Calculus II, Section 6.4, #24
Work

A tank is full of water. Find the work required to pump the water out of the spout.[1]

As we have done many times, and as we will continue to do with virtually all applications, we will analyze what happens to one small part of the object, write the Riemann sum for all the small parts, and finally add up all the small parts with integration.

For the tank in Figure 1, we will find an expression for the radius at any depth y in the tank, find the volume of a representative slice with that radius, find the mass of the representative slice, find the force on the slice, and finally form the integral for the work.

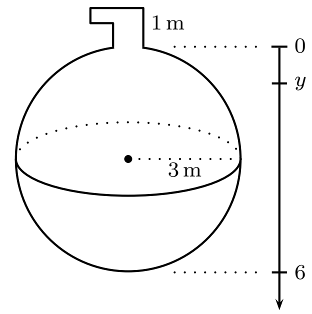


Figure 1: *A tank filled with water.*

In Figure 2, we've drawn a cross-section of the tank to assist in finding the radius of the representative slice. We've placed a radius r at a depth of y m, and we'll use the Pythagorean Theorem to find r in terms of y.

$$\begin{aligned}(3-y)^2 + r^2 &= 3^2\\ 9 - 6y + y^2 + r^2 &= 9\\ r^2 &= 6y - y^2\end{aligned}$$

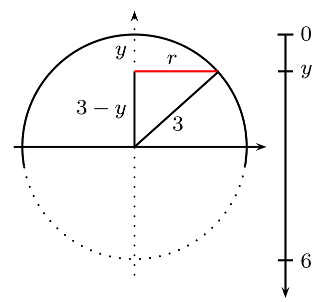


Figure 2: *Cross-section of the tank.*

and since r is a length and therefore nonnegative,

$$r = \sqrt{6y - y^2}$$

In Figure 3, we've sketched a representative slice. As usual, the thickness of the representative slice is Δy. We compute

$$\text{Volume}_{\text{slice}} = \pi\left(\sqrt{6y-y^2}\right)^2 \Delta y$$

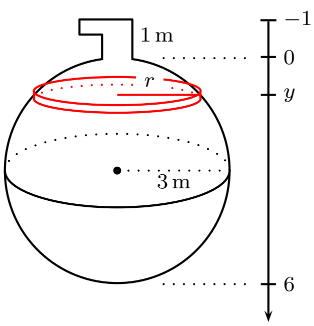


Figure 3: *A representative slice.*

We are working with SI (metric) m-kg-s units. The density of water in the m-kg-s system is $1000\,\text{kg/m}^3$. So

$$\text{Mass}_{\text{slice}} = 1000\pi\left(\sqrt{6y-y^2}\right)^2 \Delta y$$

From Newton's Second Law, $F = ma$, where $a = 9.8\,\text{m/s}^2$ is the acceleration due to gravity, so the force[2] on the slice is given by

$$\text{Force}_{\text{slice}} = 9.8 \cdot 1000\pi\left(\sqrt{6y-y^2}\right)^2 \Delta y$$

As the water is pumped up and out of the tank's spout, each slice moves a distance of $y - (-1) = y + 1$ m, so the work done on each slice is

$$\text{Work}_{\text{slice}} = 9.8 \cdot 1000\pi\,(y+1)\left(\sqrt{6y-y^2}\right)^2 \Delta y$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 459, #24.

[2] In the SI system, mass is separate from the force (as it should be), whereas in the (lame) U.S. customary system of ft-lb-s, the weight *is* a force.

