

# MATH 151A Homework 3

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## Question 1

- (a) Notice that we have  $x_i = 1, 2, 3$  and  $f(x_i) = 1, \frac{1}{2}, \frac{1}{3}$ . Thus, it follows that the Lagrange interpolation formula is

$$\begin{aligned} P(x) &= f(x_0) \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1) \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2) \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\ &= 1 \cdot \frac{(x-2)(x-3)}{(1-2)(1-3)} + \frac{1}{2} \cdot \frac{(x-1)(x-3)}{(2-1)(2-3)} + \frac{1}{3} \frac{(x-1)(x-2)}{(3-1)(3-2)} \\ &= \frac{1}{6}x^2 - x + \frac{11}{6} \end{aligned}$$

- (b) To use Neville's method, we first have the following:

$$P_0(x) = 1 \quad P_1(x) = \frac{1}{2} \quad P_2(x) = \frac{1}{3}$$

Then the next iteration of Neville's method generates:

$$\begin{aligned} P_{0,1}(x) &= \frac{P_0(x)(x-x_1) - P_1(x)(x-x_0)}{x_0 - x_1} = \frac{3}{2} - \frac{1}{2}x \\ P_{1,2}(x) &= \frac{P_1(x)(x-x_2) - P_2(x)(x-x_1)}{x_1 - x_2} = \frac{5}{6} - \frac{1}{6}x \end{aligned}$$

Finally, we have the complete polynomial

$$P_{0,1,2}(x) = \frac{P_{0,1}(x)(x-x_2) - P_{1,2}(x)(x-x_0)}{x_0 - x_2} = \frac{1}{6}x^2 - x + \frac{11}{6}$$

- (c) Use the forward difference formula:

$$P_{0,1,2}(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1)$$

Firstly, we have

$$f[x_0] = f(x_0) = 1 \quad f[x_1] = f(x_1) = \frac{1}{2} \quad f[x_2] = f(x_2) = \frac{1}{3}$$

Secondly, we have

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = -\frac{1}{2} \quad f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = -\frac{1}{6}$$

Finally, we have  $f[x_0, x_1, x_2] = \frac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2} = \frac{1}{6}$ . Thus, we have

$$P_{0,1,2}(x) = 1 - \frac{1}{2}(x - 1) + \frac{1}{6}(x - 1)(x - 2) = \frac{1}{6}x^2 - x + \frac{11}{6}$$

## Question 2

We have two interval, so the cubic spline gives two cubic equation to estimate the function:

$$S_0(x) = a_0 + b_0(x + 1) + c_0(x + 1)^2 + d_0(x + 1)^3$$

$$S_1(x) = a_1 + b_1x + c_1x^2 + d_1x^3$$

By the condition of the cubic spline, we have the following equations

$$\begin{cases} S_0(x_0) = a_0 = f(x_0) = 1 \\ S_0(x_1) = a_0 + b_0 + c_0 + d_0 = f(x_1) = 1 \\ S_1(x_1) = a_1 = f(x_1) = 1 \\ S_1(x_2) = a_1 + b_1 + c_1 + d_1 = f(x_2) = 2 \\ b_0 + 2c_0 + 3d_0 = b_1 \quad \left(\text{by } S'_0(x_1) = S'_1(x_1)\right) \\ 2c_0 + 6d_0 = 2c_1 \quad \left(\text{by } S''_0(x_1) = S''_1(x_1)\right) \\ 2c_0 = 0 \quad \left(\text{by } S''_0(x_0) = 0\right) \\ 2c_1 + 6d_1 = 0 \quad \left(\text{by } S''_1(x_2) = 0\right) \end{cases}$$

Solving the above equation gives us the following

$$S_0(x) = 1 - \frac{1}{4}(x + 1) + \frac{1}{4}(x + 1)^3$$

$$S_1(x) = 1 + \frac{1}{2}x + \frac{3}{4}x^2 - \frac{1}{4}x^3$$

Thus,

$$S(x) = \begin{cases} 1 - \frac{1}{4}(x + 1) + \frac{1}{4}(x + 1)^3 & x \in [-1, 0) \\ 1 + \frac{1}{2}x + \frac{3}{4}x^2 - \frac{1}{4}x^3 & x \in [0, 1] \end{cases}$$

### Question 3

We first have the following:

$$\begin{aligned} H(x_i) &= \sum_{j=0}^n f(x_j)H_{n,j}(x_i) + \sum_{j=0}^n f'(x_j)\hat{H}_{n,j}(x_i) \\ &= \sum_{j=0}^n f(x_j)(1 - 2(x_i - x_j)L'_{n,j}(x_j))L_{n,j}^2(x_i) + \sum_{j=0}^n f'(x_j)((x_i - x_j)L_{n,j}^2(x_i)) \end{aligned}$$

Since we have  $L_{n,j}^2(x_i) = 0$  if  $j \neq i$  and  $L_{n,j}^2(x_i) = 1$  if  $j = i$ . Thus,  $f'(x_j)((x_i - x_j)L_{n,j}^2(x_i)) = 0 \quad \forall x_i$  and  $(1 - 2(x_i - x_j)L'_{n,j}(x_j))L_{n,j}^2(x_i)$  equals to 0 when  $j \neq i$  and equals to 1 when  $j = i$ . Therefore,  $H(x_i) = f(x_i) \times 1 + 0 = f(x_i)$ .

