

代码说明

第一题

$$\begin{aligned} Pdf_B &= c \times x_i^{-4.5} \\ X_i &\sim N(m, \sigma^2), \therefore Pdf_s = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i-m)^2}{2\sigma^2}} \\ \therefore f_s &= \frac{n_s}{n_s + n_b}, \therefore L_k = \prod_{i=1}^n \left(\frac{n_s}{n_s + n_b} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i-m)^2}{2\sigma^2}} + \frac{n_b}{n_s + n_b} c x_i^{-4.5} \right) \\ \ln L_k &= \sum_{i=1}^n \ln \left(\frac{n_s}{n_s + n_b} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i-m)^2}{2\sigma^2}} + \frac{n_b}{n_s + n_b} c x_i^{-4.5} \right) \end{aligned}$$

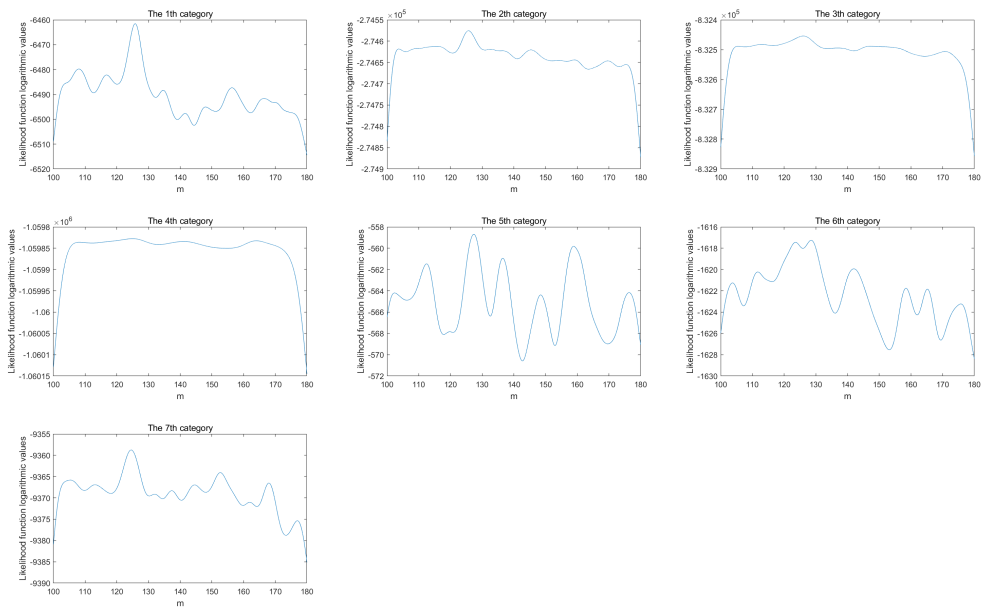
计算要求（续）

$$\begin{aligned} L &= \prod_{i=1}^N \left(\sum_{k=0}^6 f s_k \times pdf_s + f b \times pdf_b \right) \\ &= \prod_{i=1}^N (f s \times pdf_s + f b \times pdf_b) \\ \ln L &= \sum_{i=1}^N \ln (f s \times pdf_s + f b \times pdf_b) \\ &= \sum_{i=1}^N \ln \left(f s \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i-m)^2}{2\sigma^2}} + f b \times c x_i^{-4.5} \right) \end{aligned}$$

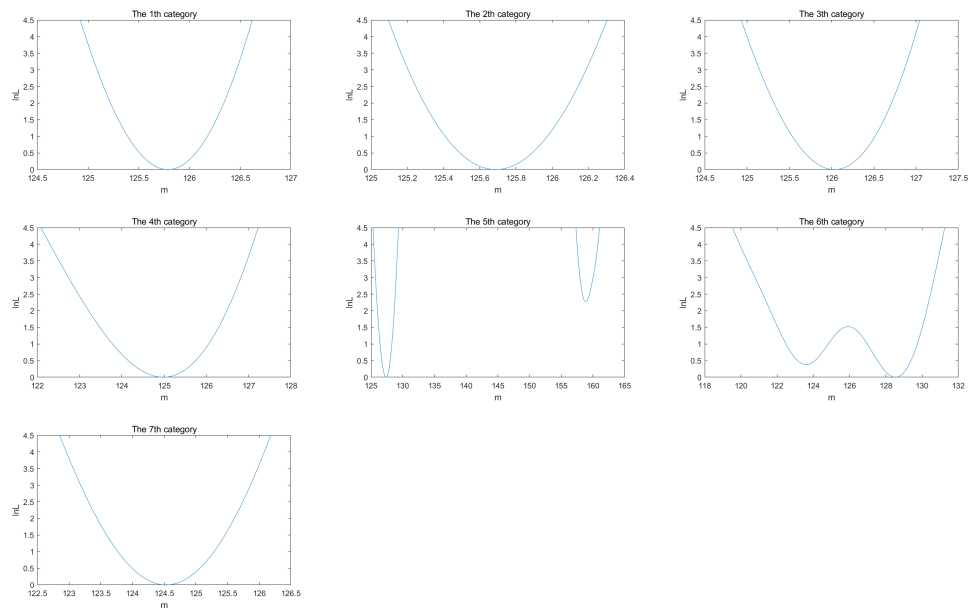
$$\text{其中 } \sigma = \sqrt{\frac{\sum_{k=0}^7 n s_k \sigma_k^2}{n s}}$$

