Rutherford scattering

In this project we model Rutherford scattering. When shall find out how the scattered angle depends on the impact parameter. Take the position, velocity, and acceleration of the alpha particle as $\vec{r} = (x, y)$, $\vec{v} = (v_x, v_y)$, $\vec{a} = (a_x, a_y)$. The gold nucleus is at rest and is located at the origin $(0, 0)$.

1. The acceleration of the alpha particle is due to the electrostatic repulsion:
$$m\frac{d^2\vec{r}}{dt^2} = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{e}}_r \tag{1}$$
where Z_1 and Z_2 are the atomic numbers of the alpha particle and the gold nucleus respectively, and the radial unit vector is
$$\hat{\mathbf{e}}_r = \frac{\vec{r}}{r} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{r} \tag{2}$$

2. Let v_0 be the initial speed of the alpha particle. A natural length scale of this problem is the closest distance of approach r_m in a head-on collision. At the closest distance, the potential energy is equal to the initial kinetic energy:
$$\frac{1}{2}mv_0^2 = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r_m} \tag{3}$$
Choose a time scale T such that $v_0 = r_m/T$. Then
$$T = \sqrt{\frac{2\pi m\epsilon_0 r_m^3}{Z_1 Z_2 e^2}} \tag{4}$$
With r_m as the unit of length and T as the unit of time, the initial speed of the alpha particle is $V_0 = 1$.

3. Scale the equation of motion with $t = T\tau$ and $\vec{r} = r_m\vec{R}$:
$$\frac{d^2\vec{R}}{d\tau^2} = \frac{1}{2R^2}\,\hat{\mathbf{e}}_r \tag{5}$$
Hence the two components of (5) are
$$A_X = \frac{X}{2R^3} \quad , \quad A_Y = \frac{Y}{2R^3} \tag{6}$$
where $R = \sqrt{X^2 + Y^2}$.

4. Iterate the equation of motion by using
$$X(n{+}1) = X(n) + V_X(n)\Delta\tau \quad , \quad V_X(n{+}1) = V_X(n) + A_X(n)\Delta\tau \tag{7}$$
and likewise for the Y component. The initial conditions of the alpha particle are $X = -k$, $Y = b$, $V_x = 1$, $V_y = 0$. The collision is characterized by the impact parameter b, where $b = 0$ represents a head-on collision. Here k is a sufficiently large distance so that the particle is not affected by the gold nucleus initially.

5. Take $k = 10$, $b = 1$, and use (7) to generate a table of X and Y versus τ. Start with a time step $\Delta\tau = 0.005$ or smaller. End your iteration when R is sufficient large, say 10. At the final position, find the deflection angle θ by

$$\theta = \tan^{-1}\frac{V_Y}{V_X} \quad (V_X > 0) \quad , \quad \theta = \pi - \tan^{-1}\frac{V_Y}{|V_X|} \quad (V_X < 0) \tag{8}$$

6. Check that your calculation actually conserves the total energy and angular momentum reasonably well, i.e.,

$$E = \frac{1}{2}(V_X^2 + V_Y^2) + \frac{1}{2R} \quad , \quad L = (\vec{R} \times \frac{\vec{P}}{m})_Z = XV_Y - YV_X \tag{9}$$

   If not, make $\Delta\tau$ smaller. Plot the trajectory[1] (i.e., a graph of Y versus X).

7. Repeat the calculation for $b = 0.1$, 0.25, 0.5, 1, and 2. Theory predicts that for a given v_0, the deflection angle and the impact parameter satisfy the following relation:

$$\tan\frac{\theta}{2} = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 m v_0^2}\frac{1}{b} \tag{10}$$

   Plot a graph of $\tan(\theta/2)$ versus $1/b$ to verify the proportional relation.

[1] If you use Excel to plot the trajectory, use a scatter plot and make sure that the scales of the x and y axes are the same, otherwise the trajectory will appear distorted.