Computer Simulations and Risk Assessment – Lecture 6

Fall 2019

Brandeis International Business School

Course Information - Schedule

Class Date	Text Chapters
Aug. 30, 2019 – L1	• Course Introduction/Python Installation • Introduction to Quantitative Finance Career • Python basics
Sep. 6, 2019 – L2	• Advanced Python Topics
Sep. 13, 2019 – L3	• Advanced Python Topics
Sep. 20, 2019 – L4	• Sourcing and handling Data • Stylized financial data analysis using Python
Sep. 27, 2019 – L5	• Value at Risk
Oct. 4, 2019 – L6	• Conditional Value at Risk (Expected Shortfall) + Mid-term Review
Oct. 11, 2019	• Mid-term
Oct. 18, 2019 – L7	• Modeling Volatility I
Oct. 25, 2019 – L8	• Modeling Volatility II
Nov. 1, 2019 – L9	• Practical application case Studies I
Nov. 8, 2019 – L10	• Practical application case Studies II
Nov. 15, 2019 – L11	• Back Testing + Conditional risk prediction
Nov. 22, 2019 – L12	• Research project presentation
Dec. 6, 2019 – L13	• Final Review

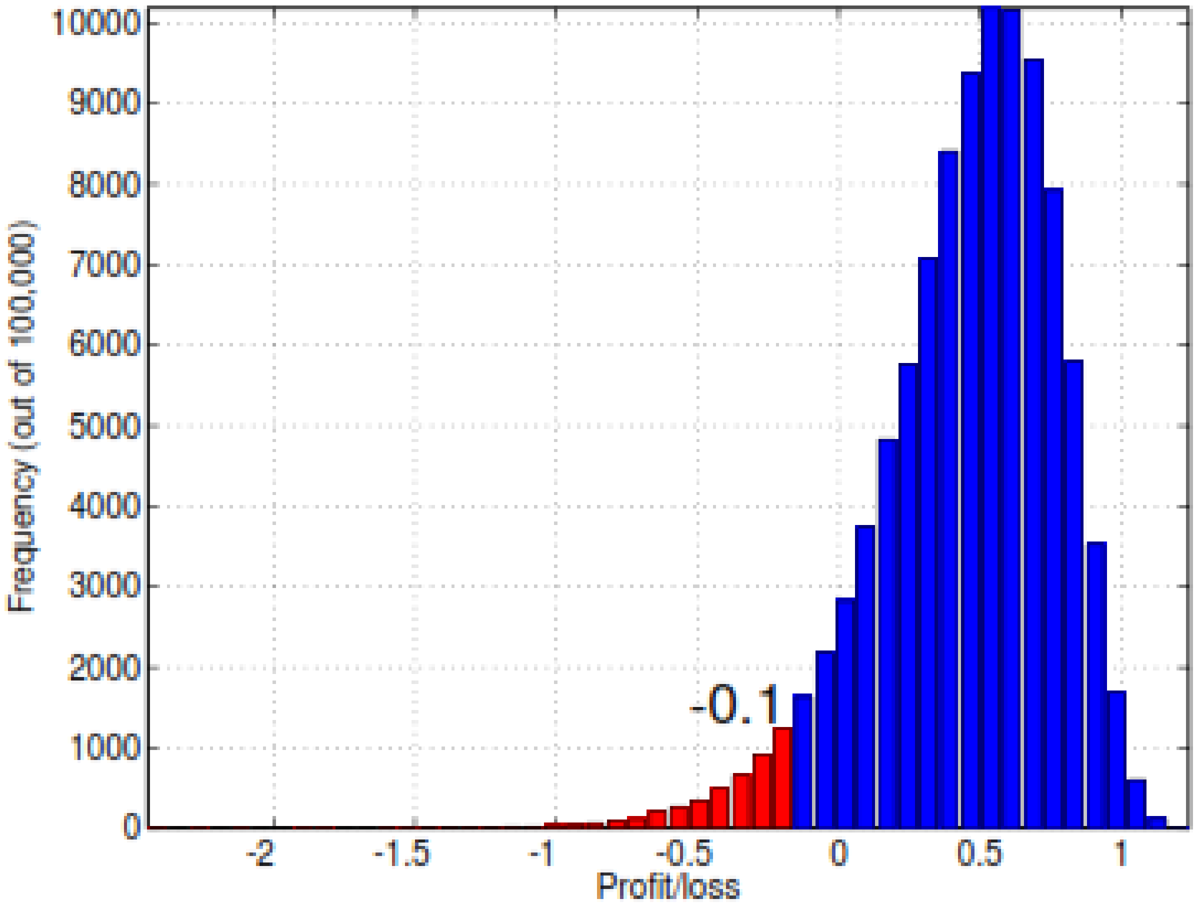
Expected Shortfall and the Impact of Holding Period

