UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH1853
Tutorial 6

1. Suppose that X follows $N(3, 9)$, find
(a) $P(2 < X < 5)$, (b) **Demo**. $P(X > 0)$, and (c) $P(|X - 3| > 6)$.

2. A machine produces tubes of length 1m. Assume the length of the tubes follows the normal distribution $N(1, 0.04)$. If the length of a tube has a deviation less than 0.1m from the mean (Class A), the profit is 100 dollars. If the deviation more than 0.1m but less than 0.2 from the mean (Class B), the profit is 50 dollars. However, if the deviation is more than 0.2 (Class C), then it incurs a loss of 80 dollars. If 1000 tubes are produced, what will be expected profit?

3. **Demo**. A biased coin is tossed 200 times (probability of getting a head is 0.3). Find the probability that the number of heads obtained is between 48 and 52 by using normal approximation.

4. In a factory producing 1,000 transistors a day, the probability of producing a defective transistor was found to be 0.04.
(a) Find the probability of obtaining less than 30 defective transistors in a day's production, using a normal approximation to simplify the calculation.
(b) Find the probability that there are exactly 3 days out of 5 working days in a week that the number of defective transistor per day is less than 30.
(c) If the number of defective transistors per day for the 30 days of last month was taken and the mean over these 30 observations was calculated and denoted by $\bar{X}_{30}$. Find the probability that this mean is less than 30, that is $P(\bar{X}_{30} < 30)$.

5. In truckloads of fruit, on the average 5% of the cases are unsaleable on arrival. If a truckload consists of 200 cases, what proportion of truckloads would have more than 10% unsaleable cases?
(a) Write the exact Binomial result down.
(b) Use Poisson distribution $Poi(\lambda)$ with $\lambda = np = 200 \times 0.05$ to get an approximation.
(c) Use normal approximations to evaluate the probability.

6. In a public exam, a sample of 100 scores yields an average of 57.4. From the past experience, the standard deviation is around 10. Construct the 90% and 95% confidence intervals for the average score μ.

7. (a) The hourly wage of employees in a certain service industry is believed to follow the Normal Distribution $N(40, 5^2)$ which has a mean μ of 40 dollars and a standard deviation σ of 5 dollars. The hourly wage of an employee is said to be reasonable if it is between 35 dollars and 45 dollars. A group of 10 employees was selected randomly. Find the probability that more than 7 employees of the selected group are having a reasonable hourly wage.
(b) Later a survey was conducted by selecting a group of 64 employees randomly. The average hourly wage of the selected group of employees is found to be 41 dollars. Assuming that the standard deviation ($\sigma = 5$) remains the same, construct a 95% confidence interval for the average hourly wage μ.

1. $X \sim N(3, 3^2) = N(\mu, \sigma^2)$, so
(a) $P(2 < X < 5) = P(-\frac{1}{3} < \frac{X-3}{3} < \frac{2}{3}) = P(-\frac{1}{3} < Z < \frac{2}{3})$ where $Z \sim N(0,1)$. Usually the table $P(Z \leq x)$ is given, thus

$$\begin{aligned} P(-\frac{1}{3} < Z < \frac{2}{3}) &= P(Z < \frac{2}{3}) - P(Z \leq -\frac{1}{3}) \\ &= P(Z < \frac{2}{3}) - \big(1 - P(Z \leq \frac{1}{3})\big) . \end{aligned}$$

Using linear approximations:

$$\frac{P(Z < \frac{2}{3}) - P(Z \leq 0.66)}{\frac{2}{3} - 0.66} = \frac{P(Z \leq 0.67) - P(Z \leq 0.66)}{0.67 - 0.66} ,$$

i.e.,

$$\frac{P(Z < \frac{2}{3}) - 0.7454}{\frac{2}{3} - 0.66} = \frac{0.7486 - 0.7454}{0.67 - 0.66} .$$

