3. Prove Beta-Binomial Conjugation 王肇綎 313551170

likelihood: $m \sim \text{Binomial}(N, p)$, $P(m|N,p) = \binom{N}{m} p^m (1-p)^{N-m}$

prior: $p \sim \text{Beta}(a, b)$, $P(p|a,b) = \frac{\gamma(a+b)}{\gamma(a)\gamma(b)} p^{a-1} (1-p)^{b-1}$

posterior: $P(p | N, m, a, b)$

$$= \frac{\text{likelihood} \times \text{prior}}{\text{marginal}} = \frac{P(m|N,p)\, P(p|a,b)}{P(D)}$$

$$= \frac{\binom{N}{m} p^m (1-p)^{N-m} \, p^{a-1} (1-p)^{b-1} \frac{\gamma(a+b)}{\gamma(a)\gamma(b)}}{\int_0^1 \binom{N}{m} \theta^m (1-\theta)^{N-m} \theta^{a-1} (1-\theta)^{b-1} \frac{\gamma(a+b)}{\gamma(a)\gamma(b)}\, d\theta}$$

$$= \frac{p^{m+a-1} (1-p)^{N-m+b-1}}{\int_0^1 \theta^{m+a-1} (1-\theta)^{N-m+b-1}\, d\theta}$$

(Beta(a+m, N-m+b))

$$1 = \int_0^1 \beta(\theta | m+a, N-m+b)\, d\theta = \int_0^1 \frac{\gamma(a+b+N)}{\gamma(m+a)\gamma(N-m+b)} \theta^{m+a-1} (1-\theta)^{N-m+b-1}\, d\theta$$

$$= \frac{\gamma(a+b+N)}{\gamma(m+a)\gamma(N-m+b)} \int_0^1 \theta^{m+a-1} (1-\theta)^{N-m+b-1}\, d\theta$$

$$\Rightarrow \int_0^1 \theta^{m+a-1} (1-\theta)^{N-m+b-1}\, d\theta = \frac{\gamma(m+a)\gamma(N-m+b)}{\gamma(a+b+N)}$$

$$= \frac{\gamma(a+b+N)}{\gamma(m+a)\gamma(N-m+b)} p^{m+a-1}(1-p)^{N-m+b-1} \quad \sim Beta(a+m, N-m+b)$$

4. Prove Gamma - Poisson Conjugation

likelihood: $X \sim Poisson(N, \lambda)$, $P(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$

prior: $\lambda \sim Gamma(\alpha, \beta)$, $P(\lambda|\alpha,\beta) = \frac{\beta^\alpha}{\gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$, $\lambda > 0$

posterior: $P(x|\lambda, \alpha, \beta)$

$$= \frac{\text{likelihood} \times \text{prior}}{\text{marginal}} = \frac{P(x|\lambda)\,P(\lambda|\alpha,\beta)}{P(D)}$$

$$= \frac{\frac{\lambda^x e^{-\lambda}}{x!} \frac{\beta^\alpha}{\gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}}{\int_0^\infty \frac{\theta^x e^{-\theta}}{x!} \frac{\beta^\alpha}{\gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}\, d\theta}$$

$$= \frac{\lambda^{x+\alpha-1} e^{-(\beta+1)\lambda}}{\int_0^\infty \theta^{x+\alpha-1} e^{-(\beta+1)\lambda}\, d\theta}$$

$$1 = \int_0^\infty \frac{(\beta+1)^{x+\alpha}}{\gamma(x+\alpha)} \theta^{x+\alpha-1} e^{-(\beta+1)\theta}\, d\theta \quad (Gamma(x+\alpha, \beta+1))$$

$$= \frac{(\beta+1)^{x+\alpha}}{\gamma(x+\alpha)} \int_0^\infty \theta^{x+\alpha-1} e^{-(\beta+1)\theta}\, d\theta$$

$$\Rightarrow \int_0^\infty \theta^{x+\alpha-1} e^{-(\beta+1)\theta}\, d\theta = \frac{\gamma(x+\alpha)}{(\beta+1)^{x+\alpha}}$$

$$= \frac{(\beta+1)^{x+\alpha}}{\gamma(x+\alpha)} \lambda^{x+\alpha-1} e^{-(\beta+1)\lambda}$$

$\sim$ Gamma$(x+\alpha, \beta+1)$