- **Expected Shortfall and Calculation**
- **VaR may violate risk subadditivity**
- **Holding periods and scaling**

Issues for VaR: It Doesn't Tell the Whole Story

- VaR is just a quantile marking a threshold point on the distribution
- VaR does not tell us anything about left tail beyond it
- Might ignore a lot of risk if the tail is fat or irregular
- What is the right probability level to use? 5%, 1%. It is not always the smaller the probability the better

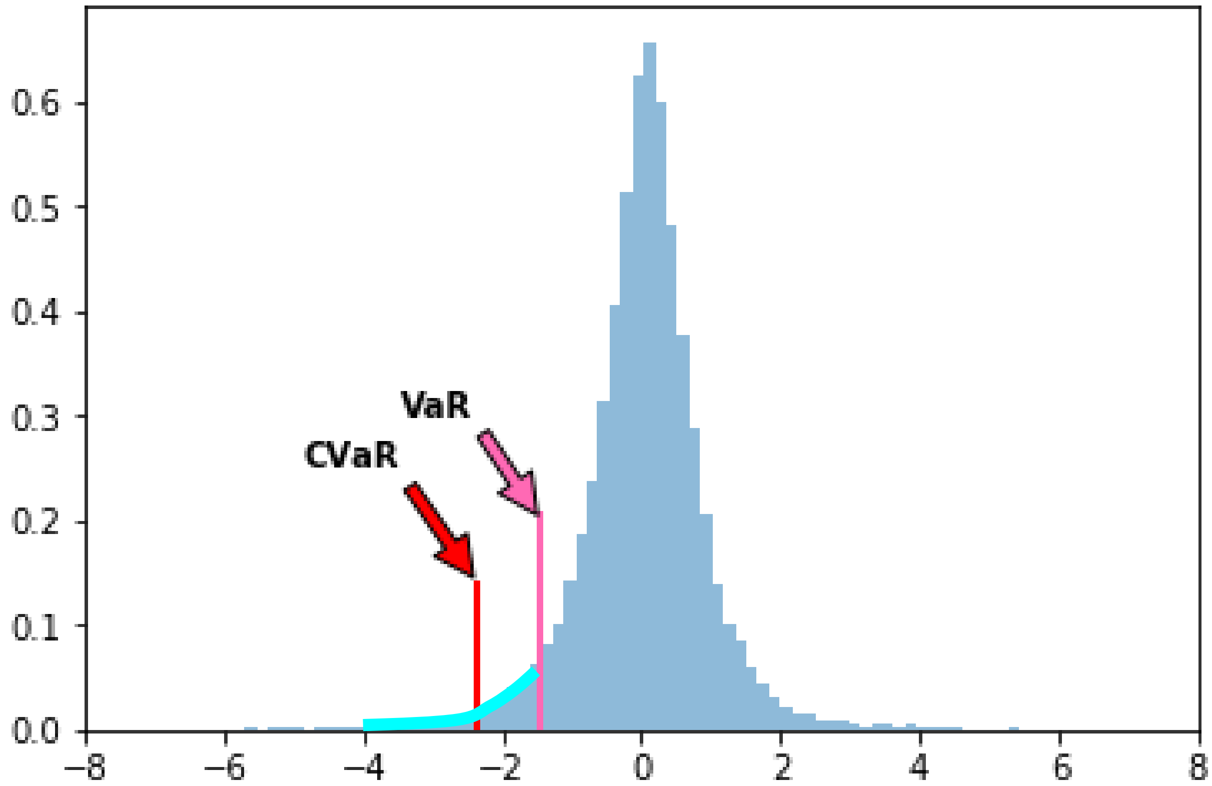
VaR Ignores the Left Tail



- Doesn't tell us the magnitudes of loss and associated probabilities left of the cutoff point associated with VaR

Solution: Expected Shortfall

- CVaR (Expected shortfall): Mean loss given that VaR loss is exceeded – the mean of returns that are less than VaR, or the weighted average of areas under the cyan line



Expected Shortfall

- Definition of expected shortfall

$$p = \Pr(X \leq VaR(p)) = \int_{-\infty}^{-VaR(p)} f_q(x)dx$$

$$Expected\ Shortall = -E(Q|Q \leq -VaR(p)) = -\frac{1}{p}\int_{-\infty}^{-VaR(p)} x f_q(x)dx$$

