MATH 151B Homework 1

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Question 1

(a) Let $D = \{(t, y) \,|\, 1 \le t \le 2, -\infty < y < \infty\}$. Obviously, $f(t, y) = \frac{dy}{dt} = \frac{1+y}{t}$ is continous for all $(t, y) \in D$.
Moreover, since

$$\left|\frac{\partial f}{\partial y}(t, y)\right| = \left|\frac{\partial}{\partial y}\frac{1+y}{t}\right| = \left|\frac{1}{t}\right| \le 1, \quad \text{for all } (t, y) \in D$$

Thus, the f satisfy Lipschitz condition on D in the variable y, and the IVP

$$\frac{dy}{dt} = \frac{1+y}{t}, \quad 1 \le t \le 2, \quad y(1) = 2$$

is well-posed. ∎

(b) Let $D = \{(t, y) \,|\, 0 \le t \le 1, -\infty < y < \infty\}$. Obviously, $f(t, y) = \frac{dy}{dt} = y\cos(t)$ is continous for all $(t, y) \in D$.
Moreover, since

$$\left|\frac{\partial f}{\partial y}(t, y)\right| = \left|\cos(t)\,\frac{\partial}{\partial y}y\right| = |\cos(t)| \le 1, \quad \text{for all } (t, y) \in D$$

Thus, the f satisfy Lipschitz condition on D in the variable y, and the IVP

$$\frac{dy}{dt} = y\cos(t), \quad 0 \le t \le 1, \quad y(0) = 1$$

is well-posed. ∎

Question 2

(a) Using the Euler's Method, we have the following

$$\begin{cases} w_{i+1} = w_i + hf(t_i, w_i) \\ w_0 = \alpha = 2 \end{cases}$$

Then, we have $y(1.5) = w_1 = w_0 + hf(t_0, w_0) = 2 + 0.5 \times f(1, 2) = 2 + \frac{1}{2} \times \frac{1+1}{1+2} = \frac{7}{3}$.
Secondly, we have $y(2) = w_2 = w_1 + hf(t_1, w_1) = \frac{7}{3} + \frac{1}{2} \times f\left(1.5, \frac{7}{3}\right) = \frac{65}{24}$.
Thus, we have $\boxed{y(2) = \frac{65}{24} \approx 2.70833}$

(b) For $h = 0.5$, $y(2) \approx 2.708333$.
For $h = 0.2$, $y(2) \approx 2.729166$.
For $h = 0.1$, $y(2) \approx 2.735541$.
For $h = 0.01$, $y(2) \approx 2.741057$.

h	0.5	0.2	0.1	0.01
$y(2)$	2.708333	2.729166	2.735541	2.741057
Error (e)	0.033324	0.012491	0.006116	0.00060

(c) The exact result of $y(2) = \sqrt{2^2 + 2 \times 2 + 6} - 1 \approx 2.741657$. From the summary table above, we can see that when h get closer to 0, the approximation of $y(2)$ generated by Euler's method will get closer to the exact value of $y(2)$.

Question 3

(a) We have the following by the chain rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = -y^2e^{-t} + (2ye^{-t})(y^2e^{-t}) = -y^2e^{-t} + 2y^3e^{-2t}$$

(b) Euler's method give us the following formula

$$\begin{cases} w_{i+1} = w_i + hf(t_i, w_i) \\ w_0 = \alpha = 1 \end{cases}$$

Then, we have $y(0.5) = w_1 = w_0 + hf(t_0, w_0) = 1 + 0.5 \times f(0, 1) = \frac{3}{2}$.
Secondly, we have $y(1) = w_2 = w_1 + hf(t_1, w_1) = \frac{3}{2} + 0.5 \times f\left(0.5, \frac{3}{2}\right) \approx 2.18235$.
Thus, we have approximation from Euler's method: $\boxed{y(1) \approx 2.18235}$.

Taylor method of order 2 give us the following formula

$$\begin{cases} w_{i+1} = w_i + hw_i^2e^{-t_i} + \frac{h^2}{2}(-w_i^2e^{-t_i} + 2w_i^3e^{-2t_i}) \\ w_0 = \alpha = 1 \end{cases}$$

Then, we have $y(0.5) = w_1 = 1 + 0.5 \times 1^2e^0 + \frac{0.5^2}{2}(-1^2e^0 + 2 \times 1^3e^0) = \frac{13}{8}$.
Secondly, we have
$y(1) = w_2 = \frac{13}{8} + 0.5 \times \left(\frac{13}{8}\right)^2 e^{-0.5} + \frac{0.5^2}{2}(-\left(\frac{13}{8}\right)^2 e^{-0.5} + 2 \times \left(\frac{13}{8}\right)^3 e^{-2\times 0.5}) \approx 2.62025$.
Thus, we have approximation from Taylor's method of order 2: $\boxed{y(1) \approx 2.62025}$.

(c) When $h = 0.5$, $y(1)$ approximated by Euler's method is $\boxed{2.182347}$, $y(1)$ approximated by Taylor method of order 2 is $\boxed{2.620252}$.
When $h = 0.1$, $y(1)$ approximated by Euler's method is $\boxed{2.531887}$, $y(1)$ approximated by Taylor method of order 2 is $\boxed{2.711460}$.
When $h = 0.01$, $y(1)$ approximated by Euler's method is $\boxed{2.695519}$, $y(1)$ approximated by

Taylor method of order 2 is $\boxed{2.718205}$.

The output is generated by the following code:

```
fprintf("h = 0.5, Euler's method approximation result is y(1)=%f, Taylor method of order 2 approximation result is y(1)=%f.\n", euler(0.5), taylor(0.5));
fprintf("h = 0.1, Euler's method approximation result is y(1)=%f, Taylor method of order 2 approximation result is y(1)=%f.\n", euler(0.1), taylor(0.1));
fprintf("h = 0.01, Euler's method approximation result is y(1)=%f, Taylor method of order 2 approximation result is y(1)=%f.\n", euler(0.01), taylor(0.01));

function y1 = euler(h)
    y = 1;
    ts = linspace(0,1,int32(1/h)+1);
    for t = ts(1:end-1)
        y = y + h * (y^2*exp(-t));
    end
    y1 = y;
end

function y1 = taylor(h)
    y = 1;
    ts = linspace(0,1,int32(1/h)+1);
    for t = ts(1:end-1)
        y = y + h * (y^2*exp(-t)) + h^2/2 * (-y^2*exp(-t) + 2*y^3*exp(-2*t));
    end
    y1=y;
end
```

(d) The exact solution of $y(1) = e^1 = e \approx 2.718282$.
The result from Euler's method is summarized in the following table:

h	0.5	0.1	0.01
$y(1)$	2.182347	2.531887	2.695519
Error (e)	0.535935	0.186395	0.022763

The result from Taylor method of order 2 is summarized in the following table:

h	0.5	0.1	0.01
$y(1)$	2.620252	2.711460	2.718205
Error (e)	0.098030	0.006822	0.000077

We can see that the while the error of the estimation decrease for both method as h becomes smaller. The error of Taylor method of order 2 is much smaller than the error of Euler's Method. Moreover, the error of Taylor method of order 2 decrease much faster as h decrease than the error of Euler's method.