We add up all the slices and take the limit as the number of slices increases without bound. The limit of the Riemann sum is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 9800\pi\,(y+1)\left(6y - y^2\right)\Delta y$$

We create slices from $y = 0$ to $y = 6$ (there are no slices in the spout), so we get

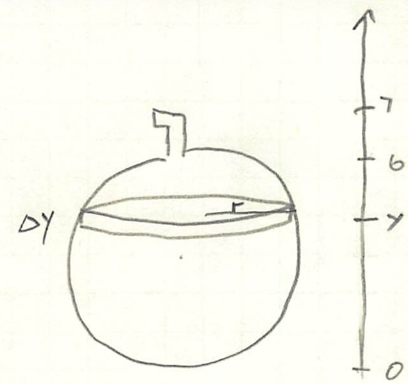
$$\begin{aligned}
W &= \int_{y=0}^{y=6} 9800\pi\,(y+1)\left(6y - y^2\right)\, \mathrm{d}y \\
&= 9800\pi \int_{y=0}^{y=6} 6y^2 - y^3 + 6y - y^2 \,\mathrm{d}y \\
&= 9800\pi \int_{y=0}^{y=6} -y^3 + 5y^2 + 6y \,\mathrm{d}y \\
&= 9800\pi \left[-\frac{y^4}{4} + \frac{5y^3}{3} + 3y^2\right]_{y=0}^{y=6} \\
&= 9800\pi \left[\left(-\frac{6^4}{4} + \frac{5\cdot 6^3}{3} + 3\cdot 6^2\right) - \left(-\frac{0^4}{4} + \frac{5\cdot 0^3}{3} + 3\cdot 0^2\right)\right] \\
&= 9800\pi \left[-\frac{1296}{4} + \frac{1080}{3} + 108 + 0 - 0 - 0\right] \\
&= 9800\pi \cdot 144 \\
&\approx 4{,}433{,}416 \\
\text{or } &\approx 4.4 \times 10^6
\end{aligned}$$

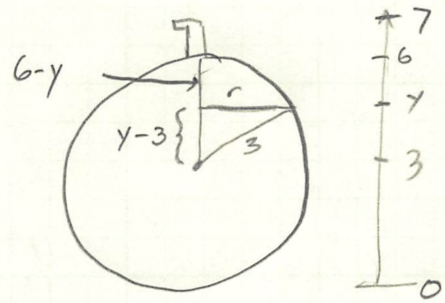
Thus, the amount of work to pump all the water out of this tank is about 4.4×10^6 Newton $\cdot$ m or about 4.4×10^6 J.

In our solution to this problem, we placed $y = 0$ at the top of the spherical portion of the tank, and $y = 6$ at the bottom. Students often ask about the “best” placement for the coordinates, and the honest answer is that it just doesn’t matter. As long as we are consistent with the algebraic expressions that arise from the geometry, the end result will be the same. On the following pages, we’ll set up the problem three additional ways, and yet the value for the work will be the same each time.[3]

[3] Yes, my fingers were crossed as I typed that sentence. It did not make the typing any easier.

Version II





From the Pythag. Thm.,

$$(y-3)^2 + r^2 = 3^2$$

$$y^2 - 6y + 9 + r^2 = 9$$

$$r^2 = 6y - y^2$$

Since r is a length, $r \geq 0$, so

$$r = \sqrt{6y - y^2}$$

$$V_{slice} = \pi\left(\sqrt{6y-y^2}\right)^2 \Delta y$$

$$M_{slice} = 1000\,\pi\left(\sqrt{6y-y^2}\right)^2 \Delta y$$

$$F_{slice} = 9.8 \cdot 1000\,\pi\left(\sqrt{6y-y^2}\right)^2 \Delta y$$

Each slice travels a distance of $7-y$, so

$$W_{slice} = 9800\pi\,(6y-y^2)(7-y)\,\Delta y$$

We create slices from $y=0$ to $y=6$, so the work is

$$W = \int_{y=0}^{y=6} 9800\,\pi\,(6y-y^2)(7-y)\,dy$$

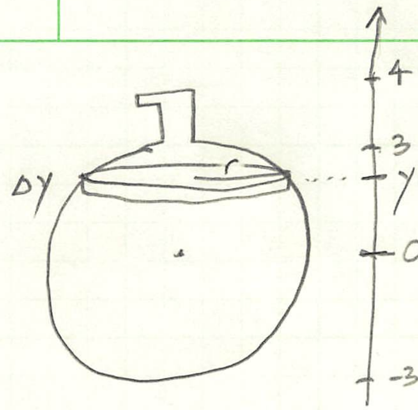
From my TI-83,

fnInt(9800 * π * (6y − y^2)(7−y), y, 0, 6)

4433415.553

So $W \approx 4.4 \times 10^6$ J.