- Assuming x follows normal distribution, with mean of μ and standard deviation of σ, we have

$$f_q(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

Expected Shortfall under normal distribution

- Convert X from a normal distribution to a standard normal distribution, z= $\frac{x-\mu}{\sigma}$ (or x= μ +z σ), which leads to $\phi\,(z) = \frac{1}{\sqrt{2\pi}} exp\left[-\frac{z^2}{2}\right]$, for z, the expected shortfall is

$$ES0 = -\frac{1}{p}\int_{-\infty}^{-VaR0(p)} z\frac{1}{\sqrt{2\pi}} exp\left[-\frac{z^2}{2}\right] dz$$

$$= \frac{1}{p}\left[-\frac{1}{\sqrt{2\pi}} exp\left[-\frac{z^2}{2}\right]\right]_{-\infty}^{-VaR0(p)}$$

$$= -\frac{\frac{1}{\sqrt{2\pi}}exp\left[-\frac{VaR0(p)^2}{2}\right]}{p} - 0 = -\frac{\phi(-VaR0(p))}{p}$$

- Converting back to x from z (ES is a x in this case, and ES0 is a z):

$$x= \mu + z\,\sigma \rightarrow ES = \mu + ES0*\sigma = \mu - \frac{\phi(-VaR0(p))}{p} *\sigma$$

Where ϕ is the pdf, and $VaR0(p)$ is the VaR at p% for a standard normal distribution

Expected Shortfall

- **For a more formal prove of the formula for expected shortfall, please see the following online prove:**

http://blog.smaga.ch/expected-shortfall-closed-form-for-normal-distribution/

Expected Shortfall for Normal Distribution

- $R_{t+1} \sim N(\mu, \sigma^2)$, e.g., $R_{t+1} \sim N(\mu = 0.12, \sigma^2 = 0.2^2)$
- Assume $\Phi(x)$ and $\phi(x)$ are Normal(0,1) CDF and PDF
- VaR(5%) for a regular normal distribution:

$R^*(p) = \sigma * \Phi^{-1}(p) + \mu = stats.norm.ppf(p, \mu, \sigma)$, which means

$$VaR0(p) = \Phi^{-1}(p) = \frac{R^*(p) - \mu}{\sigma}$$

- This means Expected shortfall

$$\tilde{R}(p) = E(R | R \leq R^*(p)) = \mu - \sigma \frac{\phi\left(\frac{R^*(p) - \mu}{\sigma}\right)}{p}$$

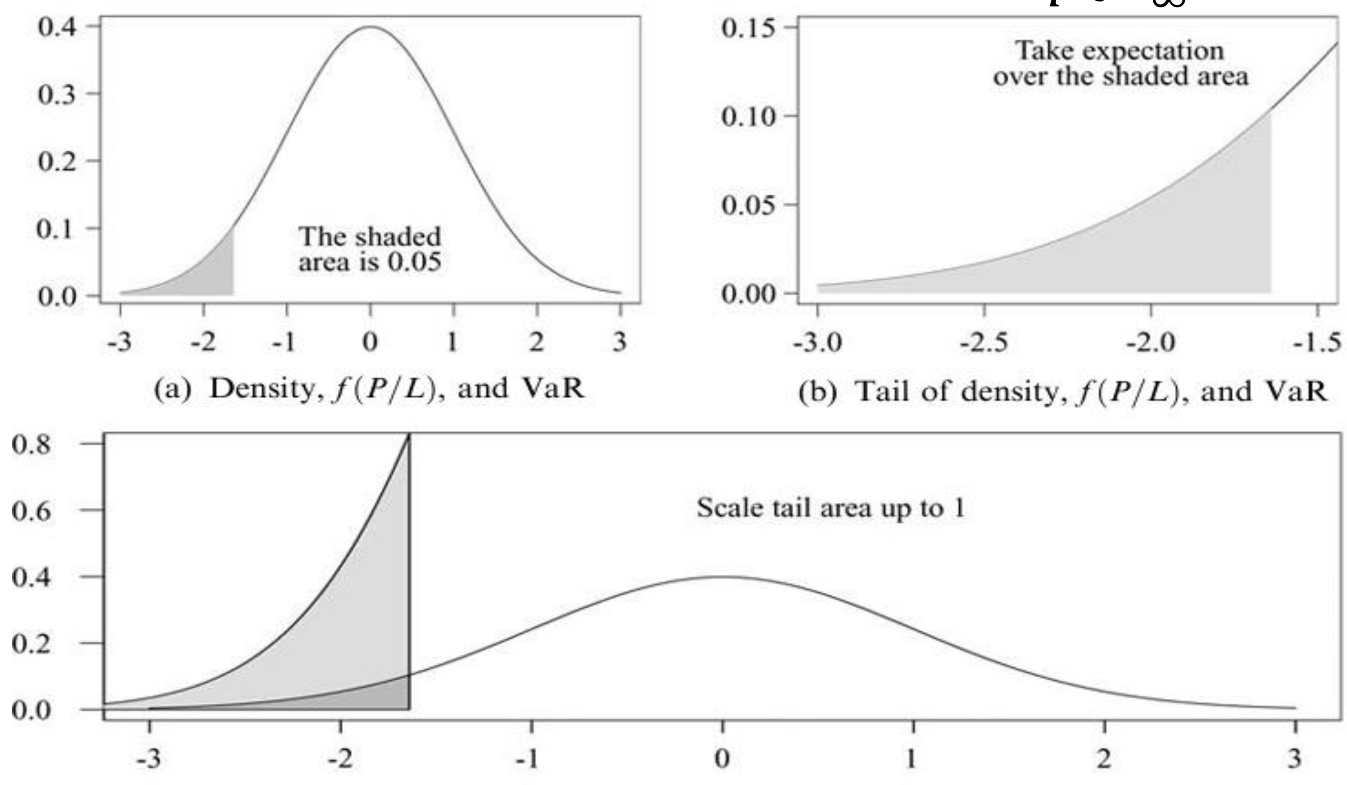
$$\text{ES(p)} = -((1+\tilde{R}(p))\, P_t - P_t) = -P_t \tilde{R}(p)$$

