MATH 151B Homework 2

Zheng Wang (404855295)

April 25, 2019

Question 1

(a) First of all, we derive the term $T^{(4)}(t_i, w_i)$ as the following:

$$\begin{aligned} T^{(4)}(t_i, w_i) &= f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{6} f''(t_i . w_i) + \frac{h^3}{24} f'''(t_i, w_i) \\ &= (t_i^2 - 1) + \frac{h}{2}(2t_i) + \frac{h^2}{6} \cdot 2 + \frac{h^3}{24} \cdot 0 \\ &= t_i^2 + ht_i + \frac{h^2}{3} - 1 \end{aligned}$$

Thus, the Taylor Method of order 4 gives us the following:

$$\begin{cases} w_0 = 0 \\ w_{i+1} = w_i + h(t_i^2 + ht_i + \frac{h^2}{3} - 1) \quad \text{for each } i = 0,\ 1, ...,\ N-1 \end{cases}$$

(b) Using $h = 1$, we have the following from the Taylor Method of order 4:

$$\begin{aligned} y(1) = w_1 &= y(0) + h \cdot (t_0^2 + ht_0 + \frac{h^2}{3} - 1) \\ &= 0 + 1 \times (0^2 + 1 \times 0 + \frac{1^2}{3} - 1) \\ &= \boxed{-\frac{2}{3}} \end{aligned}$$

To find the exact solution, we have the following process:

$$\frac{dy}{dt} = t^2 - 1 \quad \Longrightarrow \quad \int dy = \int t^2 - 1\ dt \quad \Longrightarrow \quad y = \frac{t^3}{3} - t + C$$

Now, since $y(0) = 0$, we have $C = 0$. Thus, the solution to the IVP is $\boxed{y = \frac{t^3}{3} - t}$.

So, $y(1) = \frac{1}{3} - 1 = \boxed{-\frac{2}{3}}$, and the error is 0. The error is zero because the original funtion $y(t)$ is a polynomial of degree 3. Thus, the corresponding local truncation error when we use Talyor's method of degree 4 is $\tau_1(h) = \frac{h^5}{5!} y^{(5)}(\xi_i) = 0$, where $\xi_i \in (0, 1)$. So, the result is exact.

Question 2

(a) Using the multivariable version of Taylor Method, we have the following (R_1 is the remainder term):

$$\begin{aligned} & a_1 f(t,y) + a_2 f(t+\alpha, y+\beta f(t,y)) \\ &= a_1 f(t,y) + a_2[f(t,y) + \alpha \cdot f_t(t,y) + \beta \cdot f(t,y) \cdot f_y(t,y) + R_1(t+\alpha, y+\beta f(t,y))] \\ &= (a_1+a_2) f(t,y) + a_2\alpha \cdot f_t(t,y) + a_2\beta \cdot f(t,y) \cdot f_y(t,y) + a_2 R_1 \end{aligned}$$

When align the coefficient, we leave R_1 out and use:
$a_1 f(t,y) + a_2 f(t+\alpha, y+\beta f(t,y)) \approx (a_1+a_2) f(t,y) + a_2\alpha \cdot f_t(t,y) + a_2\beta \cdot f(t,y) \cdot f_y(t,y)$
By align the coefficient, we have the following:

$$\begin{cases} a_1 + a_2 = 1 \\ a_2 \cdot \alpha = \frac{h}{2} \\ a_2 \cdot \beta = \frac{h}{2} \end{cases}$$

Thus, one way of choosing the coefficient is:

$$\begin{cases} a_1 = 0 \\ a_2 = 1 \\ \alpha = \frac{h}{2} \\ \beta = \frac{h}{2} \end{cases}$$

(b) By setting $a_1 = \frac{1}{2}$, we then have the following solution:

$$\begin{cases} a_1 = \frac{1}{2} \\ a_2 = \frac{1}{2} \\ \alpha = h \\ \beta = h \end{cases}$$

Then, the approximation of $T^{(2)}(t,y)$ break downs to $\frac{1}{2}f(t,y) + \frac{1}{2}f(t+h, y+hf(t,y))$. Thus, the modified Euler Method is given as below:

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))] \quad \text{for each } i = 0, 1, ..., N-1 \end{cases}$$

