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Exponential Gradient with Momentum for Online Portfolio Selection

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ABSTRACT

Online portfolio selection is a fundamental research problem, which has drawn extensive investigations in both machine learning and computational finance communities. The evolution of electronic trading has contributed to the growing prevalence of High-Frequency Trading (HFT) in recent years. Generally, HFT requires trading strategies to be fast in execution. However, the existing online portfolio selection strategies fail to either satisfy the demand for high execution speed or make effective utilization of historical data. In response, we propose a framework named *Exponential Gradient with Momentum* (EGM) which integrates EG with an acknowledged optimization method in stochastic learning, i.e., momentum. Specifically, momentum boosts the performance of EG by making full use of historical information. Most essentially, EGM can execute with only constant memory and running time in the number of assets per trading period, thus overcoming the drawback of most online strategies. The theoretical analysis reveals that EGM bounds the regret sublinearly. The extensive experiments conducted on four real-world datasets demonstrate that EGM outperforms relevant strategies with respect to comprehensive evaluation metrics.

1. Introduction

Portfolio selection, which aims to optimize the allocation of wealth over a collection of assets, is a problem of great concern in both academia and industry. On the one hand, the earliest study of portfolio selection can be traced back to the construction of Mean–Variance model that concentrates on optimizing single-period portfolio selection by trading off the expected return (mean) and the risk (variance) (Markowitz, 1952, 1959; Markowitz & Todd, 2000). On the other hand, capital growth theory is increasingly included in recent researches on portfolio selection, mainly targeting the maximization of the expected growth rate of multiple periods (Hakansson & Ziemba, 1995; Kelly, 1956; Rotando & Thorp, 1992; Thorp & Kassouf, 1967; Ziemba, 2005). Particularly, the application of capital growth theory has promoted extensive studies on online portfolio selection problem in machine learning and computational finance communities (Li & Hoi, 2012; MacLean et al., 2011).

The evolution of electronic trading has contributed to the growing prevalence of High-Frequency Trading (HFT) in recent years (Baron et al., 2019; Kirilenko et al., 2017; Van Kervel & Menkveld, 2019). This is caused by multiple reasons, including the reduction of latency (Gomber & Haferkorn, 2015), the decrease in trading costs, and the increase in price efficiency (Brogaard et al., 2018). Typically, HFT is required to complete the transaction of a massive amount of orders within a second (Brogaard et al., 2014). In general, high-frequency traders with the fastest execution speed appear to be more profitable than their counterparts with a slower speed. Despite considerable researches on the problem of online portfolio selection, it is still difficult to construct HFT strategies that can fully meet the requirement on execution speeds.

In the attempt to seek qualified HFT strategies, we start to pay attention to Exponential Gradient (EG), which is claimed to run with constant memory and computing time per asset in each trading period (Helmbold et al., 1998). However, EG utilizes only the information from the last trading period. Thus, one of the feasible solutions is incorporating more historical information to enhance the performance of EG without significantly increasing time and space complexity. In the artificial intelligence community, it is ubiquitous to see the use of boosting methods in Gradient Descent (GD), which leverages historical gradients to ensure a faster updating speed. Among them, momentum is a classical boosting method for the gradient-based optimization of stochastic learning. Other momentum-oriented methods include RMSProp (Tieleman & Hinton, 2012) and Adam (Kingma & Ba, 2014), etc.

In this paper, we address the problem of online portfolio selection in the context of high-frequency trading by proposing a framework named *Exponential Gradient with Momentum* (EGM). The characteristics and contributions of the proposed framework are summarized as follows:

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- We propose a novel framework, i.e., EGM, for online portfolio selection. To the best of our knowledge, this is the first EG-based framework that incorporates momentum-oriented boosting methods widely leveraged by GD-based algorithms.
- The proposed EGM framework enhances the performance of EG by making full use of historical information. More importantly, EGM is characterized with not only its sublinear regret bound, but also its efficiency in execution speed.
- We conduct extensive experiments on four real-world datasets to demonstrate that strategies in our EGM framework not only outperform competing strategies in terms of multiple evaluation criteria, but also have promising scalability of transaction costs.

