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Implications of Conformal Invariance in Field Theories for General Dimensions

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The requirements of conformal invariance for two and three point functions for general dimension d on flat space are investigated. A compact group theoretic construction of the three point function for arbitrary spin fields is presented and it is applied to various cases involving conserved vector operators and the energy momentum tensor. The restrictions arising from the associated conservation equations are investigated. It is shown that there are, for general d, three linearly independent conformal invariant forms for the three point function of the energy momentum tensor, although for $d = 3$ there are two and for $d = 2$ only one. The form of the three point function is also demonstrated to simplify considerably when all three points lie on a straight line. Using this the coefficients of the conformal invariant three point functions are calculated for free scalar and fermion theories in general dimensions and for abelian vector fields when $d = 4$. Ward identities relating three and two point functions are also discussed. This requires careful analysis of the singularities in the short distance expansion and the method of differential regularisation is found convenient. For $d = 4$ the coefficients appearing in the energy momentum tensor three point function are related to the coefficients of the two possible terms in the trace anomaly for a conformal theory on a curved space background.

1 Introduction

There has been an enormous literature in recent years devoted to conformal field theories in two dimensions. The starting point of such discussions [1,2] is almost invariably the operator product expansion for $T(z) = T_{zz}(x)$, where $T_{\mu\nu}(x)$ is the traceless two dimensional energy momentum tensor and $z = x_1 + ix_2$, which has the form

$$T(z)T(w) \sim \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}T'(w)\,. \tag{1.1}$$

An alternative and equivalent result for this operator product expansion is given by the conformal invariant forms for the two and three point functions of $T(z)$,

$$\begin{aligned}\langle T(z)\,T(w)\rangle &= \frac{\frac{1}{2}c}{(z-w)^4}\,,\\ \langle T(z_1)\,T(z_2)\,T(z_3)\rangle &= \frac{c}{(z_1-z_2)^2(z_2-z_3)^2(z_3-z_1)^2}\,.\end{aligned} \tag{1.2}$$

Writing $T(z) = \sum_n L_n z^{-n-2}$ then (1.1) is also equivalent to the Virasoro algebra where where c is the central charge. When $c \leq 1$, there is a complete classification of unitary conformal field theories but also for $c > 1$ there is a extremely large set of explicitly constructed examples of conformal field theories with very rich mathematical structure. Besides string theory these are of relevance to two dimensional statistical physics systems at critical points defined by vanishing β functions where the trace of the energy momentum tensor vanishes and the systems enjoy conformal invariance. The work on conformal field theories in two dimensions has also led to further understanding of non conformally invariant quantum field theories. Motivated by the definition of c in terms of the two point function of the energy momentum tensor as in (1.2) in conformal field theories Zamolodchikov [3] showed how to define for non conformal theories a quantity $C(t)$, where $t = \ln a + \text{const.}$ for a some distance scale, which is monotonically decreasing under renormalisation flow for t increasing and is stationary only at critical points where the β function vanishes. As $t \to \infty$ and the critical point is approached $C(\infty)$ becomes equal to the Virasoro central charge of the conformal field theory associated with the critical point.

Although the conformal group in two dimensions is infinite dimensional, and additional powerful constraints such as modular invariance are present, there remains a clear motivation for studying conformal field theories in three and four dimensions which are respectively relevant for realistic statistical physics systems at their critical points and for physical quantum field theories. However conformal field theories in dimensions $d > 2$ are relatively far less well explored although much work was undertaken twenty or so years ago [4,5,6,7,8,9,10,11,12] under the influence of the discovery of scaling in deep inelastic scattering and more recently there have been discussions motivated in part by the wish to extend two dimensional results [13,14,15,16]. In three dimensions there should be many non-trivial examples corresponding to the different universality classes of statistical physics systems at their various critical points. Nevertheless, it is very difficult to construct explicitly solvable examples except in some approximation based on free field theories. In four dimensions it is likely that conformal field theories are rather rare, based on the strong evidence of triviality for most quantum field theories when the cut off is removed. Possible examples are $N = 4$ supersymmetric gauge theories when the β function is identically zero and also $SU(N)$ gauge theories in the $N \to \infty$ limit with large numbers of fermions in the fundamental representation. If the number of fermion representations $f = \frac{11}{2}N - k$ with $k \ll N$ then there is an infra-red stable fixed point where $\beta(g_*) = 0$ for $g_*^2 N = \mathrm{O}(N^{-1})$ which should be perturbatively accessible [17,18]. For $d = 4 - \varepsilon$, and also $d = 2 + \varepsilon$, there are of course examples of conformal field theories whose critical exponents may be determined as an expansion in ε.

In general for $d > 2$ it is not clear what are the crucial parameters specifying a conformal field theory analogous to c when $d = 2$. The most obvious generalisation of c is in terms of the overall scale of the two point function of the energy momentum tensor which has a unique form in the conformal limit. Indeed, Cappelli *et al.* [19] and also Shore [20] considered this in an attempt to generalise the Zamolodchikov c-theorem to $d > 2$. This endeavour is unfortunately not fully successful in that the supposed candidate for $C(t)$ is not monotonic under renormalisation flow. An alternative approach considered by Cardy [21], and pursued subsequently by Jack and Osborn [18,22], is through the trace of the energy momentum tensor $T_{\mu\nu}$ on curved space which is a c-number in the conformal limit. In two dimensions for conformal theories $g^{\mu\nu}T_{\mu\nu} \propto R$, the scalar curvature, where the coefficient of proportionality provides an alternative definition of the Virasoro central charge c. In four dimensions, assuming conformal invariance, $g^{\mu\nu}T_{\mu\nu}$ contains two terms proportional to F, G where F, G are dimension 4 scalars constructed from the Riemann tensor. The coefficient of F is related to the two point function of the energy momentum tensor while the coefficient of G, which is the Euler density, was suggested by Cardy, and subsequently analysed in perturbation theory by Jack and Osborn, as a candidate for a c-theorem. Although the perturbative evidence is encouraging [18] there are no general results for the desired positivity conditions which ensure a monotonic renormalisation flow.

In this paper the intention is to initiate an analysis of three point functions of the energy momentum tensor and other operators in dimensions $d > 2$, extending the results in (1.1) and (1.2). In conformal field theories three point functions are essentially unique since the conformal group is transitive on three points. We will show that there are three linearly independent forms for the conformally invariant three point function for the energy momentum tensor in arbitrary dimension. However in three dimensions this is restricted to two and for $d = 2$ to one (and the coefficient is just the central charge c again). In four dimensions the three coefficients present in the general conformally invariant expression may be related to the coefficients of F, G in the expansion of the trace of the energy momentum tensor on curved space [23].

The analysis of the consequences of conformal invariances rapidly becomes tedious with the proliferation of indices. In the next section we describe a group theoretic approach for conformal invariant three point functions which is based on a paper of Mack [10] some time ago. This gives a relatively compact construction and allows a simple analysis of the conditions for conservation of the energy momentum tensor $\partial_\mu T_{\mu\nu} = 0$ and also for vector currents V_μ when we require $\partial_\mu V_\mu = 0$. The construction is shown to be both necessary and sufficient for any conformal invariant three point function by showing how the complete three point function is determined by the the leading term in the operator product expansion for two operators. In section 3 the general construction is applied to various cases involving $T_{\mu\nu}$ and V_μ and also scalar operators of arbitrary dimension. In section 4 an alternative approach based on requiring the three points to lie on a straight line is described. In this case the conformal invariant form simplifies and a straightforward algebraic approach is feasible. It is also possible to connect specific cases to the more general analysis of the previous section. The discussion of this collinear configuration is further advantageous since it is then feasible to derive fairly simply the forms of the three point function in the examples of conformal field theories in general dimensions provided by free scalars and fermions. This is undertaken in section 5 where we calculate the coefficients defining the three point function of the energy momentum tensor in these cases as well as for other examples discussed earlier. For $d = 4$ we also consider the conformal field theory defined by free vector fields. These three field theories give rise to three linearly independent forms for the energy momentum tensor three point function so that the general form may be realised as a linear combination of the expressions for free scalars, fermions and vectors.

In section 6 we discuss the Ward identities relating two and three point functions. These are derived by considering formulae for diffeomorphism invariance and scale invariance on curved space and then reducing to flat space. To verify these identities using our general results it is necessary to analyse carefully the form of the three point function as two points become close together (equivalent to an operator product expansion) and to ensure that the singular short distance terms are well defined distributions. This is achieved by adapting the technique of differential regularisation [24]. As a result of the Ward identity there is a linear relation between the three coefficients defining the general three point function of the energy momentum tensor and the scale of the unique two point function in a conformal theory. Explicit forms for the leading and next to leading terms involving scalar and conserved vector fields and also the energy momentum tensor in the operator product expansion of the energy momentum tensor with the same operators are found. In section 7 we also construct a Hamiltonian operator H from the energy momentum tensor. Knowing the action of H enables us to relate any three point function containing $T_{\mu\nu}$ and the corresponding two point function without $T_{\mu\nu}$. The same formulae as in section 6 are obtained but without the need for careful regularisation of the coefficient functions in the short distance operator product expansion. In section 8 the effects of c-number contributions, depending on external fields and the metric in a curved space background, to the trace of the energy momentum tensor are considered when $d = 4$. These correspond to anomalies in the simple conformal Ward identities and reflect the existence of singularities in the three point functions when all points are coincident. Using dimensional regularisation the anomalous terms are considered for the three point function of the energy momentum tensor with two scalar fields with dimension $\eta \approx 2$ and with two conserved currents and for the three point function of the energy momentum tensor itself. In the last case the coefficients in the general three point function are related to the coefficients of the F and G terms in the trace of the energy momentum tensor on curved space. In section 9 we show that in some cases an alternative form for three point functions involving vector currents and the energy momentum tensor, in which their conservation is manifest, is possible. This is achieved by pulling out derivatives from the expression for the three point function. In the final section 10 some general remarks about our results and possible directions for future investigations are presented. In appendix A some formulae used in the discussion of the energy momentum tensor three point function are collected.

2 Conformal Invariance for Two and Three Point Functions

Conformal transformations may be defined as coordinate transformations preserving the infinitesimal euclidean length element up to a local scale factor,

$$x_\mu \to x'_\mu(x) = (gx)_\mu \, , \quad dx'_\mu dx'_\mu = \Omega^g(x)^{-2} dx_\mu dx_\mu \, . \tag{2.1}$$

For any such conformal transformation g we may define a local orthogonal transformation by

$$\mathcal{R}^g_{\mu\alpha}(x) = \Omega^g(x) \frac{\partial x'_\mu}{\partial x_\alpha} \, , \quad \mathcal{R}^g_{\mu\alpha}(x)\mathcal{R}^g_{\nu\alpha}(x) = \delta_{\mu\nu} \, , \tag{2.2}$$

which in d dimensions is an element of $O(d)$, $\mathcal{R}^{g'}(gx)\mathcal{R}^g(x) = \mathcal{R}^{g'g}(x)$, $\mathcal{R}^g(x)^{-1} = \mathcal{R}^{g^{-1}}(gx)$.

Besides conventional constant rotations and translations forming the group $O(d) \ltimes T_d$

$$x'_\mu = R_{\mu\nu} x_\nu + a_\mu \, , \qquad R_{\mu\alpha} R_{\nu\alpha} = \delta_{\mu\nu} \, , \tag{2.3}$$

for which $\Omega^g(x) = 1$, there are also constant scale transformations forming the dilatation group D

$$x'_\mu = \lambda x_\mu \, , \quad \Omega^g(x) = \lambda^{-1} \tag{2.4}$$

and special conformal transformations

$$x'_\mu = \frac{x_\mu + b_\mu x^2}{\Omega^g(x)} \, , \quad \Omega^g(x) = 1 + 2b{\cdot}x + b^2 x^2 \, . \tag{2.5}$$

The set of conformal transformations $\{g\}$ defined by (2.1) forms the conformal group which is isomorphic to $O(d+1,1)$. It is crucial to note for our subsequent calculations that the full conformal group may be generated just by combining rotations and translations, as in (2.3), with an inversion through the origin represented by the discrete element i, $i^2 = 1$,

$$x'_\mu = (ix)_\mu = \frac{x_\mu}{x^2} \, , \quad \mathcal{R}^i_{\mu\nu}(x) = I_{\mu\nu}(x) \equiv \delta_{\mu\nu} - 2\,\frac{x_\mu x_\nu}{x^2} \, , \quad \Omega^i(x) = x^2 \, . \tag{2.6}$$

Inversions are not elements of the component of the conformal group connected to the identity, since $\det I = -1$, but special conformal transformations in (2.3) are formed by considering an inversion, a translation and then another inversion.

For a quasi-primary quantum field $\mathcal{O}(x)$ of scale dimension η then a finite dimensional representation under conformal transformations (2.1) is induced by a representation of the little group $\big(O(d) \otimes D\big) \ltimes T_d$ [4] if $\mathcal{O} \to T(g)\mathcal{O}$ where

$$\big(T(g)\mathcal{O}\big)^i(x') = \Omega^g(x)^\eta D^i{}_j(\mathcal{R}^g(x))\mathcal{O}^j(x) \, , \tag{2.7}$$

with $\mathcal{R}^g(x)$ as given by (2.2). The index i here denotes the components in some representation of the rotation group $O(d)$ so that for $R_{\mu\nu}$ any orthogonal rotation matrix $D^i{}_j(R)$ is the corresponding element in this representation acting on the fields $\mathcal{O}^i$*. For two such fields $\mathcal{O}_1^{i_1}(x_1)$ and $\mathcal{O}_2^{i_2}(x_2)$ of equal scale dimension, $\eta_1 = \eta_2 = \eta$, we may define a conformally invariant two point function by

$$\langle \mathcal{O}_1^{i_1}(x_1)\, \mathcal{O}_2^{i_2}(x_2) \rangle = \frac{1}{(x_{12}^2)^\eta}\, P^{i_1 i_2}(x_{12}) \, , \quad x_{12} = x_1 - x_2 \ , \tag{2.8}$$

if $P^{i_1 i_2}(x)$ is required to satisfy

$$D_1{}^{i_1}{}_{j_1}(\mathcal{R}(x_1)) D_2{}^{i_2}{}_{j_2}(\mathcal{R}(x_2)) P^{j_1 j_2}(x_{12}) = P^{i_1 i_2}(x'_{12}) \, , \quad P^{i_1 i_2}(\lambda x) = P^{i_1 i_2}(x) \ , \tag{2.9}$$

using $x'^2_{12} = x_{12}^2 / \big(\Omega^g(x_1)\Omega^g(x_2)\big)$. A solution of this condition is provided by

$$P^{i_1 i_2}(x_{12}) = D_1{}^{i_1}{}_{j_1}(I(x_{12}))\, g^{j_1 i_2} \ , \tag{2.10}$$

where $g^{i_1 i_2}$ is an invariant tensor for the representations D_1 and D_2, i.e.

$$D_1{}^{i_1}{}_{j_1}(R) D_2{}^{i_2}{}_{j_2}(R) g^{j_1 j_2} = g^{i_1 i_2} \text{ for all } R \, . \tag{2.11}$$

Since $D(R)D(I(x))D(R)^{-1} = D(I(Rx))$ to verify that (2.10) satisfies (2.9) it is sufficient to demonstrate that $D(I(x_1))D(I(x_{12}))D(I(x_2)) = D(I(x'_{12}))$ which follows from, by direct calculation,

$$I_{\mu\alpha}(x_1) I_{\alpha\beta}(x_{12}) I_{\beta\nu}(x_2) = I_{\mu\nu}(x'_{12}) \, , \quad x'_{12} = \frac{x_1}{x_1^2} - \frac{x_2}{x_2^2} \ . \tag{2.12}$$

For fields of differing spins then of course there is no tensor $g^{i_1 i_2}$ satisfying (2.11) and the two point function is zero.

The above results show that $D(I(x_{12}))$ acts effectively as a parallel transport matrix between x_1 and x_2 for local conformal rotations. This is crucial in constructing an analogous formula for three point functions, adapting some results of Mack [10] in the context of operator product expansions. Since conformal transformations map any three points into any other three points the three point function is also essentially unique in general dimension d. Our discussion for arbitrary representations for the fields $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ is based on writing

$$\begin{aligned} \langle \mathcal{O}_1^{i_1}(x_1)\, \mathcal{O}_2^{i_2}(x_2)\, \mathcal{O}_3^{i_3}(x_3) \rangle = {} & \frac{1}{(x_{12}^2)^{\delta_{12}}\, (x_{23}^2)^{\delta_{23}}\, (x_{31}^2)^{\delta_{31}}} \\ & \times D_1{}^{i_1}{}_{j_1}(I(x_{13})) D_2{}^{i_2}{}_{j_2}(I(x_{23}))\, t^{j_1 j_2 i_3}(X_{12}) \, , \end{aligned} \tag{2.13}$$

where $t^{i_1 i_2 i_3}(X)$ is a homogeneous tensor satisfying

$$\begin{gathered} D_1{}^{i_1}{}_{j_1}(R) D_2{}^{i_2}{}_{j_2}(R) D_3{}^{i_3}{}_{j_3}(R)\, t^{j_1 j_2 j_3}(X) = t^{i_1 i_2 i_3}(RX) \text{ for all } R \ , \\ t^{i_1 i_2 i_3}(\lambda X) = \lambda^q t^{i_1 i_2 i_3}(X) \end{gathered} \tag{2.14}$$

* We assume that if $\mathcal{O}^i$ forms a representation of $SO(d)$ it may be extended to $O(d)$. In this case $\bar{D}(R) \equiv D(I^{-1}RI) \simeq D(R)$ if $\det I = -1$. For some representations this is not possible but such complications [9] are not important here.

and

$$X_{12} = -X_{21} = \frac{x_{13}}{x_{13}^2} - \frac{x_{23}}{x_{23}^2}\,, \quad X_{12}^2 = \frac{x_{12}^2}{x_{13}^2 x_{23}^2}\ . \tag{2.15}$$

By virtue of (2.13,14,15) the scaling properties of the fields are satisfied if

$$\begin{aligned} \delta_{12} &= \tfrac{1}{2}(\eta_1 + \eta_2 - \eta_3 + q)\,, \\ \delta_{23} &= \tfrac{1}{2}(\eta_2 + \eta_3 - \eta_1 - q)\,, \\ \delta_{31} &= \tfrac{1}{2}(\eta_3 + \eta_1 - \eta_2 - q)\,. \end{aligned} \tag{2.16}$$

By a rescaling $t^{i_1 i_2 i_3}(X) \to (X^2)^{-p} t^{i_1 i_2 i_3}(X)$ then $q \to q - 2p$ so that given the form of X_{12}^2 in (2.15) the expression (2.13) is unchanged. Clearly it is possible to set $q = 0$ or $q = 1$ which is sometimes convenient later. The verification that (2.13) is in accord with the required transformation properties of the fields, as given by (2.7), depends on (2.14) together with the essential result that for an inversion

$$I_{\mu\nu}(x_3) X_{12\,\nu} = \frac{1}{x_3^2}\, X'_{12\,\mu}\ , \tag{2.17}$$

where X'_{12} is formed from x'_1, x'_2, x'_3 as in (2.15).

The expression (2.13) for the three point function is asymmetric in its treatment of the external fields. Nevertheless this is only apparent. If we use

$$I_{\mu\alpha}(x_{13}) I_{\alpha\nu}(x_{23}) = I_{\mu\alpha}(x_{12}) I_{\alpha\nu}(X_{13})\,, \quad I_{\mu\alpha}(x_{23}) I_{\alpha\nu}(x_{13}) = I_{\mu\alpha}(x_{21}) I_{\alpha\nu}(X_{23})\,, \tag{2.18}$$

as well as

$$I_{\mu\alpha}(x_{23}) X_{12\,\alpha} = \frac{x_{12}^2}{x_{13}^2}\, X_{13\,\mu}\,, \quad I_{\mu\alpha}(x_{13}) X_{12\,\alpha} = \frac{x_{12}^2}{x_{23}^2}\, X_{32\,\mu}\,, \tag{2.19}$$

then we may show, by virtue of (2.14),

$$\begin{aligned} & D_1{}^{i_1}{}_{j_1}(I(x_{13})) D_2{}^{i_2}{}_{j_2}(I(x_{23}))\, t^{j_1 j_2 i_3}(X_{12}) \\ &\quad = \Big(\frac{x_{12}^2}{x_{13}^2}\Big)^q D_1{}^{i_1}{}_{j_1}(I(x_{12})) D_3{}^{i_3}{}_{j_3}(I(x_{32}))\, \tilde{t}^{\,j_1 i_2 j_3}(X_{13})\,, \\ &\quad = \Big(\frac{x_{12}^2}{x_{23}^2}\Big)^q D_2{}^{i_2}{}_{j_2}(I(x_{21})) D_3{}^{i_3}{}_{j_3}(I(x_{31}))\, \hat{t}^{\,i_1 j_2 j_3}(X_{32})\,, \\ &\quad \tilde{t}^{\,i_1 i_2 i_3}(X) = D_1{}^{i_1}{}_{j_1}(I(X))\, t^{j_1 i_2 i_3}(X)\,, \quad \hat{t}^{\,i_1 i_2 i_3}(X) = D_2{}^{i_2}{}_{j_2}(I(X))\, t^{i_1 j_2 i_3}(X)\,. \end{aligned} \tag{2.20}$$

Hence we may equivalently write an analogous representation for $\langle \mathcal{O}_1^{i_1}(x_1)\, \mathcal{O}_2^{i_2}(x_2)\, \mathcal{O}_3^{i_3}(x_3)\rangle$ to that given by (2.13) and (2.16) with on the r.h.s $2 \leftrightarrow 3$ and $t^{j_1 j_2 i_3} \to \tilde{t}^{\,j_1 i_2 j_3}$ or $1 \leftrightarrow 3$ and $t^{j_1 j_2 i_3} \to \hat{t}^{\,i_1 j_2 j_3}$. If the three point function is symmetric, for all fields $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ belonging to the same representation, then it is necessary that

$$t^{i_2 i_1 i_3}(X) = t^{i_1 i_2 i_3}(-X)\,, \quad D^{\,i_1}{}_{j_1}(I(X))\, t^{j_1 i_2 i_3}(X) = t^{i_3 i_1 i_2}(-X)\ . \tag{2.21}$$

We are here mostly interested in applying this formalism to cases involving vector fields $V_\mu(x)$, of dimension $d-1$, and the energy momentum tensor $T_{\mu\nu}(x)$, which is symmetric and traceless and of dimension d. These satisfy the conservation equations

$$\partial_\mu V_\mu = 0\,, \qquad \partial_\mu T_{\mu\nu} = 0\ . \tag{2.22}$$

The two point functions, in accord with the general result in (2.8) and (2.10), are

$$\begin{gathered} \langle V_\mu(x)\, V_\nu(0)\rangle = \frac{C_V}{x^{2(d-1)}}\, I_{\mu\nu}(x)\,, \quad \langle T_{\mu\nu}(x)\, T_{\sigma\rho}(0)\rangle = \frac{C_T}{x^{2d}}\, \mathcal{I}_{\mu\nu,\sigma\rho}(x)\,, \\ \mathcal{I}_{\mu\nu,\sigma\rho}(x) = \tfrac{1}{2}\big(I_{\mu\sigma}(x) I_{\nu\rho}(x) + I_{\mu\rho}(x) I_{\nu\sigma}(x)\big) - \frac{1}{d}\,\delta_{\mu\nu}\delta_{\sigma\rho}\ . \end{gathered} \tag{2.23}$$

C_V, C_T are constants determining the overall scale of these two point functions. $\mathcal{I}_{\mu\nu,\sigma\rho}$ represents the inversion operator on symmetric traceless tensors. Corresponding to $I(x)^2 = 1$ we have

$$\mathcal{I}_{\mu\nu,\alpha\beta}(x)\, \mathcal{I}_{\alpha\beta,\sigma\rho}(x) = \tfrac{1}{2}\big(\delta_{\mu\sigma}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\sigma}\big) - \frac{1}{d}\,\delta_{\mu\nu}\delta_{\sigma\rho} \equiv \mathcal{E}_{\mu\nu,\sigma\rho}\,, \tag{2.24}$$

with $\mathcal{E}$ representing the projection operator onto the space of symmetric traceless tensors. It is easy to verify that (2.23) is consistent with the conservation equations (2.22) (see also below).

