MATH 151A Homework 1

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Question 1

(a) First, I will show that the limit p^* is 0.
Let $g(x) = x - \frac{1}{2}\ln(x+1)$, Then we have $g(0) = 0$ and $g'(x) = 1 - \frac{1}{2} \cdot \frac{1}{x+1} \geq 0 \quad \forall x > 0$. Since g and g' both exist and are continous, it follows from Mean Value Theorem that there exist $\xi \in (0, x)$ such that $g'(\xi) = \frac{g(x)-g(0)}{x-0} = \frac{g(x)}{x}$. So, $g(x) = x \cdot g'(\xi) \geq 0$ and $x \geq \frac{1}{2}\ln(x+1)$.
Claim: $0 < p_n \leq 2^{-n}$.
proof. First, $0 < p_1 \leq 1$. Then, suppose $0 < p_n \leq 2^{-n}$, then, we have $0 < p_{n+1} = \frac{1}{2}\ln(p_n + 1) \leq \frac{1}{2}p_n \leq 2^{-(n+1)}$. So, by induction, $0 < p_n \leq 2^{-n}$ and it follows from squeeze theorem that $\lim_{n\to\infty} p_n = p^* = 0$.

Then, compute the convergence order and asymptotic rate of convergence.

$$\lim_{n\to\infty} \left|\frac{p_{n+1}}{(p_n)^\alpha}\right| = \lim_{p_n\to 0} \frac{\frac{1}{2}\ln(p_n+1)}{p_n} \cdot \frac{1}{(p_n)^{\alpha-1}} = \frac{1}{2} \cdot \lim_{p_n\to 0} \frac{1}{(p_n)^{\alpha-1}} = \frac{1}{2} \quad \text{when } \alpha = 1$$

Thus, $\boxed{\alpha = 1}$ and $\boxed{\lambda = \frac{1}{2}}$, and the sequence converges linearly. ■

(b) First, I will show that the limit p^* is 1.
We have

$$\lim_{n\to\infty} 1 + 2^{1-n} + \frac{1}{(n+2)^n} = 1 + \lim_{n\to\infty} 2^{1-n} + \lim_{n\to\infty} \frac{1}{(n+2)^n} = 1 + 0 + 0 = 1$$

Thus, $p^* = 1$.

Then, we show that the convergence is linear by compute λ and α as follows:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{p_{n+1} - 1}{(p_n - 1)^\alpha}\right| &= \lim_{n\to\infty} \left|\frac{2^{-n} + \frac{1}{(n+3)^{n+1}}}{(2^{1-n} + \frac{1}{(n+2)^n})^\alpha}\right| \\
&= \lim_{n\to\infty} \frac{2^{(\alpha-1)n} + \frac{2^{\alpha n}}{(n+3)^{n+1}}}{\left(2 + \frac{2^n}{(n+2)^n}\right)^\alpha} \\
&= \frac{\lim_{n\to\infty} 2^{(\alpha-1)n} + \lim_{n\to\infty} \frac{2^{\alpha n}}{(n+3)^{n+1}}}{\left(2 + \lim_{n\to\infty} \frac{2^n}{(n+2)^n}\right)^\alpha} \\
&= \frac{\lim_{n\to\infty} 2^{(\alpha-1)n}}{2^\alpha} = \frac{1}{2} \quad \text{when } \alpha = 1
\end{aligned}$$

Thus, $\boxed{\alpha = 1}$ and $\boxed{\lambda = \frac{1}{2}}$, and convergence is linear. ∎

Question 2

First, we show that $p^* = 0$ since

$$\lim_{n\to\infty} 10^{(-2^n)} = \lim_{n\to\infty} \left(10^{-1}\right)^{2^n} = \lim_{n\to\infty} \left(\frac{1}{10}\right)^{2^n} = 0$$

Then, we have

$$\lim_{n\to\infty} \left| \frac{\left(\frac{1}{10}\right)^{2^{n+1}}}{\left(\left(\frac{1}{10}\right)^{2^n}\right)^{\alpha}} \right| = \lim_{n\to\infty} \left| \frac{\left(\frac{1}{10}\right)^{2^n \cdot 2}}{\left(\frac{1}{10}\right)^{2^n \cdot \alpha}} \right| = \lim_{n\to\infty} \left| \left(\frac{1}{10}\right)^{(2-\alpha)\cdot 2^n} \right| = 1 \quad \text{when } \alpha = 2$$