The remaining paper is organized as follows: Section 2 describes the online portfolio selection problem, and Section 3 reviews related works. In Section 4, we propose the EGM framework and analyze both its regret and complexity. On this basis, we evaluate the effectiveness of the strategies on real-world datasets in Section 5. In Section 6, we discuss major contributions and practical implication of our research. Finally, Section 7 summarizes the findings and presents limitations as well as the future research direction.

2. Problem setting

In this section, we describe the online portfolio selection problem with symmetric proportional transaction cost.

Considering a self-financed, discrete-time and no-margin investment environment with N assets for T trading periods. The period can be chosen arbitrarily, such as second, minute, hour and day etc. At tth period, the performance of assets can be described by a vector of price relatives, denoted by $\mathbf{x}^t = (x_1^t, x_2^t, \ldots, x_N^t)^\top$ where x_i^t is the closing price of ith asset at tth period divided by its closing price at the last period. The portfolio, which reflects the investment decision at tth period, is denoted by a weight vector $\mathbf{w}^t = (w_1^t, w_2^t, \ldots, w_N^t)^\top$ with the constrain that $\mathbf{w}^t \in \omega = \{\mathbf{w} | \mathbf{w} \geq \mathbf{0}, \mathbf{w}^\top \mathbf{1} = 1\}$ where $\geq$ denotes element-wide $\geq$ and $\mathbf{0}/\mathbf{1}$ denotes a N-dimensional vector of zeros/ones. The ith element of $\mathbf{w}^t$ specifies the proportion of total portfolio wealth invested in ith asset at tth period. Given a portfolio $\mathbf{w}^t$ and the price relatives $\mathbf{x}^t$, the portfolio wealth from tth period to the next will increase , regardless of transaction costs, by a factor of $s_t = \sum_{i=1}^N w_i^t x_i^t$ which is referred to as $\mathbf{w}^t \cdot \mathbf{x}^t$ in the remaining paper.

We adopt a symmetric proportional transaction cost model with cost ratio $c \in [0, 1]$, where $c = c_s = c_b$ is the cost of both selling c_s and buying c_b (Huang et al., 2015). Hence, the portfolio should pay the transaction cost $C^t = c|\hat{\mathbf{w}}^t - \mathbf{w}^{t+1}|_1 s_t$ at the beginning of the $(t+1)$-th period, where $|\cdot|_1$ denotes the L1-norm, and $\hat{\mathbf{w}}^t$ denotes the *current portfolio* at the end of the tth period which is computed by $\hat{\mathbf{w}}^t = (\frac{w_1^t x_1^t}{\mathbf{w}^t \cdot \mathbf{x}^t}, \frac{w_2^t x_2^t}{\mathbf{w}^t \cdot \mathbf{x}^t}, \ldots, \frac{w_N^t x_N^t}{\mathbf{w}^t \cdot \mathbf{x}^t})^\top$. Therefore, the single-period growth rate at the end of the tth period is denoted as $s_t^c = s_t - C^t$.

At the end of tth period, the portfolio strategy observes previous vectors of price relatives $\mathbf{x}^1, \ldots, \mathbf{x}^t$. Based on this information, the strategy computes a portfolio for the next period. Thus, from $t = 1, \ldots, T$, a sequence of price relative vectors $\mathbf{x}^1, \ldots, \mathbf{x}^T$ is observed and a sequence of portfolios is produced. Finally, the wealth will increase by a factor of $S_T = \prod_{t=1}^T s_t^c$. Since this paper focuses on multi-period investment, *cumulative wealth* is defined as:

$$\mathrm{CW} = I \times S_T = I \prod_{t=1}^{T} s_t^c, \tag{1}$$

where I stands for the initial wealth. Simply, the logarithmic return is computed as $\log(\mathrm{CW})$. The trader aims to design a portfolio strategy to maximize CW.

In order to focus on the main issue of online portfolio problem, we make the following assumptions, which are also widely adopted in other works (Li & Hoi, 2012), to simplify the problem setting from real-world setting: (i) Market liquidity: assuming that one is able to buy and sell any quantity of any asset at its closing prices. (ii) Impact cost: assuming any portfolio selection strategy will not influence the market behavior.