For a three point function involving a vector field we may write from (2.13), for $\eta_\pm = \eta_2 \pm \eta_3$,

$$\begin{aligned} \langle V_\mu(x_1)\, \mathcal{O}_2^{i_2}(x_2)\, \mathcal{O}_3^{i_3}(x_3)\rangle = {}& \frac{1}{x_{12}^{\,d-1+\eta_- +q}\; x_{23}^{\,\eta_+ -d+1-q}\; x_{31}^{\,d-1-\eta_- -q}} \\ & \times I_{\mu\nu}(x_{13}) D_2{}^{i_2}{}_{j_2}(I(x_{23}))\, t_\nu{}^{j_2 i_3}(X_{12})\,. \end{aligned} \tag{2.25}$$

To verify current conservation we may use

$$\begin{aligned} \partial_\mu\Big(\frac{1}{(x^2)^\lambda}\, I_{\mu\nu}(x)\Big) &= 2(\lambda-d+1)\,\frac{x_\nu}{(x^2)^{\lambda+1}}\,, \\ \partial_{1\mu}\Big(\frac{1}{(x_{13}^2)^\lambda\,(x_{12}^2)^{d-1-\lambda}}\, I_{\mu\nu}(x_{13})\Big) &= 2(\lambda-d+1)\,\frac{1}{(x_{13}^2)^{\lambda+1}\,(x_{12}^2)^{d-1-\lambda}}\,\frac{X_{12\,\nu}}{X_{12}^2}\,, \\ \partial_{1\mu}\, X_{12\,\nu} &= \frac{1}{x_{13}^2}\, I_{\mu\nu}(x_{13})\ . \end{aligned} \tag{2.26}$$

Hence current conservation in (2.25) requires

$$\Big(\partial_\mu - (d-1+\eta_- + q)\frac{X_\mu}{X^2}\Big) t_\mu{}^{i_2 i_3}(X) = 0\ . \tag{2.27}$$

It is easy to see that this is invariant under rescaling of $t_\mu{}^{i_2 i_3}(X)$ by some power of X^2 with the appropriate change in q.

For the energy momentum tensor we may also write

$$\langle T_{\mu\nu}(x_1)\, \mathcal{O}_2^{i_2}(x_2)\, \mathcal{O}_3^{i_3}(x_3)\rangle = \frac{1}{x_{12}{}^{d+\eta_-+q}\, x_{23}{}^{\eta_+-d-q}\, x_{31}{}^{d-\eta_--q}} \\ \times \mathcal{I}_{\mu\nu,\sigma\rho}(x_{13}) D_2{}^{i_2}{}_{j_2}(I(x_{23}))\, t_{\sigma\rho}{}^{j_2 i_3}(X_{12})\,. \tag{2.28}$$

In this case for determining the corresponding conservation equation

$$\begin{aligned}\partial_\mu\Big(\frac{1}{(x^2)^\lambda}\,\mathcal{I}_{\mu\nu,\sigma\rho}(x)\Big) &= 2(\lambda-d)\,\frac{x_\nu}{(x^2)^{\lambda+1}}\,\Big(\tfrac{1}{2}\big(x_\sigma I_{\nu\rho}(x)+x_\rho I_{\nu\sigma}(x)\big)+\frac{1}{d}\,x_\nu\delta_{\sigma\rho}\Big)\,,\\ \partial_{1\mu}\Big(\frac{1}{(x_{13}^2)^\lambda\,(x_{12}^2)^{d-\lambda}}\,\mathcal{I}_{\mu\nu,\sigma\rho}(x_{13})\Big) &= 2(\lambda-d)\,\frac{1}{(x_{13}^2)^{\lambda+1}\,(x_{12}^2)^{d-\lambda}}\,\frac{1}{X_{12}^2}\\ &\quad\times\Big(\tfrac{1}{2}\big(I_{\nu\rho}(x_{13})X_{12\,\sigma}+I_{\nu\sigma}(x_{13})X_{12\,\rho}\big)-\frac{1}{d}\,I_{\nu\alpha}(x_{13})X_{12\,\alpha}\,\delta_{\sigma\rho}\Big)\,.\end{aligned} \tag{2.29}$$

Assuming $t_{\mu\nu}{}^{i_2 i_3}$ is symmetric and traceless with respect to μ,ν then it is easy to obtain the condition

$$\Big(\partial_\mu - (d+\eta_-+q)\frac{X_\mu}{X^2}\Big)t_{\mu\nu}{}^{i_2 i_3}(X) = 0\ . \tag{2.30}$$

For both (2.27) and (2.30) the solutions may be restricted to homogeneous polynomials in X if q is a sufficiently large integer.

The construction described above is manifestly sufficient for forming conformally invariant three point functions. It is also necessary. To show this we assume the existence of an operator product expansion where for $x_1 \to x_2$ the contributions arising from the leading singular term for each quasi-primary operator $\mathcal{O}^i$ appearing in the expansion are of the form

$$\mathcal{O}_1^{i_1}(x_1)\, \mathcal{O}_2^{i_2}(x_2) \sim A^{i_1 i_2}{}_i(x_{12})\mathcal{O}^i(x_2)\,. \tag{2.31}$$

For conformal invariance, if $\mathcal{O}^i$ on the r.h.s. of (2.31) has dimension η and transforms as in (2.7), we require

$$\begin{aligned} D_1{}^{i_1}{}_{j_1}(R)D_2{}^{i_2}{}_{j_2}(R)A^{j_1 j_2}{}_j(x)D^j{}_i(R) &= A^{i_1 i_2}{}_i(Rx) \text{ for all } R\ ,\\ A^{i_1 i_2}{}_i(\lambda x) &= \lambda^{\eta-\eta_1-\eta_2}A^{i_1 i_2}{}_i(x)\,.\end{aligned} \tag{2.32}$$

To use this result we consider the conformal transformation on the general three point function arising from an inversion through the point z

$$x \to x' = \frac{x-z}{(x-z)^2}\,. \tag{2.33}$$

Using the transformation properties given by (2.6) and (2.7)

$$\begin{aligned}&\langle \mathcal{O}_1^{i_1}(x_1)\, \mathcal{O}_2^{i_2}(x_2)\, \mathcal{O}_3^{i_3}(x_3)\rangle\\ &= x_1'^{\,2\eta_1} x_2'^{\,2\eta_2} x_3'^{\,2\eta_3}\, D_1{}^{i_1}{}_{j_1}(I(x_1'))D_2{}^{i_2}{}_{j_2}(I(x_2'))D_3{}^{i_3}{}_{j_3}(I(x_3'))\langle \mathcal{O}_1^{j_1}(x_1')\, \mathcal{O}_2^{j_2}(x_2')\, \mathcal{O}_3^{j_3}(x_3')\rangle\,.\end{aligned} \tag{2.34}$$

Now, following similar arguments of Cardy [13], if we let $z \to x_3$ then $x'_{13} \sim x'_{23} \sim -x'_3$ and hence $|x'_{12}| \ll |x'_{13}|, |x'_{23}|$ so that we may use the operator product expansion (2.31). Assuming that the operator $\mathcal{O}^i$ has a non zero two point function with $\mathcal{O}^{i_3}$, and hence $\eta = \eta_3$, then applying the results (2.8,9,10) for the two point function to this case it is easy to see that

$$\begin{aligned} \langle \mathcal{O}_1^{j_1}(x'_1)\, \mathcal{O}_2^{j_2}(x'_2)\, \mathcal{O}_3^{j_3}(x'_3)\rangle &\sim A^{j_1 j_2}{}_j(x'_{12})\, \langle \mathcal{O}^j(x'_2)\, \mathcal{O}_3^{j_3}(x'_3)\rangle\,, \\ \langle \mathcal{O}^j(x)\, \mathcal{O}_3^k(0)\rangle &= \frac{1}{x^{2\eta_3}}\, D_3{}^k{}_{j'}(I(x)) g^{jj'}\,. \end{aligned} \tag{2.35}$$

Inserting (2.35) in (2.34) and taking the limit, when $x'_1 \to x_{13}/x_{13}^2,\ x'_2 \to x_{23}/x_{23}^2$ and also $x'_{12} \to X_{12}$ as defined by (2.15), gives finally

$$\langle \mathcal{O}_1^{i_1}(x_1)\, \mathcal{O}_2^{i_2}(x_2)\, \mathcal{O}_3^{i_3}(x_3)\rangle = \frac{1}{x_{13}^{2\eta_1}\, x_{23}^{2\eta_2}}\, D_1{}^{i_1}{}_{j_1}(I(x_{13})) D_2{}^{i_2}{}_{j_2}(I(x_{23}))\, A^{j_1 j_2}{}_j(X_{12}) g^{j i_3}\,, \tag{2.36}$$

so that the whole three point function is determined by the leading singular operator product coefficient. Comparing with (2.13) shows that the two expressions are equivalent if

$$t^{i_1 i_2 i_3}(X) = (X^2)^{\frac{1}{2}(\eta_1+\eta_2-\eta_3+q)}\, A^{i_1 i_2}{}_i(X) g^{i i_3}\,. \tag{2.37}$$

By virtue of (2.32) $t^{i_1 i_2 i_3}(X)$ satisfies the conditions required in (2.14). For a conserved vector field V_μ or the energy momentum tensor $T_{\mu\nu}$ it is not difficult to see that (2.27) and (2.30) are the same as

$$\partial_\mu A_\mu{}^i{}_j(X) = 0\,, \qquad \partial_\mu A_{\mu\nu}{}^i{}_j(X) = 0\,, \tag{2.38}$$

for the corresponding coefficients of the leading singular term in the operator product expansion.

If the operator product expansion (2.31) is extended to general set of quasi-primary operators, labelled by $a, b, \ldots$, so that

$$\mathcal{O}_a^i(x)\mathcal{O}_b^j(0) \sim A_{ab,c}{}^{ij}{}_k(x)\mathcal{O}_c^k(0)\,, \quad A_{ab,c}{}^{ij}{}_k(x) = A_{ba,c}{}^{ji}{}_k(-x) = \mathrm{O}(x^{\eta_c-\eta_a-\eta_b})\,, \tag{2.39}$$

and if the two point functions in this operator basis are diagonal,

$$\langle \mathcal{O}_a^i(x)\, \mathcal{O}_b^j(0)\rangle = \frac{1}{x^{2\eta_a}}\, D_a{}^i{}_k(I(x)) g_a^{kj}\, \delta_{ab}\,, \tag{2.40}$$

then using (2.20) consistency of different short distance limits determining the three point function requires

$$A_{ac,b}{}^{ik}{}_\ell(x) g_b^{\ell j} = (x^2)^{\eta_b-\eta_c} D_a{}^i{}_\ell(I(x)) A_{ab,c}{}^{\ell j}{}_m(x) g_c^{mk}\,. \tag{2.41}$$

This condition is essentially equivalent to associativity of the operator product expansion in this case*.

* For two vector currents and an axial current similar conditions were obtained by Crewther [25].

3 Applications of General Formalism

We here apply the previous general results to various specific cases involving a scalar field $\mathcal{O}$ of dimension η, the conserved vector current V_μ of dimension $d-1$ and the energy momentum tensor $T_{\mu\nu}$ of dimension d.

For the three point function for energy momentum tensor and two scalar fields it is easy to see that we may write

$$\begin{aligned}\langle T_{\mu\nu}(x_1)\,\mathcal{O}(x_2)\,\mathcal{O}(x_3)\rangle &= \frac{1}{x_{12}^{\,d}\,x_{23}^{\,2\eta-d}\,x_{31}^{\,d}}\,t_{\mu\nu}(X_{23}) \\ &= \frac{1}{x_{12}^{\,d}\,x_{23}^{\,2\eta-d}\,x_{31}^{\,d}}\,\mathcal{I}_{\mu\nu,\sigma\rho}(x_{13})\,t_{\sigma\rho}(X_{12})\,,\end{aligned} \tag{3.1}$$

where we have taken $t_{\mu\nu}$ to be homogeneous of degree zero. It is easy to see that the conditions for tracelessness and the conservation equation coincide to give

$$t_{\mu\nu}(X) = a\,h^1_{\mu\nu}(\hat{X})\,,\quad h^1_{\mu\nu}(\hat{X}) = \hat{X}_\mu\hat{X}_\nu - \frac{1}{d}\,\delta_{\mu\nu}\,,\quad \hat{X}_\mu = \frac{X_\mu}{\sqrt{X^2}}\,. \tag{3.2}$$

For two energy momentum tensors it is natural to write from (2.13)

$$\langle T_{\mu\nu}(x_1)\,T_{\sigma\rho}(x_2)\,\mathcal{O}(x_3)\rangle = \frac{1}{x_{12}^{\,2d-\eta}\,x_{23}^{\,\eta}\,x_{31}^{\,\eta}}\,\mathcal{I}_{\mu\nu,\alpha\beta}(x_{13})\mathcal{I}_{\sigma\rho,\gamma\delta}(x_{23})t_{\alpha\beta\gamma\delta}(X_{12})\,, \tag{3.3}$$

with $t_{\alpha\beta\gamma\delta}(X)$ again homogeneous of degree zero. A general expansion consistent with the required symmetries and tracelessness is given by

$$\begin{aligned} t_{\alpha\beta\gamma\delta}(X) &= a\,h^1_{\alpha\beta}(\hat{X})h^1_{\gamma\delta}(\hat{X}) + b\,h^2_{\alpha\beta\gamma\delta}(\hat{X}) + c\,h^3_{\alpha\beta\gamma\delta}\,, \\ h^2_{\mu\nu\sigma\rho}(\hat{X}) &= \hat{X}_\mu\hat{X}_\sigma\delta_{\nu\rho} + (\mu\leftrightarrow\nu,\sigma\leftrightarrow\rho) - \frac{4}{d}\hat{X}_\mu\hat{X}_\nu\delta_{\sigma\rho} - \frac{4}{d}\hat{X}_\sigma\hat{X}_\rho\delta_{\mu\nu} + \frac{4}{d^2}\delta_{\mu\nu}\delta_{\sigma\rho}\,, \\ h^3_{\mu\nu\sigma\rho} &= \delta_{\mu\sigma}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\sigma} - \frac{2}{d}\,\delta_{\mu\nu}\delta_{\sigma\rho}\,, \end{aligned} \tag{3.4}$$

where $h^1_{\mu\nu}(\hat{X})$ is defined in (3.2). The conservation equation from (2.30) becomes

$$\Big(\partial_\mu - (2d-\eta)\frac{X_\mu}{X^2}\Big)t_{\mu\nu\sigma\rho}(X) = 0\,, \tag{3.5}$$

which gives two linear relations between a, b, c

$$\begin{aligned} & a + 4b - \tfrac{1}{2}(d-\eta)(d-1)(a+4b) - d\eta\,b = 0\,, \\ & a + 4b + d(d-\eta)b + d(2d-\eta)c = 0\,. \end{aligned} \tag{3.6}$$

In consequence the conformal invariant form for the $\langle TT\mathcal{O}\rangle$ three point function is unique up to an overall constant.

For vector currents we write

$$\langle V^a_\mu(x_1)\,V^b_\nu(x_2)\,V^c_\omega(x_3)\rangle = \frac{f^{abc}}{x_{12}{}^{d}\,x_{23}{}^{d-2}\,x_{31}{}^{d-2}}\,I_{\mu\alpha}(x_{13})I_{\nu\beta}(x_{23})t_{\alpha\beta\omega}(X_{12})\,, \tag{3.7}$$

where a, b, c are to be regarded here as group indices with f^{abc} a corresponding totally antisymmetric structure constant and we have set $q = 1$. From (2.21) and (2.27) we require

$$\begin{gathered} I_{\mu\alpha}(X)t_{\alpha\nu\omega}(X) = -t_{\omega\mu\nu}(X)\,, \quad t_{\mu\nu\omega}(X) = t_{\nu\mu\omega}(X)\,, \\ \Big(\partial_\mu - d\,\frac{X_\mu}{X^2}\Big)t_{\mu\nu\omega}(X) = 0\,. \end{gathered} \tag{3.8}$$

It is easy to see that the general solution is

$$t_{\mu\nu\omega}(X) = a\,\frac{X_\mu X_\nu X_\omega}{X^2} + b\big(X_\mu\delta_{\nu\omega} + X_\nu\delta_{\mu\omega} - X_\omega\delta_{\mu\nu}\big)\,, \tag{3.9}$$

for independent constants a, b. By using results such as (2.18,19) the $\langle VVV\rangle$ three point function may then be expressed more symmetrically as

$$\begin{aligned} \langle V^a_\mu(x_1)\,V^b_\nu(x_2)\,V^c_\omega(x_3)\rangle = {}& \frac{f^{abc}}{x_{12}{}^{d-2}\,x_{23}{}^{d-2}\,x_{31}{}^{d-2}}\bigg\{(a-2b)X_{23\,\mu}\,X_{31\,\nu}\,X_{12\,\omega} \\ & - b\Big(\frac{1}{x_{23}^2}X_{23\,\mu}I_{\nu\omega}(x_{23}) + \frac{1}{x_{13}^2}X_{31\,\nu}I_{\mu\omega}(x_{13}) + \frac{1}{x_{12}^2}X_{12\,\omega}I_{\mu\nu}(x_{12})\Big)\bigg\}\,, \end{aligned} \tag{3.10}$$

which is in accord with previous results found over twenty years ago [7].

For the three point function involving two vector currents and the energy momentum tensor we may write

$$\begin{aligned} \langle T_{\mu\nu}(x_1)\,V^a_\sigma(x_2)\,V^b_\rho(x_3)\rangle &= \frac{\delta^{ab}}{x_{12}{}^{d}\,x_{13}{}^{d}\,x_{23}{}^{d-2}}\,I_{\sigma\alpha}(x_{21})I_{\rho\beta}(x_{31})t_{\mu\nu\alpha\beta}(X_{23}) \\ &= \frac{\delta^{ab}}{x_{12}{}^{d}\,x_{13}{}^{d}\,x_{23}{}^{d-2}}\,\mathcal{I}_{\mu\nu,\gamma\delta}(x_{13})I_{\sigma\alpha}(x_{23})\tilde t_{\gamma\delta\alpha\rho}(X_{12})\,, \\ \tilde t_{\mu\nu\sigma\rho}(X) = I_{\sigma\alpha}(X)t_{\mu\nu\alpha\rho}(X)\,, &\quad t_{\mu\nu\sigma\rho} = t_{\mu\nu\rho\sigma} = t_{\nu\mu\sigma\rho}\,, \quad t_{\mu\mu\sigma\rho} = 0\,, \end{aligned} \tag{3.11}$$

with $t_{\mu\nu\sigma\rho}(X)$ homogeneous of degree zero in X. The conservation equations for the vector currents and the energy momentum tensor require

$$\begin{gathered} \Big(\partial_\sigma - (d-2)\frac{X_\sigma}{X^2}\Big)t_{\mu\nu\sigma\rho}(X) = 0\,, \\ \Big(\partial_\mu - d\,\frac{X_\mu}{X^2}\Big)\tilde t_{\mu\nu\sigma\rho}(X) = 0\,, \quad \Big(\partial_\sigma - d\,\frac{X_\sigma}{X^2}\Big)\tilde t_{\mu\nu\sigma\rho}(X) = 0\,, \end{gathered} \tag{3.12}$$

where the second equation for $\tilde t$ follows from that for t given its definition in (3.11). A general expression for $t_{\mu\nu\sigma\rho}(X)$ satisfying the constraints in (3.11), with the definitions in

(3.2) and (3.4), and the corresponding form for $\tilde{t}_{\mu\nu\sigma\rho}(X)$ are given by

$$\begin{aligned}
t_{\mu\nu\sigma\rho}(X) &= a\, h^1_{\mu\nu}(\hat{X})\delta_{\sigma\rho} + b\, h^1_{\mu\nu}(\hat{X})h^1_{\sigma\rho}(\hat{X}) + c\, h^2_{\mu\nu\sigma\rho}(\hat{X}) + e\, h^3_{\mu\nu\sigma\rho}\,, \\
\tilde{t}_{\mu\nu\sigma\rho}(X) &= \frac{1}{d^2}\big(d(d-2)a - 2(d-1)(b+4c) - 4de\big)h^1_{\mu\nu}(\hat{X})\delta_{\sigma\rho} \\
&\quad - \frac{1}{d}\big(2da + (d-2)(b+4c)\big)h^1_{\mu\nu}(\hat{X})h^1_{\sigma\rho}(\hat{X}) \\
&\quad + e\big(h^3_{\mu\nu\sigma\rho} - h^2_{\mu\nu\sigma\rho}(\hat{X})\big) - (e+c)\tilde{h}_{\mu\nu\sigma\rho}(\hat{X})\,, \\
\tilde{h}_{\mu\nu\sigma\rho}(\hat{X}) &= \hat{X}_\mu\hat{X}_\sigma\delta_{\nu\rho} + \hat{X}_\nu\hat{X}_\sigma\delta_{\mu\rho} - \hat{X}_\mu\hat{X}_\rho\delta_{\nu\sigma} - \hat{X}_\nu\hat{X}_\rho\delta_{\mu\sigma}\,.
\end{aligned} \tag{3.13}$$

Using results such as (A.2) and (A.4) (3.12) gives the relations

$$da - 2b + 2(d-2)c = 0\,, \quad b - d(d-2)e = 0\,. \tag{3.14}$$

Hence a, b may be found in terms of c, e and there are two linearly independent amplitudes in this case.

With the experience gained in the above simpler cases we may now turn to consider the three point function for three energy momentum tensors which is the main aim of this paper. According to the general formalism we may write

$$\langle T_{\mu\nu}(x_1)\, T_{\sigma\rho}(x_2)\, T_{\alpha\beta}(x_3)\rangle = \frac{1}{x_{12}^{\,d}\, x_{13}^{\,d}\, x_{23}^{\,d}}\, \mathcal{I}_{\mu\nu,\mu'\nu'}(x_{13})\mathcal{I}_{\sigma\rho,\sigma'\rho'}(x_{23})\, t_{\mu'\nu'\sigma'\rho'\alpha\beta}(X_{12})\,, \tag{3.15}$$

with $t_{\mu\nu\sigma\rho\alpha\beta}(X)$ homogeneous of degree zero in X, symmetric and traceless on each pair of indices $\mu\nu$, $\sigma\rho$ and $\alpha\beta$ and from (2.21) satisfying

$$t_{\mu\nu\sigma\rho\alpha\beta}(X) = t_{\sigma\rho\mu\nu\alpha\beta}(X)\,. \tag{3.16a}$$

$$\mathcal{I}_{\mu\nu,\mu'\nu'}(X)t_{\mu'\nu'\sigma\rho\alpha\beta}(X) = t_{\alpha\beta\mu\nu\sigma\rho}(X)\,, \tag{3.16b}$$

The conservation equations require just

$$\Big(\partial_\mu - d\,\frac{X_\mu}{X^2}\Big)t_{\mu\nu\sigma\rho\alpha\beta}(X) = 0\ . \tag{3.17}$$

Defining

$$\begin{aligned}
h^4_{\mu\nu\sigma\rho\alpha\beta}(\hat{X}) &= h^3_{\mu\nu\sigma\alpha}\hat{X}_\rho\hat{X}_\beta + (\sigma\leftrightarrow\rho, \alpha\leftrightarrow\beta) \\
&\quad - \frac{2}{d}\,\delta_{\sigma\rho}h^2_{\mu\nu\alpha\beta}(\hat{X}) - \frac{2}{d}\,\delta_{\alpha\beta}h^2_{\mu\nu\sigma\rho}(\hat{X}) - \frac{8}{d^2}\,\delta_{\sigma\rho}\delta_{\alpha\beta}h^1_{\mu\nu}(\hat{X})\,, \\
h^5_{\mu\nu\sigma\rho\alpha\beta} &= \delta_{\mu\sigma}\delta_{\nu\alpha}\delta_{\rho\beta} + (\mu\leftrightarrow\nu, \sigma\leftrightarrow\rho, \alpha\leftrightarrow\beta) \\
&\quad - \frac{4}{d}\,\delta_{\mu\nu}h^3_{\sigma\rho\alpha\beta} - \frac{4}{d}\,\delta_{\sigma\rho}h^3_{\mu\nu\alpha\beta} - \frac{4}{d}\,\delta_{\alpha\beta}h^3_{\mu\nu\sigma\rho} - \frac{8}{d^2}\,\delta_{\mu\nu}\delta_{\sigma\rho}\delta_{\alpha\beta}\,,
\end{aligned} \tag{3.18}$$

a general expansion for $t_{\mu\nu\sigma\rho\alpha\beta}(X)$ compatible with (3.16a) has the form

$$
\begin{aligned}
t_{\mu\nu\sigma\rho\alpha\beta}(X) = {} & a\, h^5_{\mu\nu\sigma\rho\alpha\beta} + b\, h^4_{\alpha\beta\mu\nu\sigma\rho}(\hat{X}) + b'\big(h^4_{\mu\nu\sigma\rho\alpha\beta}(\hat{X}) + h^4_{\sigma\rho\mu\nu\alpha\beta}(\hat{X})\big) \\
& + c\, h^3_{\mu\nu\sigma\rho} h^1_{\alpha\beta}(\hat{X}) + c'\big(h^3_{\sigma\rho\alpha\beta} h^1_{\mu\nu}(\hat{X}) + h^3_{\mu\nu\alpha\beta} h^1_{\sigma\rho}(\hat{X})\big) \\
& + e\, h^2_{\mu\nu\sigma\rho}(\hat{X}) h^1_{\alpha\beta}(\hat{X}) + e'\big(h^2_{\sigma\rho\alpha\beta}(\hat{X}) h^1_{\mu\nu}(\hat{X}) + h^2_{\mu\nu\alpha\beta}(\hat{X}) h^1_{\sigma\rho}(\hat{X})\big) \\
& + f\, h^1_{\mu\nu}(\hat{X}) h^1_{\sigma\rho}(\hat{X}) h^1_{\alpha\beta}(\hat{X})\,.
\end{aligned}
\tag{3.19}
$$

In this case it is necessary to be more systematic in solving (3.16b) and (3.17). From the results in (A.1) (3.16b) gives

$$
b + b' = -2a\,, \quad c' = c\,, \quad e + e' = -4b' - 2c\,, \tag{3.20}
$$

so that a, b, c, e, f may be regarded as independent. Then using (A.2) imposition of (3.17) gives in addition

$$
\begin{aligned}
d^2 a + 2(b + b') - (d-2)b' - dc + e' = 0\,, \\
d(d+2)(2b' + c) + 4(e + e') + f = 0\,.
\end{aligned}
\tag{3.21}
$$

Hence there remain three undetermined coefficients which may be taken as a, b, c ($f = (d+4)(d-2)(4a+2b-c)$, $e' = -(d+4)(d-2)a - (d-2)b + dc$, $e = (d+2)(da+b-c)$).