计算结果

0.01扫描作图—— $\ln L$



0.01扫描作图—— $2(\ln L(\hat{m}) - \ln L)$



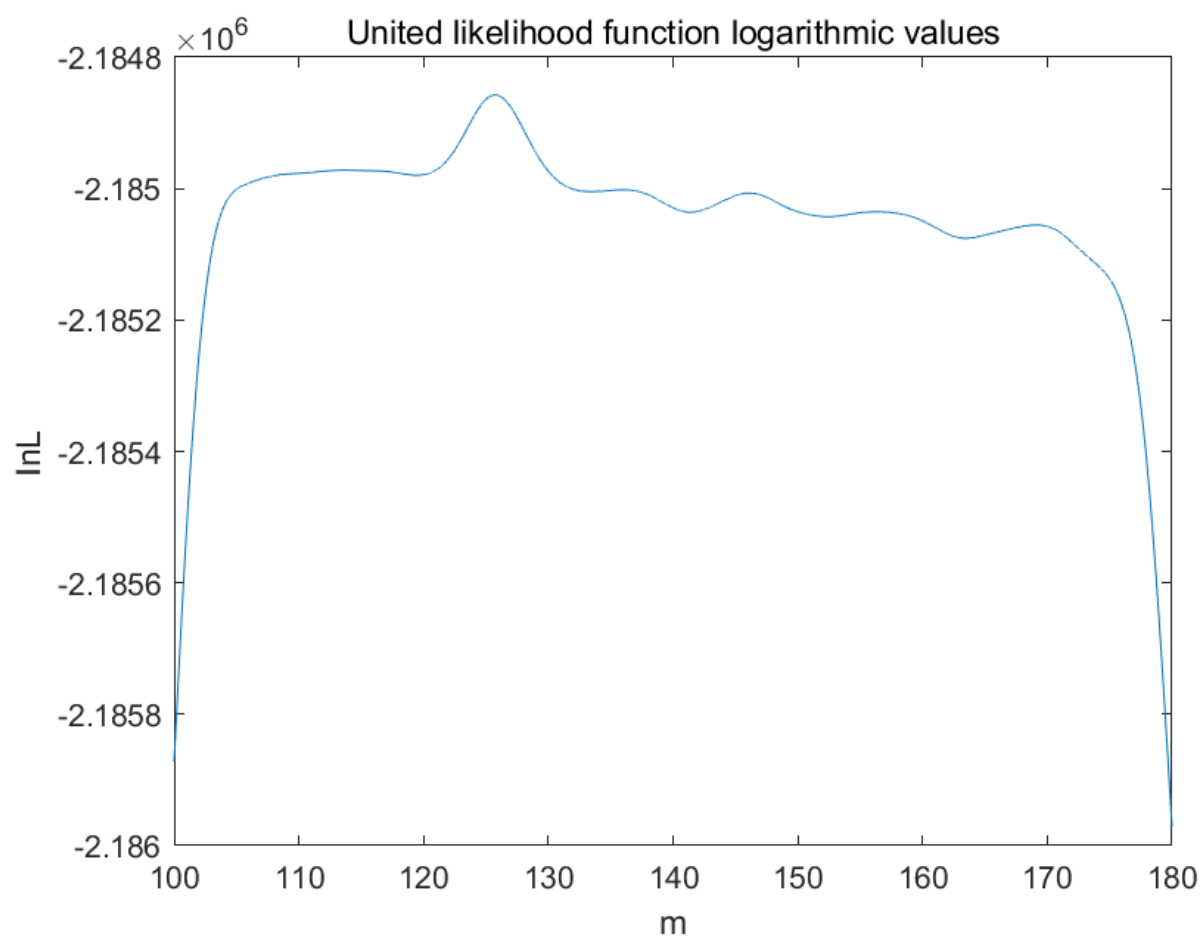
精细求解 \hat{m}_k 和68.3%CL区间端点

k	0	1	2	3	4	5	6
\hat{m}_k	125.7864	125.6887	126.0348	124.9765	127.3645	128.5342	124.5286