3. Background review

In this section, we mainly review the benchmarks and related work of online portfolio selection problem. Then, we briefly introduce momentum-oriented boosting methods.

3.1. Online portfolio selection

Online portfolio selection problem shares the analogical aim with Kelly gambling problem which maximizes the expected logarithmic return of a portfolio (Kelly, 1956).

3.1.1. Benchmarks

There are mainly two types of conventional benchmarks. One is Buy-And-Hold (BAH). This type of strategy starts investment with an initial portfolio and holds the portfolio until the end. A BAH strategy is called Uniform BAH (UBAH) if the initial portfolio is an uniform proportion vector. The other one is Constant Rebalanced Portfolios (CRP) which manipulates portfolios in an opposite way. CRP rebalances the portfolio to a fixed portfolio for every trading period. Apparently, a fixed portfolio $\mathbf{b}$, which achieves the maximal return among all $\mathbf{w} \in \omega$, can be calculated from hindsight. A CRP strategy with $\mathbf{b}$ is referred to as Best CRP (BCRP), which is claimed to be the optimal strategy in an i.i.d. market (Cover & Thomas, 1991).

3.1.2. Follow-The-Winner

In this paper, we mainly investigate *Follow-The-Winner* (FTW) category (Li & Hoi, 2012) which seeks to keep up with BCRP with the idea of increasing the proportion of well-performed assets in the history. Cover (1991) proposed the widely acknowledged Universal Portfolio (UP) strategy which achieves a regret of $O(N \log T)$ with a time complexity of $O(T^N)$.

As one type of FTW approach, the Exponential Gradient-type strategies have drawn more and more attentions since their computational efficiency. In general, these strategies develop iterative algorithms for the following optimization problem:

$$\mathbf{w}^{t+1} = \arg\max_{\mathbf{w} \in \omega} \ \eta \log \mathbf{w} \cdot \mathbf{x}^t - D(\mathbf{w}, \mathbf{w}^t), \tag{2}$$

where $D(\mathbf{w}, \mathbf{w}^t)$ denotes a regularization term. Helmbold et al. (1997) proposed Gradient Projection (GP) and Expectation Maximization with L2-norm and χ^2 regularization respectively, and GP can bound the regret with $O(\sqrt{TN})$. Exponential Gradient (EG) was proposed with relative entropy regularization (Helmbold et al., 1998). The regret of EG is bounded by $O(\sqrt{T \log N})$ with the computing cost of $O(N)$ per period.

There are other types of FTW approach, such as Follow The Leader (Gaivoronski & Stella, 2000, 2003) and Follow The Regularized Leader (FTRL) (Agarwal et al., 2006; Hazan & Kale, 2012). Compared with EG-type strategies, FTRL focuses on similar but different optimization problem:

$$\mathbf{w}^{t+1} = \arg\max_{\mathbf{w} \in \omega} \ \sum_{\tau=1}^{t} \log \mathbf{w} \cdot \mathbf{x}^\tau - \frac{\lambda}{2} R(\mathbf{w}), \tag{3}$$

where λ denotes the trade-off parameter and $R(\mathbf{w})$ is a regularization term on $\mathbf{w}$. In each period, FTRL's objective function enables strategies to take all historical information into account. However, this also costs considerable storage and time complexity. Online Newton Step (ONS) is a FTRL strategies with L2-norm regularization, which is proved to have a $O(N^{1.5} \log(TN))$ regret bound and $O(N^3)$ time complexity per period (Agarwal et al., 2006).

3.1.3. Follow-The-Loser

There is another category of strategies, i.e., *Follow-The-Loser* (FTL), to address online portfolio selection problem. FTL approaches transfer the wealth from winning assets to losers, which are totally different from FTW approaches. Although FTL approaches can achieve impressive performance in empirical studies, they lack theoretical guarantees.