- Python Code in sample code L6_ES.py:
  - R_star = stats.norm.ppf(p, mu_R, sigma_R)
  - R_Tilde = -sigma_R*stats.norm.pdf((R_star - mu_R) / sigma_R) / p + mu_R

Expected Loss When Loss is worse than VaR

$$p = \Pr(X \le VaR(p)) = \int_{-\infty}^{-VaR(p)} f_q(x)dx$$

$$Expected\ Shortall = -E(Q|Q \le -VaR(p)) = -\frac{1}{p}\int_{-\infty}^{-VaR(p)} x f_q(x)dx$$



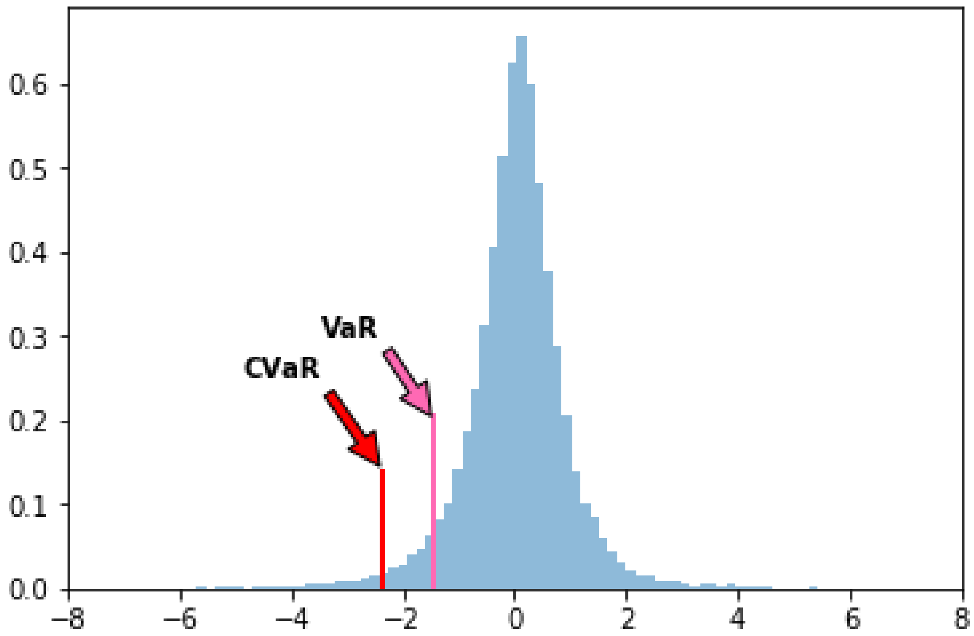
(a) Density, $f(P/L)$, and VaR

(b) Tail of density, $f(P/L)$, and VaR

(c) Blow up the tail. The darker shading has area p, whilst the entire shaded area has area 1

Expected Loss When Loss is worse than VaR

- Expected shortfall: Expected loss given that VaR loss is exceeded – there is more information on the left tail
- Also known as Expected tail loss (ETL) or conditional Value-at-Risk (CVaR)



