MATH 151B Homework 4

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Question 1

Using the equation of the Newton's backward-difference formula, we have the following by seting $t = t_i + sh$:

$$\begin{aligned}
y(t_{i+1}) &\approx y(t_i) + \int_{t_i}^{t_{i+1}} \sum_{k=0}^{3} (-1)^k \binom{-s}{k} \nabla^K f(t_i, y(t_i))\ dt \\
&= y(t_i) + \sum_{k=0}^{3} \nabla^k f(t_i, y(t_i)) h (-1)^k \int_0^1 \binom{-s}{k}\ ds \qquad \text{(By taking } dt = h\ ds\text{)} \\
&= y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i)) + \frac{5}{12}\nabla^2 f(t_i, y(t_i)) + \frac{3}{8}\nabla^3 f(t_i, y(t_i))\right] \\
&= y(t_i) + h\left[\frac{55}{24} f(t_i, y(t_i)) - \frac{59}{24} f(t_{i-1}, y(t_{i-1})) + \frac{37}{24} f(t_{i-2}, y(t_{i-2})) - \frac{3}{8} f(t_{i-3}, y(t_{i-3}))\right]
\end{aligned}$$

By aligning the coefficient of above equation with the equation given in the question. i.e.

$$y(t_i) + h\left[a \cdot f(t_i, y(t_i)) + b \cdot f(t_{i-1}, y(t_{i-1})) + c \cdot f(t_{i-2}, y(t_{i-2})) + d \cdot f(t_{i-3}, y(t_{i-3}))\right]$$

We have

$$\begin{cases} a = \frac{55}{24} \\ b = -\frac{59}{24} \\ c = \frac{37}{24} \\ d = -\frac{3}{8} \end{cases}$$

■

Question 2

(a) We first expand $y(t_{i+1})$:

$$\begin{aligned}
y(t_{i+1}) &= y(t_i) + h \cdot y'(t_i) + \frac{h^2}{2} \cdot y''(t_i) + \mathcal{O}(h^3) \\
&= y(t_i) + h \cdot f(t_i, y(t_i)) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t}(t_i, y(t_i)) + \frac{\partial f}{\partial y}(t_i, y(t_i)) \cdot f(t_i, y(t_i))\right) + \mathcal{O}(h^3)
\end{aligned}$$

Then, we expand the equation $w_{i+1} = w_i + a \cdot f(t_{i+1}, w_{i+1}) + b \cdot f(t_i, w_i)$:

$$\begin{aligned}
w_{i+1} &\approx w_i + a \cdot \left(f(t_i, w_i) + h \cdot \frac{\partial f}{\partial t}(t_i, w_i) + (w_{i+1} - w_i) \cdot \frac{\partial f}{\partial y}(t_i, w_i) \right) + bf(t_i, w_i) \\
&\approx w_i + a \cdot \left(f(t_i, w_i) + h \cdot \frac{\partial f}{\partial t}(t_i, w_i) + hf(t_i, w_i) \cdot \frac{\partial f}{\partial y}(t_i, w_i) \right) + b \cdot f(t_i, w_i) \\
&= w_i + (a+b) \cdot f(t_i, w_i) + ah \cdot \left(\frac{\partial f}{\partial t}(t_i, w_i) + \frac{\partial f}{\partial y}(t_i, w_i) \cdot f(t_i, w_i) \right) \\
&= y(t_i) + h \cdot f(t_i, y(t_i)) + \frac{h^2}{2} \left(\frac{\partial f}{\partial t}(t_i, y(t_i)) + \frac{\partial f}{\partial y}(t_i, y(t_i)) \cdot f(t_i, y(t_i)) \right)
\end{aligned}$$

By aligning the coefficient, we obtaint the following equation system:

$$\begin{cases} a + b = h \\ ah = \frac{h^2}{2} \end{cases}$$

Therefore, $\boxed{a = \frac{h}{2},\ b = \frac{h}{2}}$. ■

(b) We first let $u(t) = y(t),\ v(t) = y'(t)$. Then the we can obtain the system of IVP as the following:

$$\begin{cases} u'(t) = v & u(0) = 0 \\ v'(t) = 4u + 6e^{-t} & v(0) = 0 \end{cases}$$

Now, call $f_{(u)}(t, u, v) = \frac{du}{dt} = v$ and $f_{(v)}(t, u, v) = \frac{dv}{dt} = 4u + 6e^{-t}$. Denoting U_i be the estimate of $u(t_i)$ and V_i be the estimate of $v(t_i)$, we obtian the following by using generalization of the Mid-point method:

$$\begin{aligned}
U_{i+1} &= U_i + h \cdot f_{(u)} \left(t_i + \frac{h}{2},\ U_i + \frac{h}{2} f_{(u)}(t_i, U_i, V_i),\ V_i + \frac{h}{2} f_{(v)}(t_i, U_i, V_i) \right) \\
&= U_i + h \cdot \left(V_i + \frac{h}{2} f_{(v)}(t_i, U_i, V_i) \right) && \text{(Since } f_{(u)}(t, u, v) = v\text{)} \\
&= U_i + h \cdot \left(V_i + \frac{h}{2} \cdot \left(4U_i + 6e^{-t_i} \right) \right) && \text{(Since } f_{(v)}(t, u, v) = 4u + 6e^{-t}\text{)} \\
&= U_i + h \cdot \left(V_i + 2hU_i + 3he^{-t_i} \right)
\end{aligned}$$

and that $U_0 = u(0) = 0$.
Likewise, we can obtian the following for V_{i+1}:

$$\begin{aligned}
V_{i+1} &= V_i + h \cdot f_{(v)} \left(t_i + \frac{h}{2},\ U_i + \frac{h}{2} f_{(u)}(t_i, U_i, V_i),\ V_i + \frac{h}{2} f_{(v)}(t_i, U_i, V_i) \right) \\
&= V_i + h \cdot \left(4 \left(U_i + \frac{h}{2} f_{(u)}(t_i, U_i, V_i) \right) + 6e^{-\left(t_i + \frac{h}{2}\right)} \right) \\
&= V_i + h \cdot \left(4U_i + 2hV_i + 6e^{-\left(t_i + \frac{h}{2}\right)} \right)
\end{aligned}$$

and that $V_0 = v(0) = 0$.