4 Collinear Frame

The form of the conformally invariant three point function simplifies considerably if the three points are constrained to lie on a straight line. In this configuration the invariance group is restricted to $O(d-1)$ rotations in the perpendicular plane, scale transformations and also translations and special conformal transformations, as in (2.5), along the line. The latter allow any three points to be mapped to any other three points preserving cyclic order. With these constraints it is not difficult to see in general that, for n_μ a unit vector,

$$
\begin{aligned}
& \langle \mathcal{O}^{i_1}(x)\, \mathcal{O}^{i_2}(y)\, \mathcal{O}^{i_3}(z) \rangle \big|_{x_\mu = \hat{x} n_\mu, y_\nu = \hat{y} n_\nu, z_\omega = \hat{z} n_\omega} \\
& \qquad = \frac{1}{(\hat{x}-\hat{y})^{\eta_1+\eta_2-\eta_3}\, (\hat{x}-\hat{z})^{\eta_1+\eta_3-\eta_2}\, (\hat{y}-\hat{z})^{\eta_2+\eta_3-\eta_2}}\, \mathcal{A}^{i_1 i_2 i_3}\,,
\end{aligned}
\tag{4.1}
$$

where $\mathcal{A}^{i_1 i_2 i_3}$ is a constant $O(d-1)$ invariant tensor and we suppose $\hat{x} > \hat{y} > \hat{z}$. An alternative construction of conformally invariant three point functions is then to analyse the constraints in a collinear frame where the form is particularly simple and then to obtain the results for a general configuration by conformal transformation. The calculations are very straightforward except that in order to impose conservation equations, such as (2.22) it is necessary to consider an infinitesimal conformal transformation away from the collinear configuration. We illustrate this procedure for some of the cases considered in the previous section here since it is useful to consider the collinear form in making the connection with specific conformally invariant models later.

The first non trivial case to consider is that for $\langle TT\mathcal{O}\rangle$ where for $x_\mu = (\hat{x}, \mathbf{0})$, $y_\nu = (\hat{y}, \mathbf{0})$, $z_\omega = (\hat{z}, \mathbf{0})$ we may write

$$\begin{gathered}\langle T_{\mu\nu}(x)\, T_{\sigma\rho}(y)\, \mathcal{O}(z)\rangle \equiv T^{TT\mathcal{O}}_{\mu\nu\sigma\rho}(x,y,z) = \frac{1}{(\hat{x}-\hat{y})^{2d-\eta}\,(\hat{x}-\hat{z})^{\eta}\,(\hat{y}-\hat{z})^{\eta}}\,\mathcal{A}^{TT\mathcal{O}}_{\mu\nu\sigma\rho}\,, \\ \mathcal{A}^{TT\mathcal{O}}_{\mu\nu\sigma\rho} = \mathcal{A}^{TT\mathcal{O}}_{\sigma\rho\mu\nu} = \mathcal{A}^{TT\mathcal{O}}_{\nu\mu\sigma\rho}\,.\end{gathered} \tag{4.2}$$

$\mathcal{A}^{TT\mathcal{O}}$ may be decomposed as

$$\begin{aligned}&\mathcal{A}^{TT\mathcal{O}}_{1111} = \alpha\,, \quad \mathcal{A}^{TT\mathcal{O}}_{ij11} = \beta\,\delta_{ij}, \quad \mathcal{A}^{TT\mathcal{O}}_{i1k1} = \gamma\,\delta_{ik}, \\ &\mathcal{A}^{TT\mathcal{O}}_{ijk\ell} = \delta\,\delta_{ij}\delta_{k\ell} + \epsilon(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})\,,\end{aligned} \tag{4.3}$$

where $i, j\ldots$ denote components orthogonal to the 1 direction. It is easy to see that the tracelessness condition $\mathcal{A}^{TT\mathcal{O}}_{\mu\mu\sigma\rho} = 0$ requires

$$\alpha + (d-1)\beta = 0\,, \quad \beta + (d-1)\delta + 2\epsilon = 0\,. \tag{4.4}$$

We now consider an infinitesimal conformal transformation $x_\mu \to x_\mu + \delta x_\mu$,

$$\begin{aligned}\delta x_i &= \xi_i\big((\hat{y}+\hat{z})x_1 - x^2 - \hat{y}\hat{z}\big) + 2x_i\,\xi_j x_j\,, \\ \delta x_1 &= -\,\xi_i x_i\big(\hat{y}+\hat{z}-2x_1\big)\,,\end{aligned} \tag{4.5}$$

such that

$$(\hat{y}, \mathbf{0}) \to (\hat{y}, \mathbf{0})\,, \quad (\hat{z}, \mathbf{0}) \to (\hat{z}, \mathbf{0})\,, \quad (\hat{x}, \mathbf{0}) \to (\hat{x}, \delta\mathbf{x})\,, \quad \delta\mathbf{x} = (\hat{x}-\hat{y})(\hat{x}-\hat{z})\boldsymbol{\xi}\,. \tag{4.6}$$

In this case from (2.2)

$$\mathcal{R}_{\mu\nu}(x) \approx \begin{pmatrix} 1 & -\xi_j(\hat{y}+\hat{z}-2x_1) \\ \xi_i(\hat{y}+\hat{z}-2x_1) & \delta_{ij}\end{pmatrix}\,, \quad \Omega(x) \approx 1 - 2\xi_i x_i\,. \tag{4.7}$$

Using this transformation, with $x'_\mu = (\hat{x}, \delta\mathbf{x})$ and evaluating $\mathcal{R}_{\mu\nu}(x)$ at $(\hat{x}, \mathbf{0})$ and $(\hat{y}, \mathbf{0})$, it is straightforward to find to first order in $\delta\mathbf{x}$

$$\begin{aligned}T^{TT\mathcal{O}}_{i111}(x', y, z) ={}& \Big(\frac{1}{\hat{x}-\hat{y}} + \frac{1}{\hat{x}-\hat{z}}\Big)\big(\delta x_i T^{TT\mathcal{O}}_{1111}(x,y,z) - \delta x_j T^{TT\mathcal{O}}_{ij11}(x,y,z)\big) \\ &- \Big(\frac{1}{\hat{x}-\hat{y}} - \frac{1}{\hat{x}-\hat{z}}\Big)\big(\delta x_k T^{TT\mathcal{O}}_{i1k1}(x,y,z) + \delta x_\ell T^{TT\mathcal{O}}_{i11\ell}(x,y,z)\big)\,, \\ T^{TT\mathcal{O}}_{i1k\ell}(x', y, z) ={}& \Big(\frac{1}{\hat{x}-\hat{y}} + \frac{1}{\hat{x}-\hat{z}}\Big)\big(\delta x_i T^{TT\mathcal{O}}_{11k\ell}(x,y,z) - \delta x_j T^{TT\mathcal{O}}_{ijk\ell}(x,y,z)\big) \\ &+ \Big(\frac{1}{\hat{x}-\hat{y}} - \frac{1}{\hat{x}-\hat{z}}\Big)\big(\delta x_k T^{TT\mathcal{O}}_{i11\ell}(x,y,z) + \delta x_\ell T^{TT\mathcal{O}}_{i1k1}(x,y,z)\big)\,, \\ T^{TT\mathcal{O}}_{ijk1}(x', y, z) ={}& \Big(\frac{1}{\hat{x}-\hat{y}} + \frac{1}{\hat{x}-\hat{z}}\Big)\big(\delta x_i T^{TT\mathcal{O}}_{1jk1}(x,y,z) + \delta x_j T^{TT\mathcal{O}}_{ijk\ell}(x,y,z)\big) \\ &+ \Big(\frac{1}{\hat{x}-\hat{y}} - \frac{1}{\hat{x}-\hat{z}}\Big)\big(\delta x_k T^{TT\mathcal{O}}_{ij11}(x,y,z) - \delta x_\ell T^{TT\mathcal{O}}_{i1k1}(x,y,z)\big)\,.\end{aligned} \tag{4.8}$$

Hence we may now impose current conservation, $\partial_i T_{i1} + \partial_1 T_{11} = 0$ and also $\partial_i T_{ij} + \partial_1 T_{1j} = 0$, to give in addition to (4.4)

$$2\gamma = (d-\eta)\beta\,, \quad \beta - \delta - d\epsilon = (d-\eta)\gamma\,. \tag{4.9}$$

With four independent relations given by (4.4) and (4.9) there is just one possible conformal invariant form up to an overall constant. It is easy to relate this approach to our previous results. From (3.4) in the collinear frame then $\mathcal{A}_{\mu\nu\sigma\rho}$ in (4.2) becomes

$$\mathcal{A}^{TT\mathcal{O}}_{\mu\nu\sigma\rho} = a\, h^1_{\mu\nu}(\hat{X}) h^1_{\sigma\rho}(\hat{X}) + b\, h^2_{\mu\nu\sigma\rho}(\hat{X}) + c\, h^3_{\mu\nu\sigma\rho}\,, \tag{4.10}$$

where h^1, h^2, h^3 may be decomposed as in (4.3), with now $\hat{X}_\mu = (1, \mathbf{0})$, to give

$$\gamma = b + c \quad d^2\,\delta = a + 4b - 2d\,c\,, \quad \epsilon = c\,, \tag{4.11}$$

with α, β determined by (4.4). The relations (4.9) are then equivalent to (3.6).

In a similar fashion we may discuss the $\langle TVV \rangle$ three point function which may be written in the collinear frame as

$$\begin{gathered}\langle T_{\mu\nu}(x)\, V^a_\sigma(y)\, V^b_\rho(z)\rangle = \delta^{ab} T^{TVV}_{\mu\nu\sigma\rho}(x,y,z) = \frac{\delta^{ab}}{(\hat{x}-\hat{y})^d\,(\hat{x}-\hat{z})^d\,(\hat{y}-\hat{z})^{d-2}}\, \mathcal{A}^{TVV}_{\mu\nu\sigma\rho}\,, \\ \mathcal{A}^{TVV}_{\mu\nu\sigma\rho} = \mathcal{A}^{TVV}_{\mu\nu\rho\sigma} = \mathcal{A}^{TVV}_{\nu\mu\sigma\rho}\,,\end{gathered} \tag{4.12}$$

where

$$\begin{aligned} &\mathcal{A}^{TVV}_{1111} = \alpha\,, \quad \mathcal{A}^{TVV}_{ij11} = \beta\,\delta_{ij}\,, \quad \mathcal{A}^{TVV}_{11k\ell} = \gamma\,\delta_{k\ell}\,, \quad \mathcal{A}^{TVV}_{i1k1} = \delta\,\delta_{ik}\,, \\ &\mathcal{A}^{TVV}_{ijk\ell} = \rho\,\delta_{ij}\delta_{k\ell} + \tau(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})\,. \end{aligned} \tag{4.13}$$

The tracelessness conditions are

$$\alpha + (d-1)\beta = 0\,, \quad \gamma + (d-1)\rho + 2\tau = 0\,. \tag{4.14}$$

As before we may impose the conservation equation $\partial_\mu T_{\mu\nu} = 0$ by considering an infinitesimal conformal transformation so that $x \to x'$ which is no longer on the straight line defined by y and z

$$\begin{aligned} T^{TVV}_{i111}(x',y,z) &= \Big(\frac{1}{\hat{x}-\hat{y}} + \frac{1}{\hat{x}-\hat{z}}\Big)\big(\delta x_i T^{TVV}_{1111}(x,y,z) - \delta x_j T^{TVV}_{ij11}(x,y,z)\big) \\ &\quad - \Big(\frac{1}{\hat{x}-\hat{y}} - \frac{1}{\hat{x}-\hat{z}}\Big)\big(\delta x_k T^{TVV}_{i1k1}(x,y,z) - \delta x_\ell T^{TVV}_{i11\ell}(x,y,z)\big)\,, \\ T^{TVV}_{i1k\ell}(x',y,z) &= \Big(\frac{1}{\hat{x}-\hat{y}} + \frac{1}{\hat{x}-\hat{z}}\Big)\big(\delta x_i T^{TVV}_{11k\ell}(x,y,z) - \delta x_j T^{TVV}_{ijk\ell}(x,y,z)\big) \\ &\quad + \Big(\frac{1}{\hat{x}-\hat{y}} - \frac{1}{\hat{x}-\hat{z}}\Big)\big(\delta x_k T^{TVV}_{i11\ell}(x,y,z) - \delta x_\ell T^{TVV}_{i1k1}(x,y,z)\big)\,, \\ T^{TVV}_{ijk1}(x',y,z) &= \Big(\frac{1}{\hat{x}-\hat{y}} + \frac{1}{\hat{x}-\hat{z}}\Big)\big(\delta x_i T^{TVV}_{1jk1}(x,y,z) + \delta x_j T^{TVV}_{ijk\ell}(x,y,z)\big) \\ &\quad + \Big(\frac{1}{\hat{x}-\hat{y}} - \frac{1}{\hat{x}-\hat{z}}\Big)\big(\delta x_k T^{TVV}_{ij11}(x,y,z) + \delta x_\ell T^{TVV}_{i1k1}(x,y,z)\big)\,. \end{aligned} \tag{4.15}$$

It is then easy to obtain the additional equation to (4.14)

$$\beta + \rho + d\tau = 0\,. \tag{4.16}$$

To impose $\partial_\sigma V_\sigma = 0$ we may shift $y \to y'$ and then find

$$\beta + \gamma - 2\delta = 0\,. \tag{4.17}$$

With four relations amongst six variables it is clear that there are two linearly independent conformal invariant expressions in this case which may be specified by ρ, τ. To see the connection with our earlier treatment we may use (3.13) to find in the collinear frame

$$\mathcal{A}^{TVV}_{\mu\nu\sigma\rho} = a\, h^1_{\mu\nu}(\hat{X})\delta_{\sigma\rho} + (b+8c+8e)\, h^1_{\mu\nu}(\hat{X})h^1_{\sigma\rho}(\hat{X}) - (c+2e)\, h^2_{\mu\nu\sigma\rho}(\hat{X}) + e\, h^3_{\mu\nu\sigma\rho}\,, \tag{4.18}$$

for $\hat{X}_\mu = (1, \mathbf{0})$. Comparing with (4.13) we find

$$\begin{aligned} \beta = {} & -\frac{1}{d}(a+2e) - \frac{1}{d^2}(d-1)(b+4c)\,, \quad \delta = -b-e\,, \\ \rho = {} & -\frac{1}{d}(a+2e) + \frac{1}{d^2}(b+4c)\,, \qquad \tau = e\,, \end{aligned} \tag{4.19}$$

with α, γ given by (4.14). It is easy to see then that the constraints (4.16,17) are equivalent to (3.14) (note that ρ, τ or $c = \frac{1}{2}d(\rho+\tau), e = \tau$ may be regarded as independent variables).

Finally in this section we consider again the three point function for the energy momentum tensor. In the collinear frame as above we may write

$$\begin{gathered} \langle T_{\mu\nu}(x)\, T_{\sigma\rho}(y)\, T_{\alpha\beta}(z)\rangle = \frac{1}{(\hat{x}-\hat{y})^d\,(\hat{x}-\hat{z})^d\,(\hat{y}-\hat{z})^d}\,\mathcal{A}^{TTT}_{\mu\nu\sigma\rho\alpha\beta}\,, \\ \mathcal{A}^{TTT}_{\mu\nu\sigma\rho\alpha\beta} = \mathcal{A}^{TTT}_{\sigma\rho\mu\nu\alpha\beta} = \mathcal{A}^{TTT}_{\alpha\beta\mu\nu\sigma\rho}\,, \end{gathered} \tag{4.20}$$

with $\mathcal{A}^{TTT}_{\mu\nu\sigma\rho\alpha\beta}$ also symmetric and traceless on each pair of indices $\mu\nu$, $\sigma\rho$ and $\alpha\beta$. This may therefore be decomposed as

$$\begin{aligned} & \mathcal{A}^{TTT}_{111111} = \alpha\,, \qquad \mathcal{A}^{TTT}_{ij1111} = \beta\,\delta_{ij}\,, \qquad \mathcal{A}^{TTT}_{i1k11} = \gamma\,\delta_{ik}\,, \\ & \mathcal{A}^{TTT}_{ijk\ell 11} = \delta\,\delta_{ij}\delta_{k\ell} + \epsilon(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})\,, \quad \mathcal{A}^{TTT}_{ijk1m1} = \rho\,\delta_{ij}\delta_{k\ell} + \tau(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})\,, \\ & \mathcal{A}^{TTT}_{ijk\ell mn} = r\,\delta_{ij}\delta_{k\ell}\delta_{mn} \\ & \qquad + s\big(\delta_{ij}(\delta_{km}\delta_{\ell n} + \delta_{kn}\delta_{\ell m}) + \delta_{kl}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + \delta_{mn}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})\big) \\ & \qquad + t(\delta_{ik}\delta_{jm}\delta_{\ell n} + \delta_{jk}\delta_{im}\delta_{\ell n} + \delta_{i\ell}\delta_{jm}\delta_{kn} + \delta_{j\ell}\delta_{im}\delta_{kn} \\ & \qquad\quad + \delta_{ik}\delta_{jn}\delta_{\ell m} + \delta_{jk}\delta_{in}\delta_{\ell m} + \delta_{i\ell}\delta_{jn}\delta_{km} + \delta_{j\ell}\delta_{in}\delta_{km})\,, \end{aligned} \tag{4.21}$$

where tracelessness requires

$$\begin{gathered} \alpha + (d-1)\beta = 0\,, \quad \beta + (d-1)\delta + 2\epsilon = 0\,, \quad \gamma + (d-1)\rho + 2\tau = 0\,, \\ \delta + (d-1)r + 4s = 0\,, \quad \epsilon + (d-1)s + 4t = 0\,. \end{gathered} \tag{4.22}$$

Following similar arguments to the above the requirements following from the conservation equation imply two further relations so that there are three linearly independent coefficients which may be taken as r, s, t. In addition to (4.22) the conservation equations then determine ρ, τ by

$$
\begin{aligned}
2\tau + dr + (d+4)s + 2t &= 0\,, \\
2\rho - dr + (d-2)s + 2(d+4)t &= 0\,.
\end{aligned}
\tag{4.23}
$$

To compare with our previous results we may use (3.15) and (3.19) to obtain in the collinear frame the explicitly symmetric form

$$
\begin{aligned}
\mathcal{A}^{TTT}_{\mu\nu\sigma\rho\alpha\beta} = {}& a\, h^5_{\mu\nu\sigma\rho\alpha\beta} + b\big(h^4_{\alpha\beta\mu\nu\sigma\rho}(\hat{X}) + h^4_{\mu\nu\sigma\rho\alpha\beta}(\hat{X}) + h^4_{\sigma\rho\mu\nu\alpha\beta}(\hat{X})\big) \\
& + c\big(h^3_{\mu\nu\sigma\rho} h^1_{\alpha\beta}(\hat{X}) + h^3_{\sigma\rho\alpha\beta} h^1_{\mu\nu}(\hat{X}) + h^3_{\mu\nu\alpha\beta} h^1_{\sigma\rho}(\hat{X})\big) \\
& + (e - 8a - 8b)\big(h^2_{\mu\nu\sigma\rho}(\hat{X}) h^1_{\alpha\beta}(\hat{X}) + h^2_{\sigma\rho\alpha\beta}(\hat{X}) h^1_{\mu\nu}(\hat{X}) + h^2_{\mu\nu\alpha\beta}(\hat{X}) h^1_{\sigma\rho}(\hat{X})\big) \\
& + (f + 128a + 96b - 16c - 16e)\, h^1_{\mu\nu}(\hat{X}) h^1_{\sigma\rho}(\hat{X}) h^1_{\alpha\beta}(\hat{X})\,,
\end{aligned}
\tag{4.24}
$$

after using (A.1) and (3.20). Comparing this with (4.21) we find

$$
\begin{aligned}
& d\rho = 4a + 2b - c - e\,, \qquad \tau = a + b\,, \\
& d^3 r = 16(d-2)a - 24b + 6dc + 16c + 4e - f\,, \quad ds = -4a - c\,, \quad t = a\,.
\end{aligned}
\tag{4.25}
$$

It is then straightforward to check that (4.23) is equivalent to (3.21) with (3.20) (note $d^2 r = 16a - 2db + (d+4)c$).

Although our discussion is concerned with general dimensions d it is important to recognise that $d = 2, 3$ are special cases. When $d = 2$ there is only one transverse dimension. Hence in (4.3) only $\delta + 2\epsilon$ has any significance, similarly in (4.13) for $\rho + 2\tau$ while in (4.21) the relevant quantities are $\delta + 2\epsilon$, $\rho + 2\tau$ and $r + 6s + 8t$. These restrictions reduce the number of conformal invariant forms to one for both the $\langle TVV\rangle$ and $\langle TTT\rangle$ three point functions. Less obviously when $d = 3$ then in (4.21) $\mathcal{A}^{TTT}_{ijk\ell mn}$ depends only on $r - 4t$ and $s + 2t$, rather than r, s, t being independent variables. Consequently the number of conformal invariants for $\langle TTT\rangle$ when $d = 3$ is just two.

5 Free Field Theories

For general dimension d the only completely explicit conformal field theories are those provided by free scalar or free fermion fields. In the scalar case we may write

$$
\begin{aligned}
T_{\mu\nu} &= \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{4}\,\frac{1}{d-1}\big((d-2)\partial_\mu\partial_\nu + \delta_{\mu\nu}\partial^2\big)\phi^2\,, \\
V^a_\mu &= \phi t^a_\phi \partial_\mu\phi\,, \quad (t^a_\phi)^T = -t^a_\phi\,, \quad [t^a_\phi, t^b_\phi] = f^{abc} t^c_\phi\,.
\end{aligned}
\tag{5.1}
$$

In the fermion case

$$
\begin{aligned}
T_{\mu\nu} &= \tfrac{1}{2}\bar{\psi}(\gamma_\mu \overleftrightarrow{\partial}_\nu + \gamma_\nu \overleftrightarrow{\partial}_\mu)\psi\,, \qquad \overleftrightarrow{\partial}_\mu = \tfrac{1}{2}(\partial_\mu - \overleftarrow{\partial}_\mu)\,, \\
V^a_\mu &= \bar{\psi} t^a_\psi \gamma_\mu \psi\,, \quad (t^a_\psi)^\dagger = -t^a_\psi\,, \quad [t^a_\psi, t^b_\psi] = f^{abc} t^c_\psi\,,
\end{aligned}
\tag{5.2}
$$

for $(\gamma_\mu)^\dagger = \gamma_\mu$ euclidean gamma matrices, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. The basic two point functions for the massless scalar, fermion fields are

$$\langle \phi(x)\, \phi(0)\rangle = \frac{1}{(d-2)S_d}\,\frac{1}{x^{d-2}}\,, \quad \langle \psi(x)\bar\psi(0)\rangle = \frac{1}{S_d}\,\frac{\gamma{\cdot}x}{x^d}\,, \tag{5.3}$$

for $S_d = 2\pi^{\frac{1}{2}d}/\Gamma(\frac{1}{2}d)$.