ES analytics for Normal Distribution

- **A little more details on ES analytics**

$$p = \Pr(X \leq VaR(p)) = \int_{-\infty}^{-VaR(p)} f_q(x)dx$$

$$Rescale\ p\ to\ 100\%: 1 = \int_{-\infty}^{-VaR(p)} f_{VaR}(x)dx = \frac{1}{p}\int_{-\infty}^{-VaR(p)} f_q(x)dx$$

$$Expected\ Shortall = -E(Q|Q \leq -VaR(p)) = -\frac{1}{p}\int_{-\infty}^{-VaR(p)} x f_q(x)dx$$

- **For a portfolio with a current value of 1 and standard deviation of 1:**

$$ES = -\frac{\phi(\Phi^{-1}(p))}{p}$$

VaR and ES Table for Normal/t Distribution

- **For standard normal (mean = 0, std =1) and student-t distribution (mean = 0, std =1, dof = 4)**

VaR table							
p	0.001	0.01	0.025	0.05	0.1	0.1584	0.5
normal	-3.090	-2.326	-1.960	-1.645	-1.282	-1.000	0.000
t (4 dof)	-7.173	-3.747	-2.776	-2.132	-1.533	-1.143	0.000
ES table							
p	0.001	0.01	0.025	0.05	0.1	0.1584	0.5
normal	-3.367	-2.665	-2.338	-2.063	-1.755	-1.526	-0.798
t (4 dof)	-8.952	-5.305	-4.018	-3.215	-2.507	-2.062	-1.010

Historical Expected Shortfall

- Two routes: Portfolio value based or return based
- Portfolio value (price) based approach
  - Produce historical distribution of the portfolio value (price)
  - Estimate p quantile of portfolio value distribution, or the VaR
  - Estimate the conditional mean for portfolio values below VaR
  - $ES(p) = E(P_{t+1} | P_{t+1} \leq P^*) - P_{t+1}$
- Returns based approach
  - Estimate p quantile for returns
  - Estimate the conditional mean for returns that is below return that is corresponding to VaR
  - $ES(p) = -P_t E(R_{t+1} | R_{t+1} \leq R^*)$

Expected Shortfall: Pros and Cons

- Pros
  - More/Extra information on extreme losses
  - Less easily manipulated
  - subadditive
- Cons
  - Needs tail data to do the calculation
  - Harder to explain
  - Less adoption by financial institutions
  - Harder to backtest

Impact of Holding Period

- We discussed the importance of knowing the pdf/cdf for loss/return for calculating risk
- One important determinant of the risk number we haven't discussed is the length of the holding period for the risk calculation
  - Normally, the longer the holding period for the investment, the higher the risk level, as there is more time for bad things to happen
  - But exactly how do we link risk with length of the investment holding period?

The Data Challenge for Long Holding Period

- Normally the VaR is calculated for daily holding period
  - For 1% VaR, need a few hundred daily return data to have at least a few data points on the left tail
- For 10-day (bi-weekly) VaR, the data requirement gets serious
  - For 1% VaR, need a few hundred 10-day periods, say 300, to have a decent tail coverage. This is 3000 daily data or about 12 years!
- Solution: Scaling
  - Based on assumptions that statistical theory governs how properties of distribution changes as data are added

The Square-Root-of-Time Rule of Scaling

- The statistical measurement a random variable, such as volatility and VaR, are obtained by multiplying a higher frequency measurement by the square root of the number of observations in the holding period
  - This rule applies to <u>Volatility</u> as long as the returns are IID (independent and identically distributed), regardless whether they are normally distributed or not
  - However, for <u>VaR</u>, this rule applies only if the returns are IID and normal
- E.g., $VaR(10\ day) = VaR(daily) * \sqrt{10}$

Expected Value for Random Variables

- Discrete variables:

$$E(X) = \sum_{i=1}^{T} p_i x_i$$

$$A = \{i | x_i < z\} - think\ of\ z\ as\ VaR$$

$$E(X|X < z) = \frac{\sum_{i \in A} p_i x_i}{\sum_{i \in A} p_i}$$

- Continuous variables

$$E(X) = \int_{-\infty}^{\infty} x f_x(x) dx$$

$$p = \Pr(X \leq z) = \int_{-\infty}^{z} f_x(x) dx$$

$$E(X|X \leq z) = \frac{1}{p} \int_{-\infty}^{z} x f_x(x) dx$$

Non-standardized student-t

- Standard Student-t random variable $t(v)$: mean = 0, $Std = \sqrt{\frac{v}{v-2}}$
- Set non-standard $t^* = t * \frac{\sigma}{\sqrt{v/(v-2)}} + \mu$
- $Var(t^*) = v/(v-2) * (\frac{\sigma}{\sqrt{v/(v-2)}})^2 = \sigma^2$
- $E(t^*) = \mu$

