf{C}\mathbf{B}^{-1}$

$$\begin{aligned}&= \mathbf{a} + \mathbf{D}_{11}^{-1}\mathbf{D}_{11}\mathbf{C}\mathbf{B}^{-1}(\mathbf{y} - \mathbf{b}) \\ &= \mathbf{a} + \mathbf{C}\mathbf{B}^{-1}(\mathbf{y} - \mathbf{b})\end{aligned}$$

The variance can be found similarly by taking the second partial derivative of the exponent (the Hessian matrix) with respect to $\mathbf{x}$ and applying the identity $\mathbf{D}_{11}^{-1} = \mathbf{A} - \mathbf{C}\mathbf{B}^{-1}\mathbf{C}^T$:

$$\begin{aligned}\frac{\partial^2}{\partial \mathbf{x}^2} f(\mathbf{x}, \mathbf{y}) &= \frac{\partial}{\partial \mathbf{x}}\left(-\mathbf{D}_{11}(\mathbf{x} - \mathbf{a}) - \mathbf{D}_{12}(\mathbf{y} - \mathbf{b})\right) \\ &= -\left(\mathbf{A} - \mathbf{C}\mathbf{B}^{-1}\mathbf{C}^T\right)^{-1}\end{aligned}$$

After applying equation (2), we note that the result has been proven.

Round 3

Exercise 3.1

We now have the non-zero mean noise state space model

$$\begin{aligned}\mathbf{x}_k &= \mathbf{A}\mathbf{x}_{k-1} + \mathbf{q}_{k-1} \\ \mathbf{y}_k &= \mathbf{H}\mathbf{x}_\mathbf{k} + \mathbf{r}_k \\ \mathbf{q}_{k-1} &\sim N\left(\mathbf{m}_q, \mathbf{Q}\right) \\ \mathbf{r}_k &\sim N\left(\mathbf{m}_r, \mathbf{R}\right)\end{aligned}$$

meaning

$$\begin{aligned}\mathbf{x}_k|\mathbf{x}_{k-1} &\sim N\left(\mathbf{A}\mathbf{x}_{k-1} + \mathbf{m}_q, \mathbf{Q}\right) \\ \mathbf{y}_k|\mathbf{x}_k &\sim N\left(\mathbf{H}\mathbf{x}_\mathbf{k} + \mathbf{m}_r, \mathbf{R}\right)\end{aligned}$$

By following closely the derivation of the Kalman filter equations in the course material, we get

prediction:

$$\begin{aligned}\mathbf{m}_k^- &= \mathbf{A}\mathbf{m}_{k-1} + \mathbf{m}_q \\ \mathbf{P}_k^- &= \mathbf{A}\mathbf{P}_{k-1}\mathbf{A}^T + \mathbf{Q}\end{aligned}$$

update:

$$\begin{aligned}
\mathbf{v}_k &= \mathbf{y}_k - \mathbf{H}\mathbf{m}_k^- - \mathbf{m}_r \\
\mathbf{S}_k &= \mathbf{H}\mathbf{P}_k^-\mathbf{H} + \mathbf{R} \\
\mathbf{K}_k &= \mathbf{P}_k^-\mathbf{H}^T\mathbf{S}_k^{-1} \\
\mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{K}_k\mathbf{v}_k \\
\mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{K}_k\mathbf{S}_k\mathbf{K}_k^T
\end{aligned}$$

Exercise 3.2

Here we filter a Gaussian random walk model with the Kalman filter. The model is

$$\begin{aligned}
x_k &= x_{k-1} + q_{k-1} \\
y_k &= x_k + r_k \\
q_{k-1} &\sim N(0, Q) \\
r_k &\sim N(0, R)
\end{aligned}$$

meaning

$$\begin{aligned}
x_k | x_{k-1} &\sim N(x_{k-1}, Q) \\
y_k | x_k &\sim N(x_k, R)
\end{aligned}$$

In this one-dimensional case approximating the required integrals using a grid approximation is feasible. In figure 1 the random walk signal with $n = 100$ timesteps is plotted with the means given by the Kalman filter and the grid approximation. In figure 2 the variances of the approximated Gaussian distributions are plotted for the Kalman filter and the grid approximation. The m-code is presented in listing 1.

