TUT/1853/MATH1853/4

UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH1853
Tutorial 4

1. **Demo**. A ball is in any one of n boxes and is in the i-th box with probability P_i. If the ball is in box i, a search of that box will uncover it with probability α_i. Find the conditional probability that the ball is in box j, given that a search of box i did not uncover it.

2. You know that a certain letter is equally likely to be in any one of three different folders. Let α_i be the probability that you will find your letter upon making a quick examination of folder i if the letter is, in fact, in folder i, $i = 1, 2, 3$. Suppose you look in folder 1 and do not find the letter. What is the probability that the letter is in folder 1?

3. In a lottery 10,000 tickets are sold for \$ 1 each. There are five prizes: \$ 5,000 (once), \$ 700 (once), \$ 100 (three times). What is the expected value of a ticket?

4. Suppose it is given that $E(X) = 1$ and $Var(X) = 3$. Find
   (a) $E((1 + X)^2)$ and
   (b) $Var(4 + 2X)$.

5. **Demo**. Suppose the demand of certain new product follows the uniform distribution on $[a, b]$ (where $a < b$). The probability density function takes the form:
$$f(x) = \begin{cases} K & a \le x \le b \\ 0 & \text{otherwise.} \end{cases}$$
   Here K is a positive constant to be determined.
   (a) Show that if $K = \frac{1}{b-a}$ then $f(x)$ is a probability density function.
   (b) Find the probability that the demand of a new product lies in $[a, (b + 2a)/3]$.

6. If the probability density function of X is equal to
$$f(x) = \begin{cases} ce^{-2x} & , \ x \ge 0 \\ 0 & , \ x < 0 \end{cases},$$
   find the value of c and for any $t > 0$, find $P(X > 2 + t | X > t)$.

7. Let $X_1, X_2, \ldots, X_n$ be n independent random variables sharing the same probability distribution with mean μ and variance σ^2. Let
$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}.$$

What are the values of $E(\bar{X})$ and $Var(\bar{X})$? What will happen when $n \to \infty$?

8. Determine the smallest integer n for which the probability of no 2 persons having the same birthday (365 days) in a group of n people is less than $1/2$.

9. **Demo**. A point is chosen at random on a line segment of length L. Interpret this statement, and find the probability that the ratio of the shorter to the longer segment is less than $1/4$.

10. A and B alternate rolling a pair of dice, stopping either when A rolls the sum 9 or when B rolls the sum 6. Assuming that A rolls first, find the probability that the final roll is made by A.

1. Let N_i denote the event that the ball is not found in a search of box i, and let B_j denote the event that it is in box j. Then

$$P(B_j|N_i) = \frac{P(N_i|B_j)P(B_j)}{P(N_i|B_i)P(B_i) + P(N_i|B_i')P(B_i')}$$

and

$$P(B_j|N_i) = \begin{cases} \dfrac{P_j}{(1-\alpha_i)P_i + 1 - P_i} & \text{if } j \neq i, \\ \dfrac{(1-\alpha_i)P_j}{(1-\alpha_i)P_i + 1 - P_i} & \text{if } j = i. \end{cases}$$

2. Let F_i, $i = 1, 2, 3$, be the event that the letter is in folder i;
let E be the event that a search of folder 1 does not uncover the letter. Then

$$\begin{aligned} P(F_1|E) &= \frac{P(E|F_1)P(F_1)}{\sum_{i=1}^{3} P(E|F_i)P(F_i)} \\ &= \frac{(1-\alpha_1)\cdot\frac{1}{3}}{(1-\alpha_1)\cdot\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} \\ &= \frac{1-\alpha_1}{3-\alpha_1}\ . \end{aligned}$$

3. $(1 \cdot 5000 + 1 \cdot 700 + 3 \cdot 100)/10000 = 0.6$.

4. (a) $E[(1+X)^2] = Var(1+X) + (E[1+X])^2 = Var(X) + (1+1)^2 = 3 + 2^2 = 7$.
(b) $Var(4+2X) = 4Var(X) = 4 = 12$.

5. (a) Since

$$\int_{-\infty}^{\infty} f(x)dx = \int_a^b f(x)dx = K(b-a) = 1$$

we have $K = (b-a)^{-1}$.
(b) The probability is $\int_a^{(b+2a)/3} \frac{1}{b-a} dx = \frac{b-a}{3(b-a)} = \frac{1}{3}$.

6. It's required that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Since

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_0^{\infty} ce^{-2x}\,dx \\ &= c\Big[\frac{e^{-2x}}{-2}\Big]_0^{\infty} \\ &= c \cdot \frac{1}{2}\ , \end{aligned}$$

the function $f(x)$ is a pdf if $c \cdot \frac{1}{2} = 1$ or $c = 2$. Now,

$$
\begin{aligned}
P(X > 2+t|\ X > t) &= \frac{P(X > 2+t \quad \text{and} \quad X > t)}{P(X > t)} \\
&= \frac{P(X > 2+t)}{P(X > t)} \\
&= \frac{\int_{2+t}^{\infty} 2e^{-2x}\,dx}{\int_{t}^{\infty} 2e^{-2x}\,dx} \\
&= \frac{\left[-e^{-2x}\right]_{2+t}^{\infty}}{\left[-e^{-2x}\right]_{t}^{\infty}} \\
&= \frac{e^{-2(2+t)}}{e^{-2t}} \\
&= e^{-4}\ ,
\end{aligned}
$$

which is independent of t.

7. We have $E(\bar{X}) = \frac{1}{n}(\mu + \cdots + \mu) = \mu$ which is independent of n. However, we have $Var(\bar{X}) = \frac{1}{n^2}(\sigma^2 + \cdots + \sigma^2) = \frac{\sigma^2}{n}$ which tends to zero as n goes to infinity.

8. The probability that among n persons there is no 2 persons having the same birthday is
$$P_n = 1 \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{365-(n-1)}{365}.$$
Check $P_{22} = 0.5243$ and $P_{23} = 0.4927$. Therefore the smallest n is 23.

9. Here X is uniform on $(0, L)$.
$$
\begin{aligned}
& P\left(min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) < \frac{1}{4}\right) \\
& = 1 - P\left(min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) > \frac{1}{4}\right) \\
& = 1 - P\left(\frac{X}{L-X} > \frac{1}{4}, \frac{L-X}{X} > \frac{1}{4}\right) \\
& = 1 - P(X > L/5, X < 4L/5) \\
& = 1 - P(L/5 < X < 4L/5) \\
& = 1 - 3/5 = 2/5
\end{aligned}
$$

10. Let P_A be the probability that A wins when A rolls first, and let P_B be the probability that B wins when B rolls first. Using that the sum of the dice is 9 with

probability 1/9, we obtain upon conditioning on whether A rolls a 9 that

$$P_A = \frac{1}{9} + \frac{8}{9}(1 - P_B).$$

Similarly,

$$P_B = 5/36 + 31/36(1 - P_A).$$

Solving these equations gives that $P_A = 9/19$ (and that $P_B = 45/76$.)