MATH 151B Homework 5

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Question 1

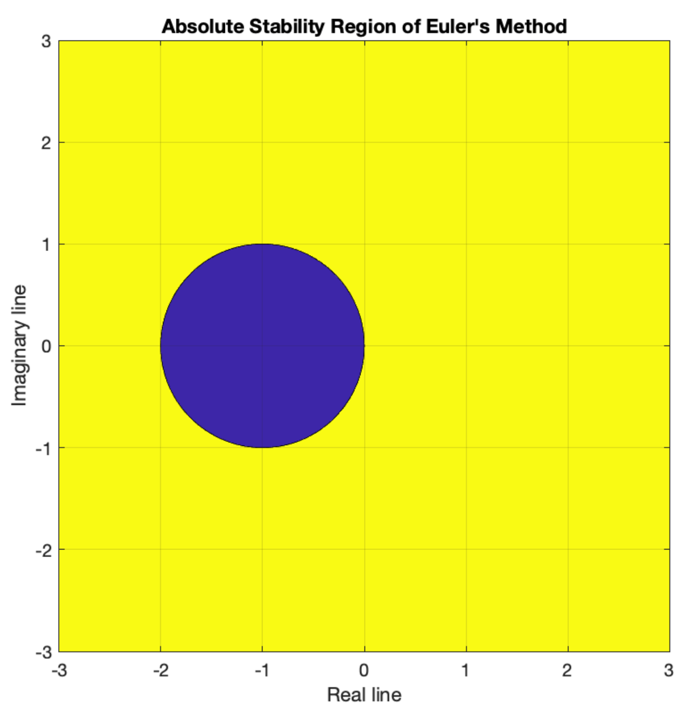
We will first discuss the region of absolute stability for **Euler's Method**. In Euler's Method, we have the following equation.

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h \cdot f(t_i, w_i) \end{cases}$$

Thus, to solve the test equation $y' = \lambda y, \quad y(0) = \alpha, \quad \text{where } \lambda < 0$, we have the following equation

$$\begin{aligned} w_{i+1} = w_i \cdot Q(\lambda h) &= w_i + h \cdot \lambda w_i \\ &= w_i \cdot (1 + h\lambda) \end{aligned}$$

Therefore, we have $Q(\lambda h) = 1 + \lambda h$. Then the following stability region (shown in blue) is obtianed.



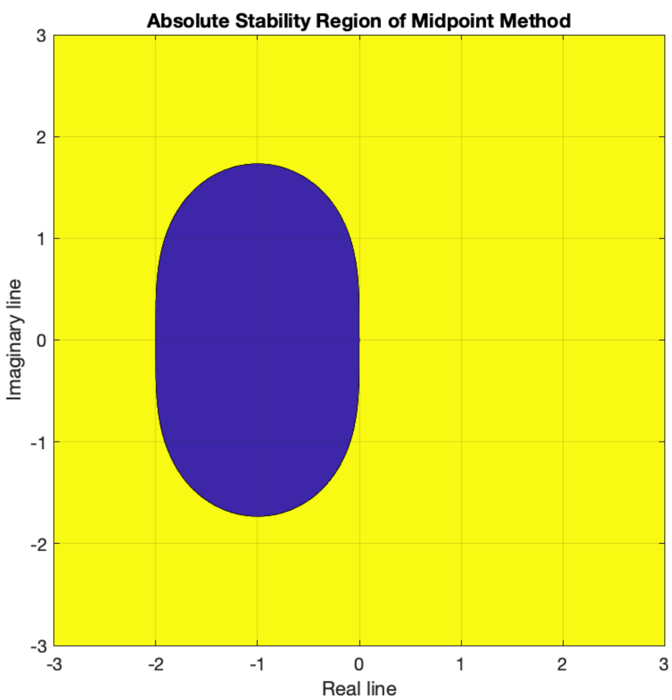
Then, for the **Midpoint Method**, we have the following equation

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h \cdot f(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)) \end{cases}$$

Therefore, we have the following when solving the test equation:

$$\begin{aligned} w_{i+1} &= w_i + h \cdot \left(\lambda \left(w_i + \frac{h}{2}\lambda w_i\right)\right) \\ &= w_i \cdot \left(1 + h\lambda + \frac{\lambda^2 h^2}{2}\right) \end{aligned}$$

Therefore, we have $Q(\lambda h) = 1 + \lambda h + \dfrac{\lambda^2 h^2}{2}$. Then we generate the following absolute stability region (colored blue):