It is easy to use these results to determine the form of the two point functions of V^a_μ and $T_{\mu\nu}$ which are compatible with (2.23) or

$$\langle V^a_\mu(x)\, V^b_\nu(0)\rangle = \delta^{ab}\,\frac{C_V}{x^{2(d-1)}}\, I_{\mu\nu}(x)\,. \tag{5.4}$$

In the scalar case, if ϕ has n_ϕ components and $\mathrm{tr}\,(t^a_\phi t^b_\phi) = -N_\phi \delta^{ab}$, then

$$C_V = \frac{N_\phi}{d-2}\,\frac{1}{S_d^{\,2}}\,, \qquad C_T = n_\phi\,\frac{d}{d-1}\,\frac{1}{S_d^{\,2}}\,, \tag{5.5}$$

while in the fermion case, if there are n_ψ Dirac fields and $\mathrm{tr}\,(t^a_\psi t^b_\psi) = -N_\psi\delta^{ab}$, then

$$C_V = N_\psi\, 2^{\frac{1}{2}d}\,\frac{1}{S_d^{\,2}}\,, \qquad C_T = n_\psi\, \tfrac{1}{2}d\, 2^{\frac{1}{2}d}\,\frac{1}{S_d^{\,2}}\,. \tag{5.6}$$

For determining three point functions it is considerably easier to evaluate them in the collinear frame where, as discussed in the previous section, the functional form is very much simpler than in the general case. Thus for the vector currents, if x, y, z lie on the 1 axis with $x_1 = \hat x, y_1 = \hat y, z_1 = \hat z$ as earlier, then from (3.7) and (3.9)

$$\begin{aligned}
\langle V^a_\mu(x)\, V^b_\nu(y)\, V^c_\omega(z)\rangle &= \frac{f^{abc}}{(\hat x-\hat y)^{d-1}\,(\hat x-\hat z)^{d-1}\,(\hat y-\hat z)^{d-1}}\,\mathcal{A}^{VVV}_{\mu\nu\omega}\,,\\
\mathcal{A}^{VVV}_{\mu\nu\omega} &= I_{\mu\alpha}(\hat X) I_{\nu\beta}(\hat X) t_{\alpha\beta\omega}(\hat X)\\
&= (a+4b)\hat X_\mu \hat X_\nu \hat X_\omega - b(\hat X_\mu \delta_{\nu\omega} + \hat X_\nu\delta_{\mu\omega} + \hat X_\omega \delta_{\mu\nu})\,,\\
\hat X_\mu = (1,\mathbf{0})\,, \qquad \mathcal{A}^{VVV}_{111} &= a+b\,, \quad \mathcal{A}^{VVV}_{ij1} = -b\,\delta_{ij}\,.
\end{aligned} \tag{5.7}$$

Hence we may determine a, b and then obtain the general expression for $\langle VVV\rangle$ from (3.7) and (3.9). For the scalar case

$$S_d^{\,3} a = \tfrac{1}{2} N_\phi\,\frac{d}{d-2}\,, \qquad S_d^{\,3} b = \tfrac{1}{2} N_\phi\,\frac{1}{d-2}\,, \tag{5.8}$$

while in the fermion case

$$a = 0\,, \qquad S_d^{\,3} b = N_\psi\, 2^{\frac{1}{2}d}\,. \tag{5.9}$$

For the determination of $\langle TVV\rangle$ for free scalar or fermion fields we again follow the prescription of restricting calculations of the three point function for these theories to the collinear configuration. It is sufficient to calculate only $\langle T_{ij} V_k V_\ell \rangle$, involving only components orthogonal to the 1 axis on which x, y, z lie, although results for components along the 1 direction provide a useful check through the general relations derived in the previous section. For the scalar case in the notation of (4.13) or (3.13)

$$\tau = e = \frac{d}{2(d-1)(d-2)}\, N_\phi\, \frac{1}{S_d^{\,3}}\,, \quad \rho = \frac{d-4}{2(d-1)(d-2)}\, N_\phi\, \frac{1}{S_d^{\,3}}\,, \quad c = \frac{d}{2(d-1)}\, N_\phi\, \frac{1}{S_d^{\,3}}\,. \tag{5.10}$$

In the fermion case we find

$$\tau = e = 0\,, \quad \rho = 2^{\frac{1}{2}d}\, N_\psi\, \frac{1}{S_d^{\,3}}\,, \quad c = \tfrac{1}{2} d\, 2^{\frac{1}{2}d}\, N_\psi\, \frac{1}{S_d^{\,3}}\,. \tag{5.11}$$

For the $\langle TTT \rangle$ three point function we may follow similar procedures with rather more labour, especially in the scalar case. It is convenient to calculate also $\langle T\phi^2\phi^2 \rangle$ and also $\langle TT\phi^2 \rangle$, where ϕ^2 is a scalar operator of dimension $d-2$, and check that these are of the required form for conformal invariance such as prescribed by (4.2,3,4) and (4.9) in the latter case. The results for scalar fields, given the form (5.1) for $T_{\mu\nu}$, are

$$\begin{gathered} r = n_\phi\, \frac{1}{8}\, \frac{1}{(d-1)^3} (d^3 + 28d - 16)\, \frac{1}{S_d^{\,3}}\,, \qquad s = n_\phi\, \frac{1}{8}\, \frac{d}{(d-1)^3} (d^2 - 8d + 4)\, \frac{1}{S_d^{\,3}}\,, \\ t = a = n_\phi\, \frac{1}{8}\, \frac{d^3}{(d-1)^3}\, \frac{1}{S_d^{\,3}}\,, \quad b = -n_\phi\, \frac{1}{8}\, \frac{d^4}{(d-1)^3}\, \frac{1}{S_d^{\,3}}\,, \quad c = -n_\phi\, \frac{1}{8}\, \frac{d^2(d-2)^2}{(d-1)^3}\, \frac{1}{S_d^{\,3}}\,. \end{gathered} \tag{5.12}$$

For fermion fields we obtain

$$r = -n_\psi\, \frac{1}{2}\, 2^{\frac{1}{2}d}\, \frac{1}{S_d^{\,3}}\,, \quad s = n_\psi\, \frac{1}{8} d\, 2^{\frac{1}{2}d}\, \frac{1}{S_d^{\,3}}\,, \quad t = a = 0\,, \quad 2b = c = -n_\psi\, \frac{1}{8} d^2\, 2^{\frac{1}{2}d}\, \frac{1}{S_d^{\,3}}\,. \tag{5.13}$$

For $d = 4$ there is an additional conformal field theory described in terms of free abelian vector fields. The energy momentum tensor is

$$T_{\mu\nu} = F_{\mu\lambda} F_{\nu\lambda} - \tfrac{1}{4}\, \delta_{\mu\nu} F_{\alpha\beta} F_{\alpha\beta} + s\, X_{\mu\nu}\,, \tag{5.14}$$

where s is the nilpotent BRS operator, $s^2 = 0$, and $X_{\mu\nu}$ contains all contributions depending on the gauge fixing term and ghost fields. This term does not contribute for correlation functions involving gauge invariant operators and physical states and for our purposes can be neglected. The basic two point function is then

$$\langle F_{\mu\nu}(x)\, F_{\sigma\rho}(0) \rangle = \frac{2}{S_4}\, \frac{1}{x^4} \big(I_{\mu\sigma}(x) I_{\nu\rho}(x) - I_{\mu\rho}(x) I_{\nu\sigma}(x) \big)\,. \tag{5.15}$$

For the energy momentum tensor two point function, comparing with (2.23), we obtain

$$C_T = 16\, \frac{1}{S_4^{\,2}}\,. \tag{5.16}$$

For the three point function (note that $\langle TT\mathcal{O}\rangle = 0$ for $\mathcal{O} = \frac{1}{4}F_{\alpha\beta}F_{\alpha\beta}$) the results are

$$r = -48\,\frac{1}{S_4^{\,3}}\,, \quad s = 32\,\frac{1}{S_4^{\,3}}\,, \quad t = a = -16\,\frac{1}{S_4^{\,3}}\,, \quad b = 0\,, \quad c = -64\,\frac{1}{S_4^{\,3}}\,. \tag{5.17}$$

It is not difficult to see that the solutions provided by (5.12,13), for $d = 4$, and (5.17) are linearly independent thereby realising the full range of possibilities for $\langle TTT\rangle$ in this case.

Calculations for three point functions of the energy momentum tensor and vector currents were also undertaken by Stanev [26] for massless free scalar, fermion and vector fields in four dimensions. Due to the complexity of the resulting formulae it is difficult to compare results in detail but we agree on the number of independent amplitudes.

6 Ward Identities and Short Distance Expansions

To derive Ward identities relating two and three point functions we assume that we may define a functional $W(g, A, J)$ depending on a background metric $g_{\mu\nu}$ and gauge field $A^a{}_\mu$ and a scalar source J such that the energy momentum tensor $T_{\mu\nu}$, vector current $V^{a\mu}$ and scalar operator $\mathcal{O}$ may be defined through functional derivatives by

$$\begin{aligned} \langle T_{\mu\nu}(x)\rangle &= -\,\frac{2}{\sqrt{g(x)}}\,\frac{\delta}{\delta g^{\mu\nu}(x)}\,W\,, \\ \langle V^{a\mu}(x)\rangle &= -\,\frac{1}{\sqrt{g(x)}}\,\frac{\delta}{\delta A^a{}_\mu(x)}\,W\,, \quad \langle \mathcal{O}(x)\rangle = -\frac{1}{\sqrt{g(x)}}\,\frac{\delta}{\delta J(x)}\,W\,. \end{aligned} \tag{6.1}$$

L_V W=0,李导数再用链式法则

Assuming W is a scalar and invariant under diffeomorphisms requires

$$\int d^dx\,\Big(-(\nabla^\mu v^\nu + \nabla^\nu v^\mu)\frac{\delta}{\delta g^{\mu\nu}} + (v^\nu\partial_\nu A^a{}_\mu + \partial_\mu v^\nu A^a{}_\nu)\frac{\delta}{\delta A^a{}_\mu} + v^\mu\partial_\mu J\,\frac{\delta}{\delta J}\Big)W = 0\,, \tag{6.2}$$

for arbitrary $v^\mu(x)$. Under local scale variations of the metric we require for arbitrary $\sigma(x)$

注意，在weyl变换下实际上右式不为0，因此6.5需要额外加上反常项。

$$\int d^dx\,\sigma\Big(2g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}} + (d-\eta)J\,\frac{\delta}{\delta J}\Big)W = 0\,, \tag{6.3}$$

since J has dimension $d-\eta$. In general this identity has anomalies reflecting the introduction of a mass scale in quantum field theories through renormalisation. In conformal field theories where the β functions vanish these anomalous effects are restricted to additional local terms in (6.3) involving scalars constructed from $g_{\mu\nu}, A^a{}_\mu$ and J of the appropriate dimension. The relevant terms for $d = 4$ will be considered later. In addition we assume invariance under local gauge transformations which implies

$$\int d^dx\,(\partial_\mu\Lambda^a + f^{abc}A^b{}_\mu\Lambda^c)\frac{\delta}{\delta A^a{}_\mu}\,W = 0\,, \tag{6.4}$$

where for simplicity we suppose that $\mathcal{O}$ is a singlet under the gauge group. (6.2,3) and (6.4) are equivalent to

$$\begin{aligned} &\nabla^\mu\langle T_{\mu\nu}\rangle + \nabla_\nu A^a{}_\mu\,\langle V^{a\mu}\rangle + \nabla_\mu\big(A^a{}_\nu\,\langle V^{a\mu}\rangle\big) + \partial_\nu J\,\langle\mathcal{O}\rangle = 0\,, \\ &g^{\mu\nu}\langle T_{\mu\nu}\rangle + (d-\eta)J\langle\mathcal{O}\rangle = 0\,, \quad \nabla_\mu\langle V^{a\mu}\rangle + f^{abc}A^b{}_\mu\langle V^{c\mu}\rangle = 0\,. \end{aligned} \tag{6.5}$$

The Ward identities that we use are then obtained by functional differentiation of (6.5) and restricting to flat space and also $A^a{}_\mu$ and J to be zero. The completely symmetric three point function of the energy momentum tensor is therefore defined by

$$\begin{aligned}
&\langle T_{\mu\nu}(x)\, T_{\sigma\rho}(y)\, T_{\alpha\beta}(z)\rangle = -8\, \frac{\delta}{\delta g^{\alpha\beta}(z)}\, \frac{\delta}{\delta g^{\sigma\rho}(y)}\, \frac{\delta}{\delta g^{\mu\nu}(x)}\, W\Big|_{g_{\mu\nu}=A^a{}_\mu=J=0} \\
&\quad = -8\, \frac{\delta}{\delta g^{\alpha\beta}(z)}\, \frac{1}{\sqrt{g(y)}}\, \frac{\delta}{\delta g^{\sigma\rho}(y)}\, \frac{1}{\sqrt{g(x)}}\, \frac{\delta}{\delta g^{\mu\nu}(x)}\, W\Big|_{g_{\mu\nu}=A^a{}_\mu=J=0} \\
&\qquad + g_{\alpha\beta}(z)\big(\delta^d(z-y)+\delta^d(z-x)\big)\langle T_{\mu\nu}(x)\, T_{\sigma\rho}(y)\rangle + g_{\sigma\rho}(y)\delta^d(y-x)\langle T_{\mu\nu}(x)\, T_{\alpha\beta}(z)\rangle\, ,
\end{aligned} \tag{6.6}$$

and similarly for other cases.

Firstly we consider the purely vector identity

$$\partial_\mu \langle V^a_\mu(x)\, V^b_\nu(y)\, V^c_\omega(z)\rangle = f^{abd}\, \delta^d(x-y)\, \langle V^d_\nu(x)\, V^c_\omega(z)\rangle - f^{acd}\, \delta^d(x-z)\, \langle V^d_\omega(x)\, V^b_\nu(y)\rangle\, . \tag{6.7}$$

From the result (3.7) we may easily find

$$\langle V^a_\mu(x)\, V^b_\nu(y)\, V^c_\omega(z)\rangle = \frac{f^{abc}}{s^{d-2}\,(x-z)^{d-2}(y-z)^d}\, I_{\nu\sigma}(s) I_{\omega\rho}(x-z)\, I_{\sigma\beta}(X) t_{\mu\beta\rho}(X)\, , \tag{6.8}$$

where

$$s_\mu = x_\mu - y_\mu\, , \qquad X_\mu = \frac{s_\mu}{s^2} - \frac{(x-z)_\mu}{(x-z)^2}\, . \tag{6.9}$$

Hence we may easily find the short distance limit

$$\langle V^a_\mu(x)\, V^b_\nu(y)\, V^c_\omega(z)\rangle \sim f^{abc}\, \frac{1}{s^d} t_{\mu\nu\rho}(s)\, \frac{I_{\rho\omega}(y-z)}{(y-z)^{2(d-1)}} \quad \text{as } x \to y\, . \tag{6.10}$$

Using the explicit form of $t_{\mu\nu\rho}(s)$, as given by (3.9) and obtained by solving (3.8), we now find taking care over the differentiation of singular functions of x

$$\partial_\mu\Big(\frac{1}{s^d} t_{\mu\nu\rho}(s)\Big) = \Big(\frac{1}{d}\, a + b\Big)\delta_{\nu\rho}\, S_d \delta^d(s)\, , \quad S_d = \frac{2\pi^{\frac12 d}}{\Gamma(\frac12 d)}\, . \tag{6.11}$$

Hence the short distance singular term for $s = x - y \to 0$ in the representation (6.8) is responsible for the first term on the right hand side of the Ward identity (6.7) (the other term of course arises from the singularities as $x \to z$) if

$$S_d\Big(\frac{1}{d}\, a + b\Big) = C_V\, , \tag{6.12}$$

given the expression (5.4) for the two point amplitude. We may trivially see that this is compatible with the results of calculations in (5.5) and (5.8,9).

For three point functions containing the energy momentum tensor we begin with that involving the scalar fields of dimension η. This has been discussed before [13] but is considered again here as a simple warm up exercise. From (6.5) it is easy to derive the identities

$$\partial_\mu \langle T_{\mu\nu}(x)\,\mathcal{O}(y)\,\mathcal{O}(z)\rangle = \partial_\nu \delta^d(x-y)\,\langle \mathcal{O}(x)\,\mathcal{O}(z)\rangle + \partial_\nu \delta^d(x-z)\,\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle\,, \qquad (6.13a)$$

$$\langle T_{\mu\mu}(x)\,\mathcal{O}(y)\,\mathcal{O}(z)\rangle = (d-\eta)\big(\delta^d(x-y)\,\langle \mathcal{O}(x)\,\mathcal{O}(z)\rangle + \delta^d(x-z)\,\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle\big)\,. \qquad (6.13b)$$

From (3.1) and (3.2), with s, X as in (6.9), the short distance limit is

$$\begin{aligned}\langle T_{\mu\nu}(x)\,\mathcal{O}(y)\,\mathcal{O}(z)\rangle &= \frac{1}{s^d\,(x-z)^d\,(y-z)^{2\eta-d}}\,t_{\mu\nu}(X)\\ &\sim A_{\mu\nu}(s)\,\frac{N}{(y-z)^{2\eta}} + B_{\mu\nu\lambda}(s)\,\frac{\partial}{\partial y_\lambda}\frac{N}{(y-z)^{2\eta}} \quad \text{as } x\to y\,,\\ A_{\mu\nu}(s)N = \frac{1}{s^d}\,t_{\mu\nu}(s) &= \frac{a}{s^d}\Big(\frac{s_\mu s_\nu}{s^2} - \frac{1}{d}\,\delta_{\mu\nu}\Big)\,, \quad B_{\mu\nu\lambda}(s) = \frac{1}{\eta}\big(\tfrac{1}{2}s^2\partial_\lambda A_{\mu\nu}(s) + d s_\lambda A_{\mu\nu}(s)\big)\,,\\ B_{\mu\nu\lambda}(s)N &= \frac{a}{2\eta s^d}\Big(s_\mu\delta_{\nu\lambda} + s_\nu\delta_{\mu\lambda} - s_\lambda\delta_{\mu\nu} + (d-2)\frac{s_\mu s_\nu s_\lambda}{s^2}\Big)\,, \end{aligned} \qquad (6.14)$$

where the coefficient N is defined by the scale of the two point function

$$\langle \mathcal{O}(x)\,\mathcal{O}(z)\rangle = \frac{N}{(x-z)^{2\eta}}\,. \qquad (6.15)$$

The result (6.14) for $s\to 0$ is then equivalent to the operator product expansion

$$T_{\mu\nu}(x)\mathcal{O}(y) \sim A_{\mu\nu}(s)\,\mathcal{O}(y) + B_{\mu\nu\lambda}(s)\,\partial_\lambda\mathcal{O}(y)\,. \qquad (6.16)$$

Since the leading singular coefficient $A_{\mu\nu}(s) = \mathrm{O}(s^{-d})$ it is not well defined as a distribution on $\mathbb{R}^d$ and requires a regularisation prescription to ensure an unambiguous evaluation of δ-function contributions to $\partial_\mu A_{\mu\nu}(s)$ and $A_{\mu\mu}(s)$ and verify the identities (6.13a,b). It is convenient here to adopt the methods of differential regularisation [24] and write a regularised expression, identical with $A_{\mu\nu}(s)$ for $s\neq 0$, as

$$\hat{A}_{\mu\nu}(s) = \frac{a}{Nd}\Big(\frac{1}{d-2}\,\partial_\mu\partial_\nu\frac{1}{s^{d-2}} + C\,\delta_{\mu\nu}\,S_d\delta^d(s)\Big)\,. \qquad (6.17)$$

Expressing $\hat{A}_{\mu\nu}(s)$ in terms of derivatives acting on $1/s^{d-2}$ then enables integration over smooth test functions on $\mathbb{R}^d$ to be defined by integration by parts. The term containing the arbitrary coefficient C is a reflection of the freedom in any regularisation of $A_{\mu\nu}(s)$. Given the form (6.17) we then calculate, using standard results such as $-\partial^2 1/s^{d-2} = (d-2)S_d\delta^d(s)$,

$$\partial_\mu\hat{A}_{\mu\nu}(s) = \frac{a}{Nd}(C-1)S_d\,\partial_\nu\delta(s)\,, \qquad \hat{A}_{\mu\mu}(s) = \frac{a}{Nd}(dC-1)\,S_d\delta^d(s)\,. \qquad (6.18)$$

On the other hand $B_{\mu\nu\lambda}(s) = \mathrm{O}(s^{-d+1})$ does not require any such regularisation as in (6.17) and unambiguously we find, either directly or from $\partial_\mu\hat{A}_{\mu\nu}(s)$,

$$\partial_\mu B_{\mu\nu\lambda}(s) = \frac{a}{N\eta d}(d-1)\,\delta_{\nu\lambda}\,S_d\delta^d(s)\,, \quad B_{\mu\mu\lambda}(s) = 0\,. \qquad (6.19)$$

Compatibility with the identities (6.13a,b), using (6.18) and (6.19) in (6.14) with the replacement $A_{\mu\nu}(s) \to \hat{A}_{\mu\nu}(s)$, then requires

$$S_d(C-1)\, a = dN\,, \quad S_d(dC-1)\, a = d(d-\eta)N\,, \quad S_d(d-1)\, a = -d\eta N\,, \tag{6.20}$$

which consistently determine a, C giving $\eta C = -(d-1-\eta)$.