Then, we have

$$H'(x) = \sum_{j=0}^n f(x_j)H'_{n,j}(x) + \sum_{j=0}^n f'(x_j)\hat{H}'_{n,j}(x)$$

Since  $H'_{n,j}(x) = 2[1 - 2(x - x_j)L'_{n,j}(x_j)]L_{n,j}(x)L'_{n,j}(x) - 2L'_{n,j}(x_j)L_{n,j}^2(x)$ ,  
 $\hat{H}'_{n,j}(x) = L_{n,j}^2(x) + 2L_{n,j}(x)L'_{n,j}(x)(x - x_j)$ .

We have  $H'_{n,j}(x_i) = 0$  for all  $x_i$ ;  $\hat{H}'_{n,j}(x_i) = 0$  when  $i \neq j$  and  $\hat{H}'_{n,j}(x_i) = 1$  when  $j = i$ . Thus,  $H'(x_i) = f'(x_i) \times 1 + 0 = f'(x_i)$ . ■

### Question 4

(a) Suppose that we have  $x_1 = x_0 - h$  and  $x_2 = x_0 + h$

$$\begin{aligned} P(x) &= f(x_0) \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + f(x_1) \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + f(x_2) \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \\ &= f(x_0) \cdot \frac{(x - x_0 + h)(x - x_0 - h)}{-h^2} + f(x_0 - h) \cdot \frac{(x - x_0)(x - x_0 - h)}{2h^2} \\ &\quad + f(x_0 + h) \frac{(x - x_0)(x - x_0 + h)}{2h^2} \end{aligned}$$

(b) The error term is  $E(x) = \frac{f'''(\xi(x))}{6}(x - x_0)(x - x_0 + h)(x - x_0 - h)$ , where  $\xi(x) \in [x_0 - h, x_0 + h]$ .

(c) We have  $f'(x) = P'(x) + E'(x)$ . Thus, it suffice to derive  $P'(x)\Big|_{x=x_0}$  and  $E'(x)\Big|_{x=x_0}$  to find  $f'(x_0)$ .

$$\begin{aligned} P'(x)\Big|_{x=x_0} &= f(x_0) \frac{2x_0 - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} + f(x_1) \frac{2x_0 - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} + f(x_2) \frac{2x_0 - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} \\ &= 0 - \frac{f(x_0 - h)}{2h} + \frac{f(x_0 + h)}{2h} \\ &= \frac{1}{2h} (f(x_0 + h) - f(x_0 - h)) \end{aligned}$$

Also, we have  $E'(x) \Big|_{x=x_0} = -h^2 \cdot \frac{f'''(\xi(x_0))}{6}$

Thus,  $f'(x_0) = \frac{1}{2h} (f(x_0 + h) - f(x_0 - h)) - \frac{h^2}{6} f'''(\xi(x_0))$ , where  $\xi(x) \in [x_0 - h, x_0 + h]$ .

(d) The statement is **TRUE**. This is because if  $f$  is a polynomial of degree less or equal 2, then  $f'''(x) = 0 \quad \forall x \in \mathbb{R}$ , then,  $E'(x_0) = 0$  and  $f'(x_0) = P'(x_0)$ .

(e) The error bound for estimating  $f'(x_0)$  is  $|E| = \left| E'(x) \Big|_{x=x_0} \right| = h^2 \cdot \frac{f'''(\xi(x_0))}{6}$ . If we in addition

assume that  $f'''(\xi(x_0)) \leq M$ , then, we have  $|E| \leq \frac{M \cdot h^2}{6}$ .

In general case (estimating  $f'(x)$  for any  $x$ ), we have

$$|E| = \frac{3x^2 + 3y^2 - h^2 - 6xx_0}{6} f'''(\xi(x)) + \frac{(x - x_0)(x - x_0 + h)(x - x_0 - h)}{6} \cdot \left( f'''(\xi(x)) \right)'$$

## Question 5 (Coding)

See file hw3\_5.m for detail.

The following is a copy of the code:

```
% MATH 151A
% Homework 3, Question 5
% Wang, Zheng (404855295)

%% input vector:
x = input('Please input a vector of x_i (e.g. [1 2 3]):\n');
y = input('Please input a vector of f(x_i) (e.g. [0 1 0]): \n');
if size(x,2) ~= size(y,2)
    error('x and y must be of same length!')
end

%% input a
a = input('Please input value of a:');

%% calculation and output
fprintf('P(a) = %12.8f\n',eval_lag_poly(x,y,a))

%% Function Toolbox
function fx = eval_lag_poly(x, y, a)
    x = x';
    y = y';
    n = size(x,1);
    X = repmat(x,1,n);
    for j=1:n
```

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        X(:,j) = X(:,j).^(j-1);
    end
    coef = X\y;
    A = repmat(a, 1, n);
    for i=1:n
        A(:,i) = A(:,i).^(i-1);
    end
    fx = A*coef;
end

```