TUT/1853/MATH1853/2

UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH1853
Tutorial 2

1. **Demo**. Let $z = 1 + i$. Find $|z|$ and $Arg(z)$.

2. Let $z = -1 - 0.5i$. Find $|z|$ and $Arg(z)$.

3. Verify that each of the two numbers $z = 1 \pm \sqrt{3}i$ satisfies the equation
$$z^2 - 2z + 4 = 0.$$

4. **Demo**. Rewrite the fraction $\dfrac{1+5i}{1-5i}$ into the form $a + bi$ where $a, b \in \mathbb{R}$.

5. Reduce the following quantity to a real number: $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$.

6. (a) Establish the identity
$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \quad \text{for} \quad (z \neq 1).$$
(b) Use (a) to derive Lagrange's trigonometric identity:
$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin(n + 1/2)\theta}{2\sin(\theta/2)}.$$

7. **Demo**. Find the set of complex numbers z for which $Re(z^2) = 0$.

8. Find the set of complex numbers z for which $\left|\dfrac{z-3}{z+3}\right| = 2$.

9. Find the set of complex numbers z such that
$$\frac{|\bar{z} + i|}{|z + i|} = \sqrt{2}.$$

10. **Demo**. Let $\omega^3 = 1$ and $\omega \neq 1$. Find the value of $\omega^{2011} + \omega^{1997} + 1$.

11. Let w be a root of $z^3 - 1 = 0$. Find the value of $|2(w^{1997} + w^{2013} + w^{2047}) - 3|$.

12. (a) Let $P = -2+i$ and $Q = 1-3i$ be two complex numbers. Show that the complex numbers on the line joining the points in the complex plane can be express as
$$z = 3t - 2 + i(1 - 4t).$$
(b) Show that the image of the line joining the two points in the complex plane under the mapping $w = z^2$ is given by
$$3 - 4t - 7t^2 + (-4 + 22t - 24t^2)i.$$

13. Show that

(a) **Demo**.

$$\tanh(-x) = -\tanh(x).$$

(b)

$$\tanh(ix) = i\tan(x).$$

1.

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

and

$$Arg(z) = \tan^{-1}(\frac{1}{1}) = \frac{\pi}{4}.$$

2. $|z| = 1.12$, $Arg(z) = 1.15\pi$.

3. If $z = 1 \pm \sqrt{3}i$, then we have

$$z^2 - 2z + 4 = (1 \pm \sqrt{3}i)^2 - 2(1 \pm 2\sqrt{3}i) + 4 = -2 \pm \sqrt{3}i - 2 \mp 2\sqrt{3}i + 4 = 0.$$

4.

$$\begin{aligned}
\frac{1+5i}{1-5i} &= \frac{1+5i}{1-5i} \cdot \frac{1+5i}{1+5i} \\
&= \frac{(1+5i)(1+5i)}{1+25} \qquad \boxed{(a+b)(a-b) = a^2 - b^2} \\
&= \frac{1+10i-25}{26} \\
&= \frac{-12}{13} + \frac{5}{13}i\ .
\end{aligned}$$

5.

$$\frac{1+2i}{3-4i} + \frac{2-i}{5i} = \frac{1+2i}{3-4i} \times \frac{3+4i}{3+4i} + \frac{2-i}{5i} \times \frac{5i}{5i} = \frac{-5+10i}{25} - \frac{5+10i}{25} = \frac{-2}{5}.$$

6. (a) If $S = 1 + z + \cdots + z^n$, then

$$S - zS = (1 + z + \cdots + z^n) - (z + z^2 + \cdots + z^{n+1}) = 1 - z^{n+1},$$

which implies that $S = \frac{1-z^{n+1}}{1-z}$, provided $z \neq 1$. That is

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

(b) Putting $z = e^{i\theta}$ into this identity, we have

$$1 + e^{i\theta} + \cdots + e^{in\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}.$$

Now, the real part of the left-hand side is clearly

$$1 + \cos\theta + \cdots + \cos n\theta.$$

To find the real part of the right-hand side, we write that side in the form

$$\frac{1 - \exp(i(n+1)\theta)}{1 - \exp(i\theta)} \frac{\exp(-i\theta/2)}{\exp(-i\theta/2)} = \frac{\exp(-i\theta/2) - \exp(i(n+1/2)\theta)}{\exp(-i\theta/2) - \exp(i\theta/2)}$$

which becomes

$$\frac{\sin\theta/2 + \sin(n+1/2)\theta + i(\cos\theta/2 - \cos(n+1/2)\theta)}{2\sin\theta/2}.$$

