MATH 151A Homework 1

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Question 1

(a) We will first show there is at least one solution for $f(x) = 0$.

Notice that $f(0.5) = -0.1 < 0$ and $f(1) = 0.3 > 0$. Also, since $f(x)$ is a polynomial function, so it is continous for $x \in \mathbb{R}$, so $f(x)$ is continous $\forall x \in [0.5, 1]$. By Intermediate Value Theorem, since $f(0.5) < 0 < f(1)$, it follows that there exist some $\xi \in (0.5, 1)$ such that $f(\xi) = 0$.

We then show that ξ is unique.

We find that $f'(x) = 2x - 0.7$ is a continous function for all $x \in \mathbb{R}$. Thus for $x \in [0.5, 1]$, $f'(x) \in [0.3, 1.3]$. Suppose that there exist more than one solution for the equation $f(x) = 0$, then we have $f(\xi) = f(\xi') = 0$, where $\xi \neq \xi'$. Therefore, since $f(x)$ is continous, differentiable on (ξ, ξ') (without loss of generality, assume that $\xi < \xi'$), and $f(\xi) = f(\xi') = 0$, by Rolle's Theorem, there exist some $x^* \in (\xi, \xi')$ such that $f'(x^*) = 0$, a contradiction. ■

(b) Claim: $|p_n - p| \leq \frac{1-0.5}{2^n}$.
proof. Since $b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \frac{1}{2^{n-1}}(b_1 - a_1)$. Also, since $p_n = \frac{1}{2}(a_n + b_n)$.

Thus,

$$|p_n - p| \leq \frac{1}{2}(b_n - a_n) = \frac{1}{2^n}(b_1 - a_1) = \frac{1 - 0.5}{2^n}$$

■

Thus, to have $p_k - p \leq 10^{-5}$, we must have $10^{-5} \geq \frac{1-0.5}{2^k}$. By solving the equation, we have $k \geq \frac{5}{\log 2} - 1 \approx 15.6$. Therefore, we must take $\boxed{k = 16}$ steps then the error will be less than 10^{-5}.

Question 2

Take $g(x) = f(x) - x$. Then if $g(x) = 0$, $f(x) = x$. There are two cases for $f(x)$, we will discuss them one by one.

Case 1

If $f(a) = a$ or $f(b) = b$. Then, we have at least one fixed point at $x = a$ or $x = b$, depending on which of the condition before is true.

Case 2

Otherwise, since $f(x) \in [a, b]$ for any $x \in [a, b]$, $f(a) > a$ and $f(b) < b$. Thus $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$. Also, we define $f(x)$ such that it is continous on $[a, b]$. Thus, by Intermediate Value Theorem, there exist some $\xi \in (a, b)$ such that $g(\xi) = 0$. In another word, there exist some $\xi \in (a, b)$ such that $f(\xi) = \xi$.

Thus, there is at least one fix point for $f(x)$. ■

Question 3

(a) $p_1 = \dfrac{p_0^2 + 3}{2p_0} = \dfrac{3^2 + 3}{2 \times 3} = 2$

$p_2 = \dfrac{p_1^2 + 3}{2p_1} = \dfrac{2^2 + 3}{2 \times 2} = 1.75$

(b) **Case 1: if $p_0 = 0$**

If $p_0 = 0$, then the sequence is not defined. Thus, the limit does not exist.

Case 2: if $p_0 \geq \sqrt{3}$ or $p_0 \leq -\sqrt{3}$

Claim: If $p_0 \geq \sqrt{3}$, then $\sqrt{3} \leq p_n \leq p_0$ for all n.
proof.: Obviously $\sqrt{3} \leq p_0 \leq p_0$. Now, suppose that $\sqrt{3} \leq p_n \leq p_0$, then $p_{n+1} = \frac{p_n}{2} + \frac{1}{2} \cdot \frac{3}{p_n}$. Since $\frac{p_n}{2} \leq \frac{p_0}{2}$, and $\frac{1}{2} \cdot \frac{3}{p_n} \leq \frac{1 \cdot 3}{2 \cdot \sqrt{3}} \leq \frac{p_0}{2}$. Thus, $p_{n+1} \leq p_0$. Moreover, since $\frac{p_n}{2} \geq \frac{\sqrt{3}}{2}$, and $\frac{1}{2} \cdot \frac{3}{p_n} \geq \frac{1 \cdot 3}{2 \cdot p_0} \geq \frac{\sqrt{3}}{2}$. Thus, $p_{n+1} \geq \sqrt{3}$. By induction, $\sqrt{3} \leq p_n \leq p_0 \quad \forall n$.
Thus, the sequence $\{p_n\}_{n=0}^{\infty}$ is bounded.