Thus, $\boxed{\alpha = 2}$ and $\boxed{\lambda = 1}$, so the sequence converges quadratically ∎

Question 3

(a) By Lagrange interpolation, we have

$$\begin{aligned} P(x) &= f(1) \cdot \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} + f(2) \cdot \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} \\ &+ f(3) \cdot \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} + f(4) \cdot \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} \\ &= -\frac{4}{3}x^3 + 10x^2 - \frac{65}{3}x + 15 \end{aligned}$$

Thus, we have $\boxed{f(x) = -\frac{4}{3}x^3 + 10x^2 - \frac{65}{3}x + 15}$

(b) At the first iteration of Neville's Method, we have the following:

$$P_0 = f(x_0) = f(1) = 2 \qquad P_1 = f(x_1) = f(2) = 1$$

$$P_2 = f(x_2) = f(3) = 4 \qquad P_3 = f(x_3) = f(4) = 3$$

At the second iteration of Neville's Method, we have the following:

$$P_{0,1} = \frac{(x-2)P_0 - (x-1)P_1}{1-2} = 3 - x$$

$$P_{1,2} = \frac{(x-3)P_1 - (x-2)P_2}{2-3} = 3x - 5$$

$$P_{2,3} = \frac{(x-4)P_2 - (x-3)P_3}{3-4} = 7 - x$$

At the third iteration of Neville's Method, we have the following:

$$P_{0,1,2} = \frac{(x-3)P_{0,1} - (x-1)P_{1,2}}{1-3} = 2x^2 - 7x + 7$$

$$P_{1,2,3} = \frac{(x-4)P_{1,2} - (x-2)P_{2,3}}{2-4} = -2x^2 + 13x - 17$$

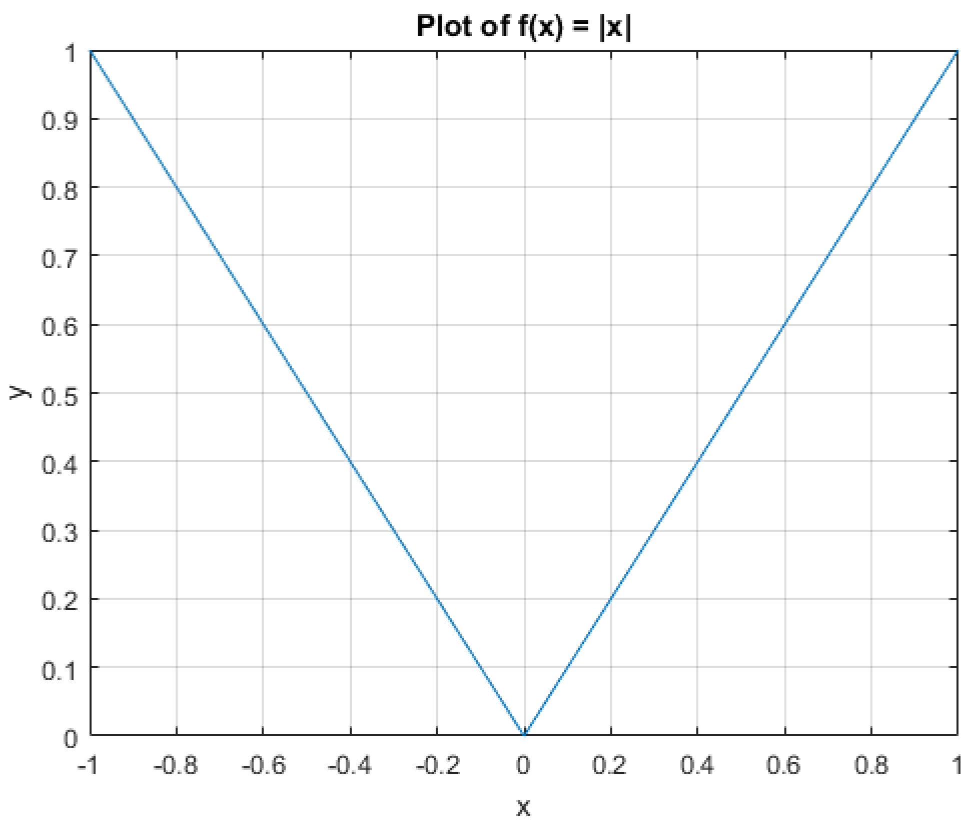
Finally, Nevilles' Method give us the ultimate polynomial:

$$P_{0,1,2,3} = \frac{(x-4)P_{0,1,2} - (x-1)P_{1,2,3}}{1-4} = -\frac{4}{3}x^3 + 10x^2 - \frac{65}{3}x + 15$$