Covariance & Correlation

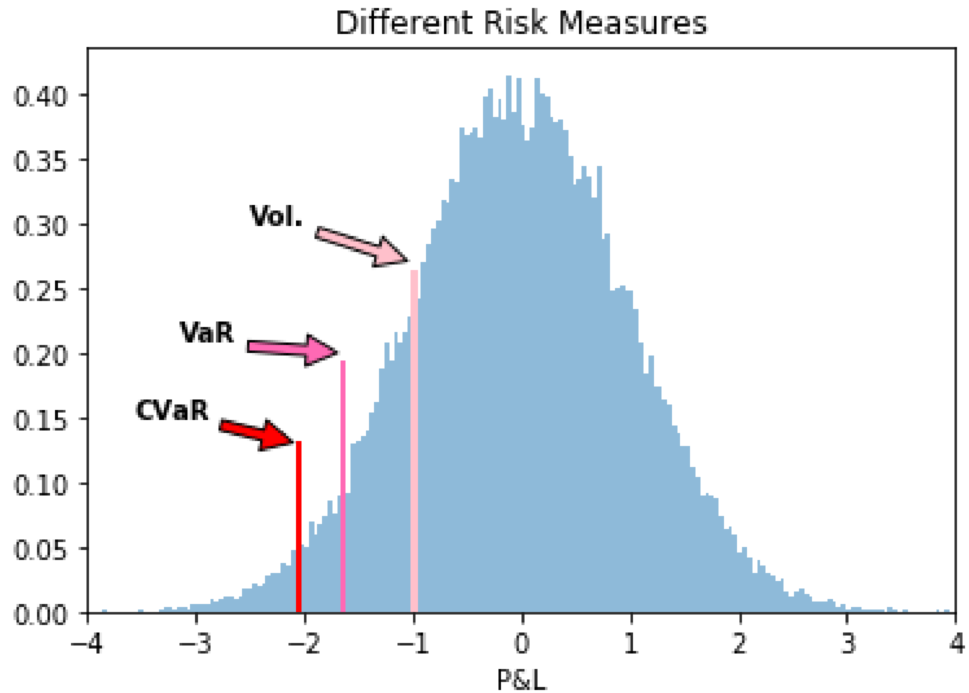
- **Risk management Q&A**
- **Revisiting Covariance and Correlation**
- **Portfolio level risk measures**
- **Intuition and Applications**

Risk Management – the question

- Remember I challenged you with the interview question of "tell me how would you forecast the risk of a given portfolio?"
- Last lecture, I talked about the core of forecasting risk is to forecast the distribution of the profit & loss or the returns of the portfolio?
- The most common risk (measure) people talk about: volatility – how do we think about volatility in the risk management context? How does it relate to potential P&L or negative returns?
- We studied VaR and CVaR. How about them? How do we explain VaR and CVaR?
- Let's look at the example of current portfolio=100, future mean=100, sigma=1,

Risk Management – The Answer

- Volatility: it measures the magnitude of loss in the sense that there is a X% probability that loss will be at or larger than μ - σ (or $1)
  - X=15.9 % for normal distribution
  - X=19.6 % for student-t distribution with degree of freedom=3
- VaR (5%) Normal distribution：The probability of loss greater than μ -1.65* σ = -$1.65 is 5%
- ES(5%) Normal distribution: The mean of all loss that's higher than VaR(5) = -$1.65 is -$2.06



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Standard Deviation, Correlation & Covariance

- Portfolio return mean and standard deviation

$$\mu_k = \frac{1}{N}\sum_{t=1}^{N} R_{t,k}$$

$$\sigma_k = \sqrt{\frac{1}{N-1}\sum_{t=1}^{N}\left(R_{t,k} - \mu_k\right)^2}$$ (Also called volatility）

