IFT 6269: Probabilistic Graphical Models	Fall 2016
Lecture 2 — September 6	
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Disclaimer: These notes have only been lightly proofread.

2.1 Probability review

2.1.1 Motivation

Question : Why do we use probability in data science ?
Answer : Probability theory is a principled framework to model uncertainty.

Question : Where does uncertainty come from ?
Answer : There are several sources :

1. it can be intrinsic to certain phenomenon (e.g. quantum mechanics) ;
2. reasoning about future events ;
3. we can only get partial information about some complex phenomenon :
   (a) e.g. throwing a dice, it is hard to fully observe the initial conditions ;
   (b) for an object recognition model, a mapping from pixels to objects can be incredibly complex.

2.1.2 Notation

Note that probability theorists and the graphical models community both use a lot of notational shorthands. The meaning of notations often has to be inferred from the context. Therefore, let's recall a few standard notations.

Random variables will be noted $X_1, X_2, X_3, \ldots$, or sometimes X, Y, Z. Usually, they will be real-valued.

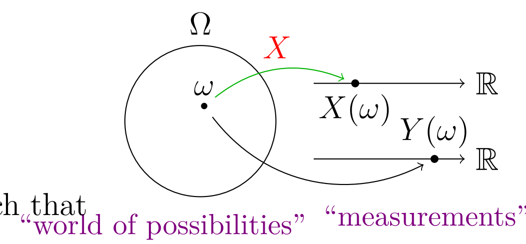
$x_1, x_2, x_3, \ldots$ (or x, y, z), will denote the realizations of the former random variables (the values the Xs can take).

Formally

Let us define Ω, a sample space of elementary events, $\{\omega_1, \omega_2, \omega_3, \dots\}$[1].

Then a random variable is a (measurable[2]) mapping $X : \Omega \mapsto \mathbb{R}$.

Then, a probability distribution P is a mapping $P : \mathcal{E} \mapsto [0,1]$, where $\mathcal{E}$ is the set of all subsets of Ω, i.e. the set of events (i.e. 2^Ω, i.e. a σ-field[3]) ; such that



$$\left.\begin{array}{l} -P(E) \geq 0 \quad \forall E \in \mathcal{E} \\ -P(\Omega) = 1 \\ -P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} (E_i) \quad \text{when } E_1, E_2, \dots \text{ are disjoint.} \end{array}\right\} \text{Kolmogorov axioms}$$

Therefore, a probability distribution on Ω induces a probability distribution on the image of X[4] : $\Omega_X \triangleq X(\Omega)$. An event $\{x\}$ for $x \in \Omega_X$ thus gets the probability

$$\begin{aligned} P_X(\{x\}) &= P(\{\omega : X(\omega) = x\}) \\ &= P(X^{-1}(\{x\})) \\ &= P\{X = x\} \quad \text{(shorthand)} \\ &= p(x) \quad \text{actually used shorthand, even more ambiguous} \end{aligned}$$

where $X^{-1}(A) \triangleq \{\omega : X(\omega) \in A\}$.

Example

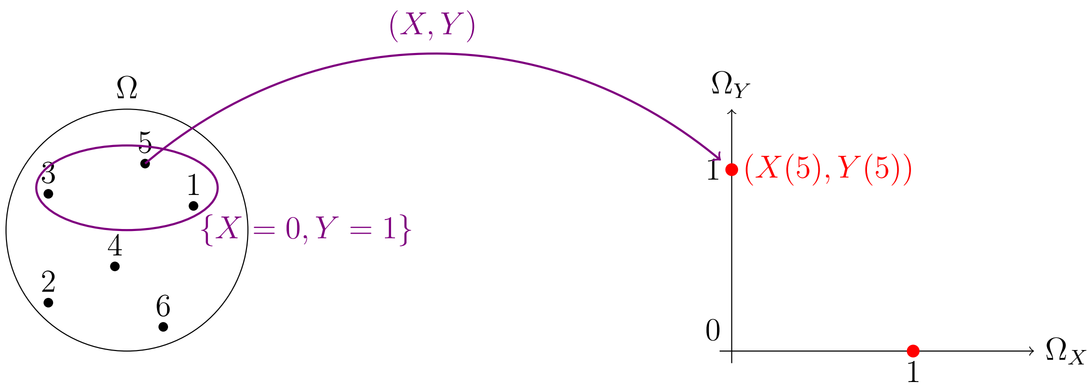
In the case of a dice roll, $\Omega = \{1, 2, \dots, 6\}$. Let's consider two random variables :
X measures whether the dice result is even.
Y measures whether the dice result is odd.