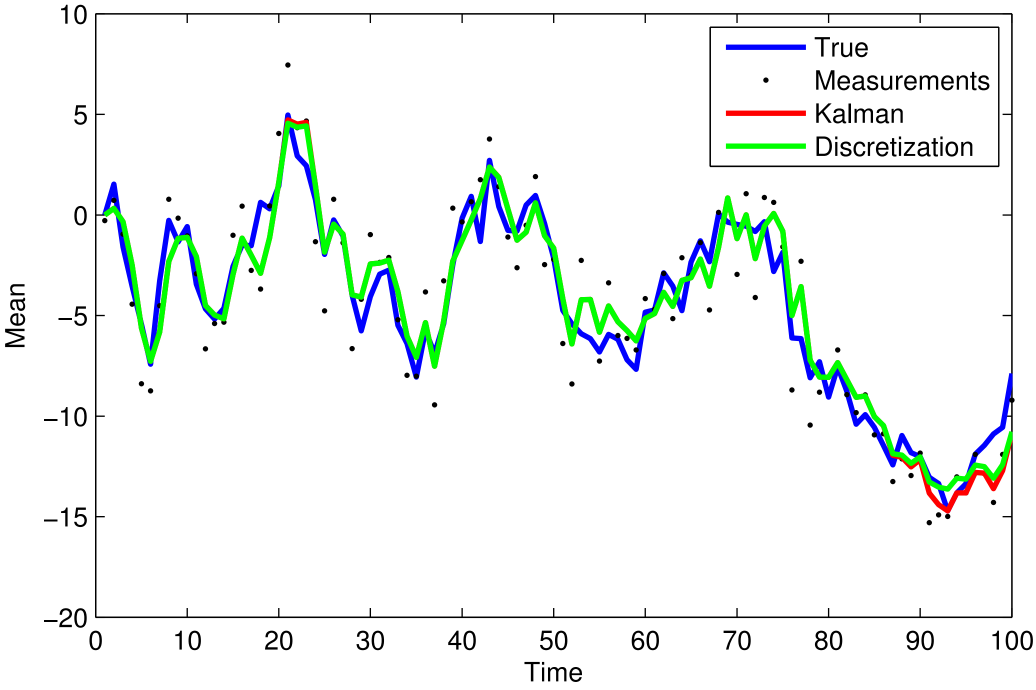


Figure 1: The true random walk signal (blue), the mean of the Kalman filter (red) and the mean of the grid approximation (green) in exercise 3.2