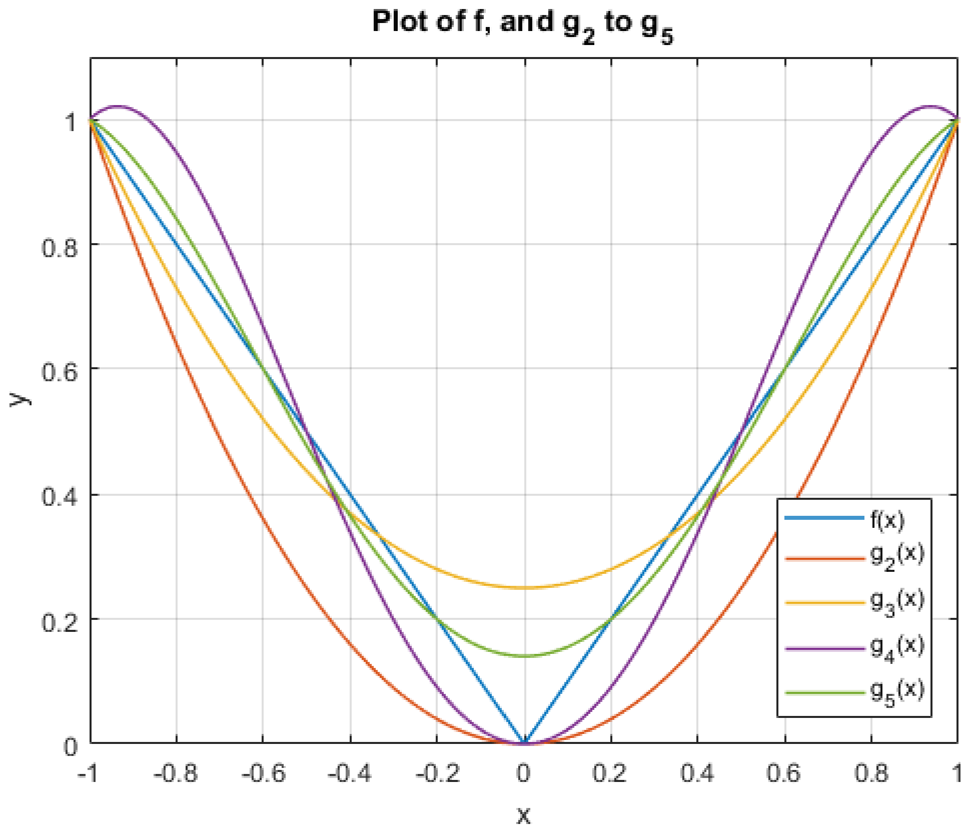
Thus, we have $\boxed{f(x) = -\frac{4}{3}x^3 + 10x^2 - \frac{65}{3}x + 15}$ from Neville's Method.

Question 4 (Coding)

