TUT/1853/MATH1853/5

UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH1853
Tutorial 5

1. At a "busy time," a telephone exchange is very near capacity, so callers have difficulty placing their calls. It may be of interest to know the number of attempts necessary in order to make a connection. Suppose that we let $p = 0.05$ be the probability of a connection during a busy time. We are interested in knowing the probability that 5 attempts are necessary for a successful call.

2. **Demo**. It is known that screws produced by a certain company will be defective with a probability 0.01 independently of each other. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10- screws is defective. What proportion of packages sold must the company replace?

3. A communications channel transmits the digits 0 and 1. However, due to static, the digit transmitted is incorrectly received with probability 0.2. Suppose that we want to transmit an important message consisting of one binary digit. To reduce the chance of error, we transmit 00000 instead of 0 and 11111 instead of 1. If the receiver of the message uses "majority" decoding, what is the probability that the message will be wrong when decoded?

4. Show that the mean and variance of the Binomial distribution are given respectively by $\mu = np$ and $\sigma^2 = npq$ where $q = 1 - p$. You may write $X = X_1 + X_2 + \cdots + X_n$ where X_i are independent Bernoulli random variables.

5. An engineering system consisting of n components is said to be a k-out-of-n system ($k \leq n$) if the system functions if and only if at least k of the n components function. Suppose that all components function independently of each other.
(a) If the i-th component functions with probability P_i, $i = 1, 2, 3, 4$, express the probability that a 2-out-of-4 system functions in terms of P_i.
(b) Repeat for a k-out-of-n system when all the P_i equal p (that is, $P_i = p$, $i = 1, 2, \cdots, n$).

6. Show that for a Poisson random variable X which follows the probability distribution $p(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots,$ we have (**Demo**.) $E(X) = \mathrm{Var}(X) = \lambda$.
[Hint: find $E(X(X-1))$.]

7. Consider an experiment that consists of counting the number of α-particles given off in a 1-second interval by 1 gram of radioactive material. If we know from past experience that, on average, 3.2 such α-particles are given off. What is the probability that no more 2 α-particles will appear?

8. **Demo**. If X is an exponential random variable with parameter $\lambda = 1$, compute the probability density function of the random variable Y defined by $Y = \log(X)$.

1. Let

$$\begin{aligned} P(s) &= \text{prob. of getting a successful connection per call} \\ X &= \text{r.v. counting the no. of connections} \\ &\qquad \text{required for the 1st success} \end{aligned}$$

Then

$$\text{“}X = 5\text{”} \qquad \text{eq.} \qquad \text{event } \{ffffs\} \ ,$$

hence

$$P(X=5) = P(ffffs) = (1-0.05)^4(0.05) = 0.041 \ .$$

2.

$$P(0\ defective) + P(1\ defective) = 0.99^{10} + 10 \cdot 0.99^9(1-0.99) = 0.995734.$$

Thus the probability of refund is $1 - 0.995734 = 0.004266$ and the proportion will be 0.4266%.

3. The message is decoded as 1 if the major of the bits in the received message is 1, otherwise 0. Then probability that the message will be wrong when decoded is

$$\binom{5}{3}(0.2)^3(0.8)^2 + \binom{5}{4}(0.2)^4(0.8) + (0.2)^5.$$

4. For the Binomial ($Bin(n,p)$) random variable X, X is equal to the sum of n independent identically distributed Bernoulli random variables X_i, i.e., $X = X_1 + X_2 + \cdots + X_n$. Hence

$$E(X) = E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = n \times p$$

and

$$Var(X) = Var(X_1 + \cdots + X_n) \overset{X_i \text{ are indep.}}{=\!=\!=} Var(X_1) + \cdots + Var(X_n) = npq \ .$$

5. (a)

$$\begin{aligned} &P_1P_2(1-P_3)(1-P_4) + P_1(1-P_2)P_3(1-P_4) + P_1(1-P_2)(1-P_3)P_4 \\ &+P_2P_3(1-P_1)(1-P_4) + (1-P_1)P_2(1-P_3)P_4 + (1-P_1)(1-P_2)P_3P_4 \\ &+P_1P_2P_3(1-P_4) + P_1P_2(1-P_3)P_4 + P_1(1-P_2)P_3P_4 + (1-P_1)P_2P_3P_4 \\ &+P_1P_2P_3P_4. \end{aligned}$$

(b) $\sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$.

6.

$$E(X) = \sum_{k=0}^{\infty} \frac{k \times \lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} e^{-\lambda} = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} = \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = \lambda.$$

Since

$$Var(X) = E(X^2) - E(X)^2 = E(X(X-1)) + E(X) - E(X)^2 = E(X(X-1)) + \lambda - \lambda^2.$$

We need to compute

$$E(X(X-1)) = \sum_{k=0}^{\infty} k(k-1) \times \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=2}^{\infty} \lambda^2 \frac{\lambda^{k-2}}{(k-2)!} e^{-\lambda} = \lambda^2.$$

Hence $Var(X) = \lambda$.

7. Suppose X is a random variable representing the number of α-particles to be emitted out of totally n atoms at the probability p. Given

$$E(X) = np = 3.2$$

since n is huge, X is just $Poi(3.2)$. Hence

$$\begin{aligned} P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &\approx \frac{3.2^0 e^{-3.2}}{0!} + \frac{3.2^1 e^{-3.2}}{1!} + \frac{3.2^2 e^{-3.2}}{2!} = 0.3799\ . \end{aligned}$$

8.

$$F_Y(y) = P(\log X \leq y) = P(X \leq e^y) = F_X(e^y).$$
$$f_Y(y) = f_X(e^y) e^y = e^y e^{-e^y}.$$