The next case to consider involves the $\langle TVV\rangle$ three point function for which (6.5) leads to the identities

$$\begin{aligned} &\partial_\mu \langle T_{\mu\nu}(x)\, V^a{}_\sigma(y)\, V^b{}_\rho(z)\rangle \\ &\qquad = \partial_\nu \delta^d(x-y)\,\langle V^a{}_\sigma(x)\, V^b{}_\rho(z)\rangle - \partial_\mu\big(\delta^d(x-y)\delta_{\nu\sigma}\langle V^a{}_\mu(x)\, V^b{}_\rho(z)\rangle\big) \\ &\qquad\quad + \partial_\nu \delta^d(x-z)\,\langle V^a{}_\sigma(y)\, V^b{}_\rho(x)\rangle - \partial_\mu\big(\delta^d(x-z)\delta_{\nu\rho}\langle V^a{}_\sigma(y)\, V^b{}_\mu(x)\rangle\big)\,, \\ &\langle T_{\mu\mu}(x)\, V^a{}_\sigma(y)\, V^b{}_\rho(z)\rangle = 0\,, \qquad \partial^y_\sigma \langle T_{\mu\nu}(x)\, V^a{}_\sigma(y)\, V^b{}_\rho(z)\rangle = 0\,. \end{aligned} \tag{6.21}$$

From (3.11), with notation as in (6.9) and neglecting group indices which are unimportant in this context, we find for the leading term in the short distance limit $s = x - y \to 0$

$$\begin{aligned} \langle T_{\mu\nu}(x)\, V_\sigma(y)\, V_\rho(z)\rangle &= \frac{1}{s^d}\, I_{\sigma\alpha}(s)\, t_{\mu\nu\alpha\beta}(X)\, \frac{I_{\beta\rho}(x-z)}{(x-z)^d (y-z)^{d-2}} \\ &\sim A_{\mu\nu\sigma\beta}(s)\, C_V \frac{I_{\beta\rho}(y-z)}{(y-z)^{2d-2}}\,, \quad A_{\mu\nu\sigma\rho}(s) C_V = \frac{1}{s^d}\, \tilde{t}_{\mu\nu\sigma\rho}(s)\,. \end{aligned} \tag{6.22}$$

In order to be in accord with (6.21) it is necessary to require that $A_{\mu\nu\sigma\rho}(s) \to \hat{A}_{\mu\nu\sigma\rho}(s)$ representing a distribution with the properties

$$\partial_\mu \hat{A}_{\mu\nu\sigma\rho}(s) = \big(\delta_{\sigma\rho}\partial_\nu - \delta_{\nu\sigma}\partial_\rho\big)\delta^d(s)\,, \tag{6.23a}$$

$$\hat{A}_{\mu\mu\sigma\rho}(s) = 0\,, \qquad \partial_\sigma \hat{A}_{\mu\nu\sigma\rho}(s) = 0\,, \tag{6.23b}$$

including δ function contributions. For $s \neq 0$ $\hat{A}_{\mu\nu\sigma\rho}(s)$ is determined by (3.13) and (3.14). To obtain a well defined distribution on $\mathbb{R}^d$ we follow the previous case and pull out derivatives in the spirit of differential regularisation so that it may be expressed in the form

$$\begin{aligned} \hat{A}_{\mu\nu\sigma\rho}(s) C_V = &- \frac{2c}{d(d-2)}\, \delta_{\sigma\rho}\, \partial_\mu \partial_\nu \frac{1}{s^{d-2}} + \frac{2e}{d(d-2)}\, \delta_{\mu\nu}\, \partial_\sigma \partial_\rho \frac{1}{s^{d-2}} \\ &- \frac{e}{d(d-4)}\, \partial_\mu\partial_\nu\partial_\sigma\partial_\rho \frac{1}{s^{d-4}} - \frac{c+de}{d(d-2)}\, \big(\delta_{\nu\rho}\, \partial_\mu\partial_\sigma + \delta_{\mu\rho}\, \partial_\nu \partial_\sigma\big)\frac{1}{s^{d-2}} \\ &+ \frac{c-(d-2)e}{d(d-2)}\, \big(\delta_{\nu\sigma}\, \partial_\mu\partial_\rho + \delta_{\mu\sigma}\, \partial_\nu\partial_\rho\big)\frac{1}{s^{d-2}} \\ &+ \big(C\, \delta_{\mu\nu}\delta_{\sigma\rho} + D\, \delta_{\mu\sigma}\delta_{\nu\rho} + E\, \delta_{\mu\rho}\delta_{\nu\sigma}\big) S_d \delta^d(s)\,, \end{aligned} \tag{6.24}$$

where C, D, E represent the potential arbitrariness in the distribution. With this form there is no ambiguity in determining the δ function contributions on differentiation or taking the trace and imposing (6.23b) gives

$$C = \frac{2}{d}\, e\,, \qquad D = E = -\frac{1}{d}(c + de) \tag{6.25}$$

and then it is easy to verify (6.23a) if

$$2S_d(c+e) = d\, C_V\,. \tag{6.26}$$

This agrees with the results of our calculations for free fields (5.5) and (5.10) or (5.6) and (5.11).

$A_{\mu\nu\sigma\rho}(s)$ clearly represents the coefficient of the leading singular term in the operator product expansion of $T_{\mu\nu}(x)$ and ${V^a}_\sigma(y)$. The next leading singular coefficient, which is $\mathrm{O}(s^{-d+1})$, also contributes δ function terms to the Ward identity (6.21) on taking the divergence although a regularisation as in $A_{\mu\nu\sigma\rho}(s) \to \hat{A}_{\mu\nu\sigma\rho}(s)$ is unnecessary in this case. This coefficient may be obtained by expanding (6.22) where the $\mathrm{O}(s^{-d+1})$ terms in the short distance limit are

$$\begin{gathered}
A_{\mu\nu\sigma\beta}(s)s_\lambda \frac{\partial}{\partial y_\lambda} C_V \frac{I_{\beta\rho}(y-z)}{(y-z)^{2d-2}} - \mathcal{B}_{\mu\nu\sigma\beta\lambda}(s)\, 2(y-z)_\lambda C_V \frac{I_{\beta\rho}(y-z)}{(y-z)^{2d}}\,,\\
\mathcal{B}_{\mu\nu\sigma\beta\lambda}(s) = \tfrac{1}{2}s^2\partial_\lambda A_{\mu\nu\sigma\beta}(s) + s_\lambda A_{\mu\nu\sigma\beta}(s) + s_\sigma A_{\mu\nu\lambda\beta}(s) - \delta_{\sigma\lambda}\, s_\alpha A_{\mu\nu\alpha\beta}(s)\,.
\end{gathered} \tag{6.27}$$

It is crucial that $\mathcal{B}_{\mu\nu\sigma\rho\lambda}(s)$ satisfies

$$\mathcal{B}_{\mu\nu\sigma\lambda\lambda}(s) = 0\,, \tag{6.28}$$

which may be obtained from $s^2\partial_\sigma A_{\mu\nu\sigma\rho}(s) = 0$ and $A_{\mu\nu\sigma\rho}(s) = I_{\sigma\alpha}(s)I_{\rho\beta}(s)A_{\mu\nu\beta\alpha}(s)$. The relation (6.28) is necessary in order for the terms in (6.27) to be written in accord with an operator product expansion,

$$\begin{gathered}
T_{\mu\nu}(x)V_\sigma(y) \sim \hat{A}_{\mu\nu\sigma\rho}(s)\, V_\rho(y) + B_{\mu\nu\sigma\rho\lambda}(s)\,\partial_\lambda V_\rho(y)\,,\\
B_{\mu\nu\sigma\rho\lambda}(s) = A_{\mu\nu\sigma\rho}(s)s_\lambda - \frac{1}{d}\delta_{\rho\lambda}A_{\mu\nu\sigma\beta}(s)s_\beta + \frac{d-1}{d(d-2)}\mathcal{B}_{\mu\nu\sigma\rho\lambda}(s) - \frac{1}{d(d-2)}\mathcal{B}_{\mu\nu\sigma\lambda\rho}(s)\,,
\end{gathered} \tag{6.29}$$

where we have required $B_{\mu\nu\sigma\lambda\lambda}(s) = 0$. The Ward identities contained in (6.20) are satisfied since

$$\begin{aligned}
\partial_\mu B_{\mu\nu\sigma\rho\lambda}(s) &= -\big(\delta_{\sigma\rho}\delta_{\nu\lambda} - \delta_{\nu\sigma}\delta_{\rho\lambda}\big)\delta^d(s)\,,\\
\partial_\sigma B_{\mu\nu\sigma\rho\lambda}(s) &= \hat{A}_{\mu\nu\lambda\rho}(s) - \frac{1}{d}\delta_{\rho\lambda}\hat{A}_{\mu\nu\beta\beta}(s)\,.
\end{aligned} \tag{6.30}$$

To show these we use (6.22a,b) (note $\partial_\sigma(s_\sigma\hat{A}_{\mu\nu\lambda\rho}(s)) = 0$) to obtain

$$\partial_\sigma\mathcal{B}_{\mu\nu\sigma\rho\lambda} = 0\,, \quad \partial_\mu\mathcal{B}_{\mu\nu\sigma\rho\lambda} = -(d-1)\hat{A}_{\lambda\nu\sigma\rho} - \hat{A}_{\rho\nu\sigma\lambda} + \delta_{\rho\lambda}\hat{A}_{\mu\nu\sigma\mu}\,, \tag{6.31}$$

including possible δ function contributions. The second equation depends on

$$\begin{aligned}
&(s_\mu\partial_\lambda - s_\lambda\partial_\mu)\hat{A}_{\mu\nu\sigma\rho}(s)\\
&\quad = -d\hat{A}_{\lambda\nu\sigma\rho}(s) - \hat{A}_{\sigma\nu\lambda\rho}(s) + \delta_{\sigma\lambda}\hat{A}_{\mu\nu\mu\rho}(s) - \hat{A}_{\rho\nu\sigma\lambda}(s) + \delta_{\rho\lambda}\hat{A}_{\mu\nu\sigma\mu}(s)\,,
\end{aligned} \tag{6.32}$$

which is a consequence of the infinitesimal version of the rotational covariance equation as in (2.32).

It remains to analyse similarly the three point function of the energy momentum tensor for which from (6.5), given the definition (6.6), the crucial identities are

$$
\begin{aligned}
&\partial_\mu \langle T_{\mu\nu}(x)\, T_{\sigma\rho}(y)\, T_{\alpha\beta}(z)\rangle \\
&\quad = \partial_\nu \delta^d(x-y)\, \langle T_{\sigma\rho}(x)\, T_{\alpha\beta}(z)\rangle + \big\{ \partial_\sigma \big(\delta^d(x-y)\langle T_{\rho\nu}(x)\, T_{\alpha\beta}(z)\rangle\big) + \sigma \leftrightarrow \rho \big\} \\
&\quad + \partial_\nu \delta^d(x-z)\, \langle T_{\sigma\rho}(y)\, T_{\alpha\beta}(x)\rangle + \big\{ \partial_\alpha \big(\delta^d(x-z)\langle T_{\beta\nu}(x)\, T_{\sigma\rho}(y)\rangle\big) + \alpha \leftrightarrow \beta \big\}\,, \\
&\langle T_{\mu\mu}(x)\, T_{\sigma\rho}(y)\, T_{\alpha\beta}(z)\rangle = 2\big(\delta^d(x-y) + \delta^d(x-z)\big)\, \langle T_{\sigma\rho}(y)\, T_{\alpha\beta}(z)\rangle\,.
\end{aligned}
\tag{6.33}
$$

As before the δ function contributions arise from the leading short distance singularities at coincident points. From (3.15) for $x \to y$

$$
\begin{aligned}
\langle T_{\mu\nu}(x)\, T_{\sigma\rho}(y)\, T_{\alpha\beta}(z)\rangle &= \frac{1}{s^d}\, \mathcal{I}_{\sigma\rho,\sigma'\rho'}(s)\, t_{\sigma'\rho'\alpha'\beta'\mu\nu}(X)\, \frac{\mathcal{I}_{\alpha'\beta'\alpha\beta}(x-z)}{(x-z)^d (y-z)^d} \\
&\sim A_{\mu\nu\sigma\rho\alpha'\beta'}(s)\, C_T \frac{\mathcal{I}_{\alpha'\beta'\alpha\beta}(y-z)}{(y-z)^{2d}}\,,
\end{aligned}
\tag{6.34a}
$$

$$
A_{\mu\nu\sigma\rho\alpha\beta}(s) C_T = \frac{1}{s^d}\, t_{\mu\nu\sigma\rho\alpha\beta}(s)\,, \tag{6.34b}
$$

where C_T is the scale of the two point function as in (2.23). In order to satisfy (6.33) we require the associated regularised distribution to satisfy

$$
\partial_\mu \hat{A}_{\mu\nu\sigma\rho\alpha\beta}(s) = \big(\mathcal{E}_{\sigma\rho,\alpha\beta}\, \partial_\nu + \mathcal{E}_{\nu\rho,\alpha\beta}\, \partial_\sigma + \mathcal{E}_{\nu\sigma,\alpha\beta}\, \partial_\rho\big)\delta^d(s)\,, \tag{6.35a}
$$

$$
\hat{A}_{\mu\mu\sigma\rho\alpha\beta}(s) = 2\mathcal{E}_{\sigma\rho,\alpha\beta}\, \delta^d(s)\,, \tag{6.35b}
$$

where $\mathcal{E}$, defined in (2.24), represents the identity on the space of symmetric traceless tensors. From the original definition (6.34b) also

$$
\hat{A}_{\mu\nu\sigma\rho\alpha\beta}(s) = \hat{A}_{\sigma\rho\mu\nu\alpha\beta}(s)\,, \quad \hat{A}_{\mu\nu\sigma\rho\alpha\alpha}(s) = 0\,. \tag{6.36}
$$

As before we represent $\hat{A}_{\mu\nu\sigma\rho\alpha\beta}(s) = \mathrm{O}(s^{-d})$ in a differentially regularised form in terms of derivatives acting on functions of $\mathrm{O}(s^{-d+2})$ which are unambiguous as distributions. It is convenient for this purpose then to define the following set of tensors $H^i_{\mu\nu\sigma\rho\alpha\beta}(s)$ which are traceless and symmetric on $\mu\nu$, $\sigma\rho$ and $\alpha\beta$,

$$
\begin{aligned}
H^1_{\mu\nu\sigma\rho\alpha\beta}(s) &= \Big(\partial_\mu\partial_\nu - \frac{1}{d}\,\delta_{\mu\nu}\partial^2\Big)\, \frac{1}{s^{d-2}} h^1_{\sigma\rho}(\hat{s}) h^1_{\alpha\beta}(\hat{s})\,, \\
H^2_{\mu\nu\sigma\rho\alpha\beta}(s) &= \Big(\delta_{\sigma\alpha}\partial_\rho\partial_\beta + (\sigma \leftrightarrow \rho, \alpha \leftrightarrow \beta) \\
&\qquad\qquad - \frac{4}{d}\,\delta_{\sigma\rho}\partial_\alpha\partial_\beta - \frac{4}{d}\,\delta_{\alpha\beta}\partial_\sigma\partial_\rho + \frac{4}{d^2}\,\delta_{\sigma\rho}\delta_{\alpha\beta}\partial^2\Big)\, \frac{1}{s^{d-2}} h^1_{\mu\nu}(\hat{s})\,, \\
H^3_{\mu\nu\sigma\rho\alpha\beta}(s) &= h^3_{\sigma\rho\alpha\beta}\, \Big(\partial_\mu\partial_\nu - \frac{1}{d}\,\delta_{\mu\nu}\Big) \frac{1}{s^{d-2}}\,, \\
H^4_{\mu\nu\sigma\rho\alpha\beta}(s) &= \Big(h^3_{\mu\nu\sigma\alpha}\partial_\rho\partial_\beta + (\sigma \leftrightarrow \rho, \alpha \leftrightarrow \beta) - \frac{2}{d}\,\delta_{\sigma\rho}\big(h^3_{\mu\nu\lambda\alpha}\partial_\lambda\partial_\beta + (\alpha \leftrightarrow \beta)\big) \\
&\qquad - \frac{2}{d}\,\delta_{\alpha\beta}\big(h^3_{\mu\nu\lambda\sigma}\partial_\lambda\partial_\rho + (\sigma \leftrightarrow \rho)\big) + \frac{8}{d^2}\,\delta_{\sigma\rho}\delta_{\alpha\beta}\Big(\partial_\mu\partial_\nu - \frac{1}{d}\,\delta_{\mu\nu}\Big)\Big) \frac{1}{s^{d-2}}\,,
\end{aligned}
\tag{6.37}
$$

where $\hat{s} = s/\sqrt{s^2}$. With this basis we can express after some labour $A_{\mu\nu\sigma\rho\alpha\beta}(s)$, which is given by (3.19) for $s \neq 0$ with the coefficients written in terms of a, b, c by virtue of (3.20,21), as a distribution consistent with (6.36) in the form

$$\begin{aligned}
\hat{A}_{\mu\nu\sigma\rho\alpha\beta}(s)C_T = {}& \frac{d-2}{d+2}(4a+2b-c)H^1_{\alpha\beta\mu\nu\sigma\rho}(s) + \frac{1}{d}(da+b-c)H^2_{\alpha\beta\mu\nu\sigma\rho}(s) \\
& - \frac{d(d-2)a-(d-2)b-2c}{d(d+2)}\big(H^2_{\mu\nu\sigma\rho\alpha\beta}(s) + H^2_{\sigma\rho\mu\nu\alpha\beta}(s)\big) \\
& + \frac{2da+2b-c}{d(d-2)}H^3_{\alpha\beta\mu\nu\sigma\rho}(s) - \frac{2(d-2)a-b-c}{d(d-2)}H^4_{\alpha\beta\mu\nu\sigma\rho}(s) \\
& - 2\,\frac{(d-2)a-c}{d(d-2)}\big(H^3_{\mu\nu\sigma\rho\alpha\beta}(s) + H^3_{\sigma\rho\mu\nu\alpha\beta}(s)\big) \\
& + \frac{(d-2)(2a+b)-dc}{d(d^2-4)}\big(H^4_{\mu\nu\sigma\rho\alpha\beta}(s) + H^4_{\sigma\rho\mu\nu\alpha\beta}(s)\big) \\
& + \big(C\, h^5_{\mu\nu\sigma\rho\alpha\beta} + D(\delta_{\mu\nu}h^3_{\sigma\rho\alpha\beta} + \delta_{\sigma\rho}h^3_{\mu\nu\alpha\beta})\big)S_d\delta^d(s)\,.
\end{aligned} \tag{6.38}$$

C, D are arbitrary coefficients until the imposition of the Ward identities. From the trace identity (6.35b) it is easy to see that (note $h^3_{\sigma\rho\alpha\beta} = 2\mathcal{E}_{\sigma\rho,\alpha\beta}$)

$$d\,D = C_T\,. \tag{6.39}$$

With the results in the appendix in (A.6) we now find

$$\begin{aligned}
\partial_\mu \hat{A}_{\mu\nu\sigma\rho\alpha\beta}(s)C_T = {}& \Big(\frac{2}{d^2}(d-1)\big((d-2)a-c\big) - \frac{4}{d}\,C + D\Big)h^3_{\sigma\rho\alpha\beta}\partial_\nu\, S_d\delta^d(s) \\
& + \Big(\frac{1}{d}\big(2(d-2)a-b-c\big) + C\Big)\big(h^3_{\nu\sigma\alpha\beta}\partial_\rho + h^3_{\nu\rho\alpha\beta}\partial_\sigma\big)\, S_d\delta^d(s) \\
& + \Big(-\frac{2}{d^2}\big(2(d-2)a-b-c\big) - \frac{2}{d}C + D\Big)\delta_{\sigma\rho}h^3_{\mu\nu\alpha\beta}\partial_\mu\, S_d\delta^d(s) \\
& + \Big(-\frac{(d-2)(2a+b)-dc}{d(d+2)} + C\Big) \\
& \qquad \times \Big(h^3_{\sigma\rho\nu\alpha}\partial_\beta + h^3_{\sigma\rho\nu\beta}\partial_\alpha - \frac{2}{d}\,\delta_{\alpha\beta}h^3_{\sigma\rho\nu\mu}\partial_\mu\Big)S_d\delta^d(s)\,.
\end{aligned} \tag{6.40}$$

Comparison with (6.35a) then gives the equations

$$\begin{aligned}
-\frac{(d-2)(2a+b)-dc}{d(d+2)} + C &= 0\,, \\
S_d\Big(\frac{2}{d^2}(d-1)\big((d-2)a-c\big) - \frac{4}{d}\,C + D\Big) &= \tfrac{1}{2}C_T\,, \\
S_d\Big(\frac{1}{d}\big(2(d-2)a-b-c\big) + C\Big) &= \tfrac{1}{2}C_T\,,
\end{aligned} \tag{6.41}$$

apart from one which follows trivially given (6.39). It is then a simple exercise to check their consistency and to finally obtain

$$4S_d\,\frac{(d-2)(d+3)a-2b-(d+1)c}{d(d+2)} = C_T\,. \tag{6.42}$$

It may readily be verified that this is in agreement with the results for free field theories in (5.5) and (5.12), (5.6) and (5.13) or for $d = 4$ (5.16) and (5.17).

Using our results we may write an operator product expansion for the energy momentum tensor, generalising (1.1) to arbitrary d, which takes the form

$$T_{\mu\nu}(x)T_{\sigma\rho}(y) \sim C_T \frac{\mathcal{I}_{\mu\nu,\sigma\rho}(s)}{s^{2d}} + \hat{A}_{\mu\nu\sigma\rho\alpha\beta}(s)T_{\alpha\beta}(y) + B_{\mu\nu\sigma\rho\alpha\beta\lambda}(s)\partial_\lambda T_{\alpha\beta}(y)\,, \tag{6.43}$$

where the c-number term corresponds to the two point function as in (2.23). The next to leading operator term may also be found as in our discussion of the similar $T_{\mu\nu}(x)V_\sigma(y)$ case. By expanding (6.34a) we find

$$\begin{aligned} B_{\mu\nu\sigma\rho\alpha\beta\lambda}(s) = {} & A_{\mu\nu\sigma\rho\alpha\beta}(s)s_\lambda \\ & + \frac{1}{(d+2)(d-1)}\Big((d+1)\mathcal{B}_{\mu\nu\sigma\rho\alpha\beta\lambda}(s) - \mathcal{B}_{\mu\nu\sigma\rho\lambda\beta\alpha}(s) - \mathcal{B}_{\mu\nu\sigma\rho\alpha\lambda\beta}(s) \\ & \qquad - d\,\delta_{\alpha\lambda}A_{\mu\nu\sigma\rho\gamma\beta}(s)s_\gamma - d\,\delta_{\beta\lambda}A_{\mu\nu\sigma\rho\alpha\gamma}(s)s_\gamma + 2\delta_{\alpha\beta}A_{\mu\nu\sigma\rho\lambda\gamma}(s)s_\gamma\Big)\,, \end{aligned} \tag{6.44}$$

where

$$\begin{aligned} \mathcal{B}_{\mu\nu\sigma\rho\alpha\beta\lambda}(s) = {} & \tfrac{1}{2}s^2\partial_\lambda A_{\mu\nu\sigma\rho\alpha\beta}(s) + s_\sigma A_{\mu\nu\lambda\rho\alpha\beta}(s) - \delta_{\sigma\lambda}\,s_{\sigma'}A_{\mu\nu\sigma'\rho\alpha\beta}(s) \\ & + s_\rho A_{\mu\nu\sigma\lambda\alpha\beta}(s) - \delta_{\rho\lambda}\,s_{\rho'}A_{\mu\nu\sigma\rho'\alpha\beta}(s)\,. \end{aligned} \tag{6.45}$$

As before it is crucial that $\mathcal{B}_{\mu\nu\sigma\rho\lambda\beta\lambda}(s) = 0$ and the corresponding condition has been imposed on $B_{\mu\nu\sigma\rho\alpha\beta\lambda}(s)$ in (6.44). The Ward identity in (6.33) is satisfied due to

$$\begin{aligned} \partial_\mu B_{\mu\nu\sigma\rho\alpha\beta\lambda}(s) = {} & -\delta_{\nu\lambda}\mathcal{E}_{\sigma\rho,\alpha\beta}\,\delta^d(s) \\ & + \frac{1}{(d+2)(d-1)}\Big(d\,\delta_{\alpha\lambda}\mathcal{E}_{\sigma\rho,\nu\beta} + d\,\delta_{\beta\lambda}\mathcal{E}_{\sigma\rho,\nu\alpha} - 2\delta_{\alpha\beta}\mathcal{E}_{\sigma\rho,\nu\lambda}\Big)\delta^d(s)\,, \end{aligned} \tag{6.46}$$

where the second line on the r.h.s. of (6.46) does not contribute in the operator product expansion (6.43). To verify (6.46) we use (6.35a,b) along with the definition (6.45) and an analogous equation to (6.32) to obtain

$$\begin{aligned} \partial_\mu \mathcal{B}_{\mu\nu\sigma\rho\alpha\beta\lambda}(s) = {} & -d\,\hat{A}_{\lambda\nu\sigma\rho\alpha\beta}(s) - \hat{A}_{\alpha\nu\sigma\rho\lambda\beta}(s) + \delta_{\alpha\lambda}\,\hat{A}_{\mu\nu\sigma\rho\mu\beta}(s) \\ & - \hat{A}_{\beta\nu\sigma\rho\alpha\lambda}(s) + \delta_{\beta\lambda}\,\hat{A}_{\mu\nu\sigma\rho\alpha\mu}(s) \\ & + \big(2\delta_{\nu\lambda}\mathcal{E}_{\sigma\rho,\alpha\beta} + (d+1)(\delta_{\sigma\lambda}\mathcal{E}_{\nu\rho,\alpha\beta} + \delta_{\rho\lambda}\mathcal{E}_{\nu\sigma,\alpha\beta}) \\ & \quad - \delta_{\nu\sigma}\mathcal{E}_{\lambda\rho,\alpha\beta} - \delta_{\nu\rho}\mathcal{E}_{\lambda\sigma,\alpha\beta} - 2\delta_{\sigma\rho}\mathcal{E}_{\nu\lambda,\alpha\beta}\big)\delta^d(s)\,. \end{aligned} \tag{6.47}$$

Of course in any conformal theory other operators may appear in the expansion (6.43). In particular any scalar operator $\mathcal{O}$ with dimension $\eta < d$ will be more significant in the short distance limit than the contribution of the energy momentum tensor itself shown in (6.43) if the $\langle TT\mathcal{O}\rangle$ three point function, whose general form was found in section 3 in (3.3,4) and (3.6), is non zero.