(c) By the formula of local truncation error of difference method, we have the following:

$$
\begin{aligned}
\tau_{i+1}(h) &= \frac{y_{i+1}-y_i}{h} - \frac{1}{2}[f(t_i,y_i) + f(t_{i+1}, y_i + hf(t_i,y_i))] \\
&= \frac{\cancel{y(t_i)} + h\cancel{y'(t_i)} + \cancel{\frac{h^2}{2}y''(t_i)} + \frac{h^3}{6}y'''(t_i) + \mathcal{O}(h^4) - \cancel{y(t_i)}}{h} \\
&\quad - \frac{1}{2}[\cancel{f(t_i,y_i) + f(t_i,y_i)} + h\cancel{\frac{\partial f}{\partial t}(t_i,y_i) + hf(t_i,y_i)}\frac{\partial f}{\partial y}(t_i,y_i) \\
&\quad + \frac{h^2}{2}\frac{\partial^2 f}{\partial t^2}(t_i,y_i) + \frac{h^2 f(t_i,y_i)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_i,y_i) + h^2 f(t_i,y_i)\frac{\partial^2 f}{\partial t \partial y}(t_i,y_i) + \mathcal{O}(h^3)] \\
&= \frac{h^2}{6}\left(\frac{\partial^2 f}{\partial t^2}(t_i,y_i) + \frac{\partial^2 f}{\partial y^2}(t_i,y_i)f(t_i,y_i)^2 + 2f(t_i,y_i)\frac{\partial^2 f}{\partial t \partial y}(t_i,y_i) + \frac{\partial f}{\partial y}(t_i,y_i)f'(t_i,y_i)\right) \\
&\quad - \frac{1}{2}\left(\frac{h^2}{2}\frac{\partial^2 f}{\partial t^2}(t_i,y_i) + \frac{h^2 f(t_i,y_i)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_i,y_i) + h^2 f(t_i,y_i)\frac{\partial^2 f}{\partial t \partial y}(t_i,y_i)\right) + \mathcal{O}(h^3) \\
&= \frac{h^2}{6}\frac{\partial f}{\partial y}(t_i,y_i)f'(t_i,y_i) - \frac{1}{12}h^2\left(\frac{\partial^2 f}{\partial t^2}(t_i,y_i) + \frac{\partial^2 f}{\partial y^2}(t_i,y_i)f(t_i,y_i)^2 + 2f(t_i,y_i)\frac{\partial^2 f}{\partial t \partial y}(t_i,y_i)\right) \\
&\quad + \mathcal{O}(h^3)
\end{aligned}
$$

By assuming that all partial derivatives are bounded, we have $\tau_{i+1}(h) \leq Mh^2$
Thus, the error is of order 2 (i.e. $\mathcal{O}(h^2)$).

(d) At the first step, we have $w_0 = 0$.
Secondly, we have $y(0.5) = w_1 = 0 + \frac{0.5}{2}(f(0,0) + f(0.5, 0.5 \times f(0,0))) = -\frac{7}{16}$.
Finally, we have $y(1) = -\frac{7}{16} + \frac{0.5}{2}(f(0.5, -\frac{7}{16}) + f(1, 0.5 \times f(0.5, -\frac{7}{16}))) = \boxed{-\frac{5}{8}}$.

Question 4

(a) The Heuns Method is implemented with the function `heun`:

```
% run a test on the functions below
disp(heun(0.1,0,1,1,@f))


% Use function the following function as a test
% the actual soltuion is y = exp(t)
function dydt = f(t,y)
    dydt = y^2*exp(-t);
end

% Heun's Method
% input h, a, b, alpha (initial condition), func
function y = heun(h,a,b,alpha,func)
    t = a;
    w = alpha;
    N = (b-a)/h;
    for i = 1:N
        K1 = h/3 * func(t,w);
```

```
        K2 = 2/3 * h * func(t + h/3, w + K1);
        K3 = 3 * func(t + 2/3*h, w + K2);
        w = w + h/4 * ( func(t,w) + K3 );
        t = a + i*h;
    end
    y = w;
end
```