In general, based on the *mean reversion* assumption (Lo & MacKinlay, 1990) and adopting *Passive-Aggressive* learning (Crammer et al., 2006), FTL approaches estimate the *mean* in various ways to produce the portfolios. Li et al. (2012) proposed Passive Aggressive Mean Reversion (PAMR), which simply regards price relative of the last period as the mean and has achieved great performance in empirical studies. Then Li et al. (2015) proposed Online Moving Average Reversion (OLMAR), which estimates the mean by the moving average of historical data. Following works include Robust Median Reversion (RMR) (Huang et al., 2016), Gaussian Weighting Reversion (GWR) (Cai & Ye, 2019), and Vector Autoregressive Weighting Reversion (VAWR) (Cai, 2020), which use L1-median and fixed Gaussian function, vector autoregressive moving average to estimate the mean respectively.

3.2. Momentum

Exponential Weighted Moving Average (EWMA) is a special type of moving average that allocates a greater weight on more recent data. EWMA has been applied in financial field to observe and forecast assets, such as seasonals and trends (Holt, 2004). Furthermore, deep learning engineers train the neural networks based model more efficiently by incorporating EWMA, which is usually called momentum in the artificial intelligence community. In this paper, we also refer to these sort of techniques as *momentum*, including but not limited to Adaptive Gradient (AdaGrad) (Duchi et al., 2011), Root Mean Square Prop (RMSProp) (Tieleman & Hinton, 2012) and Adam (Kingma & Ba, 2014).

4. Methodology

In this section, we propose a framework, called Exponential Gradient with Momentum and EGM for short, for online portfolio selection in the context of high-frequency trading. We first explain motivation of incorporating momentum and then present explicit update rules for all strategies in EGM framework. In addition, we analyze the regret bound of EGM. Last but not least, we analyze the time and space complexities of our proposed framework, and compare them with other relevant strategies.

4.1. Motivation

Firstly, we review the derivation of EG, and analyze its drawback. As we introduced in Section 3.1.2, EG produces its update rule by focusing on optimizing Eq. (2) with relative entropy $D_{re}(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{N} u_i \log u_i/v_i$. Hence, the objective function of EG is:

$$J(\mathbf{w}) = \eta \log \mathbf{w} \cdot \mathbf{x}^t - D_{re}(\mathbf{w}, \mathbf{w}^t). \tag{4}$$

Both terms of J depend nonlinearly on $\mathbf{w}$, which makes it hard to find the exact maximizer. Considering that J satisfying a Lipschitz Condition, EG replaces the first term by applying first-order Taylor Expansion at $\mathbf{w}^t$. Then the approximation of J is:

$$\hat{J}(\mathbf{w}) = \eta \left[\log \mathbf{w}^t \cdot \mathbf{x}^t + \frac{\mathbf{x}^t \cdot (\mathbf{w} - \mathbf{w}^t)}{\mathbf{w}^t \cdot \mathbf{x}^t}\right] - D_{re}(\mathbf{w}, \mathbf{w}^t). \tag{5}$$

Finally, applying Lagrange Multiplier to $\hat{J}$ with the constraint that $\sum_i w_i = 1$ and setting partial derivatives to zero lead to the EG's update rule as follows:

$$w_i^{t+1} = \frac{w_i^t \exp \eta g_i^t}{Z_t}, \tag{6}$$

Algorithm 1 EGM framework

Input: T: the number of trading periods, N: the number of assets, $\mathbf{x}^1, \dots, \mathbf{x}^T$: the sequence of price relative vectors
Output: $\mathbf{w}^1, \dots, \mathbf{w}^T$: the sequence of portfolio weight vectors.

1: Initialize $\mathbf{v}^1 = (0, \dots, 0)^\top$ and $\mathbf{w}^1 = (1/N, \dots, 1/N)^\top$.
2: **for** $t = 1 \to T-1$ **do**
3: **for** $i = 1 \to N$ **do**
4: Update portfolio weights:

$$w_i^{t+1} = \begin{cases} \dfrac{w_i^t \exp \eta g_i^t}{Z_t} & \text{EG} \\ \dfrac{w_i^t \exp \eta v_i^{t+1}}{Z_t^e} & \text{EGE} \\ \dfrac{w_i^t \exp \frac{\eta g_i^t}{\sqrt{m_i^{t+1}+\epsilon}}}{Z_t^r} & \text{EGR} \\ \dfrac{w_i^t \exp \frac{\eta v_i^{t+1}}{\sqrt{m_i^{t+1}+\epsilon}}}{Z_t^a} & \text{EGA} \end{cases}$$