7 Hamiltonian

The discussion of Ward identities in the previous section relating two and three point functions requires a careful treatment of the singularities of the three point function at coincident points. An alternative derivation which does not require an analysis of the short distance limit may be given by constructing the Hamiltonian H operator from the energy momentum tensor. If H is assumed to generate translations along the 1 direction then it can be written as

$$H = - \int d^d x \, \delta(x_1) \, T_{11}(x) \, . \tag{7.1}$$

The minus sign in the definition (7.1) is a reflection of the rotation to euclidean space, as defined with our conventions H should have a positive spectrum. Choosing the configuration specified by the points $x = (0, \mathbf{x})$, $y = (\frac{1}{2}s, \mathbf{0})$, $\bar{y} = (-\frac{1}{2}s, \mathbf{0})$ then from (3.1) we may write

$$\langle T_{11}(x) \, \mathcal{O}(y) \, \mathcal{O}(\bar{y}) \rangle = \frac{d-1}{d} \, a \, \frac{1}{(\mathbf{x}^2 + \frac{1}{4}s^2)^d} \, \frac{1}{s^{2\eta - d}} \, . \tag{7.2}$$

It is then straightforward to evaluate the integral over $\mathbf{x}$ to obtain

$$\langle \mathcal{O}(y) \, H \, \mathcal{O}(\bar{y}) \rangle = -2 \frac{d-1}{d} S_d \, a \, \frac{1}{s^{2\eta+1}} \, . \tag{7.3}$$

However since H is the generator of translations along the 1 direction

$$\langle \mathcal{O}(y) \, H \, \mathcal{O}(\bar{y}) \rangle = - \frac{\partial}{\partial s} \, \langle \mathcal{O}(y) \, \mathcal{O}(\bar{y}) \rangle = \frac{2\eta N}{s^{2\eta+1}} \, , \tag{7.4}$$

given the expression (6.15) for the two point function of the operator $\mathcal{O}$. Comparing (7.3) and (7.4) gives the same relation between a and N as found earlier in (6.20).

For the three point function involving the energy momentum tensor and two vector currents then from (3.11), using (3.13) and (3.14), we may similarly obtain in this configuration

$$\begin{aligned} \langle T_{11}(x) \, V_1(y) \, V_1(\bar{y}) \rangle = {} & \frac{1}{(\mathbf{x}^2 + \frac{1}{4}s^2)^d \, s^{d-2}} \bigg\{ 2 \frac{d-1}{d} \, c \\ & + e \Big((d-1)^2 - (d-2)(d+1) \, \frac{s^2 \mathbf{x}^2}{(\mathbf{x}^2 + \frac{1}{4}s^2)^2} \Big) \bigg\} \, , \\ \langle T_{11}(x) \, V_i(y) \, V_j(\bar{y}) \rangle = {} & \frac{1}{(\mathbf{x}^2 + \frac{1}{4}s^2)^d \, s^{d-2}} \bigg\{ -2 \frac{d-1}{d} \, c \, \delta_{ij} \\ & + e \Big((d-3)\delta_{ij} - (d-2)(d+1) \, \frac{s^2 x_i x_j}{(\mathbf{x}^2 + \frac{1}{4}s^2)^2} \Big) \bigg\} \, , \end{aligned} \tag{7.5}$$

where i, j denote components orthogonal to the 1 direction. We may then obtain

$$\begin{aligned} \langle V_1(y) \, H \, V_1(\bar{y}) \rangle = {} & -4 \frac{d-1}{d} S_d \, (c+e) \, \frac{1}{s^{2d-1}} \, , \\ \langle V_i(y) \, H \, V_j(\bar{y}) \rangle = {} & \delta_{ij} \, 4 \frac{d-1}{d} S_d \, (c+e) \, \frac{1}{s^{2d-1}} \, . \end{aligned} \tag{7.6}$$

As before this may be directly related to the two point function of the vector currents which as given in (2.23) requires

$$\langle V_1(y)\, V_1(\bar y)\rangle = - C_V\, \frac{1}{s^{2d-2}}\,, \quad \langle V_i(y)\, V_j(\bar y)\rangle = \delta_{ij}\, C_V\, \frac{1}{s^{2d-2}}\,. \tag{7.7}$$

Hence the same result (6.26) derived from the Ward identities may be obtained.

For the energy momentum tensor three point function the algebraic complications are again more severe. It is actually more convenient to start from the collinear configuration of section 4 with points at $(0,\mathbf{0})$, $y=(\frac{1}{2}s,\mathbf{0})$, $\bar y = (-\frac{1}{2}s,\mathbf{0})$ and consider the conformal transformation

$$x_\mu \to \frac{(1+\frac{1}{4}s^2b^2)x_\mu + (x^2 - \frac{1}{2}s^2 b{\cdot}x - \frac{1}{4}s^2)b_\mu}{1+2b{\cdot}x + b^2x^2}\,, \quad b_\mu = (0,\mathbf{b})\,, \tag{7.8}$$

so that $y\to y$, $\bar y \to \bar y$ and $(0,\mathbf{0}) \to (0, -\frac{1}{4}s^2\mathbf{b}) = x$. Using this we may find by transformation from (4.20) and (4.21)

$$\begin{aligned}
&\langle T_{11}(x)\, T_{ij}(y)\, T_{k\ell}(\bar y)\rangle \\
&\quad = \frac{1}{(\mathbf{x}^2+\frac{1}{4}s^2)^d s^d}\Big\{ \delta\, \delta_{ij}\delta_{k\ell} + \epsilon\big(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\big) + A\, x_ix_jx_kx_\ell \frac{s^4}{(\mathbf{x}^2+\frac{1}{4}s^2)^4} \\
&\qquad + \Big((\beta-\delta)(\delta_{ij}x_kx_\ell + \delta_{k\ell}x_ix_j) - (\gamma+\epsilon)\big(\delta_{ik}x_jx_\ell + i\leftrightarrow j, k\leftrightarrow \ell\big)\Big)\frac{s^2}{(\mathbf{x}^2+\frac{1}{4}s^2)^2}\Big\}\,, \\
&\langle T_{11}(x)\, T_{i1}(y)\, T_{k1}(\bar y)\rangle \\
&\quad = \frac{1}{(\mathbf{x}^2+\frac{1}{4}s^2)^d s^d}\Big\{ \Big(\gamma - (\gamma+\epsilon)\frac{s^2\mathbf{x}^2}{(\mathbf{x}^2+\frac{1}{4}s^2)^2}\Big)\delta_{ik} + A\, x_ix_k \frac{s^4\mathbf{x}^2}{(\mathbf{x}^2+\frac{1}{4}s^2)^4} \\
&\qquad\qquad + (\gamma+\epsilon-A)x_ix_k\frac{s^2}{(\mathbf{x}^2+\frac{1}{4}s^2)^2}\Big\}\,, \\
&\quad A = \alpha - 2\beta + 4\gamma + \delta + 2\epsilon = (d+3)(d-2)(4a+2b-c)\,.
\end{aligned} \tag{7.9}$$

Hence we obtain

$$\begin{aligned}
\langle T_{ij}(y)\, H\, T_{k\ell}(\bar y)\rangle = {}& -\frac{2S_d}{s^{2d+1}}\,\frac{1}{d}\Big\{\Big(\frac{A}{d+2} + 2\beta + (d-2)\delta\Big)\delta_{ij}\delta_{k\ell} \\
&\qquad + \Big(\frac{A}{d+2} - 2\gamma + (d-2)\epsilon\Big)\big(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\big)\Big\} \\
={}& \Big(\tfrac{1}{2}\big(\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}\big) - \frac{1}{d}\,\delta_{ij}\delta_{k\ell}\Big)\,\frac{2dC_T}{s^{2d+1}}\,, \\
\langle T_{i1}(y)\, H\, T_{k1}(\bar y)\rangle = {}& -\frac{2S_d}{s^{2d+1}}\,\frac{1}{d}\Big\{2\gamma - (d-2)\epsilon - \frac{A}{d+2}\Big\}\,\delta_{ik} = -\tfrac{1}{2}\delta_{ik}\,\frac{2dC_T}{s^{2d+1}}\,,
\end{aligned} \tag{7.10}$$

where we have used, from (4.22,23) and (4.25) $2\beta + (d-2)\delta = -2(d-1)b + (d-5)c$, $2\gamma - (d-2)\epsilon = 2(d+3)(d-2)a + 2(d-3)b - 3(d-1)c$ and the results for A in (7.9) and

C_T in (6.42). The final result (7.10) is then as expected from the general formula for the two point function of the energy momentum tensor in (2.23).

8 Scale Anomalies

The discussion in this paper has so far been based on exact conformal invariance. This is relevant for quantum field theories at critical points where β functions are zero and hence the operator contributions to the trace of the energy momentum tensor vanish. However even at such critical points for a curved space background and with other external fields there are possible c number contributions to the trace. For $d = 4$ these are local dimension four scalars and the trace may have the form

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = \tfrac{1}{2}p\, J^2 - \tfrac{1}{4}\kappa\, F^a{}_{\mu\nu}F^{a\mu\nu} - \beta_a F - \beta_b G + h\nabla^2 R\,. \tag{8.1}$$

$F^a{}_{\mu\nu}$ is the field strength formed from the gauge field $A^a{}_\mu$ while if $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature formed from the metric $g_{\mu\nu}$, with $R_{\alpha\beta}$ the Ricci tensor and R the scalar curvature, then for general d we define

$$\begin{aligned} F &= R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - \frac{4}{d-2}\,R^{\alpha\beta}R_{\alpha\beta} + \frac{2}{(d-2)(d-1)}\,R^2 = C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}\,,\\ G &= R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta}R_{\alpha\beta} + R^2 = 6\,R^{\alpha\beta}{}_{[\alpha\beta}R^{\gamma\delta}{}_{\gamma\delta]}\,,\\ C_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} - \frac{2}{d-2}\big(g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha}\big) + \frac{2}{(d-1)(d-2)}\,g_{\alpha[\gamma}g_{\delta]\beta}R\,, \end{aligned} \tag{8.2}$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, the traceless part of the Riemann tensor. In writing (8.1) we have supposed that the operator $\mathcal{O}$ which is coupled to the source J has a dimension $\eta \approx 2$, as is relevant for the case of the operator ϕ^2 in scalar field theories (if such a low dimension operator is present then generically the expansion (8.1) has several more possible terms, we neglect such complications [27] here since scalar operators are only considered as an illustrative exercise for more serious calculations involving conserved vector fields and the energy momentum tensor). Many authors have computed the trace of the energy momentum tensor on curved space for free fields [23] and found

$$\begin{aligned} \beta_a &= -\frac{1}{64\pi^2}\,\frac{1}{10}\big(\tfrac{1}{3}n_\phi + 2n_\psi + 4n_V\big)\,,\\ \beta_b &= \frac{1}{64\pi^2}\,\frac{1}{90}\big(n_\phi + 11n_\psi + 62n_V\big)\,, \end{aligned} \tag{8.3}$$

with the notation of section 5 where now n_V denotes the number of free vector fields. In general $g^{\mu\nu}\langle T_{\mu\nu}\rangle$ might be expected to contain a term $\propto R^2$ but such a contribution can be shown to be absent, at the critical point, due to integrability conditions for W [27,28]. The coefficient h in (8.1) is undetermined in general since it may be modified at will by the addition of an arbitrary local term $\propto \int d^4x\sqrt{g}\,R^2$ to W. If present the trace identity for the two point function on flat space becomes

$$\langle T_{\mu\mu}(x)\,T_{\sigma\rho}(0)\rangle = 2h(\partial_\sigma\partial_\rho - \delta_{\sigma\rho}\partial^2)\partial^2\delta^4(x)\,. \tag{8.4}$$

For simplicity we take $h = 0$ henceforth.

The result for the trace anomaly in (8.1) is intimately connected with the renormalisation group equation since, for μ an arbitrary renormalisation mass scale, then by dimensional analysis for general d

$$\left(\mu\frac{\partial}{\partial\mu} + \int d^d x \left(2g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}} + (d-\eta)J\,\frac{\delta}{\delta J}\right)\right)W = 0\,. \tag{8.5}$$

Hence (8.1) implies, for $d = 4$,

$$\mu\frac{\partial}{\partial\mu}\langle V^a{}_\mu(x)\, V^b{}_\nu(0)\rangle = \, -\,\kappa\,\delta^{ab}S_{\mu\nu}\delta^4(x)\,, \tag{8.6a}$$

$$\mu\frac{\partial}{\partial\mu}\langle T_{\mu\nu}(x)\, T_{\sigma\rho}(0)\rangle = \, -\,4\beta_a\,\Delta^T_{\mu\nu\sigma\rho}\delta^4(x)\,, \tag{8.6b}$$

where we define for arbitrary d

$$\begin{aligned} S_{\mu\nu} &= \partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2\,, \qquad \partial_\mu S_{\mu\nu} = 0 \\ \Delta^T_{\mu\nu\sigma\rho} &= \tfrac{1}{2}\big(S_{\mu\sigma}S_{\nu\rho} + S_{\mu\rho}S_{\nu\sigma}\big) - \frac{1}{d-1}S_{\mu\nu}S_{\sigma\rho}\,, \quad \Delta^T_{\mu\mu\sigma\rho} = 0\,. \end{aligned} \tag{8.7}$$

Even for a conformal field theory a dependence on a mass scale μ arises when the two point functions are carefully defined as distributions for $d = 4$. From (2.23) or (5.4) we may write for general d

$$\begin{aligned} \langle V^a{}_\mu(x)\, V^b{}_\nu(0)\rangle &= \, -\,\delta^{ab}\frac{C_V}{2(d-2)(d-1)}\, S_{\mu\nu}\frac{1}{x^{2d-4}}\,, \\ \langle T_{\mu\nu}(x)\, T_{\sigma\rho}(0)\rangle &= \frac{C_T}{4(d-2)^2 d(d+1)}\Delta^T_{\mu\nu\sigma\rho}\frac{1}{x^{2d-4}}\,. \end{aligned} \tag{8.8}$$

However $(x^2)^{-\lambda}$ is a singular function with poles at $\lambda = \frac{1}{2}d + n$, $n = 0, 1, \ldots$

$$\frac{1}{(x^2)^\lambda} \sim \frac{1}{d+2n-2\lambda}\,\frac{1}{2^{2n}n!}\,\frac{\Gamma(\frac{1}{2}d)}{\Gamma(\frac{1}{2}d+n)}\, S_d(\partial^2)^n\delta^d(x)\,. \tag{8.9}$$

Consequently x^{-2d+4} although defined by analytic continuation in d is singular when $d = 4$. For $d = 4$ using differential regularisation ensures a sensible distribution through writing

$$\mathcal{R}\frac{1}{x^4} = -\frac{1}{4}\partial^2\frac{1}{x^2}\big(\ln\mu^2 x^2 + a\big)\,, \tag{8.10}$$

with a an arbitrary constant (which may be absorbed into μ). With this prescription

$$\mu\frac{\partial}{\partial\mu}\mathcal{R}\frac{1}{x^4} = 2\pi^2\delta^4(x)\,. \tag{8.11}$$

Using this regularisation in (8.8) when $d = 4$ then (8.6a,b) give

$$\kappa = \frac{\pi^2}{6}\, C_V\,, \qquad \beta_a = -\frac{\pi^2}{640}\, C_T\,. \tag{8.12}$$

It is easy to check that the results of section 5, (5.5,6) and (5.16), are in accord with (8.3).

For three point functions (8.1) leads to modified trace identities involving extra pieces containing δ functions for all three points coincident. Such terms arise due to the need to subtract singularities of the schematic form

$$\frac{1}{\big((x-y)^2\big)^{\delta_{xy}}\big((y-z)^2\big)^{\delta_{yz}}\big((z-x)^2\big)^{\delta_{zx}}} \sim \frac{1}{d+n-(\delta_{xy}+\delta_{yz}+\delta_{zx})}\, \partial^{2n}\delta^d(x-y)\delta^d(x-z)\,, \tag{8.13}$$

for $n = 0, 1, \ldots$, although detailed formulae are more complicated than (8.9)*. For illustrative purposes we consider first the $\langle T\mathcal{O}\mathcal{O}\rangle$ three point function when (8.1) implies

$$\begin{aligned}\langle T_{\mu\mu}(x)\, \mathcal{O}(y)\, \mathcal{O}(z)\rangle = (d-\eta)\big(&\delta^d(x-y)\, \langle \mathcal{O}(x)\, \mathcal{O}(z)\rangle + \delta^d(x-z)\, \langle \mathcal{O}(x)\, \mathcal{O}(y)\rangle\big) \\ &+ p\, \delta^d(x-y)\delta^d(x-z)\,,\end{aligned} \tag{8.14}$$

replacing (6.13b). We show below that the additional term in (8.14) involving p is necessary for any d when $\eta \approx \frac{1}{2}d$.

To verify (8.14) we start from (6.17) which with (6.20) may be written as

$$\hat{A}_{\mu\nu}(s) = -\frac{\eta}{(d-2)(d-1)S_d}\Big(\partial_\mu\partial_\nu - \frac{1}{d}\,\delta_{\mu\nu}\partial^2\Big)\frac{1}{s^{d-2}} + \frac{d-\eta}{d}\,\delta_{\mu\nu}\delta^d(s)\,. \tag{8.15}$$

Based on this form the result provided by (3.1) and (3.2) for the complete three point function may be re-expressed as

$$\begin{aligned}&\langle T_{\mu\nu}(x)\, \mathcal{O}(y)\, \mathcal{O}(z)\rangle \\ &\quad = \frac{d\eta N}{(d-2)^2(d-1)S_d}\Big(\partial_\mu\partial_\nu - \frac{1}{d}\,\delta_{\mu\nu}\partial^2\Big)\bigg(\frac{1}{(x-y)^{d-2}(x-z)^{d-2}(y-z)^{2\eta-d+2}}\bigg) \\ &\quad\quad - \frac{2\eta N}{(d-2)^2 S_d}\bigg(\Big(\partial_\mu\partial_\nu - \frac{1}{d}\,\delta_{\mu\nu}\partial^2\Big)\frac{1}{(x-y)^{d-2}}\,\frac{1}{(x-z)^{d-2}(y-z)^{2\eta-d+2}} \\ &\quad\quad\quad + \Big(\partial_\mu\partial_\nu - \frac{1}{d}\,\delta_{\mu\nu}\partial^2\Big)\frac{1}{(x-z)^{d-2}}\,\frac{1}{(x-y)^{d-2}(y-z)^{2\eta-d+2}}\bigg) \\ &\quad\quad + \frac{d-\eta}{d}\,\delta_{\mu\nu}\big(\delta^d(x-y) + \delta^d(x-z)\big) N\mathcal{R}\frac{1}{(y-z)^{2\eta}} \\ &\quad\quad + \frac{1}{d}\, p\, \delta_{\mu\nu}\delta^d(x-y)\delta^d(x-z)\,,\end{aligned} \tag{8.16}$$

which is consistent as $x \to y$ and also $x \to z$ with the regularised short distance expansion represented by $\hat{A}_{\mu\nu}(s)$ in (8.15) and agrees with (3.1,2) for non coincident points. Each

* For a particular case see ref. [29].

of the terms on the r.h.s of (8.16) involving derivatives are well defined distributions for $\eta \approx \frac{1}{2}d$, the first since the derivative operators act on a non singular expression for $\eta \approx \frac{1}{2}d$ and may be integrated by parts as in differential regularisation and the second since, although there is a potential singularity when $\eta = \frac{1}{2}d$, the coefficient of this apparent singularity can only be proportional to $\delta_{\mu\nu}\delta^d(x-y)\delta^d(x-z)$ but such a factor cannot be present since this term has been arranged to be traceless. In the piece corresponding to the second term in $\hat{A}_{\mu\nu}(s)$ in (8.15) we also introduce the regularised expression, for $\eta \approx \frac{1}{2}d$,

$$\begin{aligned}\mathcal{R}\frac{1}{x^{2\eta}} &= \frac{1}{x^{2\eta}} - \frac{\mu^{2\eta-d}}{d-2\eta}\, S_d\delta^d(x) \\ &= -\frac{1}{d-2\eta}\partial^2\Big(\frac{1}{2\eta-2}\,\frac{1}{x^{2\eta-2}} - \frac{\mu^{2\eta-d}}{d-2}\,\frac{1}{x^{d-2}}\Big) \\ &= -\frac{1}{2(d-2)}\,\partial^2\frac{1}{x^{d-2}}\Big(\ln\mu^2x^2 + \frac{2}{d-2}\Big) \quad \text{for } \eta\to\tfrac{1}{2}d\,,\end{aligned} \tag{8.17}$$

which coincides with (8.10) for $d=4, \eta=2$ taking $a=1$. The last term on the r.h.s. of (8.16) reflects the ambiguity of representing the whole three point function as a well defined distribution, its coefficient has been chosen to correspond with the extra term in the trace identity (8.14). The identity (8.14) is then automatically satisfied if instead of (6.15) we now take for the two point function the regularised form

$$\langle\mathcal{O}(x)\,\mathcal{O}(z)\rangle = N\,\mathcal{R}\frac{1}{(x-z)^{2\eta}}\,. \tag{8.18}$$

The resulting expression given by (8.16) for $\langle T_{\mu\nu}(x)\,\mathcal{O}(y)\,\mathcal{O}(z)\rangle$ is hence a well defined distribution on $\mathbb{R}^{3d}$ for $\eta \approx \frac{1}{2}d$ including the limiting case $\eta = \frac{1}{2}d$. If we now consider the divergence of (8.16) standard calculations for $\eta \neq \frac{1}{2}d$ give

$$\begin{aligned}\partial_\mu\langle T_{\mu\nu}(x)\,\mathcal{O}(y)\,\mathcal{O}(z)\rangle &= \frac{\eta N}{d}\Big(\partial_\nu\delta^d(x-y)\,\frac{1}{(x-z)^{2\eta}} + \partial_\nu\delta^d(x-z)\,\frac{1}{(x-y)^{2\eta}}\Big) \\ &\quad - \frac{d-\eta}{d}N\Big(\delta^d(x-y)\,\partial_\nu\frac{1}{(x-z)^{2\eta}} + \delta^d(x-z)\,\partial_\nu\frac{1}{(x-y)^{2\eta}}\Big) \\ &\quad + \frac{d-\eta}{d}\,\partial_\nu\big(\delta^d(x-y)+\delta^d(x-z)\big)N\mathcal{R}\frac{1}{(y-z)^{2\eta}} \\ &\quad + \frac{1}{d}\,p\,\partial_\nu\big(\delta^d(x-y)\delta^d(x-z)\big) \\ &= \partial_\nu\delta^d(x-y)\,N\mathcal{R}\frac{1}{(x-z)^{2\eta}} + \partial_\nu\delta^d(x-z)\,N\mathcal{R}\frac{1}{(x-y)^{2\eta}} \\ &\quad + \frac{1}{d}\big(p-\mu^{2\eta-d}S_dN\big)\partial_\nu\big(\delta^d(x-y)\delta^d(x-z)\big)\,,\end{aligned} \tag{8.19}$$

using the definition of $\mathcal{R}$ in (8.17), where the final result allows taking the limit $\eta\to\frac{1}{2}d$ in each term without singularities. Assuming the regularised form (8.18) for $\langle\mathcal{O}\mathcal{O}\rangle$ then requiring the Ward identity (6.13a) therefore determines finally $p = \mu^{2\eta-d}S_dN$ (the same relation also follows from $\mu\frac{\partial}{\partial\mu}\langle\mathcal{O}(x)\mathcal{O}(0)\rangle = p\delta^d(x)$). In the above derivation of (8.19) η may be regarded as a convenient regularisation parameter, as in analytic regularisation. Alternatively, if for instance $\eta = d-2$, the subtraction introduced in (8.17) and subsequent discussion is equivalent to using dimensional regularisation.