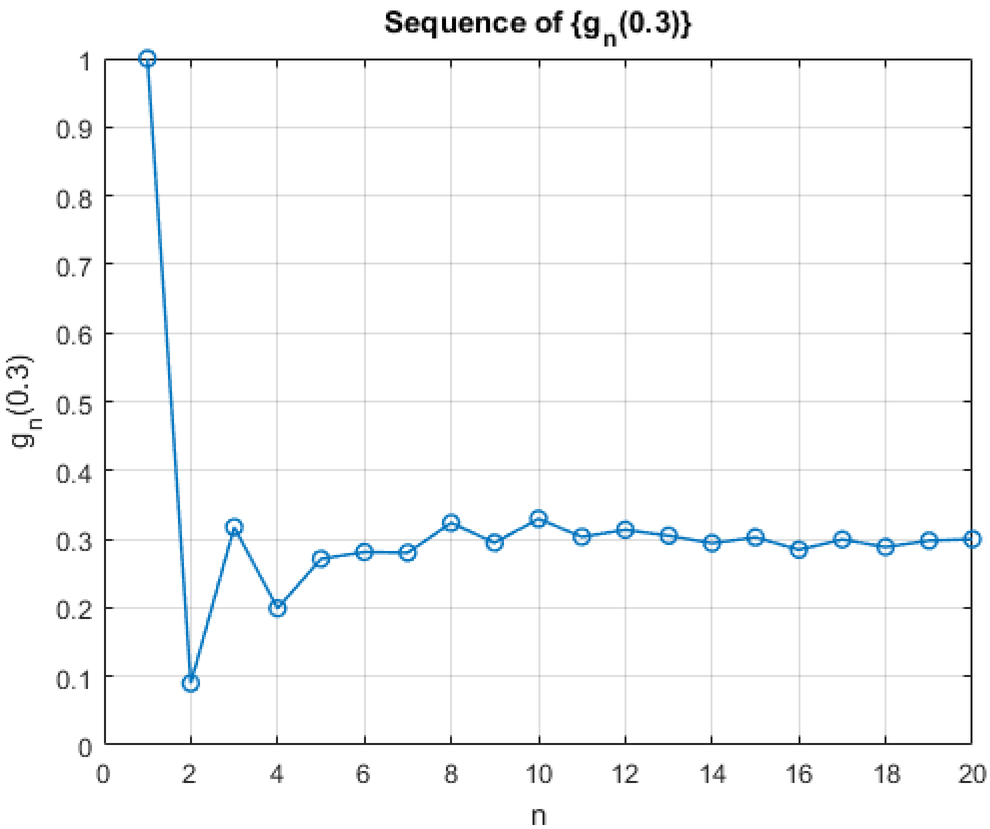
(a) I obtain the following graph from MATLAB:



(b) I obtain the following graph from MATLAB:


Plot of f, and g2 to g5
y
x
1
0.8
0.6
0.4
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
f(x)
g2(x)
g3(x)
g4(x)
g5(x)

(c) I obtain the following graph:



Code for question 4

```
% % MATH 151A HOMEWORK 2
% % QUESTION 4
% % Wang, Zheng
% % Results are recorded in homework2.pdf

%% (a) plot the graph
figure;
fplot(@f, [-1,1]);
xlabel('x');
ylabel('y');
title('Plot of f(x) = |x|');
grid on;

%% (b) Plot
sequence(5)
figure;
fplot(@f, [-1,1],'Linewidth', 1.1);
hold on;

for g=2:5
    [fx,x] = plt_seq(solv(sequence(g),f(sequence(g))));
    plot(fx,x,'Linewidth', 1.1)
end

xlabel('x');
ylabel('y');
legend({'f(x)','g_2(x)', 'g_3(x)','g_4(x)','g_5(x)'},'Location','southeast')
title('Plot of f, and g_2 to g_5');
grid on;
hold off;

%% (c) sequence of g_n
result = ones(1,20);
for n=1:20
    result(1,n) = eval_func( solv(sequence(n),f(sequence(n))) );
end

figure;
plot(1:20, result, 'o-', 'Linewidth', 1.1);
xlabel('n');
ylabel('g_n(0.3)');
title('Sequence of \{g_n(0.3)\}');
grid on;
```

```
%% Function declaration
function y = f(x)
    y = abs(x);
end

function x_nk = sequence(n)
    x_nk_t = ones(n+1,1);
    for k=0:n
        x_nk_t(k+1,1) = -1 + (2*k)/n;
    end
    x_nk = x_nk_t;
end

function coef = solv(x, y)
    n = size(x,1);
    X = repmat(x,1,n);
    for j=1:n
        X(:,j) = X(:,j).^(j-1);
    end
    coef = X\y;
end

function [x, fx] = plt_seq(coef)
    x = sequence(100);
    degree = size(coef,1);
    X = repmat(x,1,degree);
    for i=1:degree
        X(:,i) = X(:,i).^(i-1);
    end
    fx = X*coef;
    x = x';
    fx = fx';
end

function fx = eval_func(coef)
    x = 0.3;
    degree = size(coef,1);
    X = repmat(x,1,degree);
    for i=1:degree
        X(:,i) = X(:,i).^(i-1);
    end
    fx = X*coef;
end
```