# Assignment: Bayes Filter

## 1. Assignment Overview

### 1.1. Description

Consider a household robot equipped with a camera. It operates in an apartment with two rooms: a living room and a bedroom. The robot runs an artificial neural network that can recognize a living room in the camera image. Further, the robot can perform a switch-room action, i.e., it moves to the living room if it is in the bedroom, and vice versa. Neither the recognition nor the motion controller is perfect.

From previous experience, you know that the robot succeeds in moving from the living room to the bedroom with a probability of 0.7, and with a probability of 0.8 in the other direction.

The probability that the neural network indicates that the robot is in the living room although it is in the bedroom is given, and the probability that the network correctly detects the living room is given.

Unfortunately, you have no knowledge about the current location of the robot.

However, after performing the switch-room action, the neural network indicates that the robot is not in the living room. After performing the switch-room action for the second time, the network again indicates not seeing a living room.

Use the Bayes filter algorithm to compute the probability that the robot is in the bedroom after performing the two actions.

### 1.2. Action Model

From the given conditions about action model we could learn that,

$$P(x_{t+1} = bedroom|x_t = living\ \ room,\ u_{t+1} = switch) = 0.7$$

$$P(x_{t+1} = living\ \ room|x_t = bedroom,\ \ u_{t+1} = switch) = 0.8$$

So that

$$P(x_{t+1} = living\ \ room|x_t = living\ \ room, u_{t+1} = switch) = 0.3$$

$$P(x_{t+1} = bedroom|x_t = bedroom, u_{t+1} = switch) = 0.2$$

## 1.3. Sensor Model

Besides that we could also get the sensor model

$$P(z = bedroom|x = bedroom) = 0.7$$

$$P(z = living\ \ room|x = living\ \ room) = 0.9$$

So that

$$P(z = living\ \ room|x = bedroom) = 0.3$$

$$P(z = bedroom|x = living\ \ room) = 0.1$$

## 1.4. Measurement

According to the description, we have known the following measurement from robot camera network,

$$z_1 = bedroom$$

$$z_2 = bedroom$$

because the neural network indicates that the robot is not in the living room after performing the switch-room action two times. (we default that $\mathrm{z_0}$ is the initialization observation before switch action)

## 1.5. Prior Knowledge Distribution

The last part we need to determine for the calculation is Prior Knowledge Distribution:

$$P(x_0 = living\ \ room)$$

$$\mathrm{P(x_0 = bedroom)}$$

We could appropriate a prior distribution by considering the action model bellow, that the robot can only move at the start from bedroom or living room to bedroom after first moving action with the following two probabilistic distributuion,

$$P(x_1 = bedroom|x_0 = living\ \ room,\ u_1 = switch) = 0.7$$

$$P(x_1 = bedroom|x_0 = bedroom, u_1 = switch) = 0.2$$

And the sum of two prior distribution should be 1, so that we make an approximate assumption that

$$Bel(x_0 = living\ \ room) = 0.75$$

$$Bel(x_0 = bedroom) = 0.25$$

# 2. Bayes Formula

Based on the discrete Bayes Filter theory and the Markov Independence Assumption, we can rewrite the Bayesian formula:

$$Bel(x_t) = P(x_t|z_{1\ldots t}, u_{1\ldots t}) = \frac{P(z_t|x_t, z_{1\ldots t-1}, u_{1\ldots t}) \cdot P(x_t|z_{1\ldots t-1}, u_{1\ldots t})}{P(z_t|z_{1\ldots t-1}, u_{1\ldots t})}$$

We make normalization on denominator part that

$$Bel(x_t) = \alpha \cdot P(z_t|x_t, z_{1\ldots t-1}, u_{1\ldots t}) \cdot P(x_t|z_{1\ldots t-1}, u_{1\ldots t})$$