For the $\langle TVV\rangle$ three point function then for $d=4$ (8.1) now leads to a modified trace identity instead of (6.21)

$$\langle T_{\mu\mu}(x)\, V^a{}_\sigma(y)\, V^b{}_\rho(z)\rangle = \delta^{ab}\kappa\big(\partial_\rho\delta^4(x-y)\partial_\sigma\delta^4(x-z)-\delta_{\sigma\rho}\partial_\lambda\delta^4(x-y)\partial_\lambda\delta^4(x-z)\big)\,. \tag{8.20}$$

To derive this result we use dimensional regularisation since d is the only variable parameter in $\langle TVV\rangle$ consistent with maintaining conformal invariance. According to (8.13) $\langle TVV\rangle$ has a potential singularity as a distribution on $\mathbb{R}^{3d}$ proportional to $\partial^{2n}\delta^d(x-y)\delta^d(x-z)$ for $d=2n+2$. For $d=4-\varepsilon$ a regularised form of the three point function is given by

$$\begin{aligned}\langle T_{\mu\nu}(x)\, V_\sigma(y)\, V_\rho(z)\rangle = {}& I_{\sigma\alpha}(x-y)I_{\rho\beta}(x-z)\,\frac{t_{\mu\nu\sigma\rho}(X)}{(x-y)^d(x-z)^d(y-z)^{d-2}}\\ &-\big(A_{\mu\nu\sigma\beta}(x-y)-\hat{A}_{\mu\nu\sigma\beta}(x-y)\big)\,C_V\,\frac{I_{\beta\rho}(y-z)}{(y-z)^{2d-2}}\\ &-\big(A_{\mu\nu\alpha\rho}(x-z)-\hat{A}_{\mu\nu\alpha\rho}(x-z)\big)\,C_V\,\frac{I_{\alpha\sigma}(z-y)}{(z-y)^{2d-2}}\\ &+\frac{\mu^{-\varepsilon}S_4}{\varepsilon}\,\frac{C_V}{12}\,D_{\mu\nu\sigma\rho}(x-y,x-z)\,.\end{aligned} \tag{8.21}$$

The second and third terms on the r.h.s. of (8.21) subtract the singular contributions for any d arising as $x\to y$ and $x\to z$ respectively and replace them by the leading regularised coefficients in the short distance expansion of $T_{\mu\nu}(x)V_\sigma(y)$ and $T_{\mu\nu}(x)V_\rho(z)$ as given by (6.24). The last term is necessary to subtract singular pieces present as $\varepsilon\to 0$ according to (8.13) which are then purely local with support only for $x=y=z$. This satisfies the obvious symmetry requirements $D_{\mu\nu\sigma\rho}(s,t) = D_{\nu\mu\sigma\rho}(s,t) = D_{\mu\nu\rho\sigma}(t,s)$. From (5.4) and (8.8,9) the regularised two point function using dimensional regularisation is easily seen to be

$$\langle V_\mu(x)\, V_\nu(0)\rangle = C_V\,\frac{I_{\mu\nu}(x)}{x^{2d-2}} + \frac{\mu^{-\varepsilon}S_4}{\varepsilon}\,\frac{C_V}{12}\,S_{\mu\nu}\delta^d(x)\,, \tag{8.22}$$

with $S_{\mu\nu}$ as in (8.7). The expression (8.22) clearly agrees with (8.6a), for $\varepsilon\to 0$, given the relation between κ and C_V again. The last term in (8.21) can now be determined by the Ward identities reflecting conservation of $T_{\mu\nu}$ and V_σ. In order to satisfy (6.21) it is necessary that this term should generate the ε poles required by the regularised two point function as in (8.22) since in section 5 we have verified that without this term the expression given by (8.21) obeys the conservation Ward identities with the unregularised form for the $\langle VV\rangle$ two point functions. Hence $D_{\mu\nu\sigma\rho}(x-y,x-z)$ is required to satisfy

$$\begin{aligned}\partial_\mu D_{\mu\nu\sigma\rho}(x-y,x-z) = {}& \partial_\nu\delta^d(x-y)S_{\sigma\rho}\delta^d(x-z) - \delta_{\nu\sigma}\partial_\mu\delta^d(x-y)S_{\mu\rho}\delta^d(x-z)\\ &+\partial_\nu\delta^d(x-z)S_{\sigma\rho}\delta^d(x-y) - \delta_{\nu\rho}\partial_\mu\delta^d(x-z)S_{\mu\sigma}\delta^d(x-y)\,,\\ \frac{\partial}{\partial y_\sigma}&D_{\mu\nu\sigma\rho}(x-y,x-z) = 0\,.\end{aligned} \tag{8.23}$$

This has a unique solution consistent with the symmetry requirements

$$\begin{aligned}D_{\mu\nu\sigma\rho}(s,t) = {}& \big(\delta_{\mu\sigma}\partial_\rho\delta^d(s)\partial_\nu\delta^d(t) + \delta_{\mu\rho}\partial_\nu\delta^d(s)\partial_\sigma\delta^d(t) - \delta_{\sigma\rho}\partial_\mu\delta^d(s)\partial_\nu\delta^d(t) + (\mu\leftrightarrow\nu)\big)\\ &- \delta_{\mu\nu}\partial_\rho\delta^d(s)\partial_\sigma\delta^d(t) + \big(\delta_{\mu\nu}\delta_{\sigma\rho} - \delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}\big)\partial_\lambda\delta^d(s)\partial_\lambda\delta^d(t)\,.\end{aligned} \tag{8.24}$$

It is easy to see that

$$D_{\mu\mu\sigma\rho}(s,t) = \varepsilon\big(\partial_\rho\delta^d(s)\partial_\sigma\delta^d(t) - \delta_{\sigma\rho}\partial_\lambda\delta^d(s)\partial_\lambda\delta^d(t)\big)\,. \tag{8.25}$$

Hence applying this result in (8.21) shows that the regularised $\langle TVV\rangle$ three point function is in accord with the modified trace identity (8.20) for $d=4$ if κ is given by (8.12) once more.

As before we finally consider the three point function for the energy momentum tensor although the calculational details are more intricate. By virtue of (8.1) the trace identity is modified when $d=4$ from the result in (6.33) to

$$\begin{aligned}\langle T_{\mu\mu}(x)\,T_{\sigma\rho}(y)\,T_{\alpha\beta}(z)\rangle = {}& 2\big(\delta^4(x-y)+\delta^4(x-z)\big)\,\langle T_{\sigma\rho}(y)\,T_{\alpha\beta}(z)\rangle\\ &- 4\big(\beta_a\,\mathcal{A}^F_{\sigma\rho,\alpha\beta}(x-y,x-z) + \beta_b\,\mathcal{A}^G_{\sigma\rho,\alpha\beta}(x-y,x-z)\big)\,,\end{aligned} \tag{8.26}$$

where we assume $h=0$ and we define, for general d,

$$\begin{aligned}\frac{\delta}{\delta g^{\sigma\rho}(y)}\,\frac{\delta}{\delta g^{\alpha\beta}(z)}\,F(x)\,\Big|_{g_{\mu\nu}=0} &= 2\,\frac{d-3}{d-2}\,\mathcal{A}^F_{\sigma\rho\alpha\beta}(x-y,x-z)\,,\\ \frac{\delta}{\delta g^{\sigma\rho}(y)}\,\frac{\delta}{\delta g^{\alpha\beta}(z)}\,G(x)\,\Big|_{g_{\mu\nu}=0} &= \mathcal{A}^G_{\sigma\rho\alpha\beta}(x-y,x-z)\,.\end{aligned} \tag{8.27}$$

The d dependent coefficients in the definition of $\mathcal{A}^F_{\sigma\rho\alpha\beta}$ are not essential but are chosen for later convenience. The particular form of $\mathcal{A}^F_{\sigma\rho,\alpha\beta}(s,t)$ is not required here but it is clearly symmetric $\mathcal{A}^F_{\sigma\rho,\alpha\beta}(s,t) = \mathcal{A}^F_{\alpha\beta,\sigma\rho}(t,s)$ as is $\mathcal{A}^G_{\sigma\rho,\alpha\beta}(s,t)$. By explicit calculation, with $\Delta^T_{\sigma\rho\alpha\beta}$ defined in (8.7),

$$\begin{aligned}\int d^dx\,\mathcal{A}^F_{\sigma\rho,\alpha\beta}(x-y,x-z) &= \Delta^T_{\sigma\rho\alpha\beta}\delta^d(y-z)\,,\\ \mathcal{A}^G_{\sigma\rho,\alpha\beta}(x-y,x-z) &= -24\big\{\mathcal{E}^A_{\sigma\alpha\gamma\kappa,\rho\beta\delta\lambda}\partial_\kappa\partial_\lambda\big(\partial_\gamma\delta^d(x-y)\partial_\delta\delta^d(x-z)\big) + \sigma\leftrightarrow\rho\big\}\,,\end{aligned} \tag{8.28}$$

where $\mathcal{E}^A_{\alpha\beta\gamma\delta,\mu\nu\sigma\rho}$ is the projector for totally antisymmetric four index tensors, for any $T_{\alpha\beta\gamma\delta}$ then $T_{[\alpha\beta\gamma\delta]} = \mathcal{E}^A_{\alpha\beta\gamma\delta,\mu\nu\sigma\rho}T_{\mu\nu\sigma\rho}$ (if $d=4$ then we may write $\mathcal{E}^A_{\alpha\beta\gamma\delta,\mu\nu\sigma\rho} = \frac{1}{24}\epsilon_{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\sigma\rho}$).

The anomalous additional terms in (8.26) arise due to extra singularities present in the three point function when $d=4$. As in the $\langle TVV\rangle$ case it is convenient to use dimensional regularisation and subtract the singular poles in $\varepsilon = d-4$. The possible counterterms for the $\langle TTT\rangle$ three point function may be formed from

$$\begin{aligned}\frac{\delta}{\delta g^{\mu\nu}(x)}\,\frac{\delta}{\delta g^{\sigma\rho}(y)}\,\frac{\delta}{\delta g^{\alpha\beta}(z)}\int d^dx\sqrt{g}\,F\,\Big|_{g_{\mu\nu}=0} &= \frac{d-3}{d-2}\,D^F_{\mu\nu\sigma\rho\alpha\beta}(x,y,z)\,,\\ \frac{\delta}{\delta g^{\mu\nu}(x)}\,\frac{\delta}{\delta g^{\sigma\rho}(y)}\,\frac{\delta}{\delta g^{\alpha\beta}(z)}\int d^dx\sqrt{g}\,G\,\Big|_{g_{\mu\nu}=0} &= \tfrac{1}{2}\,D^G_{\mu\nu\sigma\rho\alpha\beta}(x,y,z)\,,\end{aligned} \tag{8.29}$$

where explicit forms for $D^F_{\mu\nu\sigma\rho\alpha\beta}$ and $D^G_{\mu\nu\sigma\rho\alpha\beta}$ are not essential at this stage, although we may obtain

$$\begin{aligned} & D^G_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) \\ & \quad = -60\big\{\mathcal{E}^A_{\mu\sigma\alpha\gamma\kappa,\nu\rho\beta\delta\lambda}\,\partial_\gamma\partial_\delta\delta^d(x-y)\,\partial_\kappa\partial_\lambda\delta^d(x-z) + \sigma\leftrightarrow\rho, \alpha\leftrightarrow\beta\big\}\,, \end{aligned} \tag{8.30}$$

where $\mathcal{E}^A_{\mu\sigma\alpha\gamma\kappa,\nu\rho\beta\delta\lambda}$ corresponds to antisymmetrisation of five index tensors.

D^F and D^G are clearly symmetric local functions of thee points, from their definitions in (8.29), and satisfy important identities. From Weyl rescaling and invariance under diffeomorphisms

$$\frac{2}{\sqrt{g}}\,g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int d^dx\sqrt{g}\,F = \varepsilon F\,, \quad \nabla^\mu\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^dx\sqrt{g}\,F = 0\,, \tag{8.31}$$

we may derive, using (8.28),

$$\begin{aligned} & D^F_{\mu\mu\sigma\rho\alpha\beta}(x,y,z) \\ & \quad = \varepsilon\,\mathcal{A}^F_{\sigma\rho,\alpha\beta}(x-y,x-z) - 2\big(\delta^d(x-y)+\delta^d(x-z)\big)\Delta^T_{\sigma\rho\alpha\beta}\delta^d(y-z)\,, \\ & \partial_\mu D^F_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) \\ & \quad = \,-\,\partial_\nu\delta^d(x-y)\,\Delta^T_{\sigma\rho\alpha\beta}\delta^d(x-z) - \big\{\partial_\sigma\big(\delta^d(x-y)\,\Delta^T_{\rho\nu\alpha\beta}\delta^d(y-z)\big) + \sigma\leftrightarrow\rho\big\} \\ & \qquad -\,\partial_\nu\delta^d(x-z)\,\Delta^T_{\alpha\beta\sigma\rho}\delta^d(x-y) - \big\{\partial_\alpha\big(\delta^d(x-y)\,\Delta^T_{\beta\nu\sigma\rho}\delta^d(y-z)\big) + \alpha\leftrightarrow\beta\big\}\,, \end{aligned} \tag{8.32}$$

which imply also $\mathcal{A}^F_{\sigma\sigma,\alpha\beta}(s,t)=0$, $\partial^s_\sigma\mathcal{A}^F_{\sigma\rho,\alpha\beta}(s,t)=0$. Similarly, or from the explicit form (8.30) using $5\mathcal{E}^A_{\mu\sigma\alpha\gamma\kappa,\mu\rho\beta\delta\lambda} = \varepsilon\,\mathcal{E}^A_{\sigma\alpha\gamma\kappa,\rho\beta\delta\lambda}$,

$$D^G_{\mu\mu\sigma\rho\alpha\beta}(x,y,z) = \varepsilon\,\mathcal{A}^G_{\sigma\rho,\alpha\beta}(x-y,x-z)\,, \quad \partial_\mu D^G_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) = 0\,. \tag{8.33}$$

As is well known for $d=4$ $\int d^4x\sqrt{g}\,G$ is a topological invariant, proportional to the Euler number. In consequence $D^G_{\mu\nu\sigma\rho\alpha\beta}$ vanishes when $d=4$. However this involves the vanishing of antisymmetric five index tensors which is valid strictly only when $d=4$ and so $D^G_{\mu\nu\sigma\rho\alpha\beta}$ is a legitimate and in fact necessary counterterm in the context of dimensional regularisation when relations dependent on specific integer dimensions should not imposed in a consistent treatment*.

Using the previous results we may now write a regularised form for the energy mo-

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* Note that the first variation of $\int d^dx\sqrt{g}\,G$ defines a tensor $H_{\mu\nu} = -15\,R^{\alpha\beta}{}_{[\alpha\beta}R^{\gamma\delta}{}_{\gamma\delta}\,g_{\mu]\nu} = 2R^{\alpha\beta\gamma}{}_\mu R_{\alpha\beta\gamma\nu} - 8R^{\alpha\beta}R_{\alpha\mu\beta\nu} + 2RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}G$ which is a direct analogue of the usual Einstein tensor $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$. $H_{\mu\nu}=H_{\nu\mu}$ is zero for $d=4$ and satisfies $g^{\mu\nu}H_{\mu\nu}=\frac{1}{2}\varepsilon G$, $\nabla^\mu H_{\mu\nu}=0$.

mentum tensor three point function, analogous to (8.21), as

$$
\begin{aligned}
&\langle T_{\mu\nu}(x)\, T_{\sigma\rho}(y)\, T_{\alpha\beta}(z)\rangle \\
&\quad = \mathcal{I}_{\sigma\rho,\sigma'\rho'}(x-y)\,\mathcal{I}_{\alpha\beta\alpha'\beta'}(x-z)\,\frac{t_{\sigma'\rho'\alpha'\beta'\mu\nu}(X)}{(x-y)^d(x-z)^d(y-z)^d} \\
&\quad - \big(A_{\mu\nu\sigma\rho\gamma\delta}(x-y) - \hat{A}_{\mu\nu\sigma\rho\gamma\delta}(x-y)\big)\,\tfrac{1}{2}C_T\left(\frac{I_{\gamma\delta\alpha\beta}(y-z)}{(y-z)^{2d}} + \frac{I_{\gamma\delta\alpha\beta}(x-z)}{(x-z)^{2d}}\right) \\
&\quad - \big(A_{\mu\nu\alpha\beta\gamma\delta}(x-z) - \hat{A}_{\mu\nu\alpha\beta\gamma\delta}(x-z)\big)\,\tfrac{1}{2}C_T\left(\frac{I_{\gamma\delta\sigma\rho}(z-y)}{(z-y)^{2d}} + \frac{I_{\gamma\delta\sigma\rho}(x-y)}{(x-y)^{2d}}\right) \\
&\quad - \big(A_{\sigma\rho\alpha\beta\gamma\delta}(y-z) - \hat{A}_{\sigma\rho\alpha\beta\gamma\delta}(y-z)\big)\,\tfrac{1}{2}C_T\left(\frac{I_{\gamma\delta\mu\nu}(z-x)}{(z-x)^{2d}} + \frac{I_{\gamma\delta\mu\nu}(y-x)}{(y-x)^{2d}}\right) \\
&\quad - \frac{\mu^{-\varepsilon}}{\varepsilon}\,4\big(\beta_a D^F_{\mu\nu\sigma\rho\alpha\beta}(x,y,z) + \beta_b D^G_{\mu\nu\sigma\rho\alpha\beta}(x,y,z)\big)\,.
\end{aligned}
\tag{8.34}
$$

By virtue of (8.32) and (8.33) this is in accord with the conservation Ward identity in (6.33) and the modified trace identity (8.26) in the limit $d \to 4$ if

$$
\langle T_{\mu\nu}(x)\, T_{\sigma\rho}(0)\rangle = C_T\,\frac{\mathcal{I}_{\mu\nu,\sigma\rho}(x)}{x^{2d}} + \frac{\mu^{-\varepsilon}}{\varepsilon}\,4\beta_a\,\Delta^T_{\mu\nu\sigma\rho}\delta^d(x)\,. \tag{8.35}
$$

This has the appropriate form for the dimensionally regularised two point function for the energy momentum tensor, based on (8.8,9), so long as the same relation obtained earlier in (8.12) between β_a and C_T holds. The regularised expression (8.34) leads to a renormalisation group equation for the three point function

$$
\mu\frac{\partial}{\partial\mu}\langle T_{\mu\nu}(x)\, T_{\sigma\rho}(y)\, T_{\alpha\beta}(z)\rangle = 4\beta^a\, D^F_{\mu\nu\sigma\rho\alpha\beta}(x,y,z)\,, \tag{8.36}
$$

in the limit $d \to 4$ when the apparent $D^G_{\mu\nu\sigma\rho\alpha\beta}$ contribution disappears. Clearly β_a arises from the scale dependence introduced in the regularisation of the two and three point functions of the energy momentum tensor even in the conformal limit whereas β_b has a rather different significance [30]. Using dimensional regularisation the coefficients β_a, β_b can both in principle be determined by analysing the singular poles in ε of the three point function according to an appropriate version of (8.13). Assuming the general counterterm is just a linear combination of $D^F_{\mu\nu\sigma\rho\alpha\beta}$ and $D^G_{\mu\nu\sigma\rho\alpha\beta}$, as exhibited in (8.34), then β_a and β_b may be found as expressions linear in the three parameters a, b, c which are discussed in sections 3 and 6 and which specify the general three point function of the energy momentum tensor. Due to its complexity we have not undertaken this calculation here. However the result for β_a may alternatively be found given (8.12) and (6.42) when $d = 4$. Using this and also the results for free fields in section 5 with (8.3) gives

$$
\beta_a = -\frac{\pi^4}{64\times 30}(14a - 2b - 5c)\,, \quad \beta_b = \frac{\pi^4}{64\times 90}(9a - 2b - 10c)\,. \tag{8.37}
$$

9 Derivative Relations

In sections 2 and 3 we have endeavoured to derive expressions for conformally invariant three point functions involving conserved vector currents and the energy momentum tensor. In obtaining these results the conservation equations such as in (2.22) had to be imposed as additional constraints. Here we show that in some cases it is possible to obtain alternative representations in which the conservation equations are automatic.

We consider first a general three point function involving a vector current which from (2.25), with $d-1+\eta_-+q=0$, has the form

$$\langle V_\mu(x_1)\, \mathcal{O}_2^{i_2}(x_2)\, \mathcal{O}_3^{i_3}(x_3)\rangle = \frac{1}{x_{13}^{\,2(d-1)}\, x_{23}^{\,2\eta_2}}\, I_{\mu\nu}(x_{13}) D_2{}^{i_2}{}_{j_2}(I(x_{23}))\, t_\nu{}^{j_2 i_3}(X_{12})\,, \tag{9.1}$$

where from (2.27)

$$\partial_\mu t_\mu{}^{i_2 i_3}(X) = 0\,. \tag{9.2}$$

If this equation is trivially satisfied by writing

$$t_\nu{}^{i_2 i_3}(X) = \partial_\rho t_{\nu\rho}{}^{i_2 i_3}(X)\,, \quad t_{\nu\rho}{}^{i_2 i_3}(X) = -t_{\rho\nu}{}^{i_2 i_3}(X)\,, \tag{9.3}$$

then an alternative representation to (9.1) with manifest current conservation is feasible. Since

$$\frac{\partial}{\partial X_{12\,\rho}} = x_{13}^2 I_{\sigma\rho}(x_{13})\, \frac{\partial}{\partial x_{1\,\sigma}}\,, \tag{9.4}$$

and with results such as

$$\partial_\mu\Big(\frac{1}{x^{2(d-2)}}\, \mathcal{I}^A_{\mu\nu,\sigma\rho}(x)\Big) = 0 \quad \mathcal{I}^A_{\mu\nu,\sigma\rho}(x) = \tfrac{1}{2}\big(I_{\mu\sigma}(x)I_{\nu\rho}(x) - I_{\mu\rho}(x)I_{\nu\sigma}(x)\big)\,, \tag{9.5}$$

where $\mathcal{I}^A_{\mu\nu,\sigma\rho}$ corresponds to an inversion in the representation of $O(d)$ formed by antisymmetric tensors, then we may write

$$\begin{aligned}&\langle V_\mu(x_1)\, \mathcal{O}_2^{i_2}(x_2)\, \mathcal{O}_3^{i_3}(x_3)\rangle \\ &\quad = \frac{\partial}{\partial x_{1\,\nu}}\bigg(\frac{1}{x_{13}^{\,2(d-2)}\, x_{23}^{\,2\eta_2}}\, \mathcal{I}^A_{\mu\nu,\sigma\rho}(x_{13}) D_2{}^{i_2}{}_{j_2}(I(x_{23}))\, t_{\sigma\rho}{}^{j_2 i_3}(X_{12})\bigg)\,,\end{aligned} \tag{9.6}$$

with $t_{\sigma\rho}{}^{i_2 i_3}(\lambda X) = \lambda^{-(d-2+\eta_-)} t_{\sigma\rho}{}^{i_2 i_3}(X)$.