Also, we notice that since $p_n \geq \sqrt{3}$, $\frac{3}{p_n} \leq \sqrt{3} \leq p_n$. Thus, $p_{n+1} = \frac{p_n}{2} + \frac{1}{2} \cdot \frac{3}{p_n} \leq \frac{p_n}{2} + \frac{p_n}{2} = p_n$. Thus, the sequence is monotonic and bound. So, $\{p_n\}_{n=0}^{\infty}$ is convergent, and the limit exists. Suppose the limit is L. We have:

$$L = \frac{L^2 + 3}{2L}$$

Solving the above, we see that $L = -\sqrt{3}$ or $\sqrt{3}$, but since $p_n \geq \sqrt{3}$. Thus, in this case, $L = \sqrt{3}$.

Now, if we consider $p_0 \leq -\sqrt{3}$, we can use the similar arguments as above to show that $p_0 \leq p_n \leq -\sqrt{3}$, and thus the limit L is $-\sqrt{3}$.

Case 3: $-\sqrt{3} < p_0 < \sqrt{3}, \quad p_0 \neq 0$

We will first consider the case that $0 < p_0 < \sqrt{3}$. We verify that $p_1 = \frac{p_0^2+3}{2p_0} > \sqrt{3}$. Thus, we can apply the argument for **case 2** above for sequence $\{p_n\}_{n=1}^{\infty}$, and the limit is $\sqrt{3}$. Likewise, if $-\sqrt{3} < p_0 < 0$, we have limit is $-\sqrt{3}$.

Thus, overall, $\boxed{\text{when } p_0 > 0 \text{, the limit is } \sqrt{3}}$, $\boxed{\text{when } p_0 < 0 \text{, the limit is } -\sqrt{3}}$, and when $p_0 = 0$, the limit does not exists. ■

(c) By the definition of the Newton's method, the sequence of solving the equation $x^2 - 3 = 0$ is

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = p_n - \frac{p_n^2 - 3}{2p_n} = \frac{2p_n^2 - p_n^2 + 3}{2p_n} = \frac{p_n^2 + 3}{2p_n}, \quad n \geq 1$$

and p_0 is given. This is the same as the sequence mentioned in the question. ■

Question 4

(a) The general formula for secant method is

$$p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}$$

Since we are using $f(x) = x^2 - 3$. Thus, we have

$$p_{n+1} = p_n - \frac{(p_n^2 - 3)(p_n - p_{n-1})}{p_n^2 - p_{n-1}^2}$$

If we are using $p_0 = \frac{1}{2}$ and $p_1 = 3$, we have $p_2 = 3 - \frac{(3^2-3)(3-\frac{1}{2})}{3^2-\frac{1}{2}^2} = \boxed{\frac{9}{7}}$.

Also, we have $p_3 = \frac{9}{7} - \frac{(\frac{9}{7}^2-3)(\frac{9}{7}-3)}{\frac{9}{7}^2-3^2} = \boxed{\frac{8}{5}}$.

(b) Using the Method of False Position to solve $f(x) = x^2 - 3$. Since $f(p_0) \cdot f(p_1) = -\frac{33}{2} < 0$, we have

$$p_2 = 3 - \frac{(3^2 - 3)(3 - \frac{1}{2})}{3^2 - \frac{1}{2}^2} = \frac{9}{7}$$