5: **end for**
6: **end for**
7: **return**: $\mathbf{w}^1, \dots, \mathbf{w}^T$

where $g_i^t = x_i^t / \mathbf{w}^t \cdot \mathbf{x}^t$ is the gradient term for EG, $\eta > 0$ is the learning rate and $Z_t = \sum_{j=1}^{N} w_j^t \exp \eta g_i^t$ is a normalization term to make all the elements of $\mathbf{w}^{t+1}$ sum to one (Helmbold et al., 1998).

Therefore, it is the objective function that makes EG to update only with the information (price relatives and portfolio weights) from the last period. On the one hand, this characteristic results in computational efficiency in terms of both time and space. On the other hand, the majority of historical information remains unexploited for EG and the thorough dependence on the information from the last period may lead to potential problems. For instance, EG could suffer from noisy price fluctuation from the market, especially in the context of high-frequency trading. In contrast, FTRL's objective function, Eq. (3), takes all historical price relatives into account, and ONS achieves a lower regret bound than EG. Consequently, the motivation is to boost the performance of EG by leveraging more historical information.

Based on the motivation above, we focus on the momentum which contributes to a faster updating speed of gradient-based optimization in stochastic learning by taking historical gradients into consideration without significantly increasing time and space complexities.

Although Gradient Descent (GD) and EG are both gradient-based stochastic learning methods, momentum-oriented boosting methods are frequently utilized in GD while they are definitely underestimated in EG. This is probably because GD is widely studied with the development in deep learning (LeCun et al., 2015), as GD is used to optimized deep learning models.

Kivinen and Warmuth (1997) firstly proposed EG as an optimization method and compared it with GD under the problem of online linear prediction. Kivinen and Warmuth claimed that the bounds suggest that EG is generally incomparable to GD with respect to loss, but EG can have a much smaller loss in certain conditions (Kivinen & Warmuth, 1997). Inspired by Kivinen and Warmuth's work, EG was then applied in online portfolio selection (Helmbold et al., 1998). EG is more competitive for online portfolio selection problem, which is illustrated by a simulation experiment in Section 5.1.

4.2. Update rules

Different strategies follow their own update rules. In this paper, integrating EG with EWMA, RMSProp and Adam separately, we propose three new strategies: EGE, EGR and EGA. The general computing process of EG and EGM is presented in Algorithm 1.

4.2.1. EGE

In the community of finance, EWMA, or known as EMA, is a modified type of moving average which is widely applied to observe and forecast assets (Holt, 2004). In the community of artificial intelligence, EWMA, or known as momentum, is an optimizer designed to overcome the problem of loss fluctuation with improper learning rate in mini-batch stochastic gradient descent. EWMA multiplies the historical gradients and the present gradient by a trade-off parameter γ and $1-\gamma$ respectively, and then use the sum of these two to update learning process. Formally, adopting EWMA on EG's gradient term, EGE updates as follows:

$$w_i^{t+1} = \frac{w_i^t \exp \eta v_i^{t+1}}{Z_t^e}, \tag{7}$$

where $v_i^{t+1} = \gamma_1 v_i^t + (1-\gamma_1) g_i^t$, $\gamma_1 \in [0,1]$ is a trade-off parameter for the linear combination of historical gradients and present gradient, and Z_t^e is a normalization term.

4.2.2. EGR

RMSProp is another optimizer for solving loss fluctuation problem in stochastic learning (Tieleman & Hinton, 2012). Empirical studies have shown that RMSProp usually performances well in online and non-stationary setting (Kingma & Ba, 2014), which is usually the case in financial markets. Basically, RMSProp applies EWMA to AdaGrad, which calculates the root exponential weighted mean square (Duchi et al., 2011). Consequently, EGR follows the same idea and its update rule computes as follows:

$$w_i^{t+1} = \frac{w_i^t \exp \frac{\eta g_i^t}{\sqrt{m_i^{t+1}}+\epsilon}}{Z_t^r}, \tag{8}$$

where $m_i^{t+1} = \gamma_2 m_i^t + (1-\gamma_2)(g_i^t)^2$, $\gamma_2 \in [0,1]$ for trading-off , Z_t^r for normalization, and ϵ is a small value to prevent the denominator from being zero, which is set to 10^{-8} as default in this paper.