So $P(Z < \frac{2}{3}) = 0.7475$. Also

$$\frac{P(Z \leq \frac{1}{3}) - P(Z \leq 0.33)}{\frac{1}{3} - 0.33} = \frac{P(Z \leq 0.34) - P(Z \leq 0.33)}{0.34 - 0.33} ,$$

i.e.,

$$\frac{P(Z \leq \frac{1}{3}) - 0.6293}{\frac{1}{3} - 0.33} = \frac{0.6331 - 0.6293}{0.34 - 0.33} .$$

Now, $P(Z \leq \frac{1}{3}) = 0.6306$. Therefore

$$P(-\frac{1}{3} < Z < \frac{2}{3}) = 0.7475 - (1 - 0.6306) = 0.3781 .$$

(b) $P(X > 0) = P(Z > -1) = P(Z \leq 1) = 0.8413$.
(c)

$$\begin{aligned} P(|X-3| > 6) &= P(|Z| > 2) \\ &= P(Z > 2) + P(Z < -2) \\ &= 2\big(1 - P(Z \leq 2)\big) \\ &= 2(1 - 0.9772) \\ &= 0.0456. \end{aligned}$$

2. We have $\mu = 1$ and $\sigma = 0.2$. We first find

$$P(|X-1| \leq 0.1) \quad \text{and} \quad P(|X-1| < 0.2).$$

$$P(|X-1| \leq 0.1) = P(\frac{-0.1}{0.2} \leq Z \leq \frac{0.1}{0.2}) = P(-0.5 \leq Z \leq 0.5) = 0.3830.$$
$$P(|X-1| < 0.2) = P(\frac{-0.2}{0.2} < Z < \frac{0.2}{0.2}) = P(-1.0 < Z < 1.0) = 0.6826.$$

Hence the expected number of Class A tubes is $1000 \times 0.3830 = 383$, the expected number of Class B tubes is $1000 \times (0.6826 - 0.3830) = 299.6$ and finally the expected number of Class C tubes is $1000 \times (1 - 0.6826) = 317.4$. Hence the profit will be

$$383 \times 100 + 299.6 \times 50 - 80 \times 317.4 = 27888.$$

3. We note that $\mu = 200 \times 0.3 = 60$ and $\sigma^2 = npq = 200 \times 0.3 \times 0.7 = 42$. The area of the following five rectangles can be approximated by

$$P(47.5 \leq X \leq 52.5) \quad \text{and} \quad X \sim N(60, 42)$$

or

$$P\left(\frac{47.5-60}{6.4807} \leq Z \leq \frac{52.5-60}{6.4807}\right) = P(-1.93 \leq Z \leq -1.16) = 0.9732 - 0.8770 = 0.0962.$$

If using linear approx.:

$$P\left(\frac{47.5-60}{6.4807} \leq Z \leq \frac{52.5-60}{6.4807}\right) = P(-1.9288 \leq Z \leq -1.1573) = P(1.1573 \leq Z \leq 1.9288)\ ,$$

and

$$\begin{aligned} P(Z \leq 1.1573) &= (\frac{0.8770 - 0.8749}{0.01}) \cdot (1.1573 - 1.15) + 0.8749 \\ &= 0.8764\ , \\ P(Z \leq 1.9288) &= (\frac{0.9732 - 0.9726}{0.01}) \cdot (1.9288 - 1.92) + 0.9726 \\ &= 0.9731\ , \end{aligned}$$

hence $P(47.5 \leq X \leq 52.5) = 0.0967$.