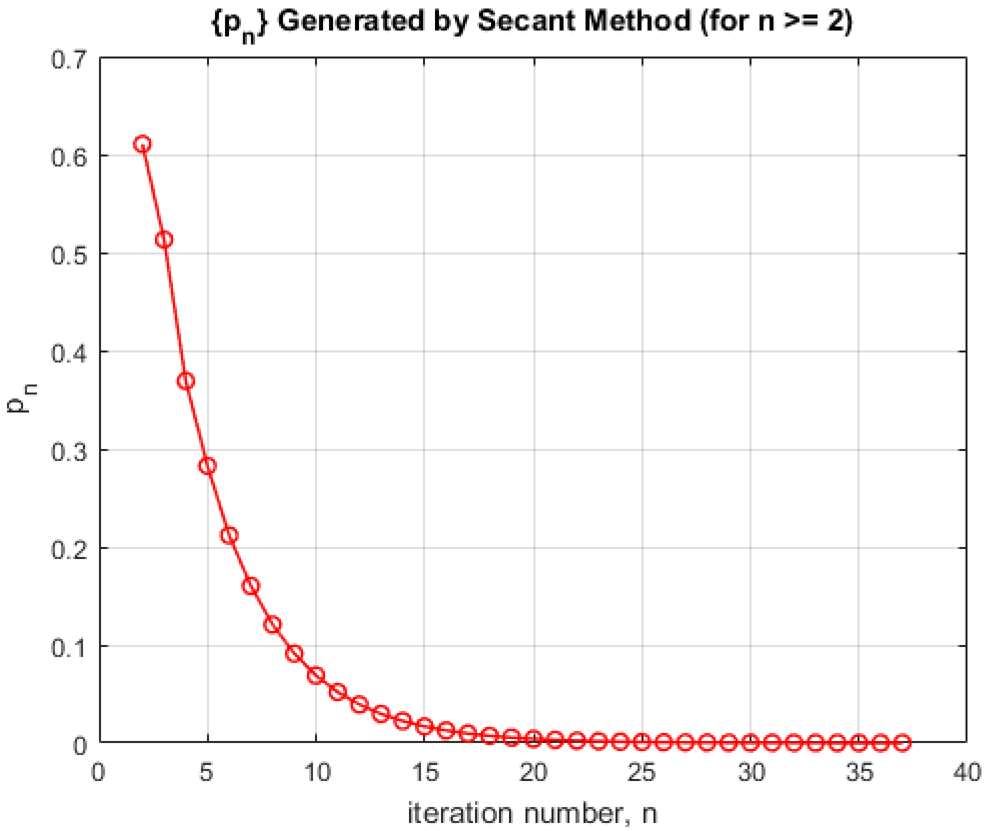
Now, since $f(p_1) \cdot f(p_2) = -\frac{396}{49} < 0$, we have

$$p_3 = \frac{9}{7} - \frac{(\frac{9}{7}^2 - 3)(\frac{9}{7} - 3)}{\frac{9}{7}^2 - 3^2} = \frac{8}{5}$$

Notice that the result is the same for both method this far. This is due to the fact that $f(p_0) \cdot f(p_1) < 0$ and $f(p_1) \cdot f(p_2) < 0$. Otherwise, there should be differences.

Question 5 (Coding)