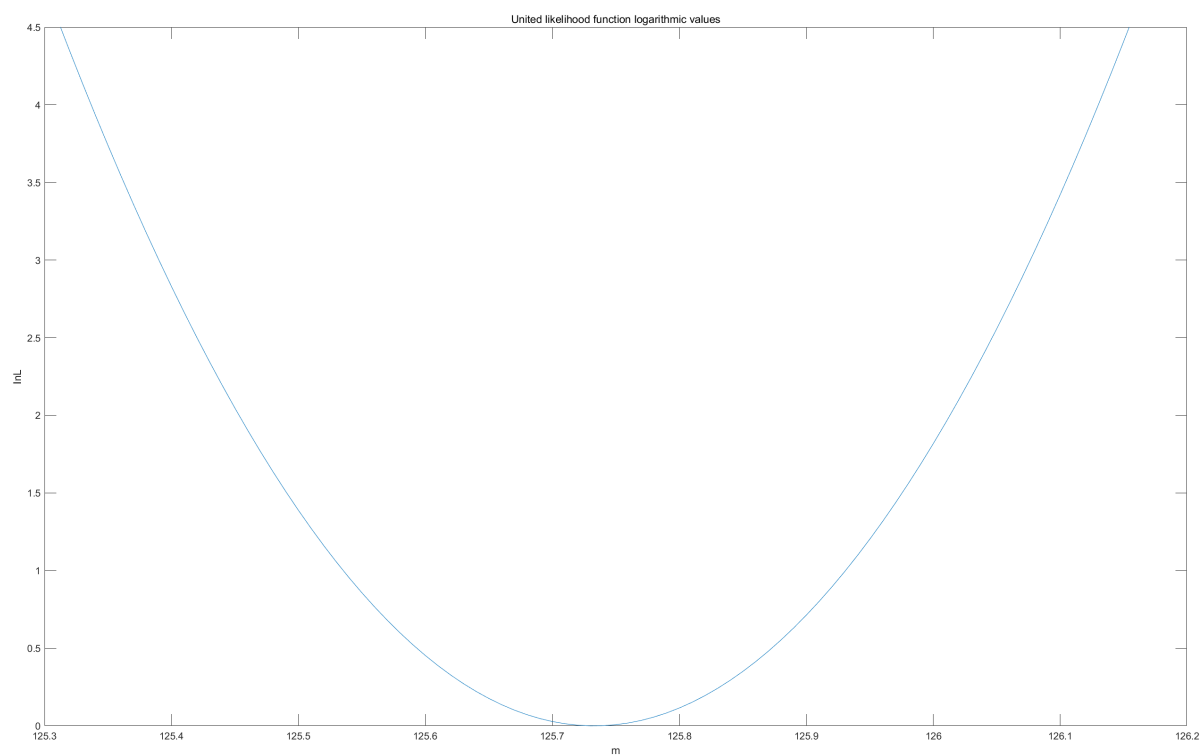
k	0	1	2	3	4	5	6
m_{lo}	125.3900	125.4088	125.5354	123.7780	126.4094	126.9747	123.7678
m_{up}	126.1758	125.9733	126.5141	126.0513	128.2738	129.7347	125.2854