4.2.3. EGA

Adam claims to take advantage of both AdaGrad and RMSProp (Kingma & Ba, 2014). The main characteristic of Adam is its power for non-stationary problems and for gradients with noise and/or sparsity features, which we consider suitable for portfolio selection problem. Therefore, incorporating with Adam, EGA updates as follows:

$$w_i^{t+1} = \frac{w_i^t \exp \frac{\eta v_i^{t+1}}{\sqrt{m_i^{t+1}}+\epsilon}}{Z_t^a}, \tag{9}$$

where Z_t^a for a normalization term, v_i^{t+1} and m_i^{t+1} follow the same definitions above.

4.3. Regret analysis

Regret is a common way to evaluate the theoretical performance of online learning algorithms. In order to analyze theoretical performance of EGM framework, we define *Regret* as the difference between the logarithmic return of the benchmark BCRP and that of a competing strategy π. The regret of π with respect to BCRP is computed as follows:

$$\begin{aligned} \text{Regret}(\pi) &= \log(\text{CW}_{\text{BCRP}}) - \log(\text{CW}_\pi) \\ &= \log(\prod_{t=1}^T \mathbf{w}_{BCRP} \cdot \mathbf{x}^t) - \log(\prod_{t=1}^T \mathbf{w}_\pi^t \cdot \mathbf{x}^t) \\ &= \sum_{t=1}^T \log(\mathbf{w}_{BCRP} \cdot \mathbf{x}^t) - \sum_{t=1}^T \log(\mathbf{w}_\pi^t \cdot \mathbf{x}^t). \end{aligned} \tag{10}$$

Apparently, the strategy that achieves the lower regret implies the better theoretical performance.

Theorem 1 (*Lower Bound of EGM*). *Assuming that all price relatives $x_i^t \in [r,1]$ where $r>0$ and $\eta \ge 1$, EGM guarantees that after T rounds, for any portfolio weight $\mathbf{u}$, we have*

$$\sum_{t=1}^T \log(\mathbf{u}\cdot\mathbf{x}^t) - \sum_{t=1}^T \log(\mathbf{w}^t\cdot\mathbf{x}^t) \le \frac{\log N}{\eta K_2} + \eta T\theta, \tag{11}$$

where $\theta = \log(K_2/K_1) + K_2/(8r^2)$ and

$$K_{1/2} = \begin{cases} \min/\max_{i,t}(v_i^{t+1}/g_i^t) & EGE \\ \min/\max_{i,t}(1/\sqrt{m_i^{t+1}}+\epsilon) & EGR \\ \min/\max_{i,t}[v_i^{t+1}/(\sqrt{m_i^{t+1}}+\epsilon)g_i^t] & EGA. \end{cases} \tag{12}$$

And setting $\eta = \sqrt{\log N/K_2 T\theta}$ and $\mathbf{u}$ to be the portfolio weight of BCRP strategy, we have

$$Regret(EGM) \le 2\sqrt{K_2 T\theta \log N}. \tag{13}$$

Note that we assume $x_i^t \in [r,1]$ in Theorem 1. We can make this assumption without loss of generality, because scaling the price relatives $\mathbf{x}_t$ by a constant c is equivalent to adding $\log c$ to the logarithmic cumulative wealth, and leaving the regret, which is the difference between logarithmic cumulative wealth, unchanged. Moreover, it is common for price relatives to have upper and lower bounds in real markets. Real-world cases can be easily transformed into our settings.