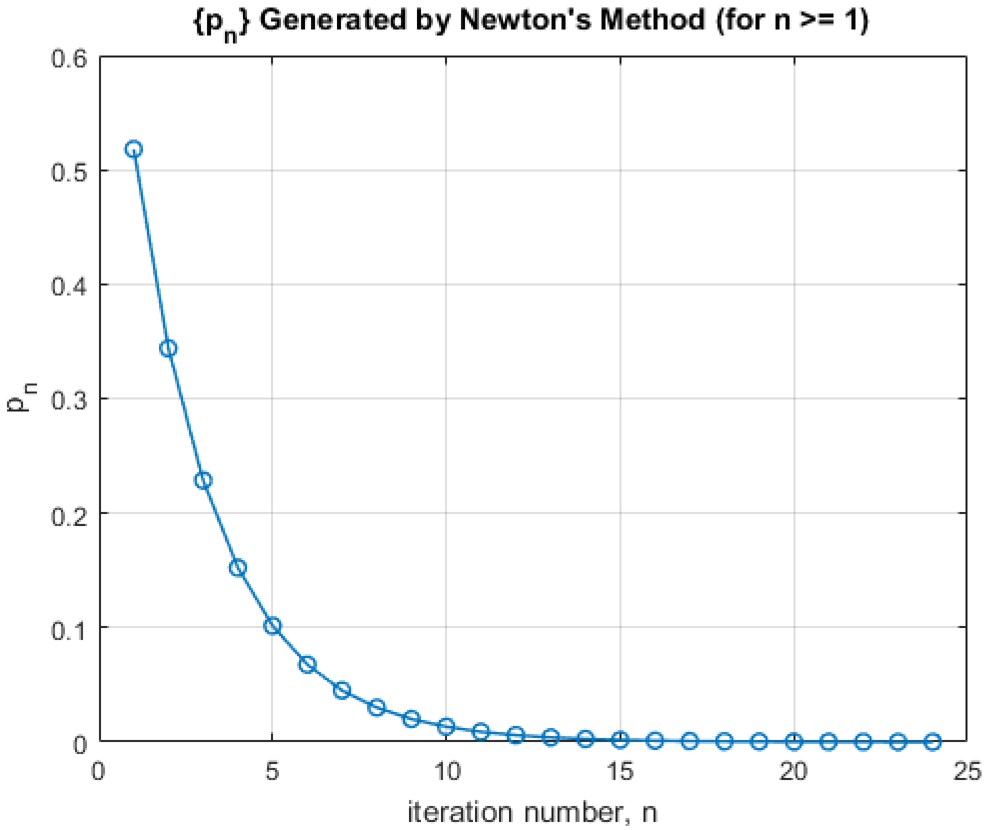
(a) The resulting graph I got is



I also got the following from the console. Thus the solution is $\boxed{2.620 \times 10^{-5}}$.

```
The solution using secant method is 2.619881e-05
```

(b) The resulting graph I got is



I also got the following from the console. Thus the solution is $\boxed{3.053 \times 10^{-5}}$.

```
The solution of Newton's method is 3.052884e-05
```

Code for Question 5

The following matlab codes are used.

```
% % MATH 151A HOMEWORK1
% % QUESTION 5
% % Wang, Zheng (404855295)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  Secant Method to solve f(x) = sin(x)-x = 0
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% initialize
i = 2;
p0 = pi/4;
p1 = 3*pi/8;
q0 = f(p0);
q1 = f(p1);
arr_p = [];
arr_it = [];

% Set the max iteration number
N = 10000;

% Set the tolerance
TO = 10^(-5);


%iteration
while i <= N
    p = p1 - q1*(p1-p0)/(q1-q0);
    % draw line segments
    plot([p0;p1], [q0;q1],'r');
    plot([p;p1],[0,q1],'r');
    plot([p;p], [0;f(p)],'r');
    % stopping condition
    if abs(p - p1) < TO
        fprintf('The solution using secant method is %e\n', p)
        break;
    end
    i = i+1;
    p0 = p1;
    q0 = q1;
    p1 = p;
    q1 = f(p);
    arr_p = [arr_p p];
```

```
    arr_it = [arr_it i-1];
end

% output the failure message if needed
if i > N
fprintf('Secant Method failed after %d iteration, the value found is %e\n',N, p)
end

% plot the graph
figure;
plot(arr_it, arr_p, 'o-r', 'Linewidth', 1.1);
title('\{p_n\} Generated by Secant Method (for n >= 2)');
xlabel('iteration number, n')
ylabel('p_n')
grid on;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  Newton's Method to solve f(x) = sin(x)-x = 0
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% initialize
i = 1;
p0 = pi/4;
q0 = f(p0);
d0 = fprim(p0);
arr_p = [];
arr_it = [];


% Set the max iteration number
N = 10000;

% Set the tolerance
TO = 10^(-5);

%iteration
while i <= N
    p = p0 - q0/d0;
    plot([p;p0], [0;q0],'r');
    plot([p;p], [0;f(p)],'r');
    if abs(p - p1) < TO
        fprintf('The solution of Newton''s method is %e\n', p)
        break
    end
    i = i+1;
    p0 = p;
```

```
    arr_p = [arr_p p0];
    arr_it = [arr_it i-1];
    q0 = f(p0);
    d0 = fprim(p);
end

% output the failure message if needed
if i > N
fprintf('Newton''s Method failed after %d iteration, the value found is %e\n',N, p)
end

% plot the graph
figure;
plot(arr_it, arr_p, 'o-', 'Linewidth', 1.1);
title('\{p_n\} Generated by Newton''s Method (for n >= 1)');
xlabel('iteration number, n')
ylabel('p_n')
grid on;

% % function declaration
function y = f(x)
    y = sin(x) - x;
end

function y = fprim(x)
    y = cos(x) - 1;
end
```