因此Category1,Category0,Category2精确度最高。

联合分布似然函数 $\ln L$



联合分布似然函数 $2(\ln L(\hat{m}) - \ln L)$

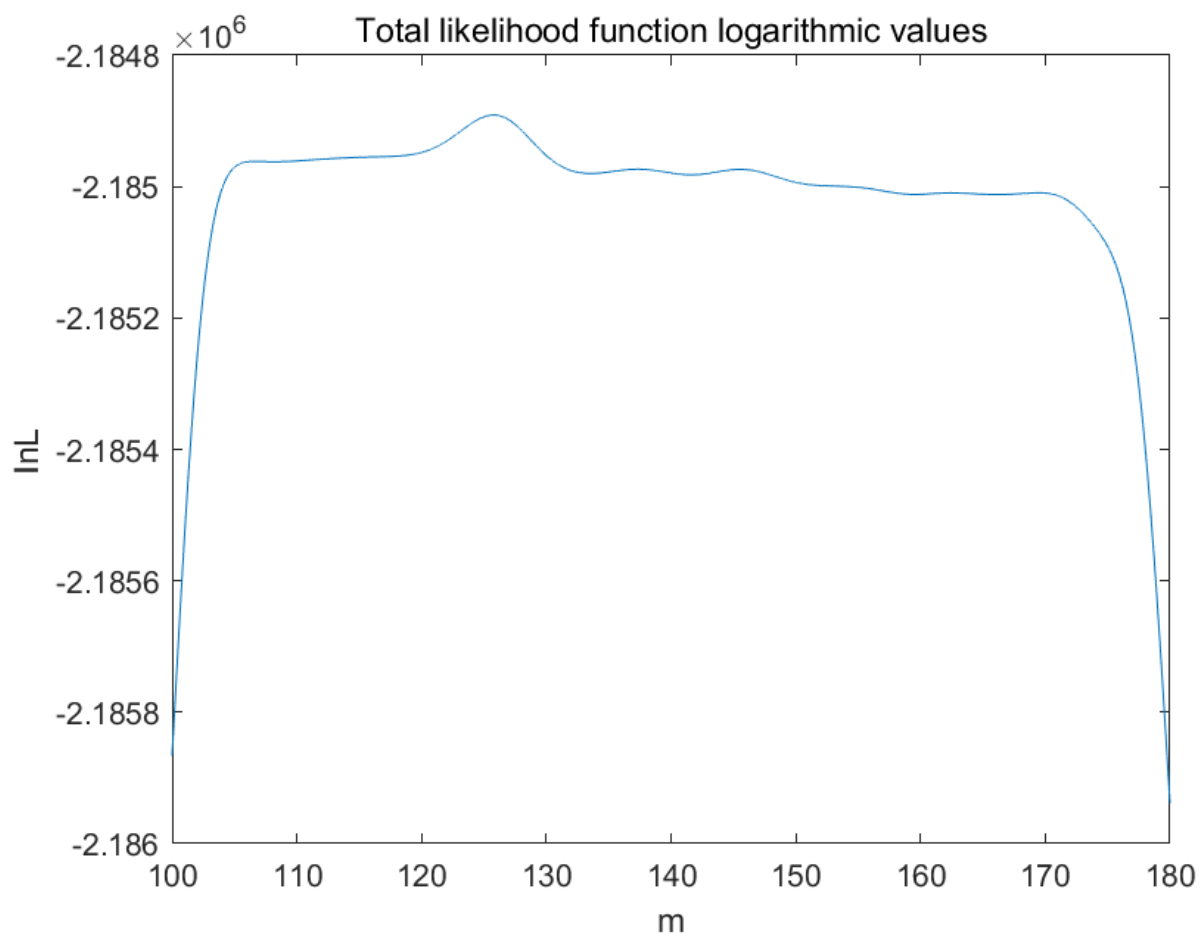


联合分布的参数值

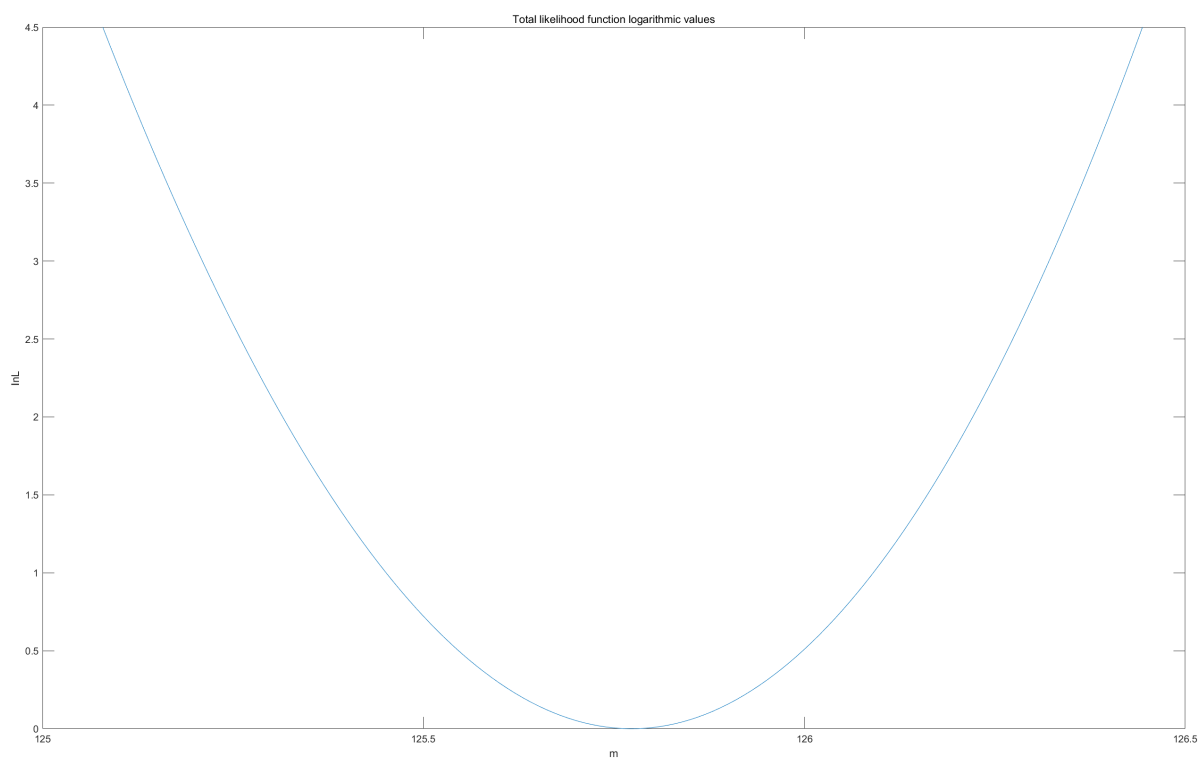
$$\hat{m} = 125.73, 68.3\%CL \rightarrow [125.54, 125.93]$$