Notice that for each step we know $t_i,\ U_i,\ V_i$, thus, we can easily solve $t_{i+1},\ U_{i+1},\ V_{i+1}$ from the above equation.

Secondly, we generalize the formula we have for the One-step implicit method. From **(a)**, we see that $w_{i+1} = w_i + \frac{h}{2} \cdot f(t_{i+1}, w_{i+1}) + \frac{h}{2} f(t_i, w_i)$. Thus, the generalization give us following:

$$\begin{cases} U_{i+1} = U_i + \frac{h}{2} \cdot f_{(u)}(t_{i+1}, U_{i+1}, V_{i+1}) + \frac{h}{2} \cdot f_{(u)}(t_i, U_i, V_i) \\ V_{i+1} = V_i + \frac{h}{2} \cdot f_{(v)}(t_{i+1}, U_{i+1}, V_{i+1}) + \frac{h}{2} \cdot f_{(v)}(t_i, U_i, V_i) \end{cases}$$

Therefore, with the formula from above, we have the code which implement the above method and solve for the IVP given.
Note: The following code consider correcting both U_{i+1} and V_{i+1}

```
% Math 151b, Homework 4, Question 2(b),2(c)
% Wang, Zheng

% test with the function given in part (a)
% the correct solution is about 3.16177
pred_cor(0.1, 0, 1, 0, 0, @dudt, @dvdt)

% du/dt
function f_u = dudt(t,u,v)
    f_u = t*0 + u*0 + v;
end

% dv/dt
function f_v = dvdt(t,u,v)
    f_v = 4*u + 6*exp(-t) + 0*v;
end

% function of predictor-corrector method
% input h, a, b, alpha_u (initial condition of u), alpha_v, f_u, f_v
function y = pred_cor(h,a,b,alpha_u,alpha_v,f_u,f_v)
    t = a;
    U = alpha_u;
    V = alpha_v;
    N = (b-a)/h;
    for i = 1:N
        % Predictor Step
        Ku_1 = U + h/2 * f_u(t,U,V);
        Kv_1 = V + h/2 * f_v(t,U,V);
        Ku_2 = f_u(t + h/2, Ku_1, Kv_1);
        Kv_2 = f_v(t + h/2, Ku_1, Kv_1);
        U_temp = U; % store U_i
        V_temp = V; % store V_i
        t_temp = t; % store t_i
        U = U + h*Ku_2; % update to U_i+1 (prediction)
        V = V + h*Kv_2; % update to V_i+1 (prediction)
        t = a + i*h;    % update to t_i+1
        % Corrector step
        % Correct U and V with the one-step implicit method and pass
```

```
39          % to next iteration
40          U = U_temp + h/2 * f_u(t,U,V) + h/2 * f_u(t_temp,U_temp,V_temp);
41          V = V_temp + h/2 * f_v(t,U,V) + h/2 * f_v(t_temp,U_temp,V_temp);
42      end
43      y = U;
44  end
```

(c) The result from the console is:

```
>> predictor_correct

ans =

    3.1798
```

Thus, the estimate of $\boxed{y(1) = 3.1798}$.
(*Note*: Another way to implement this predictor corrector method is NOT correcting V_{i+1} at all. In the code, this can be done by commenting line 41. The result obtianed is then 3.1400. In general, both method work fine for this IVP, as when h is set to be small, both methods give 3.1618, which is close to correct answer $\frac{e^2}{2} + \frac{3e^{-2}}{2} - 2e^{-1} \approx 3.16177$.)

Question 3

From the question, $f(t_{i+1}, w_{i+1}) = w_{i+1} \cdot g(t_{i+1})$, where $g(t)$ is some known function. Then, we substitute this into the equation of Adams-Moulton 3-step implicit method, we have:

$$w_{i+1} = w_i + \frac{3}{8}h \cdot g(t_{i+1}) \cdot w_{i+1} + \frac{h}{24}\left(19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})\right)$$

Next, we can move the term $\frac{3}{8}h \cdot g(t_{i+1}) \cdot w_{i+1}$ to the left hand side:

$$\left(1 - \frac{3}{8}h \cdot g(t_{i+1})\right) w_{i+1} = w_i + \frac{h}{24}\left(19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})\right)$$

Therefore, by expanding $f(t_i, w_i)$ to $f(t_{i-2}, w_{i-2})$ the explicit form is the following:

$$\boxed{w_{i+1} = \frac{w_i + \frac{h}{24}(19w_i \cdot g(t_i) - 5w_{i-1} \cdot g(t_{i-1}) + w_{i-2} \cdot g(t_{i-2}))}{1 - \frac{3}{8}h \cdot g(t_{i+1})}}$$

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