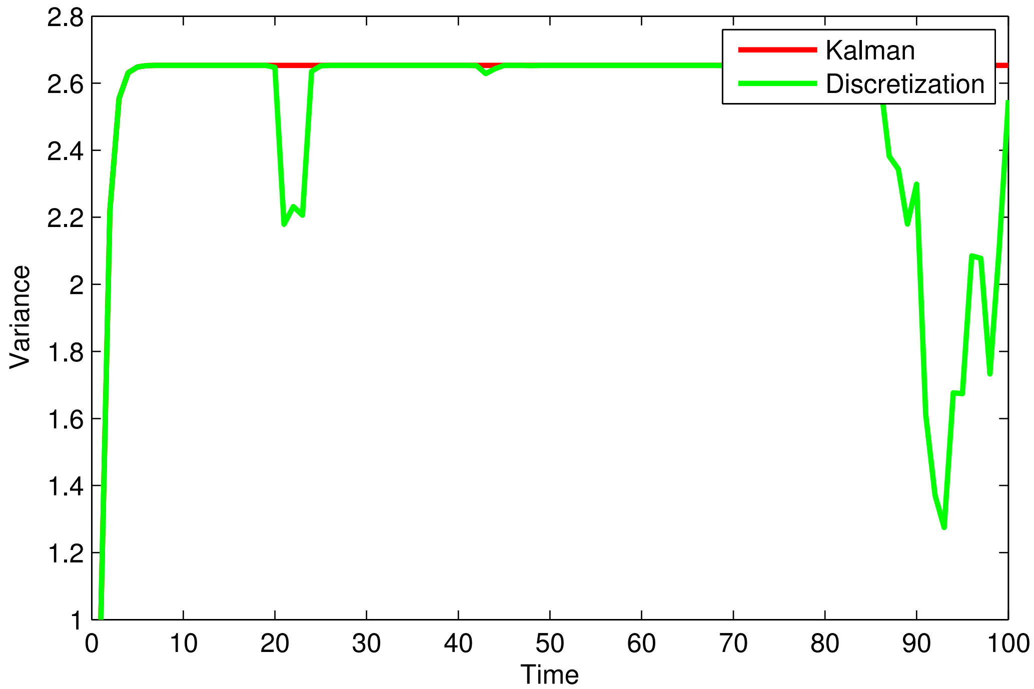


Figure 2: The true random walk signal (blue), the mean of the Kalman filter (red) and the mean of the grid approximation (green) in exercise 3.2

Listing 1: m-code in exercise 3.2

```
%% Exercise round 3, exercise 2

%true signal
N=100;
q=3;

s=cumsum([0 normrnd(0,sqrt(q),1,N-1)]);
figure(1);
clf;
true_m = plot(1:N,s);
xlabel('Time');
ylabel('Mean');
hold on;

% measurements
A = 1;
H = 1;
r = 5;
mn = 0;
n = N;
x = floor(1:N/n:N);
y = s(1,x)+normrnd(mn,sqrt(r),1,n);
measurements = plot(x,y,'.k');

% Kalman filter
m = 0;
P = 1;
m2 = 0;
P2 = 1;
ms = zeros(1,n);ms(1)=m;
ms2 = zeros(1,n);
Ps2 = zeros(1,n);
Ps = zeros(1,n);Ps(1)=P;
Ks = Ps;
for k=2:n
    % prediction
    m_= m;
    P_ = P+q;
    % update
    K = P_/(P_+r);
    m = m_+K*(y(k)-m_);
    P = P_-(P_^2/(P_+r));
    ms(k) = m;
    Ps(k) = P;
    Ks(k) = K;
end
figure(1);
kalman_m = plot(x,ms,'-r');
hold on;
mso = ms;
Pso = Ps;
rmse(x,ms)
figure(2);
kalman_P = plot(x,Ps,'-r');
xlabel('Time');
ylabel('Variance');
hold on;

% discretization
a=min(y);
b=max(y);
N = 500;
t = linspace(a,b,N);
[TX,TY] = meshgrid(t);
p_dyn = normpdf(TX,TY,sqrt(q));
m = 0;
P = 1;
```

```
p_ = normpdf(t,m,P);
ms = zeros(1,n);ms(1)=m;
Ps = zeros(1,n);Ps(1)=P;
distr = zeros(N,n);
for k=2:n
    p = sum(p_dyn.*repmat(p_',1,N));
    p = normpdf(y(k),t,sqrt(r)).*p;
    p = p/sum(p);
    p_ = p;
    distr(:,k) = p';
    m = t*p';
    ms(k) = m; % mean
    Ps(k) = (t-m).^2*p';
end;
```

Exercise 3.3

Here we consider the Kalman filter for a noisy resonator model. The model is presented as a state-space model in the exercise paper and is not reproduced here. The state $\mathbf{x}_k \in \mathbb{R}^2$ at time k consists of the location $x_k^{(1)}$ and its derivative $x_k^{(2)}$.

A)

We compare the Kalman filter solution to a base line solution, where the measurement y_k is directly used as $x_k^{(1)}$ and $x_k^{(2)}$ is calculated as a weighted average of the measurement differences. The base line solution is presented in figure 3 and the Kalman filter solution in figure 4

Unsurprisingly, the Kalman filter solution is clearly better than the baseline solution, which is directly affected by the noise. The root-mean-square errors for the solutions are presented in table 1, from where it can be seen that the Kalman filter RMSE is 44% smaller. The relevant parts of the mcode for this exercise is shown in listing 2.