4. (a) Let X denote the number of defective transistors out of 1000 for one day production. Then $X \sim Bin(1000, 0.04)$, so X is approximately $N(np, np(1-p)) = N(40, 38.4)$ (note that $38.4 > 10$), remind the continuity correction factor,

$$P(X < 30) \approx P(X \leq 29.5) = P(Z \leq -1.6944) = 0.0451\ .$$

(b) Let Y denote the number of days out of 5 working days in a week that the number of defective transistor per day is less than 30. Then $Y \sim Bin(5, 0.0451)$, and therefore

$$P(Y = 3) = \binom{5}{3} 0.0451^3 (1 - 0.0451)^2 = 0.0008 \ .$$

(c) If we write $\bar{X}_{30} = (X_1 + \cdots + X_{30})/30$ where X_i means the observation in day i. Hence $\bar{X}_{30} \sim N(40, 1.28)$.

$$P(\bar{X}_{30} < 30) = P\left(Z \leq \frac{30 - 40}{\sqrt{1.28}}\right) = P(Z \leq -8.84) = 0.000 \ .$$

5. Let X denote the number of unsaleable cases out of 200 cases. Then we are considering $P(X > 20)$ with $n = 200$, $p = 0.05$.
The exact binomial is

$$P(X > 20) = 1 - \sum_{i=0}^{20} \binom{200}{i} 0.05^i \cdot 0.95^{200-i} = 0.00116 \ .$$

Using Poisson approximation: $\lambda = np = 200 \cdot 0.05 = 10$,

$$\begin{aligned} P(X > 20) &= 1 - P(X \leq 20) \\ &= 1 - \sum_{i=0}^{20} e^{-10} \cdot \frac{10^i}{i!} \\ &= 1 - 0.99841 = 0.00159 \ . \end{aligned}$$

Using normal approximation: $\mu = np = 10$, $\sigma^2 = np(1-p) = 9.5$, so

$$P(X > 20) = P(X \geq 20.5) = P(Z > 3.4066) = 1 - 0.9997 = 0.0003 \ .$$

One should be noted that this normal approximation is not a good one because $np(1-p) = 9.5 \not\geq 10$.

6. We have $\bar{X} = 57.4$, $\sigma = 10$ and $n = 100$, thus we have
For $\beta = 0.90$, $z_{0.05} = 1.645$ and the C.I. is

$$[57.4 - 1.645 \times \frac{10}{\sqrt{100}}, \quad 57.4 + 1.645 \times \frac{10}{\sqrt{100}}] = [55.755, 59.045].$$

For $\beta = 0.95$, $z_{0.025} = 1.96$ and the C.I. is

$$[57.4 - 1.96 \times \frac{10}{\sqrt{100}}, \quad 57.4 + 1.96 \times \frac{10}{\sqrt{100}}] = [55.44, 59.36].$$

7. (a) (i) Let X be the hourly wage of an employee, $X \sim N(40, 5^2)$. The probability that his/her wage is reasonable will be

$$\begin{aligned}
P(35 \leq X \leq 45) &= P\left(\frac{35-40}{5} \leq \frac{X-40}{5} \leq \frac{45-40}{5}\right) \\
&= P(-1 \leq Z \leq 1) \\
&= 1 - 2(1 - \Phi(1)) = 1 - 2(1 - 0.8413) \\
&= 0.6826.
\end{aligned}$$

(ii) Let Y be the Bernoulli experiment (a selected employee has a reasonable hourly wage). The probability can be obtained by using Binomial distribution $Bin(10, 0.6826)$.

$$\begin{aligned}
P(Y > 7) &= P(Y = 10) + P(Y = 9) + P(Y = 8) \\
&= 0.6826^{10} + 10(0.6826)^9(1 - 0.6826) + 45(0.6826)^8(1 - 0.6826)^2 \\
&= 0.3378
\end{aligned}$$

(b) The 95% Confident Interval (C.I.) is given by

$$\left[\bar{X} - 1.96 \times \frac{\sigma}{\sqrt{n}}, \quad \bar{X} + 1.96 \times \frac{\sigma}{\sqrt{n}}\right].$$

where $\sigma = 5$, $\bar{X} = 41$ and $n = 64$. Then the required C.I. is $[39.775, 42.225]$.