Proof. We use EGE as an example. This solution can be straightforwardly adapted to the other two strategies in EGM framework.

$$\begin{aligned} \Delta_t =& D_{re}(\mathbf{u}, \mathbf{w}^{t+1}) - D_{re}(\mathbf{u}, \mathbf{w}^t) \\ =& -\sum_{i=1}^N u_i \log \frac{w_i^{t+1}}{w_i^t} = -\sum_{i=1}^N u_i \log \frac{\exp \eta v_i^{t+1}}{Z_t^e} = \log Z_t^e - \eta \mathbf{u}\cdot\mathbf{v}^{t+1} \\ \le& \log Z_t^e - \eta K_1 \frac{\mathbf{u}\cdot\mathbf{x}^t}{\mathbf{w}^t\cdot\mathbf{x}^t}. \end{aligned} \tag{14}$$

In Eq. (14), we let $k_{i,t} = v_i^{t+1}/g_i^t$ and $K_1 = \min_{i,t}(k_{i,t})$. To bound Z_t^e, we let $K_2 = \max_{i,t}(k_{i,t})$, then we have

$$\begin{aligned} Z_t^e \le& \sum_{i=1}^N w_i^t \exp \eta K_2 g_i^t = \sum_{i=1}^N w_i^t \exp \frac{\eta K_2 x_i^t}{\mathbf{w}^t\cdot\mathbf{x}^t} \\ \le& \sum_{i=1}^N w_i^t [1 - (1-\exp\frac{\eta K_2}{\mathbf{w}^t\cdot\mathbf{x}^t}) x_i^t] = 1 - \sum_{i=1}^N \mathbf{w}^t\cdot\mathbf{x}^t (1-\exp\frac{\eta K_2}{\mathbf{w}^t\cdot\mathbf{x}^t}). \end{aligned} \tag{15}$$

Note that for Eq. (15), we use the fact that $q^x \le 1-(1-q)x$ for $q>0$ and $x\in[0,1]$. Then using the fact that $\log[1-p(1-\exp x)] \le px + x^2/8$ for $p\in[0,1]$ and $x\in\mathbb{R}$, we have

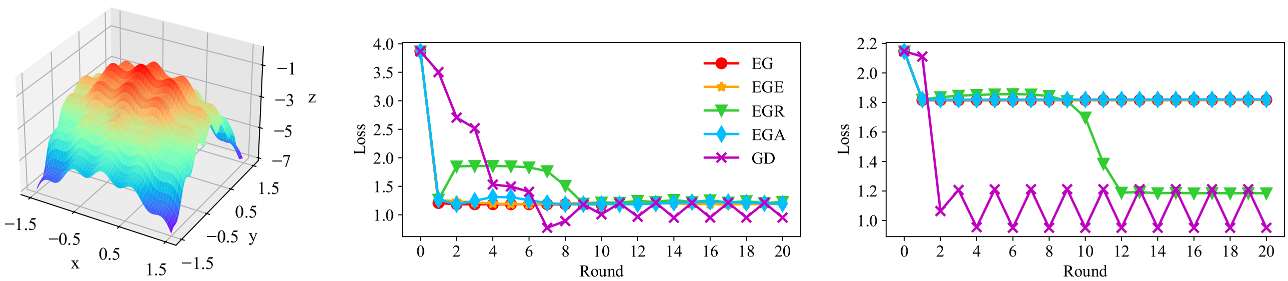
$$\log Z_t^e \le \eta K_2 + \frac{\eta^2 K_2^2}{8(\mathbf{w}^t\cdot\mathbf{x}^t)^2}. \tag{16}$$

Then combining Eq. (16) with (14) and using that fact that $1-\exp x \le -x$ for all x, we have

$$\begin{aligned} \Delta_t \le& \eta K_2 \left[1 - \left(\frac{K_1}{K_2}\cdot\frac{\mathbf{u}\cdot\mathbf{x}^t}{\mathbf{w}^t\cdot\mathbf{x}^t}\right)\right] + \frac{\eta^2 K_2^2}{8(\mathbf{w}^t\cdot\mathbf{x}^t)^2} \\ \le& -\eta K_2 \log \frac{K_1}{K_2}\cdot\frac{\mathbf{u}\cdot\mathbf{x}^t}{\mathbf{w}^t\cdot\mathbf{x}^t} + \frac{\eta^2 K_2^2}{8(\mathbf{w}^t\cdot\mathbf{x}^t)^2}. \end{aligned} \tag{17}$$