Listing 2: m-code in exercise 3.3A

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% Kalman filter solution. The estimates
% of x_k are stored as columns of
% the matrix EST2.
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

m = x0;     % Initialize to true value
P = eye(2); % Some uncertanty in covariance
EST2 = zeros(2,steps);
EST2P = zeros(2,2,steps);

Ks = EST2;
H = [1 0];
kdiff = zeros(1,steps);
for k=1:steps
  % prediction
  m_ = A*m;
  P_ = A*P*A'+Q;

  % update
  K = P_(:,1)/(P_(1,1)+r);
  Ks(:,k) = K;
  m = m_ + K*(Y(k)-m_(1));
  P = P_-(P_(1,1)+r)*(K*K');

  % Store the results
```

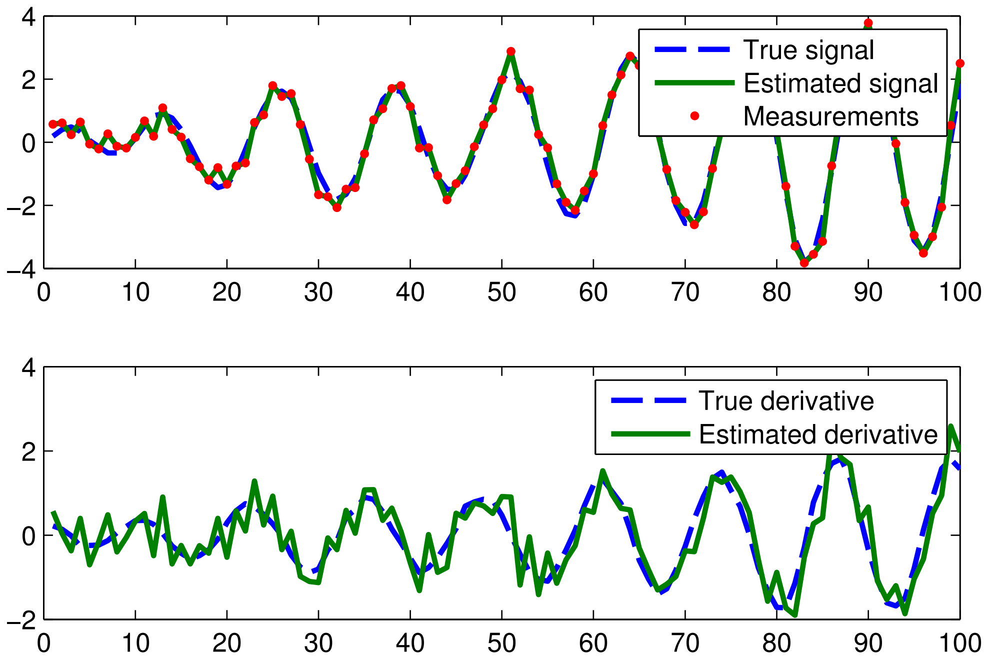


Figure 3: The base line solution in exercise 3.3

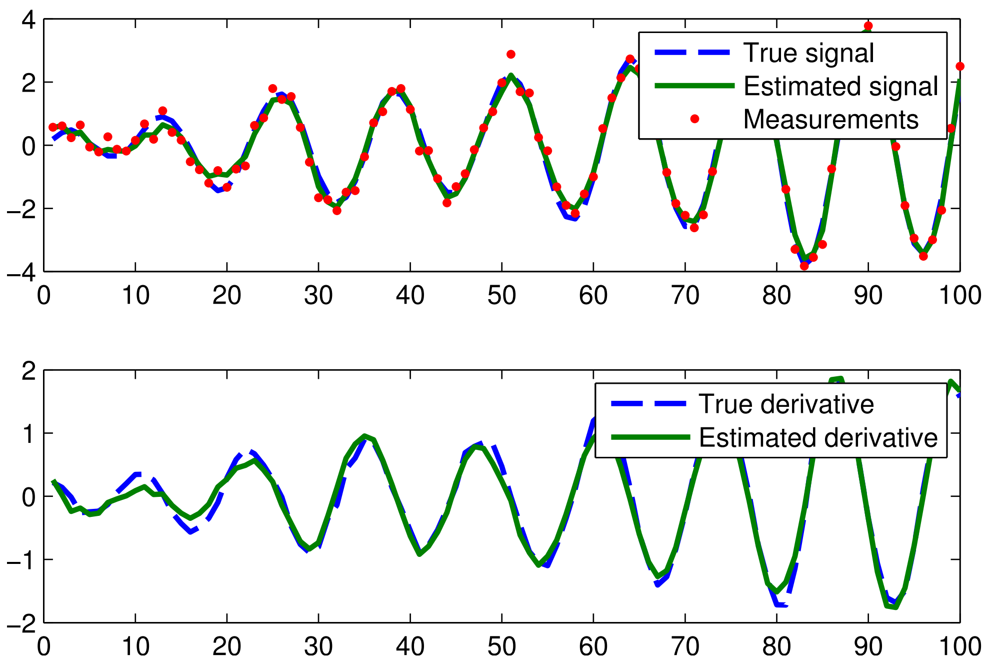


Figure 4: The Kalman filter solution in exercise 3.3