由于实验得到的是 $\hat{m} = 125.78 \pm 0.21 \rightarrow m \in [125.57, 125.99]$ 宽度基本吻合，中心值略偏差。

混合分布似然函数 $\ln L$



混合分布似然函数 $2(\ln L(\hat{m}) - \ln L)$



混合分布参数值

$$\hat{m} = 125.77, 68.3\%CL \rightarrow [125.45, 126.09]$$

相比联合分布，精度下降了64.1%。原因是相比联合分布，缺少了七组数据各自标准差等信息，对样本的描述变粗糙了。

代码

```
1 cat=cell(1,7);
2 for i=1:7
3     str=['mgg_cms2020_cat',num2str(i-1),'.txt'];
4     cat(i)={load(str)};
5 end
6 parameter=readmatrix('数据和模型.xlsx');
7 %%
8 syms c m
9 P=int(c*m^(-4.5),m,100,180);
10 c=double(solve(P==1,c));
11 %%
12 sigma=parameter(:,3);
13 ns=parameter(:,2);
14 nb=parameter(:,4);
15 N=Ns+Nb;
16 M=100:.01:180;
17 f1=figure;
18 f1.Position=[0,0,2000,2000];
19 for k=1:7
20     sj=cat{k};
21     sigma=sigma(k);
22     ns=Ns(k);
23     nb=Nb(k);
24     n=N(k);
25     lnLk=0;
26     for j=1:length(M)
27         m=M(j);
28         for i=1:length(sj)
29             lnLk=lnLk+log(ns/n*1/(2*pi*sigma^2)^(1/2).*exp(-(sj(i)-
30 m).^2./(2*sigma^2))+nb/n*c.*sj(i).^(-4.5));
31         end
32         siran(j)=lnLk;
33         lnLk=0;
34     end
35     subplot(3,3,k)
36     plot(M,siran)
37     xlabel('m')
38     ylabel('Likelihood function logarithmic values')
39     Tit=['The ',num2str(k),'th category'];
40     save([Tit,'.mat'],'siran')
41     title(Tit)
42     hatm=M(find(siran==max(siran)));
43     Hatm(k)=hatm;
44 end
45 %%
46 f2=figure;
```

```

46 f2.Position=[0,0,2000,2000];
47 for k=1:7
48     sj=cat{k};
49     sigma=Sigma(k);
50     ns=Ns(k);
51     nb=Nb(k);
52     n=N(k);
53     lnLk=0;
54     m=Hatm(k);
55     for i=1:length(sj)
56         lnLk=lnLk+log(ns/n*1/(2*pi*sigma^2)^(1/2).*exp(-(sj(i)-
m).^2./(2*sigma^2))+nb/n*c.*sj(i).^(-4.5));
57     end
58     pred=lnLk;
59     Tit=['The',num2str(k),'th category'];
60     load([Tit,'.mat'])
61     noname=2.*(pred-siran);
62     subplot(3,3,k)
63     M=100:.01:180;
64     plot(M,noname)
65     xlabel('m')
66     ylabel('lnL')
67     Tit=['The ',num2str(k),'th category'];
68     title(Tit);
69     ylim([0,4.5]);
70     jidian=find(M==m);
71     if k~=6
72
73         CLmin(k)=M(find(abs(noname(1:jidian)-1)==min(abs(noname(1:jidian)-1)))));
74
75         CLmax(k)=M(find(abs(noname(jidian:end)-1)==min(abs(noname(jidian:end)-1)))-1+jidian);
76         elseif k==6
77             left=find(M==126);right=find(M==128);
78
79             CLmin(k)=M(find(abs(noname(left:right)-1)==min(abs(noname(left:right)-1)))-1+left);
80
81             CLmax(k)=M(find(abs(noname(right:end)-1)==min(abs(noname(right:end)-1)))-1+right);
82     end
83 end
84 %% 精细求解hatm
85 for k=1:7
86     sj=cat{k};
87     sigma=Sigma(k);
88     ns=Ns(k);
89     nb=Nb(k);
90     n=N(k);
91     M=Hatm(k)-0.01:0.0001:Hatm(k)+0.01;
92     lnLk=0;
93     for j=1:length(M)
94         m=M(j);
95         for i=1:length(sj)

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92         lnLk=lnLk+log(ns/n*1/(2*pi*sigma^2)^(1/2).*exp(-(sj(i)-