- Correlation and Covariance

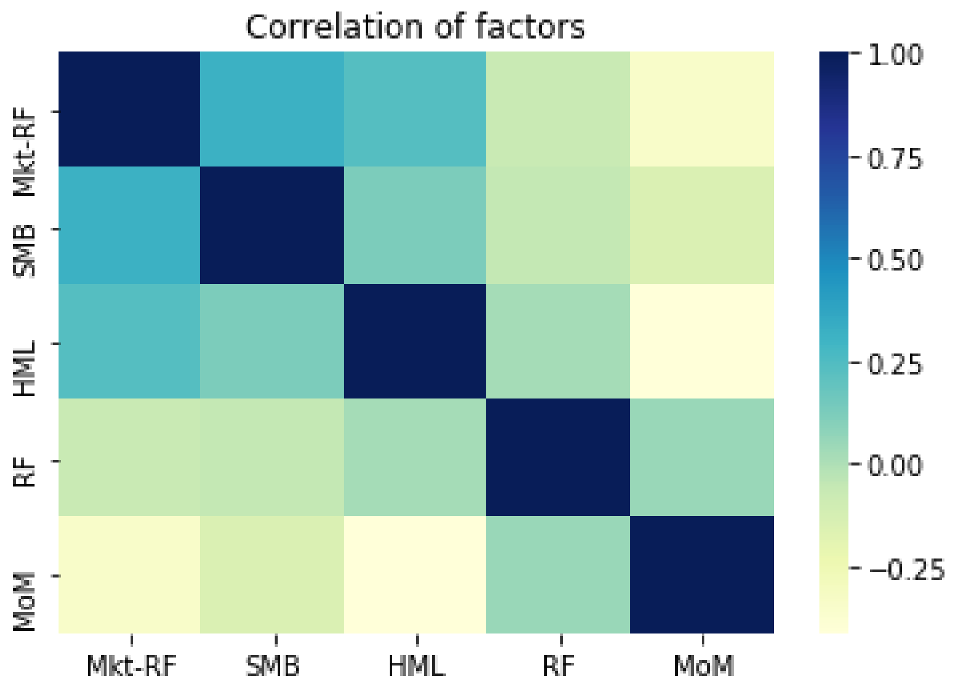
$$\rho = \frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{N}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{N}(y_i - \bar{y})^2}}$$

$$cov = \sigma_{ij}^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_{ij} - \overline{x_j}\right)\left(x_{ik} - \overline{x_k}\right)$$

Correlation

- Correlation measures how returns move together

$$\rho = \frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{N}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{N}(y_i - \bar{y})^2}}$$



Covariance Matrix Σ_t

- The covariance between two assets i and j is :

$$Cov(R_k, R_j) = \sigma_{kj}^2$$

- In the case of a three assets, the conditional matrix takes the following form

$$\Sigma_t = \begin{pmatrix} \boldsymbol{\sigma_{11}^2} & \boldsymbol{\sigma_{12}^2} & \boldsymbol{\sigma_{13}^2} \\ \boldsymbol{\sigma_{12}^2} & \boldsymbol{\sigma_{22}^2} & \boldsymbol{\sigma_{23}^2} \\ \boldsymbol{\sigma_{13}^2} & \boldsymbol{\sigma_{23}^2} & \boldsymbol{\sigma_{33}^2} \end{pmatrix} \textbf{ where } \boldsymbol{\sigma_{ij}^2 = \rho_{ij}\sigma_i\sigma_j}$$

- The diagonal terms measure the volatilities of each of the assets $\boldsymbol{\sigma_{ii}^2 = \rho_{ii}\sigma_i\sigma_i = \sigma_i^2}$ (because $\boldsymbol{\rho_{ii}}$=1)
- The off-diagonal terms is a product of the correlation of the assets and the volatilities of each of the asset $\boldsymbol{\sigma_{ij}^2 = \rho_{ij}\sigma_i\sigma_j}$

Covariance Matrix Σ_t

- Observe the covariance: the diagonal terms are always positive and in general larger in magnitude relative to the off-diagonal terms

Index	Mkt-RF	SMB	HML	RF	MoM
Mkt-RF	0.0028482	0.000540799	0.000442601	-8.92293e-06	-0.00084904
SMB	0.000540799	0.00102323	0.000138314	-4.12695e-06	-0.000219559
HML	0.000442601	0.000138314	0.00121865	2.09176e-06	-0.000679714
RF	-8.92293e-06	-4.12695e-06	2.09176e-06	6.45226e-06	6.61262e-06
MoM	-0.00084904	-0.000219559	-0.000679714	6.61262e-06	0.00221179

Formulas for Portfolio Level Risk

- Asset level

$$\mathrm{E}(R_t)=\mu_i,\ \mathrm{var}(R_i)=\sigma_i^2,\ corr(R_1, R_2)=\rho$$

- Portfolio return, $R_p = w_1 R_1 + w_2 R_2$
- Portfolio variance: $\sigma_p^2 = w_1^2\sigma_1^2 + 2w_1 w_2 \rho \sigma_1 \sigma_2 + w_2^2 \sigma_2^2$
- In the general case, with time varying, for a multi-assets portfolio