According to Markov Independence Assumption or HMM model we could do some simplication bacause $z_t$ is independent with all previous measurments and action, but only related to current state:

$$Bel(x_t) = \alpha \cdot P(z_t|x_t) \cdot P(x_t|z_{1\ldots t-1}, u_{1\ldots t})$$

And based on total probability

$$Bel(x_t) = \alpha \cdot P(z_t|x_t) \cdot \int P(x_t|x_{t-1}, z_{1\ldots t-1}, u_{1\ldots t}) \cdot P(x_{t-1}|z_{1\ldots t-1}, u_{1\ldots t-1}) dx_{t-1}$$

Applying also Markov Independence Assumption we rewrite formula continuously:

$$Bel(x_t) = \alpha \cdot P(z_t|x_t) \cdot \int P(x_t|x_{t-1}, u_t) \cdot Bel(x_{t-1}) dx_{t-1}$$

# 3. Solve Bayes State Estimation

## 3.1. State Estimation at First Time Step

So in next part we could estimate robot state probability after first action according to the bayes filter formel:

$$Bel(x_1) = \alpha_1 \cdot P(z_1|x_1) \cdot \int P(x_1|x_0, u_1) \cdot Bel(x_0) dx_0$$

Then we could solve the state estimation or belief of robot stays in living room after the first switch action like following discrete bayes formel

$$\begin{aligned}
&Bel(x_1 = living\ \ room) \\
&\quad = \alpha_1 \cdot P(z_1 = bedroom|x_1 = living\ \ room) \cdot \sum_{x_0} P(x_1 = living\ \ room|x_0, u_1) \cdot Bel(x_0) \\
&\quad = \alpha_1 \cdot P(z_1 = bedroom|x_1 = living\ \ room) \\
&\qquad\qquad \cdot \ \ (P(x_1 = living\ \ room|x_0 = living\ \ room, u_1 = switch\ \ room) \\
&\qquad\qquad \cdot Bel(x_0 = living\ \ room) \\
&\qquad\qquad + P(x_1 = living\ \ room|x_0 = bedroom, u_1 = switch\ \ room) \\
&\qquad\qquad \cdot Bel(x_0 = bedroom)) \\
&\quad = \alpha_1 \cdot 0.1 \cdot (0.3 \cdot 0.75 + 0.8 \cdot 0.25) \\
&\quad = 0.0425 \cdot \alpha_1
\end{aligned}$$

And the state estimation or belief of robot stays in bedroom after the first switch action can also be solved in fowlling way

$$\begin{aligned}
&Bel(x_1 = bedroom) \\
&\quad = \alpha_1 \cdot P(z_1 = bedroom|x_1 = bedroom) \cdot \sum_{x_0} P(x_1 = bedroom|x_0, u_1) \cdot Bel(x_0) \\
&\quad = \alpha_1 \cdot P(z_1 = bedroom|x_1 = bedroom) \\
&\qquad\qquad \cdot (P(x_1 = bedroom|x_0 = living\ \ room, u_1 = switch\ \ room) \\
&\qquad\qquad \cdot Bel(x_0 = living\ \ room) \\
&\qquad\qquad + P(x_1 = bedroom|x_0 = bedroom, u_1 = switch\ \ room) \\
&\qquad\qquad \cdot Bel(x_0 = bedroom)) \\
&\quad = \alpha_1 \cdot 0.7 \cdot (0.7 \cdot 0.75 + 0.2 \cdot 0.25) \\
&\quad = 0.4025 \cdot \alpha_1
\end{aligned}$$

According to the rules of probability distribution, these two results should be normalized with the parameter

$$\alpha_1 \approx 2.2471910$$

And then

$$Bel(x_1 = living\ \ room) \approx 0.1$$

$$Bel(x_1 = bedroom) \approx 0.9$$

This result confirms our observations at this moment, that the probability distribution of bedroom should be larger than living room.