For a comparable treatment for the energy momentum tensor we first define $\mathcal{H}^C$ as the space of tensors with the symmetries of the Weyl tensor as defined in (8.2). Thus

$$C_{\mu\sigma\rho\nu} \in \mathcal{H}^C \;\Rightarrow\; C_{\mu\sigma\rho\nu} = C_{[\mu\sigma][\rho\nu]}\,, \quad C_{\mu[\sigma\rho\nu]} = 0\,, \quad C_{\mu\sigma\rho\mu} = 0\,, \tag{9.7}$$

which implies $C_{\mu\sigma\rho\nu} = C_{\rho\nu\mu\sigma}$, so that $\mathcal{H}^C$ has dimension $\frac{1}{12}d(d+1)(d+2)(d-3)$. It is easy to see that for any $C_{\mu\sigma\rho\nu}(x) \in \mathcal{H}^C$ implies $\partial_\sigma\partial_\rho C_{\mu\sigma\rho\nu}(x) = h_{\mu\nu}(x)$ is a conserved symmetric traceless tensor. Hence starting from (2.28) written as

$$\langle T_{\mu\nu}(x_1)\, \mathcal{O}_2^{i_2}(x_2)\, \mathcal{O}_3^{i_3}(x_3)\rangle = \frac{1}{x_{13}^{\,2d}\, x_{23}^{\,2\eta_2}}\, \mathcal{I}_{\mu\nu,\sigma\rho}(x_{13}) D_2{}^{i_2}{}_{j_2}(I(x_{23}))\, t_{\sigma\rho}{}^{j_2 i_3}(X_{12})\,, \tag{9.8}$$

where (2.30) now requires $\partial_\mu t_{\mu\nu}{}^{i_2 i_3}(X)=0$, we therefore assume that it is possible to take

$$t_{\mu\nu}{}^{i_2 i_3}(X)=\partial_\alpha\partial_\beta t_{\mu\alpha\beta\nu}{}^{i_2 i_3}(X)\,,\quad t_{\mu\alpha\beta\nu}{}^{i_2 i_3}(X)\in\mathcal{H}^C\,. \tag{9.9}$$

Using (9.4) and results akin to (9.5) and in particular the symmetry properties of tensors in $\mathcal{H}^C$, as given in (9.7), we may now show that (9.8) is equivalent to

$$\begin{aligned}&\langle T_{\mu\nu}(x_1)\,\mathcal{O}_2^{i_2}(x_2)\,\mathcal{O}_3^{i_3}(x_3)\rangle\\ &\quad=\frac{\partial}{\partial x_{1\,\sigma}}\,\frac{\partial}{\partial x_{1\,\rho}}\bigg(\frac{1}{x_{13}^{2(d-2)}\,x_{23}^{2\eta_2}}\,\mathcal{I}^C_{\mu\sigma\rho\nu,\mu'\sigma'\rho'\nu'}(x_{13})D_2{}^{i_2}{}_{j_2}(I(x_{23}))\,t_{\mu'\sigma'\rho'\nu'}{}^{j_2 i_3}(X_{12})\bigg)\,,\end{aligned} \tag{9.10}$$

for $\mathcal{I}^C\in\mathcal{H}^C\times\mathcal{H}^C$ representing inversions on tensor fields belonging to the space $\mathcal{H}^C$, $\mathcal{I}^C_{\mu\sigma\rho\nu,\alpha\beta\gamma\delta}\mathcal{I}^C_{\alpha\beta\gamma\delta,\mu'\sigma'\rho'\nu'}=\mathcal{E}^C_{\mu\sigma\rho\nu,\mu'\sigma'\rho'\nu'}$ where $\mathcal{E}^C\in\mathcal{H}^C\times\mathcal{H}^C$ acts as a projection operator onto $\mathcal{H}^C$. $\mathcal{I}^C$ and $\mathcal{E}^C$ are symmetric and $\mathcal{I}^C$ may be written as $\mathcal{I}^C_{\mu\sigma\rho\nu,\alpha\beta\gamma\delta}=I_{\mu\mu'}I_{\sigma\sigma'}I_{\rho\rho'}I_{\nu\nu'}\mathcal{E}^C_{\mu'\sigma'\rho'\nu',\alpha\beta\gamma\delta}$. If $P_{\mu\sigma\rho\nu}=P_{[\mu\sigma][\rho\nu]}$, $P_{\sigma\rho}=\frac{1}{2}(P_{\mu\sigma\mu\rho}+P_{\mu\rho\mu\sigma})$ then $\mathcal{E}^C$ is defined by

$$\begin{aligned}\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\beta\gamma\delta}P_{\alpha\beta\gamma\delta}={}&\tfrac{1}{3}\big(P_{\mu\sigma\rho\nu}+P_{\rho\nu\mu\sigma}+P_{\mu[\nu\rho]\sigma}-P_{\sigma[\nu\rho]\mu}\big)\\ &-\frac{2}{d-2}\big(\delta_{\rho[\mu}P_{\sigma]\nu}-\delta_{\nu[\mu}P_{\sigma]\rho}\big)+\frac{2}{(d-2)(d-1)}\,\delta_{\rho[\mu}\delta_{\sigma]\nu}\,P_{\lambda\lambda}\,.\end{aligned} \tag{9.11}$$

The significance of such results as (9.6) and (9.10) is that in an expansion of the effective action $W(g,A,J)$ then the contributions corresponding to (9.6) involve just the abelian field strength $\partial_\mu A_\nu-\partial_\nu A_\mu$ and for (9.10) the Weyl tensor $C_{\alpha\beta\gamma\delta}$. The latter follows since in an expansion around flat space $g_{\mu\nu}=\delta_{\mu\nu}+h_{\mu\nu}$ then $C_{\mu\sigma\rho\nu}=(\Delta h)_{\mu\sigma\rho\nu}+\mathrm{O}(h^2)$ where the differential operator Δ is defined by

$$\int d^d x\,h_{\mu\nu}\,\partial_\sigma\partial_\rho C_{\mu\sigma\rho\nu}=\tfrac{1}{2}\int d^d x\,(\Delta h)_{\mu\sigma\rho\nu}\,C_{\mu\sigma\rho\nu}\quad\text{for}\quad C_{\mu\sigma\rho\nu}(x)\in\mathcal{H}^C\,. \tag{9.12}$$

Thus $\Delta_{\mu\sigma\rho\nu,\alpha\beta}=2\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}\partial_\gamma\partial_\delta$ or more explicitly from (9.11)

$$\begin{aligned}(\Delta h)_{\mu\sigma\rho\nu}={}&-\partial_\rho\partial_{[\mu}h_{\sigma]\nu}+\partial_\nu\partial_{[\mu}h_{\sigma]\rho}\\ &-\frac{1}{d-2}\Big(\delta_{\rho[\mu}(\mathcal{L}h)_{\sigma]\nu}-\delta_{\nu[\mu}(\mathcal{L}h)_{\sigma]\rho}\Big)+\frac{1}{(d-1)(d-2)}\delta_{\rho[\mu}\delta_{\sigma]\nu}(\mathcal{L}h)_{\lambda\lambda}\,,\\ (\mathcal{L}h)_{\sigma\rho}={}&-\partial^2 h_{\sigma\rho}+\partial_\sigma\partial_\lambda h_{\lambda\rho}+\partial_\rho\partial_\lambda h_{\lambda\sigma}-\partial_\sigma\partial_\rho h_{\lambda\lambda}\,.\end{aligned} \tag{9.13}$$

With these definitions it should be noted that

$$\begin{aligned}\Delta^T_{\mu\nu\sigma\rho}&=2\,\frac{d-2}{d-3}\,\partial_\alpha\partial_\beta\Delta_{\mu\alpha\beta\nu,\sigma\rho}\,,\\ \mathcal{A}^F_{\sigma\rho\alpha\beta}(x-y,x-z)&=4\,\frac{d-2}{d-3}\,\mathcal{E}^C_{\sigma\sigma'\rho'\rho,\alpha\alpha'\beta'\beta}\partial_{\sigma'}\partial_{\rho'}\delta^d(x-y)\partial_{\alpha'}\partial_{\beta'}\delta^d(x-z)\,,\end{aligned} \tag{9.14}$$

with Δ^T defined by (8.7) and $\mathcal{A}^F$ by (8.27).

As an illustration of the applicability of these results we consider the $\langle TT\mathcal{O}\rangle$ three point function, for $\mathcal{O}$ a scalar field of dimension η. The conformal invariant expression was constructed in (3.3,4) and (3.6). It is easy to see that we may write

$$\frac{1}{X^{2d-\eta}}\, t_{\alpha\beta\gamma\delta}(X) = \frac{d-3}{d-2}\, \Delta^T_{\alpha\beta\gamma\delta} \frac{K}{X^{2d-\eta-4}}\,, \tag{9.15}$$

where the coefficients a, b, c are determined in terms of K and satisfy (3.6). Hence we find

$$\begin{aligned}
\langle T_{\mu\nu}(x_1)\, T_{\sigma\rho}(x_2)\, \mathcal{O}(x_3)\rangle &= \frac{d-3}{d-2}\, \frac{1}{x_{13}^{\,2d}\, x_{23}^{\,2d}}\, \mathcal{I}_{\mu\nu,\alpha\beta}(x_{13})\mathcal{I}_{\sigma\rho,\gamma\delta}(x_{23})\, \Delta^T_{\alpha\beta\gamma\delta} \frac{K}{X_{12}^{\,2d-\eta-4}} \\
&= \partial_{1\mu'}\partial_{1\nu'} \bigg(\frac{1}{x_{13}^{\,2(d-2)}\, x_{23}^{\,2d}}\, \mathcal{I}^C_{\mu\mu'\nu'\nu,\alpha\beta\gamma\delta}(x_{13})\mathcal{I}_{\sigma\rho,\alpha\delta}(x_{23})\, \partial_{X_{12}\beta}\partial_{X_{12}\gamma} \frac{4K}{X_{12}^{\,2d-\eta-4}} \bigg) \\
&= \partial_{1\mu'}\partial_{1\nu'}\partial_{2\sigma'}\partial_{2\rho'} \bigg(\frac{1}{x_{13}^{\,\eta}\, x_{23}^{\,\eta}}\, \frac{4K}{x_{12}^{\,2d-\eta-4}}\, \mathcal{I}^C_{\mu\mu'\nu'\nu,\alpha\beta\gamma\delta}(x_{13})\mathcal{I}^C_{\sigma\sigma'\rho'\rho,\alpha\beta\gamma\delta}(x_{23}) \bigg)\,.
\end{aligned} \tag{9.16}$$

The conservation equation for the energy momentum tensor is automatically satisfied as is appropriate since there are no Ward identities in this case. It should be noted that for $\eta = 0$ this three point function does not depend on x_3 and is proportional to $\langle T_{\mu\nu}(x_1)\, T_{\sigma\rho}(x_2)\rangle$.

10 Conclusion

The results of this paper demonstrate that for general dimensions the form of operator product expansions involving the energy momentum tensor, or the conformal invariant expressions for three point functions, are rather more complicated than in two dimensions where the essential formulae take the very simple form given by (1.1,2). There are still many unanswered questions, particularly in relation to positivity constraints. It is trivial that in unitary theories the overall coefficient of the two point function C_T, and also C_V, is positive which given (6.42) and (8.12) provides one condition on the three parameters a, b, c in the general three point function of the energy momentum tensor and also requires $\beta_a < 0$ for the coefficient of the F term in the energy momentum tensor trace on a curved space background. As remarked by Cappelli, Friedan and Latorre [19] there is no presently known condition on β_b, despite the results for free fields (8.3). It would be very nice to derive further positivity conditions on a, b, c which might imply positivity of β_b. In general there are no such conditions on three point functions but this is not true when the energy momentum tensor is involved since the spectrum of the Hamiltonian H is required to be positive. To proceed it may be necessary to consider the operator product expansion in four point functions [30]. In this context it should be pointed out that it is not entirely clear if the operator product expansion in dimensions $d > 2$ is associative although this is probably essential for an algebraic operator formulation such as is used in two dimensions. This question should presumably be answerable at least in a perturbative context.

As an illustration of the potential use of some of our results we recapitulate the well known derivation of the c-theorem [2,3,19]. In two dimensions for a general massless field theory, or at distances small compared with any explicit mass scale, the two point function for the energy momentum tensor takes the regularised form

$$\langle T_{\mu\nu}(x)\, T_{\sigma\rho}(0)\rangle = S_{\mu\nu}S_{\sigma\rho}\Omega(t)\,, \qquad t = \tfrac{1}{2}\ln \mu x^2\,, \tag{10.1}$$

where μ is a renormalisation mass scale (note that in two dimensions from (8.7) we may write $S_{\mu\nu} = -\tilde{\partial}_\mu\tilde{\partial}_\nu$, $\tilde{\partial}_\mu = \epsilon_{\mu\alpha}\partial_\alpha$ and hence $\Delta^T_{\mu\nu\sigma\rho} = 0$). In terms of (10.1) we follow Zamoldchikov [3] and define

$$\begin{aligned}
F(t) &= z^4\langle T_{zz}(x)\, T_{zz}(0)\rangle = \tfrac{1}{16}\Omega''''(t) - \tfrac{3}{4}\Omega'''(t) + \tfrac{11}{4}\Omega''(t) - 3\Omega'(t)\,,\\
G(t) &= z^2x^2\langle T_{zz}(x)\, T_{\mu\mu}(0)\rangle = -\tfrac{1}{4}\Omega''''(t) + \tfrac{3}{2}\Omega'''(t) - 2\Omega''(t)\,,\\
H(t) &= (x^2)^2\langle T_{\mu\mu}(x)\, T_{\sigma\sigma}(0)\rangle = \Omega''''(t) - 4\Omega'''(t) + \Omega''(t)\,.
\end{aligned} \tag{10.2}$$

It is then easy to see that

$$C'(t) = -\tfrac{3}{2}H(t) < 0 \ \text{ for } \ C(t) = 2F(t) - G(t) - \tfrac{3}{8}H(t)\,, \tag{10.3}$$

where if $T_{\mu\mu} = \beta^i\mathcal{O}_i$ then $H = G_{ij}\beta^i\beta^j$ with $G_{ij}(t) = (x^2)^2\langle \mathcal{O}_i(x)\,\mathcal{O}_j(0)\rangle$ the Zamolodchikov metric on the space of couplings. The flow of $C(t)$ is clearly monotonic decreasing and is only stationary at the critical points where $\beta^i = 0$ where, from (1.2), $C \to c$ the Virasoro central charge of the corresponding conformal theory (as compared with our previous normalisation $T_{\mu\nu} \to -2\pi T_{\mu\nu}$ which is conventional for two dimensional conformal field theories). An essential part of Zamolodchikov's argument was that the quantity $C(t)$ is well defined in the conformal limit and hence the change ΔC in flowing between two conformal field theories is unambiguous, independent of the path. It is possible to imagine then that it may be feasible to define quantities in terms of the three point function, analogous to (10.2), which have a well defined conformal limit and also nice properties under renormalisation flow. It would probably be desirable to use the collinear configuration since then, as described in section 4, the dependence on the spatial variables x, y, z factorises into a unique form in the conformal limit. Presumably considering three point functions of the energy momentum tensor in two dimensions gives nothing new, since the scale is determine by the two point function, but for $d > 2$ there are three independent coefficients in general. For $d = 4$ one linear combination gives β_b, as shown in (8.37), which was suggested by Cardy [21] as a possible candidate for a generalisation of the c-theorem to four dimensions.

In two dimensions the Virasoro central charge may be related to physically measureable effects such as the universal finite size corrections to the Casimir energy for a strip of width L [32]. The derivation of this result is not valid for $d > 2$, due to the restricted nature of the conformal group in this case, but can be used to provide an alternative generalisation of c away from $d = 2$ which differs from that provided by the coefficient of the conformally invariant two point function for the energy momentum tensor. It would be nice if the three independent coefficients in the energy momentum tensor three point function could perhaps be related to such physically observable energies. Independent of such conjectures it should be feasible to obtain expressions for the effective action W on curved space backgrounds for conformally invariant theories where at least to third order in the curvature our results imply there are just three linearly independent forms whose coefficients are related to the three parameters in the general conformally invariant three point for the energy momentum tensor on flat space. Assuming conformal invariance is therefore quite restrictive and should lead to hopefully simpler results than that found recently for general theories [33]. The corresponding two dimensional result takes the remarkably elegant form

$$W(g) = -\frac{c}{96\pi}\int d^2x\sqrt{g}\,R\frac{1}{\nabla^2}R\,, \tag{10.4}$$

which was found by integrating the two dimensional trace anomaly [34].

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Appendix

Here we collect various results needed in the derivation of the conformal invariant form of the three point function for the energy momentum tensor. In order to impose (3.16b) we use, with h^1, h^2, h^3, h^4, h^5 defined in (3.2), (3.4) and (3.18),

$$\begin{aligned}
\mathcal{I}_{\mu\nu,\mu'\nu'}h^5_{\mu'\nu'\sigma\rho\alpha\beta} &= h^5_{\mu\nu\sigma\rho\alpha\beta} - 2h^4_{\alpha\beta\sigma\rho\mu\nu} - 2h^4_{\sigma\rho\alpha\beta\mu\nu} + 8h^2_{\sigma\rho\alpha\beta}h^1_{\mu\nu}\,,\\
\mathcal{I}_{\mu\nu,\mu'\nu'}h^4_{\mu'\nu'\sigma\rho\alpha\beta} &= h^4_{\mu\nu\sigma\rho\alpha\beta} - 4h^2_{\mu\nu\sigma\rho}h^1_{\alpha\beta} - 4h^2_{\mu\nu\alpha\beta}h^1_{\sigma\rho} + 32h^1_{\mu\nu}h^1_{\sigma\rho}h^1_{\alpha\beta}\,,\\
\mathcal{I}_{\sigma\rho,\sigma'\rho'}h^4_{\mu\nu\sigma'\rho'\alpha\beta} &= -\,h^4_{\mu\nu\sigma\rho\alpha\beta} + 4h^2_{\mu\nu\alpha\beta}h^1_{\sigma\rho}\,,\\
\mathcal{I}_{\mu\nu,\mu'\nu'}h^3_{\mu'\nu'\sigma\rho} &= h^3_{\mu\nu\sigma\rho} - 2h^2_{\mu\nu\sigma\rho} + 8h^1_{\mu\nu}h^1_{\sigma\rho}\,,\\
\mathcal{I}_{\mu\nu,\mu'\nu'}h^2_{\mu'\nu'\sigma\rho} &= -\,h^2_{\mu\nu\sigma\rho} + 8h^1_{\mu\nu}h^1_{\sigma\rho}\,,\\
\mathcal{I}_{\mu\nu,\mu'\nu'}h^1_{\mu'\nu'} &= h^1_{\mu\nu}\,.
\end{aligned} \tag{A.1}$$

Note that these results are consistent with $\mathcal{I}^2 = 1$. For the conservation equations we use

$$\begin{aligned}
\Big(\partial_\mu - d\,\frac{X_\mu}{X^2}\Big)h^1_{\mu\nu} &= 0\,,\\
\Big(\partial_\mu - d\,\frac{X_\mu}{X^2}\Big)h^2_{\mu\nu\sigma\rho} &= -\,2d\,\frac{X_\nu}{X^2}h^1_{\sigma\rho} - \frac{4}{d}\,\partial_\nu h^1_{\sigma\rho}\,,\\
\Big(\partial_\mu - d\,\frac{X_\mu}{X^2}\Big)h^3_{\mu\nu\sigma\rho} &= -\,2d\,\frac{X_\nu}{X^2}h^1_{\sigma\rho} - d\,\partial_\nu h^1_{\sigma\rho}\,,\\
\Big(\partial_\mu - d\,\frac{X_\mu}{X^2}\Big)h^4_{\mu\nu\sigma\rho\alpha\beta} &= -\,\frac{d-2}{d}\,\partial_\nu h^2_{\sigma\rho\alpha\beta} + 4\,\frac{X_\nu}{X^2}\big(h^2_{\sigma\rho\alpha\beta} - 2(d+2)\,h^1_{\sigma\rho}h^1_{\alpha\beta}\big)\\
&\quad -\,2(d+2)\big(h^1_{\alpha\beta}\partial_\nu h^1_{\sigma\rho} + h^1_{\sigma\rho}\partial_\nu h^1_{\alpha\beta}\big)\,,\\
\Big(\partial_\mu - d\,\frac{X_\mu}{X^2}\Big)h^4_{\alpha\beta\mu\nu\sigma\rho} &= -\,\frac{2}{d}\,\partial_\nu h^2_{\sigma\rho\alpha\beta} + \frac{X_\nu}{X^2}\big(2\,h^3_{\sigma\rho\alpha\beta} - d\,h^2_{\sigma\rho\alpha\beta}\big)\,,\\
\Big(\partial_\mu - d\,\frac{X_\mu}{X^2}\Big)h^5_{\mu\nu\sigma\rho\alpha\beta} &= -\,d\,\partial_\nu h^2_{\sigma\rho\alpha\beta} + \frac{X_\nu}{X^2}\big(4\,h^3_{\sigma\rho\alpha\beta} - 2d\,h^2_{\sigma\rho\alpha\beta}\big)\,.
\end{aligned} \tag{A.2}$$

In addition we need

$$
\begin{aligned}
h^2_{\mu\nu\sigma\rho}\partial_\mu h^1_{\alpha\beta} &= \frac{X_\nu}{X^2}\big(h^2_{\sigma\rho\alpha\beta} - 4h^1_{\sigma\rho}h^1_{\alpha\beta}\big) - \frac{4}{d}\, h^1_{\sigma\rho}\partial_\nu h^1_{\alpha\beta}\,, \\
h^3_{\mu\nu\sigma\rho}\partial_\mu h^1_{\alpha\beta} + h^3_{\mu\nu\alpha\beta}\partial_\mu h^1_{\sigma\rho} &= \partial_\nu h^2_{\sigma\rho\alpha\beta} + 2\,\frac{X_\nu}{X^2}\big(h^2_{\sigma\rho\alpha\beta} - 4\,h^1_{\sigma\rho}h^1_{\alpha\beta}\big) \\
&\quad - 2\,h^1_{\sigma\rho}\partial_\nu h^1_{\alpha\beta} - 2\,h^1_{\alpha\beta}\partial_\nu h^1_{\sigma\rho}\,.
\end{aligned}
\tag{A.3}
$$

Also with $\tilde{h}_{\mu\nu\sigma\rho}$ as in (3.12) we find

$$
\begin{aligned}
&\Big(\partial_\mu - d\,\frac{X_\mu}{X^2}\Big)\tilde{h}_{\mu\nu\sigma\rho} = 0\,, \\
&\Big(\partial_\sigma - d\,\frac{X_\sigma}{X^2}\Big)\tilde{h}_{\mu\nu\sigma\rho} = 2d\,\frac{X_\rho}{X^2}h^1_{\mu\nu} - 2\,\partial_\rho h^1_{\mu\nu}\,.
\end{aligned}
\tag{A.4}
$$

For applications in section 6 we also need formulae for the divergences of the functions H^i introduced in (6.30). It is convenient to define

$$
\begin{aligned}
D^1_{\sigma\rho\alpha\beta}(s) = D^1_{\alpha\beta\sigma\rho}(s) &= \Big(\partial_\sigma\partial_\rho - \frac{1}{d}\,\delta_{\sigma\rho}\partial^2\Big)\,\frac{1}{s^{d-2}}h^1_{\alpha\beta}(\hat{s})\,, \\
D^2_{\nu\sigma\rho\alpha\beta}(s) &= \Big(\partial_\sigma\partial_\rho - \frac{1}{d}\,\delta_{\sigma\rho}\partial^2\Big)\Big(\delta_{\nu\alpha}\partial_\beta + \delta_{\nu\beta}\partial_\alpha - \frac{2}{d}\,\delta_{\alpha\beta}\partial_\nu\Big)\frac{1}{s^{d-2}}\,, \\
D^3_{\sigma\rho\alpha\beta}(s) &= \Big(\delta_{\sigma\alpha}\partial_\rho\partial_\beta + (\sigma\leftrightarrow\rho, \alpha\leftrightarrow\beta) - \frac{4}{d}\delta_{\sigma\rho}\partial_\alpha\partial_\beta - \frac{4}{d}\delta_{\alpha\beta}\partial_\sigma\partial_\rho + \frac{4}{d^2}\delta_{\sigma\rho}\delta_{\alpha\beta}\partial^2\Big)\frac{1}{s^{d-2}}\,.
\end{aligned}
\tag{A.5}
$$

We may then find

$$
\begin{aligned}
\partial_\mu H^1_{\alpha\beta\mu\nu\sigma\rho}(s) &= \ -\frac{2}{d}\,\partial_\nu D^1_{\sigma\rho\alpha\beta}(s) - \frac{1}{d(d-2)}\,D^2_{\nu\alpha\beta\sigma\rho}(s)\,, \\
\partial_\mu H^2_{\mu\nu\sigma\rho\alpha\beta}(s) &= \ -\frac{2(d-1)}{d(d-2)}\,\partial_\nu D^3_{\sigma\rho\alpha\beta}(s)\,, \\
\partial_\mu H^2_{\alpha\beta\mu\nu\sigma\rho}(s) &= \frac{2}{d}(d-2)\,\partial_\nu D^1_{\sigma\rho\alpha\beta}(s) - \frac{2}{d-2}\,D^2_{\nu\alpha\beta\sigma\rho}(s)\,, \\
\partial_\mu H^3_{\mu\nu\sigma\rho\alpha\beta}(s) &= \ -\frac{1}{d}(d-1)(d-2)\,h^3_{\sigma\rho\alpha\beta}\partial_\nu\, S_d\delta^d(s)\,, \\
\partial_\mu H^3_{\alpha\beta\mu\nu\sigma\rho}(s) &= D^2_{\nu\alpha\beta\sigma\rho}(s)\,, \\
\partial_\mu H^4_{\mu\nu\sigma\rho\alpha\beta}(s) &= 2D^2_{\nu\alpha\beta\sigma\rho}(s) + 2D^2_{\nu\sigma\rho\alpha\beta}(s) - \frac{2}{d}\,\partial_\nu D^3_{\sigma\rho\alpha\beta}(s)\,, \\
\partial_\mu H^4_{\alpha\beta\mu\nu\sigma\rho}(s) &= \frac{1}{d}(d-2)\,\partial_\nu D^3_{\sigma\rho\alpha\beta}(s) \\
&\quad - (d-2)\Big(h^3_{\sigma\rho\nu\alpha}\partial_\beta + h^3_{\sigma\rho\nu\beta}\partial_\alpha - \frac{2}{d}\,\delta_{\alpha\beta}h^3_{\sigma\rho\nu\mu}\partial_\mu\Big)S_d\delta^d(s)\,.
\end{aligned}
\tag{A.6}
$$

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