The code to generate the above plots are given below:

```
% Math 151b HW5
% Question 1

% generate sequence of the real and imaginary axis
re = -3:0.01:3;
img = -3:0.01:3;

% run contour map, plot only Z from 0 to 1, shown in blue
figure;
```

```
contourf(re,img,Q_euler(re,img),[0 1]);
title("Absolute Stability Region of Euler's Method");
xlabel("Real Line")
ylabel("Imaginary Line")
grid on;
pbaspect([1 1 1]);


% run contour map, plot only Z from 0 to 1, shown in blue
figure;
contourf(re,img,Q_mid(re,img),[0 1]);
title("Absolute Stability Region of Midpoint Method");
xlabel("Real Line")
ylabel("Imaginary Line")
grid on;
pbaspect([1 1 1]);


% Generate a grid of |Q(lambda h)|
% otherwise
% real: real sequence, img: imaginary sequence
function z = Q_euler(rel,img)
    M = zeros(size(img,2), size(rel,2));
    for j = 1:size(rel,2)
        for k = 1:size(img,2)
            lambda_h = rel(j)+img(k)*1i;
            M(k,j) = abs(1+lambda_h);
        end
    end
    z = M;
end


% Generate a grid of |Q(lambda h)|
% otherwise
% real: real sequence, img: imaginary sequence
function z = Q_mid(rel,img)
    M = zeros(size(img,2), size(rel,2));
    for j = 1:size(rel,2)
        for k = 1:size(img,2)
            lambda_h = rel(j)+img(k)*1i;
            M(k,j) = abs(1+lambda_h+0.5*lambda_h^2);
        end
    end
    z = M;
end
```

Question 2

We will first derive the equation of estimating y_0'. Suppose that $x_0 = 0$ and $x_i = x_0 + ih = ih$. Then, by Lagrange interpolation, we have

$$\begin{aligned} y(x) = y(x_0) \cdot \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y(x_1) \cdot \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \\ + y(x_2) \cdot \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} + \frac{(x-x_0)(x-x_1)(x-x_2)}{6} f'''(\xi(x)) \end{aligned}$$

Therefore, by taking the derivative and evaluate at x_0, we have the following

$$\begin{aligned} y'(x_0) &= y(x_0)\left[\frac{2x_0 - x_1 - x_2}{(x_0-x_1)(x_0-x_2)}\right] + y(x_1)\left[\frac{2x_0 - x_0 - x_2}{(x_1-x_0)(x_1-x_2)}\right] \\ &\quad + y(x_2)\left[\frac{2x_0 - x_0 - x_1}{(x_2-x_0)(x_2-x_1)}\right] + \frac{1}{6} f'''(\xi_0)(x_0-x_1)(x_0-x_2) \\ &= -\frac{3}{2h} w_0 + \frac{2}{h} w_1 - \frac{1}{2h} w_2 + \frac{h^2}{3} f'''(\xi_0) \qquad \text{where } \xi_0 \in (0,h) \end{aligned}$$

Thus, the estimate of $y'(0)$ has accuracy of $\mathcal{O}(h^2)$, and by truncating the $f'''(\xi_0)$ term, we then obtain the following equation of w_0, w_1, and w_2:

$$0 = 2h \cdot y'(0) = -3w_0 + 4w_1 - w_2$$

Next, since the BVP has the form $y'' = p(x)y' + q(x)y + r(x)$, by using the cenrtal-finite-difference formula, we obtian the following for w_i, where $i = 1, 2, ..., N-1$:

$$-\left(1 + \frac{h}{2} p(x_i)\right) w_{i-1} + (2 + h^2 q(x_i)) w_i - \left(1 - \frac{h}{2} p(x_i)\right) w_{i+1} = -h^2 r(x_i)$$

For the BVP we are solving, as $p(x) = 0$, $q(x) = 4$, and $r(x) = -4x$; we simplify the above equation to the following:

$$w_{i-1} - (2 + 4h^2) w_i + w_{i+1} = -4h^2 \cdot x_i$$

Finally, as the last equation is $w_N = 1$, we can write the system in the matrix form $A\vec{w} = \vec{b}$, where