m).^2./(2*sigma^2))+nb/n*c.*sj(i).^(-4.5));
93     end
94     siran(j)=lnLk;
95     lnLk=0;
96 end
97 PreciseHatm(k)=M(find(siran==max(siran)));
98 clear siran
99 end
100 %% 精细求解置信区端点
101 for k=1:7
102     sj=cat{k};
103     sigma=Sigma(k);
104     ns=Ns(k);
105     nb=Nb(k);
106     n=N(k);
107     M=CLmin(k)-0.01:0.0001:CLmin(k)+0.01;
108     lnLk=0;
109     for j=1:length(M)
110         m=M(j);
111         for i=1:length(sj)
112             lnLk=lnLk+log(ns/n*1/(2*pi*sigma^2)^(1/2).*exp(-(sj(i)-
m).^2./(2*sigma^2))+nb/n*c.*sj(i).^(-4.5));
113         end
114         siran(j)=lnLk;
115         lnLk=0;
116     end
117     lnLk=0;
118     m=PreciseHatm(k);
119     for i=1:length(sj)
120         lnLk=lnLk+log(ns/n*1/(2*pi*sigma^2)^(1/2).*exp(-(sj(i)-
m).^2./(2*sigma^2))+nb/n*c.*sj(i).^(-4.5));
121     end
122     pred=lnLk;
123     noname=2.*(pred-siran);
124     PreciseCLmin(k)=M(find(abs(noname-1)==min(abs(noname-1))));
125 end
126 for k=1:7
127     sj=cat{k};
128     sigma=Sigma(k);
129     ns=Ns(k);
130     nb=Nb(k);
131     n=N(k);
132     M=CLmax(k)-0.01:0.0001:CLmax(k)+0.01;
133     lnLk=0;
134     for j=1:length(M)
135         m=M(j);
136         for i=1:length(sj)
137             lnLk=lnLk+log(ns/n*1/(2*pi*sigma^2)^(1/2).*exp(-(sj(i)-
m).^2./(2*sigma^2))+nb/n*c.*sj(i).^(-4.5));
138         end
139         siran(j)=lnLk;
140         lnLk=0;
141     end
142     lnLk=0;

```

```

143     m=PreciseHatm(k);
144     for i=1:length(sj)
145         lnLk=lnLk+log(ns/n*1/(2*pi*sigma^2)^(1/2).*exp(-(sj(i)-
m).^2./(2*sigma^2))+nb/n*c.*sj(i).^(-4.5));
146     end
147     pred=lnLk;
148     noname=2.*(pred-siran);
149     PreciseCLmax(k)=M(find(abs(noname-1)==min(abs(noname-1)))));
150 end
151 %%
152 f3=figure;
153 f3.Position=[0,0,2000,2000];
154 Unit=zeros(1,8001);
155 M=100:.01:180;
156 for k =1:7
157     Tit=['The',num2str(k),'th category'];
158     load([Tit,'.mat']);
159     Unit=Unit+siiran;
160 end
161 plot(M,Unit)
162 xlabel('m')
163 ylabel('lnL')
164 Tit='United likelihood function logarithmic values';
165 save([Tit,'.mat'],'Unit')
166 title(Tit)
167 unitedhatm=M(find(Unit==max(Unit)));
168 %%
169 f4=figure;
170 f4.Position=[0,0,2000,2000];
171 pred=Unit(find(Unit==max(Unit)));
172 load([Tit,'.mat'])
173 noname=2.*(pred-Unit);
174 M=100:.01:180;
175 plot(M,noname)
176 xlabel('m')
177 ylabel('lnL')
178 title(Tit)
179 ylim([0,4.5])
180 jidian=find(Unit==max(Unit));
181 unitedCLmin=M(find(abs(noname(1:jidian)-1)==min(abs(noname(1:jidian)-1)))));
182 unitedCLmax=M(find(abs(noname(jidian:end)-1)==min(abs(noname(jidian:end)-1)
))-1+jidian);
183 %% Updated
184 sj=[];
185 for i=1:7
186     sj=[sj,cat{i}'];
187 end
188 Sigma=parameter(:,3);
189 ns=parameter(:,2);
190 nb=parameter(:,4);
191 N=ns+nb;
192 sigma=(sum(ns.*Sigma.^2)/sum(ns))^(1/2);
193 fs=sum(ns)/sum(N);
194 fb=sum(nb)/sum(N);
195 M=100:.01:180;