```
  EST2(:,k) = m;
  EST2P(:,:,k) = P;
end
```

B)

Here the stationary Kalman filter, where the Kalman gain $\mathbf{K}_k$ is constant, is compared to the solutions in 3.3A. The constant for the Kalman gain was computed numerically by running the filter for a long time. The graphical solution is presented in figure 5 and the RMSE in table 1. The RMSE is a little smaller for the stationary Kalman filter solution. Comparing the graphical solutions of the ordinary Kalman filter and the stationary Kalman filter carefully, it can be seen that during the first few steps, the stationary Kalman filter makes a better estimate. But since the Kalman gain seems to converge to its constant value very quickly, the solutions are close to identical after the first ten steps.The relevant parts of the mcode for this exercise is shown in listing 3.

Listing 3: m-code in exercise 3.3B

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% Stationary Kalman filter solution.
% The estimates of x_k are stored as columns of
% the matrix EST3.
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

m3 = x0;     % Initialize to true value
K  =  [0.424975200169185;
       0.111137301539005];  % Store the stationary gain here

EST3 = zeros(2,steps);

for k=1:steps
  % Replace these with the stationary Kalman filter equations
  m3 = A*m3+K*(Y(k)-A(1,:)*m3);

  % Store the results
  EST3(:,k) = m3;
end
```

Table 1: The RMSE values in exercise 3.3

Baseline	Kalman	Stationary Kalman
0.536	0.237	0.234

Round 4

Exercise 4.1

In this exercise we consider the following non-linear state space model:

$$
\begin{aligned}
x_k &= x_{k-1} - 0.01\sin(x_{k-1}) + q_{k-1} \\
&= f(x_{k-1}) + q_{k-1} \\
y_k &= 0.5\sin(2x_k) + r_k \\
&= h(x_k) + r_k \\
q_{k-1} &\sim N\left(0, 0.01^2\right) \\
r_k &\sim N\left(0, 0.02\right)
\end{aligned}
$$

A)

To implement the extended Kalman filter for the model, we need the following derivatives:

$$
\begin{aligned}
f'(x) &= -0.01\cos(x) + 1 \\
h'(x) &= \cos(2x)
\end{aligned}
$$

In order to use the Kalman filter for this problem, $p\left(x_k | x_{k-1}\right)$ and $p\left(y_k | x_k\right)$ need to be approximated by a Gaussian distribution. In the EKF this is done by using linear approximations to the nonlinearities. The result is a Gaussian approximation to the filtering distribution. The result of applying EKF to this problem is presented in figure 6, where the signal and measurements were simulated for 200 timesteps, starting from $x_0 = \frac{2}{5}\pi$.

B)

In statistically linearized filter (SLF) the linear approximation of the EKF is replaced by statistical linearization. In order to use the SLF, we need the following expectations:

$$
\begin{aligned}
\mathrm{E}\left[f(x_{k-1})\right] &= m_{k-1} - 0.01\sin(m_{k-1})e^{-\frac{1}{2}P_{k-1}} \\
\mathrm{E}\left[f(x_{k-1})\delta x_{k-1}\right] &= P_{k-1} - 0.01\cos(m_{k-1})P_{k-1}e^{-\frac{1}{2}P_{k-1}}
\end{aligned}
$$

where expectations are with respect to $N(x_{k-1} | m_{k-1}, P_{k-1})$ and

$$
\begin{aligned}
\mathrm{E}\left[h(x_k)\right] &= 0.5\sin(2m_k^-)e^{-2P_k^-} \\
\mathrm{E}\left[h(x_k)\delta x_k\right] &= \cos(2m_k^-)P_k^- e^{-2P_k^-}
\end{aligned}
$$

where expectations are with respect to $N(x_k | m_k^-, P_k^-)$

The result of applying the SLF in the same situation as in 4.1A is presented also in figure 6. The RMSE's for both of the methods is presented in table 2 and the mcode is shown in listing 4.