Formally, $X = \mathbb{1}_{\{2,4,6\}}$, and $Y = \mathbb{1}_{\{1,3,5\}}$ where

$$\mathbb{1}_A(\omega) \triangleq \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$$

is the indicator function on A.

[1] temporarily assumed to be a countable set
[2] Wikipedia
[3] the σ-field formalism is necessary when Ω is uncountable, which happens as soon as we consider a continuous random variable.
[4] The image of X is the set of the possible outputs of X : $X(\Omega) = \{x : \exists \omega \in \Omega \ s.t. \ X(\omega) = x\}$



We can now define the joint distribution on $(X, Y) \in \Omega_X \times \Omega_Y$.

$$P_{X,Y}(\{X = x,^{5}\ Y = y\}) = P\left(X^{-1}(\{x\}) \cap Y^{-1}(\{y\})\right)$$

(X, Y) can be called a random vector, or a vector-valued random variable, with "random variable" meant in a generalized sense.

We can represent the joint distribution as a table, such as in our running example :

	$X = 0$	$X = 1$
$Y = 0$	0	$\frac{1}{2}$
$Y = 1$	$\frac{1}{2}$	0

For instance : $P(\{X = 1, Y = 0\}) = P(\{2, 4, 6\}) = \sum_{\omega \in \{2,4,6\}} p(\omega) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$.

Let's also define, in the context of a joint distribution, the marginal distribution, i.e. the distribution on components of the random vector :

$$P\{X = x\} = \sum_{y \in \Omega_Y} P\{X = x, Y = y\} \qquad \text{(sum rule)}$$

This rule is a property, deriving it from the axioms is left as an exercice for the reader.

2.1.3 Types of random variables

Discrete random variables

For a discrete random variable, Ω_X is countable. Its probability distribution on Ω_X, P_X, is fully defined by its probability mass function (aka pmf), $P_X(\{X = x\})$, for $x \in \Omega_X$. This notation is shortened as $P_X(x)$, and even as $p(x)$, "typing" x as only denoting values of the X variable. Thereby, it is possible that $p(x) \neq p(y)$ even if $x = y$, in the sense that $p(x)$ means $P_X(x)$ and $p(y)$ means $P_Y(y)$.

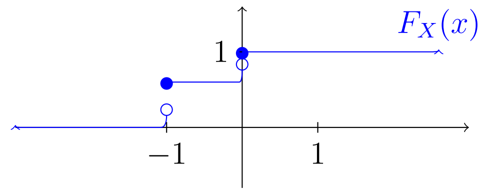
More generally, for $\Omega_X \in \mathbb{R}$, the probability distribution P_X is fully characterized by its cumulative distribution function (aka cdf) : $F_X(x) \triangleq P_X\{X \leq x\}$.

[5]This comma means and, the intersection of both events.

It has the following properties :

1. F_X is non-decreasing ;
2. $\lim_{x \to -\infty} F_X(x) = 0$;
3. $\lim_{x \to +\infty} F_X(x) = 1$.

For discrete random variables, the cumulative distribution function is piecewise constant, and has jumps.



Example of a cumulative distribution function.

Continuous random variables

For a continuous random variable, the cumulative distribution function is "absolutely continuous", i.e. is differentiable almost everywhere, and $\exists f(x)$ s.t. $F_X(x) = \int_{-\infty}^{x} f(u)du$. Said f is called the probability density function of the random variable (aka pdf). Where f is continuous, $\frac{d}{dx}F_X(x) = f(x)$.

