TUT/1853/MATH1853/1

UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH1853
Tutorial 1

1. **Demo**. Let $A = \{1, 2\}$ and $B = \{1, 3, 5, 6, 7\}$.
   (a) What are $|A|$ and $|B|$?
   (b) Find $A \cap B$.
   (c) Find $A \cup B$.
   (d) Write down all the subsets of A.

2. Let $A = \{\{1, 2\}\}$ and $B = \{\{1\}, 2, 3, 5\}$.
   (a) What are $|A|$ and $|B|$?
   (b) Find $A \cap B$.
   (c) Find $A \cup B$.
   (d) Write down all the subsets of A.

3. Let $A = \{x^2 : x \in \mathbf{Z}\}$ and $B = \{2x - 1 : x \in \mathbf{Z}\}$ where $\mathbf{Z}$ is the set of all integers. A student claims that
$$A \cap B = \{1\}$$
because if x belongs to both A and B, then we have $x^2 = 2x - 1$. Solving the equation, we can obtain one solution 1 which also belongs to $\mathbf{Z}$. Do you agree? Explain your answer.

4. Compute
$$\frac{d}{dx}\left(\int_0^x \sin(x) \cdot e^t dt\right).$$

5. **Demo**. Using the FTC and chain rule, prove the formula:
$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\, dt = f\big(g(x)\big)g'(x) - f\big(h(x)\big)h'(x)\ .$$

6. Compute
$$\frac{d}{dx}\int_{x^2}^{2} \sin(t^3)\, dt\ .$$

7. Compute
$$\int_1^2 \frac{\log_e x}{x} dx$$
by using *integration by substitution*.

8. Compute

$$\int_1^e (\log_e x)^2 dx$$

by using *integration by parts*.

9. Given that $\sigma > 0$ and

$$\int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} dy = \sqrt{2\pi}$$

compute

$$\int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx.$$

1. (a)
$$|A| = 2 \quad \text{and} \quad |B| = 5.$$
(b)
$$A \cap B = \{1\}.$$
(c)
$$A \cup B = \{1, 2, 3, 5, 6, 7\}.$$
(d) The subsets of A are
$$\phi, A, \{1\}, \{2\}.$$

2. (a) $|A| = 1$, $|B| = 4$
(b) $A \cap B = \{ \qquad \}$
(c) $A \cup B = \{\{1, 2\}, \{1\}, 2, 3, 5\}$
(d) Subsets:
$$\{\} \quad , \quad \{\{1, 2\}\}$$

3. No. Set B is the set all odd integers. Set A contains all the integers which are the square of an integer. Thus we have
$$A \cap B = \{1, 9, 25, 49, \ldots,\} = \{(2x-1)^2 : x \in \mathbf{Z}\}.$$

4. We note that
$$\frac{d}{dx}\left(\int_0^x \sin(x) \cdot e^t dt\right) = \frac{d}{dx}\left(\sin(x) \cdot \int_0^x e^t dt\right)$$
By product rule,
$$\frac{d}{dx}\left(\sin(x) \cdot \int_0^x e^t dt\right) = \cos(x) \int_0^x e^t dt + \sin(x) e^x = (e^x - 1)\cos(x) + e^x \sin(x).$$

5. Key step: we first calculate
$$\frac{d}{dx} \int_0^{g(x)} f(t)\, dt \; . \qquad \text{Set} \quad \Phi(u) = \int_0^u f(t)\, dt \; ,$$
then (let $u = g(x)$)
$$\begin{aligned} \frac{d}{dx} \int_0^{g(x)} f(t)\, dt &= \frac{d}{dx} \Phi(u) = \frac{d\Phi(u)}{du} \cdot \frac{du}{dx} \\ &= f(u) \cdot g'(x) = f\big(g(x)\big) \cdot g'(x) \; . \end{aligned}$$
Hence
$$\begin{aligned} \frac{d}{dx} \int_{h(x)}^{g(x)} f(t)\, dt &= \frac{d}{dx}\Big(\int_0^{g(x)} f(t)\, dt - \int_0^{h(x)} f(t)\, dt \Big) \\ &= f\big(g(x)\big) \cdot g'(x) - f\big(h(x)\big) \cdot h'(x) \; . \end{aligned}$$

6.

$$
\begin{aligned}
\frac{d}{dx}\int_{x^2}^{2}\sin(t^3)\,dt &= \sin(2^3)\cdot\frac{d}{dx}2-\sin\left((x^2)^3\right)\cdot\frac{d}{dx}x^2 \\
&= -2x\sin(x^6)\ .
\end{aligned}
$$

7. A good substitution can be

$$u=\log_e x,$$

and therefore

$$\frac{du}{dx}=\frac{1}{x}.$$

Since it is a definite integral, we have to take care of the upper and lower limits.

For the upper limit $x=2$, we have $u=\log_e(2)$.

For the lower limit $x=1$, we have $u=\log_e(1)=0$.

Thus

$$\int_1^2\frac{\log_e x}{x}dx=\int_0^{\log_e 2}u\ du=\left.\frac{y^2}{2}\right|_0^{\log_e 2}=\frac{(\ln 2)^2}{2}.$$

8. Since

$$\frac{d\log_e^2 x}{dx}=(2\log_e x)\cdot\frac{1}{x},$$

we have

$$
\begin{aligned}
\int_1^e(\log_e x)^2dx &= \left.x\log_e^2 x\right|_1^e-\int_1^e x\ d(\log_e^2 x) \\
&= \left.x\log_e^2 x\right|_1^e-2\int_1^e\log_e x\ dx \\
&= \left.x\log_e^2 x\right|_1^e-2[x\log_e x-x]\Big|_1^e \\
&= [x\log_e^2 x-2x\log_e x+2x]\Big|_1^e \\
&= e\log_e^2 e-2e\log_e e+2e-1\log_e^2 1+2\log_e 1-2 \\
&= e-2e+2e-2 \\
&= e-2.
\end{aligned}
$$

9. We note that if we apply integration by substitution

$$y = \frac{x-\mu}{\sigma}$$

then we get

$$\int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx = \int_{-\infty}^{\infty} \frac{(\sigma y + \mu)}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}\, dy = \underbrace{\sigma \int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}\, dy}_{=0} + \underbrace{\mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}\, dy}_{=\mu}.$$

Thus we have

$$\int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx = \mu.$$