Table 2: The RMSE values in exercise 4.1

EKF	SLF
1.83	0.84

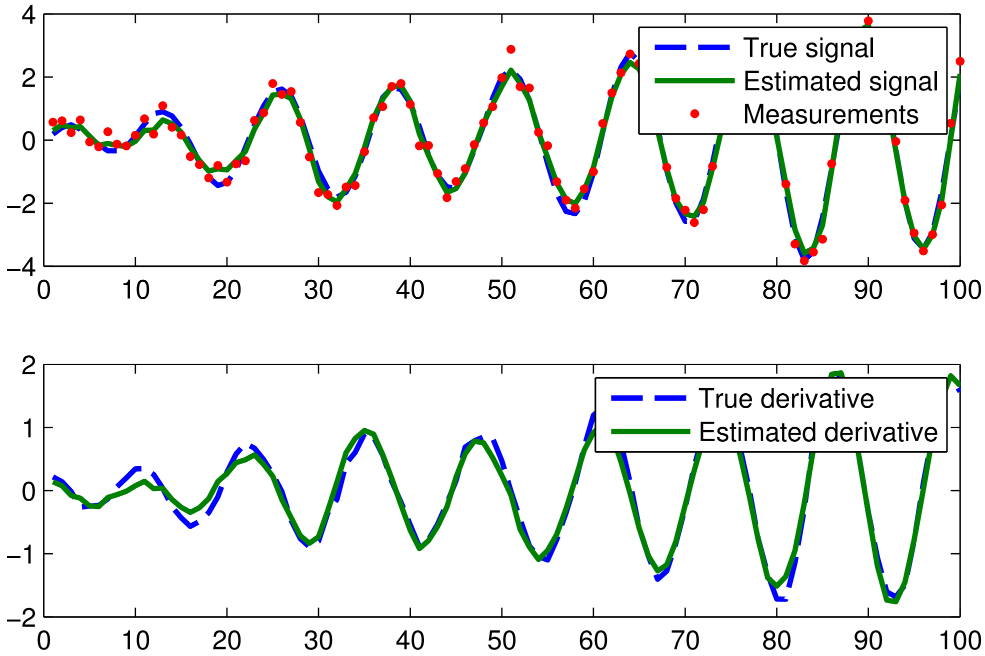


Figure 5: The stationary Kalman filter solution in exercise 3.3

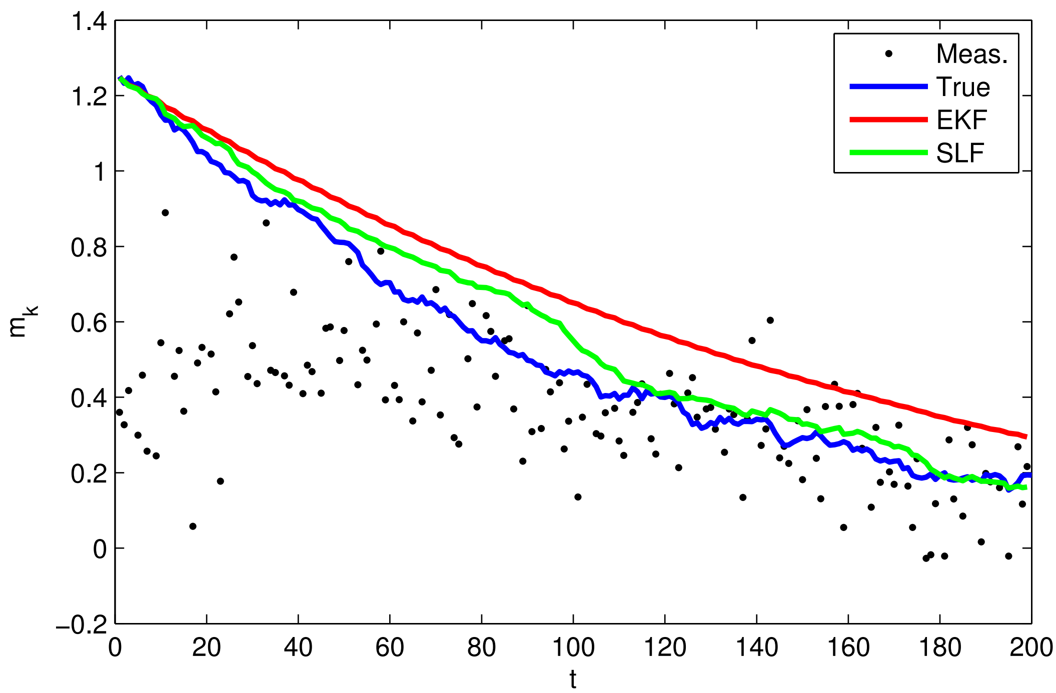


Figure 6: The true signal and the EKF and SLF approximations in exercise 4.1

Listing 4: m-code in exercise 4.1

```
%% true signal

N=200;
q=0.01^2;

s=zeros(1,N);
f=@(x)x-0.01*sin(x);
F=@(x)-0.01*cos(x);
h=@(x)0.5*sin(2*x);
H=@(x)cos(2*x);
Y=s;
x0 = 0.4*pi;
x = x0;
for k=1:N
    x = f(x)+sqrt(q)*randn;
    s(k) = x;
end

% measurements
r = 0.02;
mn = 0;
n = 150;
x = floor(1:N/n:N);
y = h(s(1,x))+normrnd(0,sqrt(r),1,n);
plot(x,y,'.k',1:N,s,'MarkerSize',8);
hold on;

%% EKF

m = x0;
P = 1;
ms = zeros(1,n);
Ps = zeros(1,n);
for k=1:n
    % prediction
    m_= f(m);
    P_ = F(m)^2*P+q;
    % update
    v = y(k)-h(m_);
