MATH 151A Homework 3

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Question 1

(a) Notice that we have $x_i = 1,\ 2,\ 3$ and $f(x_i) = 1,\ \frac{1}{2},\ \frac{1}{3}$. Thus, it follows that the Lagrange interpolation formula is

$$\begin{aligned}
P(x) &= f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\
&= 1\cdot\frac{(x-2)(x-3)}{(1-2)(1-3)} + \frac{1}{2}\cdot\frac{(x-1)(x-3)}{(2-1)(2-3)} + \frac{1}{3}\frac{(x-1)(x-2)}{(3-1)(3-2)} \\
&= \frac{1}{6}x^2 - x + \frac{11}{6}
\end{aligned}$$

(b) To use Neville's method, we first have the following:

$$P_0(x) = 1 \qquad P_1(x) = \frac{1}{2} \qquad P_2(x) = \frac{1}{3}$$

Then the next iteration of Neville's method generates:

$$P_{0,1}(x) = \frac{P_0(x)(x-x_1) - P_1(x)(x-x_0)}{x_0 - x_1} = \frac{3}{2} - \frac{1}{2}x$$

$$P_{1,2}(x) = \frac{P_1(x)(x-x_2) - P_2(x)(x-x_1)}{x_1 - x_2} = \frac{5}{6} - \frac{1}{6}x$$

Finally, we have the complete polynomial

$$P_{0,1,2}(x) = \frac{P_{0,1}(x)(x-x_2) - P_{1,2}(x)(x-x_0)}{x_0 - x_2} = \frac{1}{6}x^2 - x + \frac{11}{6}$$

(c) Use the forward difference formula:

$$P_{0,1,2}(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$$

Firstly, we have

$$f[x_0] = f(x_0) = 1 \qquad f[x_1] = f(x_1) = \frac{1}{2} \qquad f[x_2] = f(x_2) = \frac{1}{3}$$

Secondly, we have

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = -\frac{1}{2} \qquad f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = -\frac{1}{6}$$

Finally, we have $f[x_0, x_1, x_2] = \frac{f[x_0,x_1] - f[x_1,x_2]}{x_0 - x_2} = \frac{1}{6}$. Thus, we have

$$P_{0,1,2}(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{6}(x-1)(x-2) = \frac{1}{6}x^2 - x + \frac{11}{6}$$

Question 2

We have two interval, so the cubic spline gives two cubic equation to estimate the function:

$$S_0(x) = a_0 + b_0(x+1) + c_0(x+1)^2 + d_0(x+1)^3$$

$$S_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3$$

By the condition of the cubic spline, we have the following equations

$$\begin{cases} S_0(x_0) = a_0 = f(x_0) = 1 \\ S_0(x_1) = a_0 + b_0 + c_0 + d_0 = f(x_1) = 1 \\ S_1(x_1) = a_1 = f(x_1) = 1 \\ S_1(x_2) = a_1 + b_1 + c_1 + d_1 = f(x_2) = 2 \\ b_0 + 2c_0 + 3d_0 = b_1 \quad \Big(\text{by } S_0'(x_1) = S_1'(x_1)\Big) \\ 2c_0 + 6d_0 = 2c_1 \quad \Big(\text{by } S_0''(x_1) = S_1''(x_1)\Big) \\ 2c_0 = 0 \quad \Big(\text{by } S_0''(x_0) = 0\Big) \\ 2c_1 + 6d_1 = 0 \quad \Big(\text{by } S_1''(x_2) = 0\Big) \end{cases}$$

Solving the above equation gives us the following

$$S_0(x) = 1 - \frac{1}{4}(x+1) + \frac{1}{4}(x+1)^3$$

$$S_1(x) = 1 + \frac{1}{2}x + \frac{3}{4}x^2 - \frac{1}{4}x^3$$

Thus,

$$S(x) = \begin{cases} 1 - \frac{1}{4}(x+1) + \frac{1}{4}(x+1)^3 & x \in [-1, 0) \\ 1 + \frac{1}{2}x + \frac{3}{4}x^2 - \frac{1}{4}x^3 & x \in [0, 1] \end{cases}$$

Question 3

We first have the following:

$$
\begin{aligned}
H(x_i) &= \sum_{j=0}^{n} f(x_j)H_{n,j}(x_i) + \sum_{j=0}^{n} f^{'}(x_j)\hat{H}_{n,j}(x_i) \\
&= \sum_{j=0}^{n} f(x_j)(1-2(x_i-x_j)L^{'}_{n,j}(x_j))L^2_{n,j}(x_i) + \sum_{j=0}^{n} f^{'}(x_j)((x_i-x_j)L^2_{n,j}(x_i))
\end{aligned}
$$

Since we have $L^2_{n,j}(x_i)=0$ if $j\neq i$ and $L^2_{n,j}(x_i)=1$ if $j=i$. Thus, $f^{'}(x_j)((x_i-x_j)L^2_{n,j}(x_i)) = 0 \quad \forall x_i$ and $(1-2(x_i-x_j)L^{'}_{n,j}(x_j))L^2_{n,j}(x_i)$ equals to 0 when $j\neq i$ and equals to 1 when $j=i$. Therefore, $H(x_i)=f(x_i)\times 1+0=f(x_i)$.