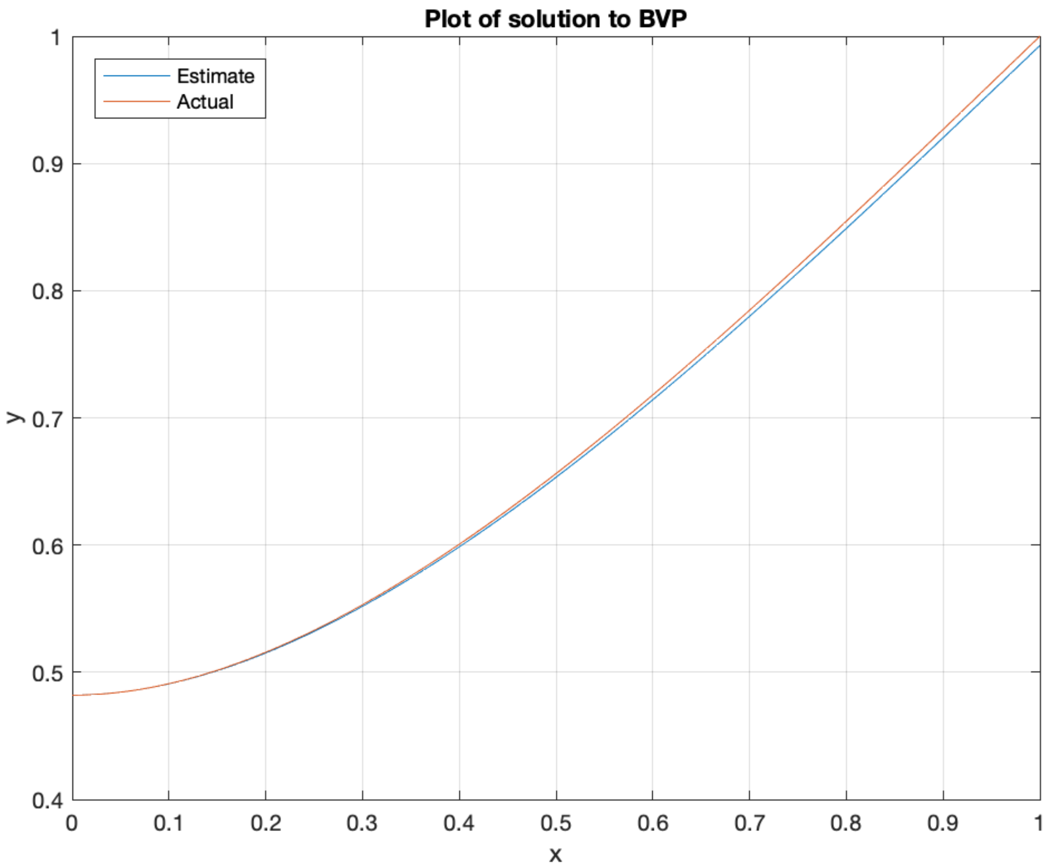
$$A = \begin{bmatrix} -3 & 4 & -1 & 0 & \dots & 0 \\ 1 & -(2+4h^2) & 1 & 0 & \dots & 0 \\ 0 & 1 & -(2+4h^2) & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & 1 & -(2+4h^2) \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-2} \\ w_{N-1} \end{bmatrix} \qquad \text{and} \qquad \vec{b} = \begin{bmatrix} 0 \\ -4h^2 x_1 \\ \vdots \\ -4h^2 x_{N-2} \\ -4h^2 x_{N-1} - 1 \end{bmatrix} \qquad \text{(Note: } w_N = 1 \text{ from question)}$$

Then we can generate the solution to the BVP with the following code:

```
% Math 151b HW5
% Question 2

N = 100;
y = solve_bvp(N);
x = linspace(0,1,N);

% show the plot
plot(x,f(x));
hold on;
plot(x,y);
grid on;
title("Plot of solution to BVP");
xlabel("x")
ylabel("y")
legend('Estimate','Actual','Location','northwest')
hold off;

% display y(0)
disp(y(1))


% actual answer
function y = f(x)
    y = (2*exp(4)*x + 2*x + exp(4-2*x) - exp(2*x))/(2+2*exp(4));
end

% function that solve the BVP in the question
% input: N - number of grids
function w = solve_bvp(N)
    h = 1/N;
    % fill A
    A = zeros(N,N);
    A(1,1) = -3;
    A(1,2) = 4;
    A(1,3) = -1;
    for i = 2:(N-1)
        A(i,i-1) = 1;
        A(i,i) = -(2+4*h^2);
        A(i,i+1) = 1;
    end
    A(N,N-1) = 1;
    A(N,N) = -(2+4*h^2);
    % fill b
    b = zeros(N,1);
    b(1) = 0;
    for i = 2:(N-1)
        b(i) = -4*h^2*(i-1)*h;
    end
    b(N) = -4*h^2*(N-1)*h - 1;
    % solve for w
    w = A\b;
end
```

The plot obtained from the code above is:



The estimate of $y(0)$ is $\boxed{0.4821}$. The console will output the following:

```
>> q2
    0.4821
```