    S = H(m_)^2*P_+r;
    K = P_*H(m_)/S;
    m = m_+K*v;
    P = P_-K^2*S;
    ms(k) = m;
    Ps(k) = P;
end
ms_ekf = ms;
Ps_ekf = Ps;
plot(x,ms,'-r');
fprintf('EKF %3.4f\n',rmse(s(x),ms));
%% SLF
es=@(m,p)m-...
        (m^3+3*m*p^2)/6+...
        (m^5+10*m^3*p^2+15*m*p^4)/120-...
        (m^7+21*m^5*p^2+105*m^3*p^4+105*m*p^6)/(7*6*5*4*3*2);

Ef=@(m,p)m-0.01*sin(m)*exp(-1*p/2);
Efdx=@(m,p)p-0.01*cos(m)*p*exp(-p/2);
Eh=@(m,p)0.5*sin(2*m)*exp(-2*p);
Ehdx=@(m,p)cos(2*m)*p*exp(-2*p);

m = x0;
P = q;
ms = zeros(1,n);
Ps = zeros(1,n);
for k=1:n
    % prediction
```

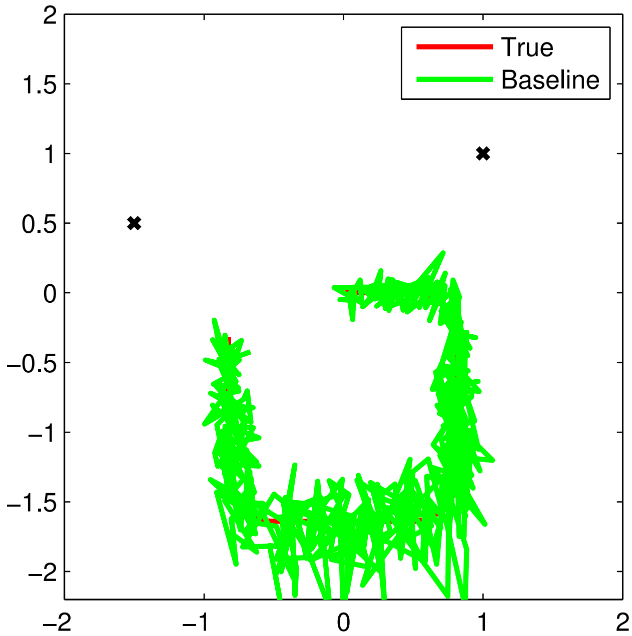
```
    m_= Ef(m,P);
    P_ = Efdx(m,P)^2/P+q;
    % update
    v = y(k)-Eh(m_,P_);
    S = Ehdx(m_,P_)^2/P_+r;
    K = Ehdx(m_,P_)/S;
    m = m_+K*v;
    P = P_-K^2*S;
    ms(k) = m;
    Ps(k) = P;
end
```