Then, we have

$$
H'(x) = \sum_{j=0}^{n} f(x_j)H^{'}_{n,j}(x) + \sum_{j=0}^{n} f^{'}(x_j)\hat{H}^{'}_{n,j}(x)
$$

Since $H^{'}_{n,j}(x) = 2[1-2(x-x_j)L^{'}_{n,j}(x_j)]L_{n,j}(x)L^{'}_{n,j}(x) - 2L^{'}_{n,j}(x_j)L^2_{n,j}(x)$,
$\hat{H}^{'}_{n,j}(x) = L^2_{n,j}(x) + 2L_{n,j}(x)L^{'}_{n,j}(x)(x-x_j)$.
We have $H^{'}_{n,j}(x_i)=0$ for all x_i; $\hat{H}^{'}_{n,j}(x_i)=0$ when $i\neq j$ and $\hat{H}^{'}_{n,j}(x_i)=1$ when $j=i$. Thus, $H^{'}(x_i)=f^{'}(x_i)\times 1+0=f^{'}(x_i)$. ■

Question 4

(a) Suppose that we have $x_1=x_0-h$ and $x_2=x_0+h$

$$
\begin{aligned}
P(x) &= f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\
&= f(x_0)\cdot\frac{(x-x_0+h)(x-x_0-h)}{-h^2} + f(x_0-h)\cdot\frac{(x-x_0)(x-x_0-h)}{2h^2} \\
&+ f(x_0+h)\frac{(x-x_0)(x-x_0+h)}{2h^2}
\end{aligned}
$$

(b) The error term is $E(x) = \dfrac{f^{'''}(\xi(x))}{6}(x-x_0)(x-x_0+h)(x-x_0-h)$, where $\xi(x)\in[x_0-h,x_0+h]$.

(c) We have $f^{'}(x)=P^{'}(x)+E^{'}(x)$. Thus, it suffice to derive $P^{'}(x)\Big|_{x=x_0}$ and $E^{'}(x)\Big|_{x=x_0}$ to find $f^{'}(x_0)$.

$$
\begin{aligned}
P^{'}(x)\Big|_{x=x_0} &= f(x_0)\frac{2x_0-x_1-x_2}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{2x_0-x_0-x_2}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{2x_0-x_0-x_1}{(x_2-x_0)(x_2-x_1)} \\
&= 0 - \frac{f(x_0-h)}{2h} + \frac{f(x_0+h)}{2h} \\
&= \frac{1}{2h}\left(f(x_0+h)-f(x_0-h)\right)
\end{aligned}
$$

Also, we have $E^{'}(x)\Big|_{x=x_0} = -h^2 \cdot \dfrac{f^{'''}(\xi(x_0))}{6}$

Thus, $\boxed{f^{'}(x_0) = \dfrac{1}{2h}\left(f(x_0+h) - f(x_0-h)\right) - \dfrac{h^2}{6} f^{'''}(\xi(x_0))}$, where $\xi(x) \in [x_0 - h, x_0 + h]$.

(d) The statement is $\boxed{\text{TRUE}}$. This is because if f is a polynomial of degree less or equal 2, then $f^{'''}(x) = 0 \quad \forall x \in \mathbb{R}$, then, $E^{'}(x_0) = 0$ and $f^{'}(x_0) = P^{'}(x_0)$.

(e) The error bound for estimating $f'(x_0)$ is $|E| = \left| E^{'}(x)\Big|_{x=x_0} \right| = h^2 \cdot \dfrac{f^{'''}(\xi(x_0))}{6}$. If we in addition assume that $f^{'''}(\xi(x_0)) \leq M$, then, we have $|E| \leq \dfrac{M \cdot h^2}{6}$.
In general case (estimating $f'(x)$ for any x), we have

$$|E| = \frac{3x^2 + 3y^2 - h^2 - 6xx_0}{6} f^{'''}(\xi(x)) + \frac{(x-x_0)(x-x_0+h)(x-x_0-h)}{6} \cdot \left(f^{'''}(\xi(x))\right)'$$

Question 5 (Coding)

See file hw3_5.m for detail.

The following is a copy of the code:

```
% MATH 151A
% Homework 3, Question 5
% Wang, Zheng

%% input vector:
x = input('Please input a vector of x_i (e.g. [1 2 3]):\n');
y = input('Please input a vector of f(x_i) (e.g. [0 1 0]): \n');
if size(x,2) ~= size(y,2)
    error('x and y must be of same length!')
end

%% input a
a = input('Please input value of a:');

%% calculation and output
fprintf('P(a) = %12.8f\n',eval_lag_poly(x,y,a))

%% Function Toolbox
function fx = eval_lag_poly(x, y, a)
    x = x';
    y = y';
    n = size(x,1);
    X = repmat(x,1,n);
    for j=1:n
```

```
        X(:,j) = X(:,j).^(j-1);
    end
    coef = X\y;
    A = repmat(a, 1, n);
    for i=1:n
        A(:,i) = A(:,i).^(i-1);
    end
    fx = A*coef;
end
```