```



```

196 f5=figure;
197 f5.Position=[0,0,2000,2000];
198 lnLk=0;
199 for j=1:length(M)
200     m=M(j);
201     for i=1:length(sj)
202         lnLk=lnLk+log(fs*1/(2*pi*sigma^2)^(1/2).*exp(-(sj(i)-
m).^2./(2*sigma^2))+fb*c.*sj(i).^(-4.5));
203     end
204     siran(j)=lnLk;
205     lnLk=0;
206 end
207 plot(M,siran)
208 xlabel('m')
209 ylabel('lnL')
210 Tit='Total likelihood function logarithmic values';
211 %save([Tit,'.mat'],'siran')
212 title(Tit)
213 Totalhatm=M(find(siran==max(siran)));
214 %%
215 f6=figure;
216 f6.Position=[0,0,2000,2000];
217 pred=max(siran);
218 noname=2.*(pred-siran);
219 M=100:.01:180;
220 plot(M,noname)
221 xlabel('m')
222 ylabel('lnL')
223 title(Tit)
224 ylim([0,4.5])
225 jidian=find(siran==max(siran));
226 TotalCLmin=M(find(abs(noname(1:jidian)-1)==min(abs(noname(1:jidian)-1))));
227 TotalCLmax=M(find(abs(noname(jidian:end)-1)==min(abs(noname(jidian:end)-1))
)-1+jidian);

```

第二题

在第*i*个实验中, 产生 n_i 个数据, $n_i \sim P(s + b)$

贝叶斯参数区间估计: 假设先验概率为 $\pi(s)$

$$p(s|n) = \frac{P(n|s)\pi(s)}{\int P(n|s')\pi(s')ds'}$$

$\pi(s)$ 为阶跃函数。

$$\begin{aligned}
 1 - \alpha &= \frac{\int_0^{s_{up}} P(n|s)\pi(s)ds}{\int_0^{\infty} P(n|s')\pi(s')ds'} \\
 &= \frac{\int_0^{s_{up}} \frac{(s+b)^n}{n!} e^{-(s+b)} ds}{\int_0^{\infty} \frac{(s+b)^n}{n!} e^{-(s+b)} ds'}
 \end{aligned}$$

$$\text{def } \int_0^a x^n e^{-x} dx = \Gamma(n+1) F_{\chi^2}(2a, 2(n+1))$$

$$\therefore s_{up} = \frac{1}{2} F_{\chi^2}^{-1}(p, 2(n+1)) - b, p = 1 - \alpha [1 - F_{\chi^2}(2b, 2(n+1))]$$