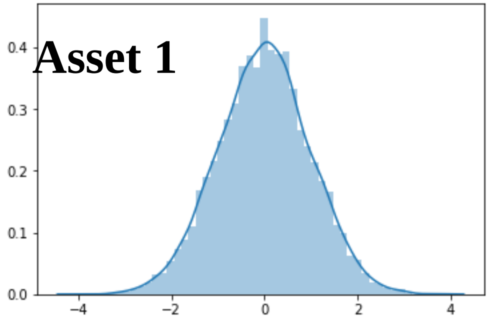
$$\text{Return: } R_{p,t} = \sum_{i=1}^{N} w_{i,t} R_{i,t}$$

$$\text{Portfolio Variance: } \Sigma_{\boldsymbol{p},\boldsymbol{t}} = \sigma_{p,t}^2 = \boldsymbol{W'\Sigma W}$$

$$= \sum_{i=1}^{N} w_{i,t}^2 \sigma_{ii,t}^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} w_{i,t} w_{j,t} \rho_{i,j,t} \sigma_{ij,t} \sigma_{ji,t}$$

Foundation of a Modern Risk System

1. For each asset (risk factor), develop assumptions about distribution of its returns



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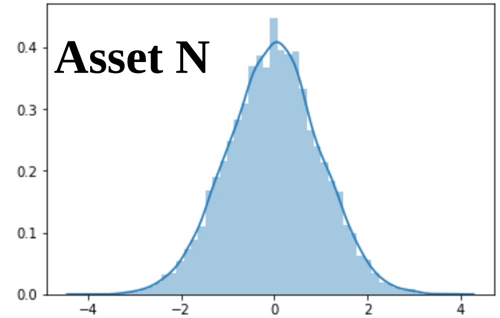
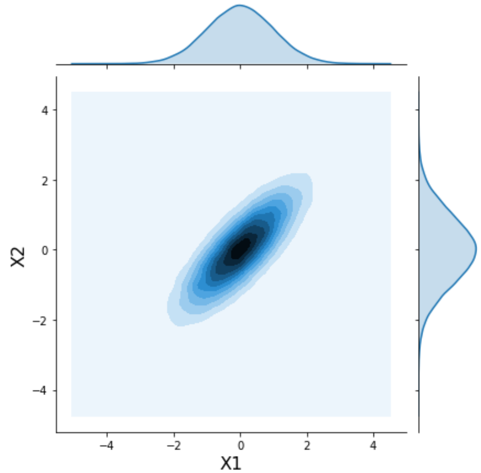


2. Incorporate assumptions about correlations of these returns



3. Combining weights (exposure to risk factors)

$\mathbf{W}'\mathbf{\Sigma}\mathbf{W}$ or through Monte Carlo Simulations

here $\mathbf{W}$ is the weights vector and $\mathbf{\Sigma}$ is the covariance matrix

Portfolio Level Risk Example

$\boldsymbol{W}$=[0.6, 0.4],

$$\Sigma = \begin{pmatrix} \mathbf{0.001} & \mathbf{0.0002} \\ \mathbf{0.0002} & \mathbf{0.002} \end{pmatrix}$$

Portfolio Variance $= \sigma_{p,t}^2 = \boldsymbol{W}'\Sigma\boldsymbol{W} =$

0.6**2*0.001+0.4**2*0.002+2*0.6*0.4*0.0002 =0.000776

Tracking Error of a Portfolio

- Realized tracking error of a fund's returns vs. its benchmark:

$$Tracking\ Error = std(r_{fund} - r_{benchmark}) = \sqrt{\frac{1}{N-1}\sum_{t=1}^{N}\left(R_{t,fund} - r_{t,benchmark}\right)^2}$$

- What about forecasting TE for the future? Tracking error of a portfolio, calculated using components of the portfolio

$$\boldsymbol{TE = \sqrt{W'_a \Sigma W_a}}$$

where $\boldsymbol{W_a}$ is the active weights vector and Σ is the covariance matrix

- Example

$\boldsymbol{W_a}$=[-0.1, 0.1],

$$\Sigma = \begin{pmatrix} \mathbf{0.001} & \mathbf{0.0002} \\ \mathbf{0.0002} & \mathbf{0.002} \end{pmatrix}$$

Projected TE $=\sqrt{\boldsymbol{W'_a \Sigma W_a}}$ **= 0.0051**

Units of the variables

- Return has the unit of %
- Variance has the unit of %^2
- Correlation has no unit
- Volatility has the unit of %

For the rest of the this class

- We develop ways to forecast the covariance matrix, e.g., exponential moving average model
- Linkage to the foundation of the risk system: These models can also be extended to calculate VaR and CVaR based on normal or student-t distribution assumptions (e.g. utilizing the VaR and ES Table for Normal/t Distribution on page 12)
- We develop the conditional simulation approach to calculate conditional covariance and tail risk
- We apply these forecasts to real-time risk modeling applications
  - Forecasting of portfolio risk
  - Tracking error optimization for the construction of exchange traded funds (ETFs)
  - Risk parity investing strategies