UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH1853
Tutorial 3

1. (a) **Demo**. Find the roots of the equation $z^{10} = 1$.
   (b) Solve the equation $z^{10} = 1 + i$.

2. From a group of 5 women and 7 men, how many different committees of 2 women and 3 men can be formed? What if 2 of the men refuse to serve on the committee together?

3. We have a supply of flowers of 3 different colours: red, white and yellow. How many ways are there to choose 4 flowers, provided we choose at least 1 flower of each colour? With the problem as above but we drop the requirement that at least one flower of each colour is to be chosen, what will be the number of ways?

4. Consider an experiment of tossing a coin three times.
   (a) Find the sample space S_1 if we wish to observe the sequences of Heads(H) and Tails(T).
   (b) Find the sample space S_2 if we wish to observe the number of Heads in the three tosses.

5. **Demo**. If two fair dice are rolled, what is the probability that the sum of the upturned faces will equal 6?

6. If it is assumed that all $\binom{52}{5}$ poker hands are equally likely,what is the probability of being dealt
   (a) **Demo**. a flush? (A hand is said to be a flush if all 5 cards are of the same suit.)
   (b) **Demo**. one pair? (This occurs when the cards have denominations a, a, b, c, d, where a, b, c, and d are all distinct.)
   (c) two pairs? (This occurs when the cards have denominations a, a, b, b, c, where a, b, and c are all distinct.)
   (d) three of a kind? (This occurs when the cards have denominations a, a, a, b, c, where a, b, and c are all distinct.)
   (e) four of a kind? (This occurs when the cards have denominations a, a, a, a, b.)

7. A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be no complete pair?

8. From a shuffled deck of 52 playing cards one card is drawn. Let A be the event that an ace is drawn, B the event that a diamond is drawn, and C the event that a red card (heart or diamond) is drawn. Which two of A, B, and C are independent? What if a joker is added to the deck? [Note: a joker is a wild card]

9. Let S be a sample space, and let $A, B \subseteq S$ be two independent events. Let $A' = S - A$ and $B' = S - B$.
(a) Show that A' and B' are independent events. [Hint: $A' \cap B' = (A \cup B)'$.]
(b) How about A and A'?

10. Andrew, Beatrix and Charles are playing with a crown. If Andrew has the crown, he throws it to Charles. If Beatrix has the crown, she throws it to Andrew or to Charles, with equal probabilities. If Charles has the crown, he throws it to Andrew or to Beatrix with equal probabilities. At the beginning of the game the crown is given to one of Andrew, Beatrix and Charles, with equal probabilities. What is the probability that, after the crown is thrown once, **Demo**. Andrew has it? that Beatrix has it? that Charles has it?

1. (a) Roots of the equation $z^{10} = 1$ are

$$z_k = e^{i \cdot \frac{2\pi}{10} k} = e^{i \cdot \frac{2\pi k}{10}} \qquad k = 0, 1, 2, \ldots, 9 \ .$$

   (b) Rewrite $1 + i$ using Euler's formula:

$$1 + i = \sqrt{2} e^{i \frac{\pi}{4}} = 2^{\frac{1}{2}} e^{i \frac{\pi}{4}} \ .$$

   So clearly

$$2^{\frac{1}{20}} e^{i \frac{\pi}{40}} \qquad \text{is a solution to } z^{10} = 2^{\frac{1}{2}} e^{i \frac{\pi}{4}} \ .$$

   To get all the (10) solutions, attach the root of unity:

$$2^{\frac{1}{20}} e^{i \frac{\pi}{40}} \cdot e^{i \cdot \frac{2\pi k}{10}} \qquad , \quad k = 0, 1, 2, \ldots, 9 \ .$$

2. We choose 2 women out of 5 and then 3 men out of 7, so there are $\binom{5}{2}\binom{7}{3} = 350$ different possible choices. If 2 of the men refuse to be together, we either chose none of them or only 1 of them, so the choices are

$$\binom{5}{2} \cdot \left[\binom{2}{0}\binom{5}{3} + \binom{2}{1}\binom{5}{2} \right] = 300.$$

3. In the first case, there are 3 ways only and they are $\{Red, White, Yellow, Red\}$, $\{Red, White, Yellow, White\}$, $\{Red, White, Yellow, Yellow\}$. Apart from the above situation, we may have (i) 4 flowers of the same colour (3 ways) (ii) three flowers in the same colour but the last one is different only (6 ways) and (iii) two flowers in the same colour and the other two are also of the same colour but different from the first two (3 ways). All together we have 15 ways.

4. (a) $S_1 = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.
   (b) $S_2 = \{0, 1, 2, 3\}$.

5. We shall solve this problem under the assumption that all of the 36 possible outcomes are equally likely. Since there are 6 possible outcomes namely, $(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)$ which result in the sum of the dice being equal to 5, the desired probability is $5/36$.

6. (a) $4\binom{13}{5}/\binom{52}{5}$.
   (b) $13\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}/\binom{52}{5}$.
   (c) $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}/\binom{52}{5}$.
   (d) $13\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}/\binom{52}{5}$.
   (e) $13\binom{4}{4}\binom{48}{1}/\binom{52}{5}$.

7. $\dfrac{\sum_{i=0}^{8}\binom{10}{i}\binom{10-i}{8-i}}{\binom{20}{8}} = \dfrac{20 \cdot 18 \cdot 16 \cdot 14 \cdot 12 \cdot 10 \cdot 8 \cdot 6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}$.

8. If we denote Clubs, Diamonds, Hearts, and Spades, respectively as C, D, H, and S, then the event A is

$$A = \{C1, D1, H1, S1\} \ .$$

Also, $B = \{D1, D2, \ldots, D13\}$ and $C = \{H1, \ldots, H13, D1, \ldots, D13\}$. Since

$$P(A \cap B) = P(\{D1\}) = 1/52, \quad P(A) = 4/52, \quad \text{and} \quad P(B) = 13/52,$$

we have $P(A \cap B) = P(A)P(B)$, so A, B are independent. Similarly, A, C are also independent; whereas B, C are not. With a Joker in the deck, no pair is independent.

9. (a) Since $A' \cap B' = (A \cup B)'$, we have

$$P(A' \cap B') = P((A \cup B)') = P(S - A \cup B) = 1 - P(A \cup B)$$

this in turns equals

$$1 - P(A) - P(B) + P(A \cap B) = (1 - P(A))(1 - P(B)).$$

Thus

$$P(A' \cap B') = P(A')P(B').$$

(b) For A and A', since $P(A \cap A') = P(\emptyset) = 0$, hence A and A' are independent when and only when $P(A) = 0$ or 1.

10. If we write $P(\text{Andrew})$ as the probability that Andrew has the crown (after the crown is thrown once), then $P(\text{Andrew})$ equals to the sum of probability

$$P(\text{Beatrix first has the crown then Andrew has it})$$

plus

$$P(\text{Charles first has the crown then Andrew has it}).$$

So

$$P(\text{Andrew}) = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{3}.$$

Similarly, we have $P(\text{Beatrix}) = 1/6$ and $P(\text{Charles}) = 1/2$.