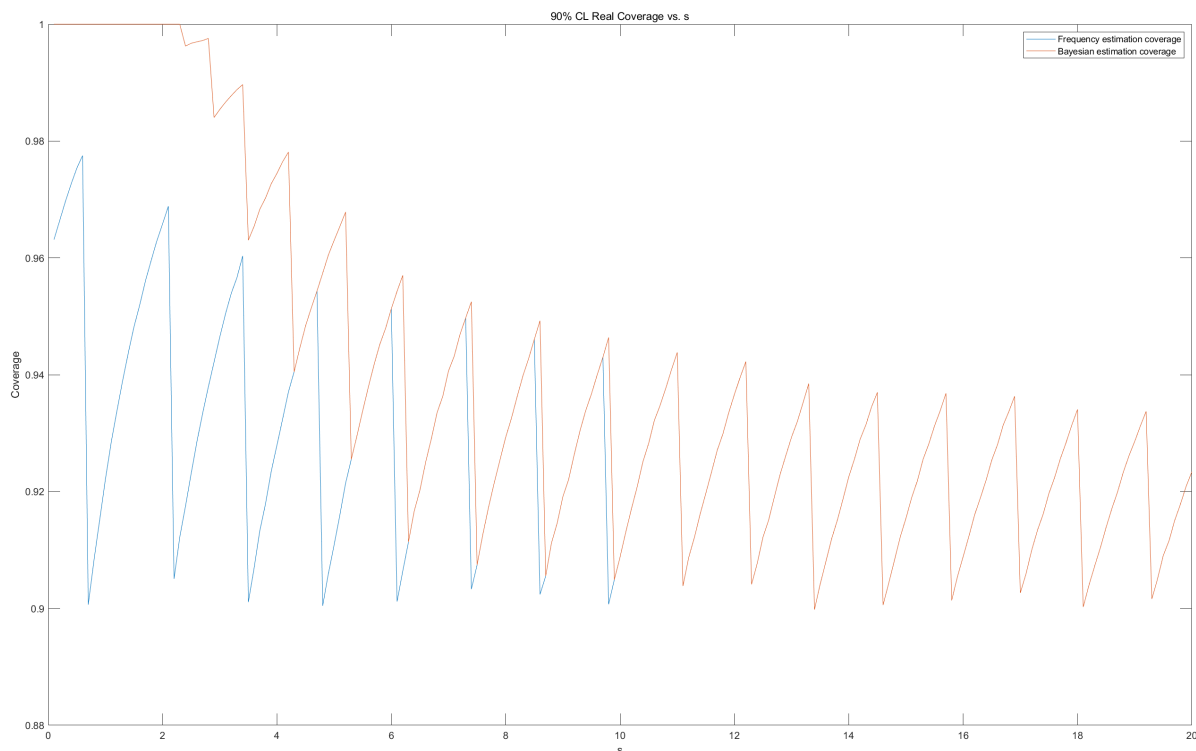
频率论估计：

$$\alpha = \sum_{k=0}^n \frac{(s_{up} + b)^k e^{-(s_{up} + b)}}{k!}$$

$$\text{Theorem } \sum_{k=0}^n \frac{\lambda^n e^{-\lambda}}{k!} = 1 - F_{\chi^2}(2\lambda, 2(n+1))$$

$$\therefore s_{up} = \frac{1}{2} F_{\chi^2}^{-1}(1 - \alpha, 2(n+1)) - b$$

计算结果：



两者均在 $s = 13.4$ 时覆盖率为89.98%低于置信度，其余点均有足够的覆盖率。

曲线的主要特征

- 基本上始终高于0.9：90%置信区间被定义为至少提供 90% 的覆盖率，无论参数 s 的真实值如何。而特别在离散分布中，由于样本离散，覆盖率一般都不会等于置信度。
- 曲线的波浪形：由图像可知，波浪的周期大致为1.原因应当是信号作为泊松过程产生，在整数点上定义，因而会在整数周期上发生覆盖率的波动。
- 贝叶斯和频率论曲线的区别：贝叶斯曲线在 $s = 10$ 之前覆盖率优于频率论，在 s 很小时覆盖率达到了100%。前期拟合较好原因是 s 较小时信号的泊松分布不明显，先验概率相对符合。后期随着似然函数起主导作用，两种估计方法差异不明显。

代码

```
1 b=3.2;  
2 s=6.8;  
3 lambda=s+b;  
4 N=poissrnd(lambda,[1,1000000]);  
5 Bsig=0;  
6 Fsig=0;  
7 for i=1:length(N)  
8     n=N(i);
```

```

9      p=1-0.1*(1-chi2cdf(2*b,2*(n+1)));
10     BCLup=1/2*chi2inv(p,2*(n+1))-b;
11     FCLup=1/2*chi2inv(0.9,2*(n+1))-b;
12     if BCLup>=s
13         Bsig=Bsig+1;
14     end
15     if FCLup>=s
16         Fsig=Fsig+1;
17     end
18 end
19 %%
20 s=0.1:0.1:20;
21 b=3.2;
22 for j=1:length(s)
23     s=s(j);
24     lambda=s+b;
25     N=poissrnd(lambda,[1,1000000]);
26     Bsig=0;
27     Fsig=0;
28     for i=1:length(N)
29         n=N(i);
30         p=1-0.1*(1-chi2cdf(2*b,2*(n+1)));
31         BCLup=1/2*chi2inv(p,2*(n+1))-b;
32         FCLup=1/2*chi2inv(0.9,2*(n+1))-b;
33         if BCLup>=s
34             Bsig=Bsig+1;
35         end
36         if FCLup>=s
37             Fsig=Fsig+1;
38         end
39     end
40     p1(j)=Bsig/1000000;
41     p2(j)=Fsig/1000000;
42 end
43 save('Byes Cover Probablity.mat','p1')
44 save('Frequency Cover Probablity.mat','p2')

```