Exercise 4.2

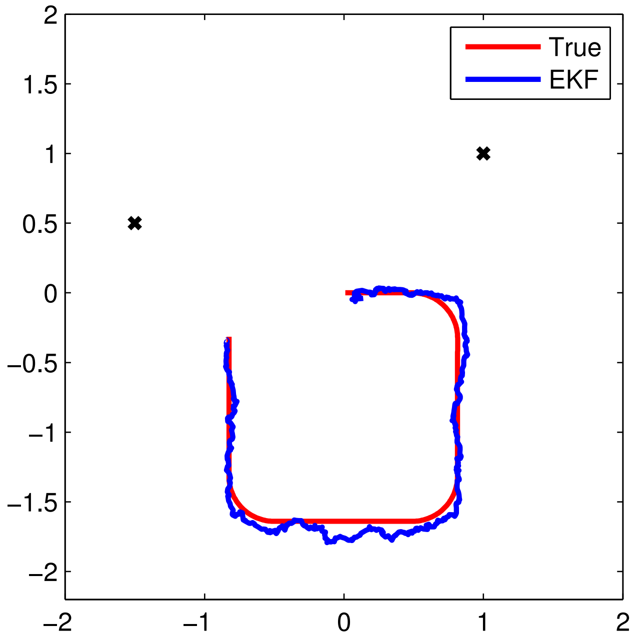
Exercise 4.3

A)

In this exercise the classical problem of bearings only taget tracking is considered. The problem setting is not reproduced here, but an EKF was implemented for an approximate solution. The graphical results are presented in figures 7a and 7b, the RMSE value in table 3 and the mcode in listing ??.



(a) Baseline approximation



(b) EKF approximation

Figure 7: The true trajectory, the baseline approximation and the EKF approximation in exercise 4.3A

Table 3: The RMSE values in exercise 4.3A

Baseline	EKF
1.019	0.441

Listing 5: m-code in exercise 4.3A

```
%%%%%%%%%%%%%%%%%%%
%%%% EKF %%%%%%%%%%
%%%%%%%%%%%%%%%%%%%
function [o,hh]=ekf()
    m = x0;             % Initialize to true value
    P = eye(4);         % Some uncertainty
    ms = zeros(4,steps);
    for k=1:steps
        %% Compute estimate here
        m_ = A*m;
        P_ = A*P*A'+ Q;
        y = Theta(:,k);
        v =  y - h(m_);
        S_ = H(m_)*P_*H(m_)'+R;
        K = P_*H(m_)'/S_;
        m = m_+K*v;
        P = P_-K*S_*K';

        ms(:,k) = m;

    end

    %% Compute error
    fprintf('EKF %3.4f\n',rmse(X,ms));
    o = ms;
    hh = showtrace('b');
    axis([-2 2 -2.5 1.5]);
end

function o=h(x)
    o = atan2(x(2)-S(2,:)',x(1)-S(1,:)');
end

function o=H(x)
    H1 = @(xx,s)-(xx(2)-s(2))/((xx-s)'*(xx-s));
    H2 = @(xx,s)(xx(1)-s(1))/((xx-s)'*(xx-s));
    o = zeros(size(S,2),4);
    for ii = 1:size(S,2)
        o(ii,1:2) = [H1(x(1:2),S(:,ii)) H2(x(1:2),S(:,ii))];
    end
end
```