Version III



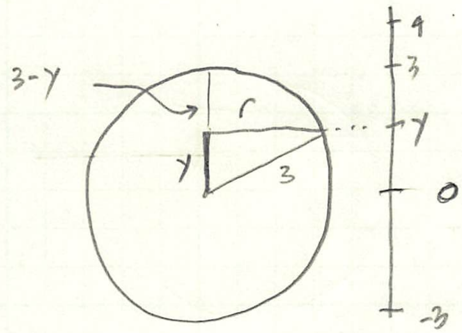
From the Pythg. Thm,

$$y^2 + r^2 = 3^2$$

$$r^2 = 9 - y^2$$

Since r is a length, $r \geq 0$, so

$$r = \sqrt{9 - y^2}$$



$$V_{slice} = \pi \left(\sqrt{9-y^2}\right)^2 \Delta y$$

$$M_{slice} = 1000\pi \left(\sqrt{9-y^2}\right)^2 \Delta y$$

$$F_{slice} = 9.8 \cdot 1000\pi \left(\sqrt{9-y^2}\right)^2 \Delta y$$

Each slice moves a distance of $4 - y$, so

$$W_{slice} = 9800\pi (9-y^2)(4-y)\Delta y$$

We create slices from $y = -3$ to $y = 3$, so the work is

$$W = \int_{y=-3}^{y=3} 9800\pi (9-y^2)(4-y)\, dy$$

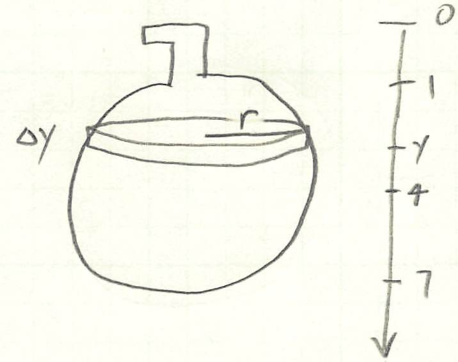
From my TI-83,

fnInt(9800 * π * $(9-y^2)(4-y)$, y, −3, 3)

4433415.553

So $W \approx 4.4 \times 10^6$ J.

Version IV



From the Pythag. Thm.,

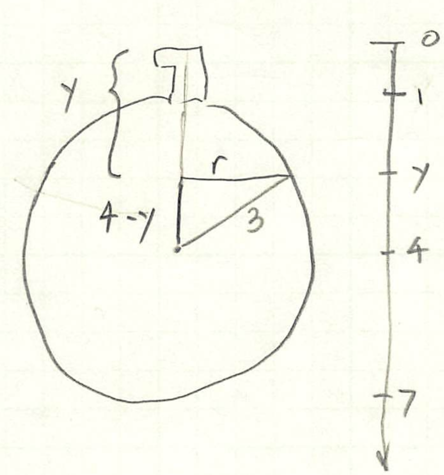
$(4-y)^2 + r^2 = 3^2$

$16 - 8y + y^2 + r^2 = 9$

$r^2 = 8y - y^2 - 7$

Since r is a length, $r \geq 0$, so

$r = \sqrt{8y - y^2 - 7}$



$V_{slice} = \pi \left(\sqrt{8y - y^2 - 7}\right)^2 \Delta y$

$M_{slice} = 1000\pi \left(\sqrt{8y - y^2 - 7}\right)^2 \Delta y$

$F_{slice} = 9.8 \cdot 1000\pi \left(\sqrt{8y - y^2 - 7}\right)^2 \Delta y$

Each slice travels a distance of y, so

$W_{slice} = 9800\pi (8y - y^2 - 7) y \Delta y$

We create slices from $y=1$ to $y=7$, so the total work is

$$W = \int_{y=1}^{y=7} 9800\pi (8y - y^2 - 7) y \, dy$$

From my TI-83,

fnInt(9800 * π * (8y - y² - 7) * y, y, 1, 7)

4433415.553

So $W \approx 4.4 \times 10^6$ J.