## 3.2. State Estimation at Second Time Step

Then we could solve the state estimation or belief of robot stays in living room after the second switch action like following discrete bayes formel

$$\begin{aligned}
&Bel(x_2 = living\ \ room) \\
&\quad = \alpha_2 \cdot P(z_2 = bedroom|x_2 = living\ \ room) \cdot \sum_{x_1} P(x_2 = living\ \ room|x_1, u_2) \cdot Bel(x_1) \\
&\quad = \alpha_2 \cdot P(z_2 = bedroom|x_2 = living\ \ room) \\
&\qquad\qquad \cdot (P(x_2 = living\ \ room|x_1 = living\ \ room, u_2 = switch\ \ room) \\
&\qquad\qquad \cdot Bel(x_1 = living\ \ room) \\
&\qquad\qquad + P(x_2 = living\ \ room|x_1 = bedroom, u_2 = switch\ \ room) \\
&\qquad\qquad \cdot Bel(x_1 = bedroom)) \\
&\quad = \alpha_2 \cdot 0.1 \cdot (0.3 \cdot 0.1 + 0.8 \cdot 0.9) \\
&\quad = 0.075 \cdot \alpha_2
\end{aligned}$$

And the state estimation or belief of robot stays in bedroom after the second switch action can also be solved in fowlling way

$$\begin{aligned}
&Bel(x_2 = bedroom) \\
&\quad = \alpha_2 \cdot P(z_2 = bedroom|x_2 = bedroom) \cdot \sum_{x_1} P(x_2 = bedroom|x_1, u_2) \cdot Bel(x_1) \\
&\quad = \alpha_2 \cdot P(z_2 = bedroom|x_2 = bedroom) \\
&\qquad\qquad \cdot (P(x_2 = bedroom|x_1 = living\ \ room, u_2 = switch\ \ room) \\
&\qquad\qquad \cdot Bel(x_1 = living\ \ room) \\
&\qquad\qquad + P(x_2 = bedroom|x_1 = bedroom, u_2 = switch\ \ room) \\
&\qquad\qquad \cdot Bel(x_1 = bedroom)) \\
&\quad = \alpha_2 \cdot 0.7 \cdot (0.7 \cdot 0.1 + 0.2 \cdot 0.9) \\
&\quad = 0.075 \cdot \alpha_2
\end{aligned}$$

The same as before, we normalize these results with

$$\alpha_2 = 4$$

So that

$$Bel(x_2 = living\ \ room) = 0.3$$

$$Bel(x_2 = bedroom) = 0.7$$

This result also confirms our observations at the second moment, that the probability distribution of bedroom should be larger than living room.

# 4. Which prior minimizes the probability?

We can analyze this problem with the help of the parametric method. First we define the prior probability distribution

$$Bel(x_0 = living\ \ room) = p$$

So the robot state estimation after first action could be rewitten as:

$$P(x_1 = living\ \ room) = \alpha_1 \cdot 0.1 \cdot (0.3 \cdot p + 0.8 \cdot (1 - p))$$

$$P(x_1 = bedroom) = \alpha_1 \cdot 0.7 \cdot (0.7 \cdot p + 0.2(1 - p))$$

And our target of analysis

$$P(x_2 = bedroom) = 0.7 \cdot \alpha_1 \cdot \alpha_2 \cdot (0.7 \cdot 0.1 \cdot (-0.5p + 0.8) + 0.2 \cdot 0.7 \cdot (0.5p + 0.2))$$

Result as

$$P(x_2 = bedroom) = 0.7 \cdot \alpha_1 \cdot \alpha_2 \cdot (0.035p + 0.084)$$

As conclusion we could know that with the prior probability distribution of "living room" is positive related to the target probability and the prior probability distribution of "bedroom" is negative related.

So if we decrease the prior probability distribution in living room

$$P(x_0 = living\ \ room)$$

and increase prior probability distribution in bedroom

$$P(x_0 = bedroom),$$

we can minimize the probability of state in "bedroom" or "not in living room" after two times actions.