MATH 151B Homework 7

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The two questions I selected are Question 2 and Question 3.

Question 2

The idea behind the inverse power method is that for any matrix A with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, the eigenvalues of $A - qI$ are $\lambda_1 - q, \lambda_2 - q, \ldots, \lambda_n - q$. Moreover, for the matrix $(A - qI)^{-1}$, the eigenvalues are $\dfrac{1}{\lambda_1 - q}, \dfrac{1}{\lambda_2 - q}, \ldots, \dfrac{1}{\lambda_n - q}$. Then when we apply the power method to find the largest eigenvalue of $(A - qI)^{-1}$, the result will be the eigenvalue λ_k that is closest to q, as the closer λ_k is to q, the larger is $\dfrac{1}{\lambda_k - q}$.

I program the following code to run the inverse power method to the matrix $\begin{bmatrix} 3 & 3 & 3 \\ 4 & 9 & 2 \\ 5 & 2 & 3 \end{bmatrix}$:

```
% MATH 151B, HOMEWORK 7, Question 2
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A = [3 3 3; 4 9 2; 5 2 3];
q_1 = 5;
q_2 = 2;
x = [1;1;1];

% q = 5
fprintf('Using q = 5\n')
[lambda, v] = power_method(A,x,q_1,10^-5,10000);
fprintf('The eigenvalue find is:\n')
disp(lambda)
fprintf('The corresponding eigenvector find is:\n')
disp(v)

% q = 2
fprintf('Using q = 2\n')
[lambda, v] = power_method(A,x,q_2,10^-5,10000);
fprintf('The eigenvalue find is:\n')
disp(lambda)
fprintf('The corresponding eigenvector find is:\n')
disp(v)

% INPUTS:
```

```matlab
% A - the matrix to be solved
% q - Find eigenvalue closest to q
% x - The initial vector
% Tol - Tolerance
% N - max iteration number
function [lambda, v] = power_method(A,x,q,Tol,N)
    n = size(A,1);
    M = A - q*eye(n);
    k = 1;
    [~, p] = max(abs(x));
    while k<=N
        y = M\x;
        % mu is the estimate of eigenvalue
        % p is the position of the largest entry
        mu = y(p);
        % update p
        [~, p] = max(abs(y));
        yp = y(p);
        err = max(abs(x-y/yp));
        x = y/yp;
        if err < Tol
            lambda = 1/mu + q;
            v = x;
            return;
        end
        k = k+1;
    end
    lambda = mu;
    v = x;
    fprintf('Reach max iteration')
end
```

I obtian the following from the console:

```
>> q2
Using q = 5
The eigenvalue find is:
    4.0000

The corresponding eigenvector find is:
    0.5294
   -0.8235
    1.0000

Using q = 2
The eigenvalue find is:
    4.0000

The corresponding eigenvector find is:
    0.5294
   -0.8235
    1.0000
```

Thus using $q = 2$ or $q = 5$ both give us the closest eigenvalue $\boxed{4}$ and eigenvector $\boxed{(0.5294, -0.8235, 1.0000)^T}$. This is expected, since the eigenvalues are $12, 4, -1$, so the closest eigenvalue to both 2 and 5 is 4.

Question 3

In the proof to this problem, we will use the following theorem:

Theorem:
For a self-adjoint matrix A (AKA symmetric), suppose the eigenvalues are all distincts, then its eigenvectors are all orthogonal.

proof:
Let $\mathbf{x}$ and $\mathbf{y}$ be eigenvectors of A. Then, we see that the following is true by definition of adjoint of A

$$\langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^T\mathbf{y}\rangle = \langle \mathbf{x}, A\mathbf{y}\rangle$$

Since $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors, say that that they are associtated with distinct eigenvalues λ and μ respectively, then we can expand out the above equation as:

$$\lambda\langle \mathbf{x}, \mathbf{y}\rangle = \langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A\mathbf{y}\rangle = \mu\langle \mathbf{x}, \mathbf{y}\rangle$$

Thus, $(\mu - \lambda)\langle \mathbf{x}, \mathbf{y}\rangle = 0$, but $(\mu - \lambda) \neq 0$, thus, we have $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T\mathbf{y} = 0$, so $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

Then we can proceed prove the statement, let $\mathbf{v_1}, \mathbf{v_2}, \ ..., \mathbf{v_n}$ be the eigenvectors associated with eigenvalues $\lambda_1, \lambda_2, \ ..., \lambda_n$ respectively. Then, for the eigenvector $\mathbf{v_1}$, we have

$$\begin{aligned} B\mathbf{v_1} &= \left(A - \frac{\lambda_1 \mathbf{v_1}\mathbf{v_1}^T}{\mathbf{v_1}^T\mathbf{v_1}}\right)\mathbf{v_1} \\ &= A\mathbf{v_1} - \frac{\lambda_1 \mathbf{v_1}\mathbf{v_1}^T\mathbf{v_1}}{\mathbf{v_1}^T\mathbf{v_1}} \\ &= (A - \lambda_1 I)\mathbf{v_1} \\ &= 0 \cdot \mathbf{v_1} \end{aligned}$$

Thus, we see that $\mathbf{v_1}$ is an eigenvector or B with eigenvalue 0. Next, for $\mathbf{v_k}$, where $k = 2, 3, \ ..., \ n$, we have the following by the **Theorem** we proved above:

$$\begin{aligned} B\mathbf{v_k} &= \left(A - \frac{\lambda_1 \mathbf{v_1}\mathbf{v_1}^T}{\mathbf{v_1}^T\mathbf{v_1}}\right)\mathbf{v_k} \\ &= A\mathbf{v_k} - \frac{\lambda_1 \mathbf{v_1}\mathbf{v_1}^T\mathbf{v_k}}{\mathbf{v_1}^T\mathbf{v_1}} \\ &= A\mathbf{v_k} + \mathbf{0} \\ &= \lambda_k \cdot \mathbf{v_k} \end{aligned}$$

Therefore, we can see that $\mathbf{v_k}$ is an eigenvector of B with eigenvalue λ_k for any $k = 2, 3, .., n$. Combine the statement above, we proved the statement in the question. ■