The probability density function is the continuous analog of the probability mass function of a discrete random variable (with sums becoming integrals). Hence :

discrete	continuous
$\sum_{x \in \Omega_X} p(x) = 1$	$\int_{\Omega_X} p(x) = 1$
$p =$ prob. mass function	$p =$ prob. density function

Note in the continuous case, as a density function, $p(x)$ can be greater than 1, on a sufficiently narrow interval. For instance, the uniform distribution on $[0, \frac{1}{2}]$:

$$p(x) = \begin{cases} 2 & \text{for } x \in [0, \frac{1}{2}] \\ 0 & \text{otherwise} \end{cases}$$

2.1.4 Other random variable basics

Expectation/mean

The expectation of a random variable is

$$\mathbb{E}[X] \triangleq \sum_{x \in \Omega_X} x\ p(x) \quad \text{or} \quad \int_{\Omega_X} x\ p(x)\, dx \quad \text{(in the continuous case)}$$

Variance

$$\begin{aligned} Var[X] &\triangleq \mathbb{E}[(X - \mathbb{E}(X))^2] \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \end{aligned}$$

Variance is a measure of the dispersion of values around the mean.

Independance

X is independant from Y, noted $X \perp\!\!\!\perp Y$, iff $p(x,y) = p(x)p(y) \;\; \forall x, y \in \Omega_X \times \Omega_Y$.
Random variables $X_1, \dots X_n$ are mutually independant iff $p(x_1, \dots x_n) = \prod_{i=1}^{n} p(x_i)$.

Conditioning

For events A and B, suppose that $p(B) \neq 0$. We define the probability of A given B,

$$P(A|B) \triangleq \frac{P(A \cap B)}{P(B)}$$

In terms of sample space, that means we look at the subspace where B happens, and in that space, we look at the subspace where A also happens.

For random variables X and Y, thus :

$$P(X = x|Y = y) \triangleq \frac{P(X = x, Y = y)}{P(Y = y)}$$

$P(Y = y) = \sum_x P(X = x, Y = y)$ is a normalization constant, necessary in order to get a real probability distribution.

By definition, we get the product rule :

$$p(x,y) = p(x|y)p(y) \qquad \text{(product rule)}$$

It is always true, with the subtle point that $p(x|y)$ is undefined if $p(y) = 0$.[6]

Bayes rule

Bayes rule is about inverting the conditioning of the variables.

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\sum_{x'} p(x', y)} \qquad \text{(Bayes rule)}$$

Chain rule

By successive application of the product rule, it is always true that :

$$\begin{aligned} p(x_1, \dots, x_n) &= p(x_{1:n-1})p(x_n|x_{1:n-1}) \\ &= \cdots \\ &= \textstyle\prod_{i=1}^{n} p(x_i|x_1, \dots, x_{i-1}) \end{aligned} \qquad \text{(Chain rule)}$$

The last part can be simplified using the conditional independance asumptions we make, like in the case of directed graphical models.

[6] In probability theory, we usually do not care what happens on sets with probability zero; so we are free to define $p(x|y)$ to be any value we want when $p(y) = 0$.

Conditional independance

X is conditionally independant of Y given Z, noted $X \perp\!\!\!\perp Y|Z$, iff

$$p(x, y|z) = p(x|z)p(y|z) \quad \forall x, y, z \in \Omega_x \times \Omega_y \times \Omega_z \ s.t.\ p(z) \neq 0$$

For instance, with Z the probability that a mother carries a genetic disease on chromosome X, X the probability for her first child to carry the disease, and Y the same probability for her second child, we can say that X is independant of Y given Z (because only the status of the mother impacts directly each child : once that is known, children's probabilities of carrying the disease are independant from each other).

As an exercise to the reader, prove that $p(x|y, z) = p(x|z)$ when $X \perp\!\!\!\perp Y|Z$.
$\because X \perp\!\!\!\perp Y|Z$
$\therefore (x, y|z) = p(x|z)p(y|z)$
Based on Bayes theorem
$p(x|y, z) = \frac{p(x,y,z)}{p(y,z)} = \frac{p(x,y|z)p(z)}{p(y|z)p(z)} = \frac{p(x|z)p(y|z)p(z)}{p(y|z)p(z)}$
$\therefore p(x|y, z) = \frac{p(x|z)p(y|z)p(z)}{p(y|z)p(z)} = p(x|z)$