The real part of this is clearly

$$\frac{1}{2} + \frac{\sin(n+1/2)\theta}{2\sin\theta/2}.$$

We then arrive at the Lagrange's trigonometric identity:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin(n+1/2)\theta}{2\sin(\theta/2)}.$$

7. Let $z = x + yi$ and we have $z^2 = x^2 - y^2 + 2xyi$. Thus $Re(z^2) = 0$ means

$$x^2 - y^2 = (x+y)(x-y) = 0.$$

The complex numbers are those on the lines $y = x$ and $y = -x$ or the set of complex numbers is given by

$$\{x + xi : x \in \mathbf{R}\} \cup \{x - xi : x \in \mathbf{R}\}.$$

8. The given equation is equivalent to $|z-3| = 2|z+3|$, i.e.,

$$\sqrt{(x-3)^2 + y^2} = 2\sqrt{(x+3)^2 + y^2}.$$

Squaring and simplifying, this becomes

$$x^2 + y^2 + 10x + 9 = 0,$$

or

$$(x+5)^2 + y^2 = 16.$$

In other words, $|z+5| = 4$, which is a circle of radius 4 with center at $-5 + 0i$. The desired answer is

$$\{z \in \mathbb{C} |\ |z+5| = 4\}\ .$$

9. Let $z = x + yi$ where $x, y \in \mathbb{R}$ then $\bar{z} = x - yi$ and we have

$$\frac{|\bar{z}+i|}{|z+i|} = \sqrt{\frac{x^2 + (1-y)^2}{x^2 + (y+1)^2}} = \sqrt{2}.$$

Hence

$$x^2 + (y-1)^2 = 2x^2 + 2(y+1)^2 \quad \text{or} \quad x^2 + y^2 + 6y + 1 = 0$$

or

$$x^2 + (y+3)^2 = 8.$$

It is a circle centered at $-3i$ with radius $\sqrt{8}$. The desired answer is

$$\{x + yi |\ x^2 + (y+3)^2 = 8\} = \{z \in \mathbb{C} |\ |z+3i| = \sqrt{8}\}\ .$$

10. Since $\omega^3 = 1$ we have

$$\omega^{2011} + \omega^{1997} + 1 = \omega^{3\cdot 670+1} + \omega^{3\cdot 665+2} + 1 = \omega^2 + \omega + 1.$$

Now we also have

$$0 = \omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1).$$

Since $\omega \neq 1$, we must have $\omega^2 + \omega + 1 = 0$.

11. First we note that since $w^3 = 1$,

$$w^{1997} = w^2,\ w^{2013} = 1,\ w^{2047} = w.$$

Thus

$$2(w^{1997} + w^{2013} + w^{2047}) - 3 = 2(w^2 + w + 1) - 3.$$

Since $w^3 - 1 = (w-1)(w^2 + w - 1) = 0$ there are two cases to consider.
If $w = 1$ then

$$2(w^2 + w + 1) - 3 = 2 \cdot 3 - 3 = 3.$$

If $w \neq 1$ then $w^2 + w + 1 = 0$ and we have

$$2(w^2 + w + 1) - 3 = -3.$$

In both cases, $|2(w^2 + w + 1) - 3| = 3$. Hence the answer is 3.

12. (a) Points P and Q have coordinates $(-2, 1)$ and $(1, -3)$. Then, the parametric equations of the line joining these points are given by

$$\frac{x - (-2)}{1 - (-2)} = \frac{y - 1}{-3 - 1} = t \text{ or } x = 3t - 2, y = 1 - 4t.$$

The equation of the line PQ can be represented by $z = 3t - 2 + i(1 - 4t)$.

(b) The curve in the w plane into which this line is mapped has the equation

$$\begin{aligned} w &= z^2 \\ &= (3t - 2 + i(1 - 4t))^2 \\ &= (3t-2)^2 - (1-4t)^2 + 2(3t-2)(1-4t)i \\ &= 3 - 4t - 7t^2 + (-4 + 22t - 24t^2)i. \end{aligned}$$

13. (a) By definition

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

we have

$$\tanh(-x) = \frac{e^{-x} - e^x}{e^{-x} + e^x} = -\frac{e^x - e^{-x}}{e^x + e^{-x}} = -\tanh(x).$$

(b) From the notes we know that

$$i\sinh(x) = \sin(ix) \quad \text{and} \quad \cosh(x) = \cos(ix).$$

By definition we have

$$\tanh(ix) = \frac{\sinh(ix)}{\cosh(ix)} = \frac{-\sin(x)}{i\cos(x)} = \frac{i\sin(x)}{\cos(x)} = i\tan(x).$$