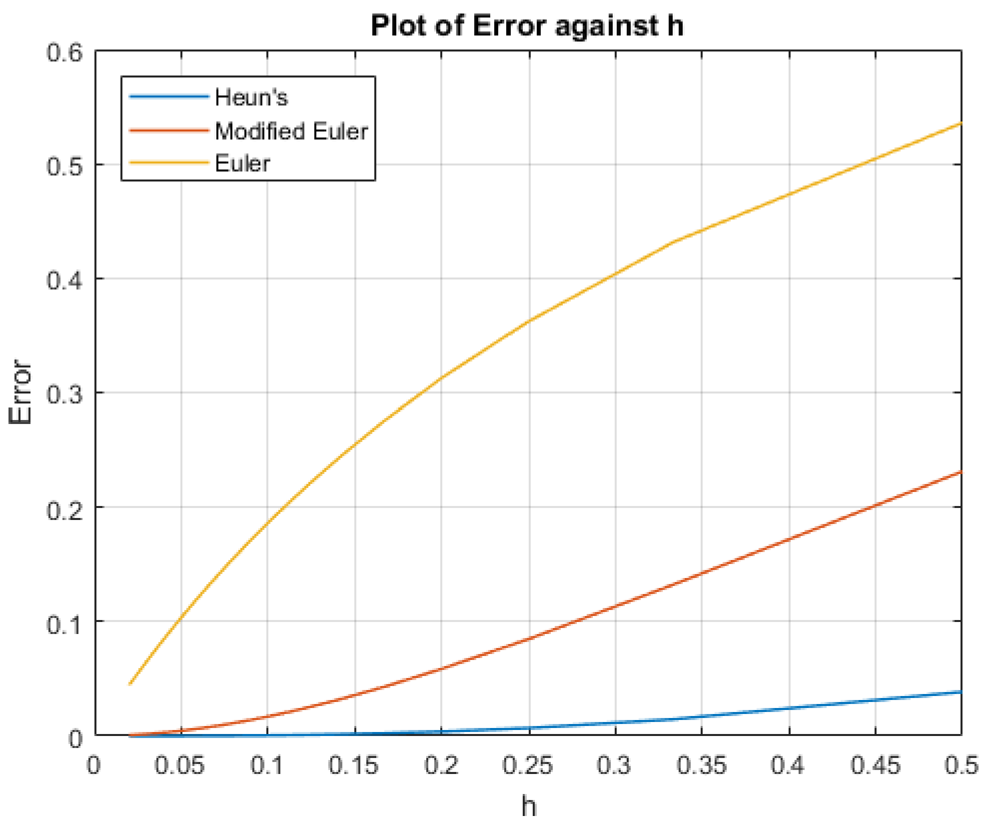
The output from the console is $\boxed{2.7177}$, which is approximately the correct result of the $y(1) = e$.

```
>> heuns
    2.7177
```

(b) I used following IVP for generating this plot:

$$y'(t) = y^2 e^{-t},\ 0 \leq t \leq 1,\ y(0) = 1$$

The plot I obtained by runing Heun's Method, Modified Euler's Method, and Euler's Method are given below:



We can see that the error of Euler's method is approximately linear, and it is the largest among the three method. The error of Modified Euler's method is smaller than Euler's method, and the increasing trending is approximately quadratic. Finally, the error of the Heun's Method is the smallest among the three method, and the increasing tending of error as h increase is approximatly cubic. Thus, this verifies that error of Euler's Method is $\mathcal{O}(h)$, the error of Modified Euler's Method is $\mathcal{O}(h^2)$, and the error of Heun's Method is $\mathcal{O}(h^3)$.

The code for generating the plot is given below:

```matlab
% collect data
hs = (2:50).^-1;
i = 1;
err_heun = zeros(1,size(hs,1));
err_modi_euler = zeros(1,size(hs,1));
err_euler = zeros(1,size(hs,1));
for h = hs
    err_heun(i) = abs(heun(h,0,1,1,@f)-sol(1));
    err_modi_euler(i) = abs(modi_euler(h,0,1,1,@f)-sol(1));
    err_euler(i) = abs(euler(h,0,1,1,@f)-sol(1));
    i = i+1;
end
% make the plot
figure;
plot(hs,err_heun,'Linewidth', 1.1);
hold on;
plot(hs,err_modi_euler,'Linewidth', 1.1);
plot(hs,err_euler,'Linewidth', 1.1);
xlabel('h');
ylabel('Error');
legend({'Heun''s','Modified Euler', 'Euler'},'Location','northwest')
title('Plot of Error against h');
grid on;
hold off;

% Use function the following function as a test
% the actual soltuion is y = exp(t)
function dydt = f(t,y)
    dydt = y^2*exp(-t);
end

% solution
function s = sol(t)
    s = exp(t);
end

% Heun's Method
% input h, a, b, alpha (initial condition), func
function y = heun(h,a,b,alpha,func)
    t = a;
    w = alpha;
    N = (b-a)/h;
    for i = 1:N
        K1 = h/3 * func(t,w);
        K2 = 2/3 * h * func(t + h/3, w + K1);
        K3 = 3 * func(t + 2/3*h, w + K2);
        w = w + h/4 * ( func(t,w) + K3 );
```

```
        t = a + i*h;
    end
    y = w;
end

% Modified Euler's Method
% input h, a, b, alpha (initial condition), func
function y = modi_euler(h,a,b,alpha,func)
    t = a;
    w = alpha;
    N = (b-a)/h;
    for i = 1:N
        K1 = h * func(t,w);
        K2 = func(t + h, w + K1);
        w = w + h/2 * ( func(t,w) + K2 );
        t = a + i*h;
    end
    y = w;
end

% Euler's Method
% input h, a, b, alpha (initial condition), func
function y = euler(h,a,b,alpha,func)
    t = a;
    w = alpha;
    N = (b-a)/h;
    for i = 1:N
        w = w + h * func(t,w);
        t = a + i*h;
    end
    y = w;
end
```