Summing Δ_t over T, then using the fact that relative entropy is non-negative, $x_i^t \ge r$ and $\eta \ge 1$, we have

$$\begin{aligned} -D_{re}(\mathbf{u}, \mathbf{w}^1) \le& D_{re}(\mathbf{u}, \mathbf{w}^{T+1}) - D_{re}(\mathbf{u}, \mathbf{w}^1) \\ \le& \eta K_2 \left(T \log \frac{K_2}{K_1} + \sum_{t=1}^T [\log(\mathbf{w}^t\cdot\mathbf{x}^t) - \log(\mathbf{u}\cdot\mathbf{x}^t)]\right) + \frac{\eta^2 K_2^2 T}{8r^2} \\ \le& \eta K_2 \left(\eta T \log \frac{K_2}{K_1} + \sum_{t=1}^T [\log(\mathbf{w}^t\cdot\mathbf{x}^t) - \log(\mathbf{u}\cdot\mathbf{x}^t)]\right) + \frac{\eta^2 K_2^2 T}{8r^2}. \end{aligned} \tag{18}$$



(a) Simulation environment (b) Far starting point (c) Near-by starting point

Fig. 1. Simulation results. (b) and (c) share the same legend.

Table 1
Complexity analysis at tth trading period.

Complexity	UP	ONS	EG	EGM
Space	$O(tN)$	$O(N^2)$	$O(N)$	$O(N)$
Time	$O(t^N)$	$O(N^3)$	$O(N)$	$O(N)$

Since $\mathbf{w}^1$ is an uniform weight vector, we get $D_{re}(\mathbf{u}, \mathbf{w}^1) \leq \log N$. Then combining this inequality with Eq. (18), rearranging and scaling, we have

$$\sum_{t=1}^{T} \log(\mathbf{u} \cdot \mathbf{x}^t) - \sum_{t=1}^{T} \log(\mathbf{w}^t \cdot \mathbf{x}^t) \leq \frac{\log N}{\eta K_2} + \eta T \left(\log \frac{K_2}{K_1} + \frac{K_2}{8r^2} \right). \quad \square \tag{19}$$

Based on our theoretical analysis, all strategies in EGM are able to achieve $O(\sqrt{T \log N})$ regret, which is equivalent to EG. Therefore, our proposed EGM framework can leverage more historical information without significantly influencing the theoretical performance.

4.4. Complexity analysis

In order to trade with high frequency, HFT strategies are required to be fast in execution. We compare both time and space complexities of relevant strategies and report the results in Table 1. As it is shown in Table 1, UP has extraordinarily high computational complexity with respect to both time and space. Firstly, UP's complexities involve t, which means they will grow larger as the investment goes on. Secondly, UP's time complexity is exponential in the number of assets. As a result, it is not eligible to regard UP as a HFT strategy in spite of the fact that UP theoretically achieves the lowest regret bound among all the strategies in the table. ONS develops an incrementally updating strategy which reduces the complexity to polynomial in the number of assets. However, a complexity of $O(N^2)$ still costs considerable computational resource when the number of assets is large. EG achieve the complexity in constant which is the theoretically optimal result of online updating strategies, because the strategy have to load price relatives of all assets.

Incorporated with momentum, the EGM framework is able to leverage more historical information in a incremental way. More importantly, these three strategies in EGM are still efficient to compute. To be specific, EGM requires a constant memory and maintain the time complexity in $O(N)$, which makes it possible for EGM to be HFT strategies.

5. Experiments

In this section, we mainly focus on the comparison studies. Firstly, we conduct an optimization experiment on a simulation environment with GD and strategies in EGM (i.e., EGE, EGR, and EGA), which provides an intuitive interpretation of difference between GD and EG. Then, we introduce testing datasets, competing portfolio strategies and criteria of evaluation. At last, we report and analyze the results of comparison studies, and also discuss about transaction costs.

Table 2
Summary of the testing datasets.

#	Dataset	Time frame	N	T
1.	NYSE (O)	07/03/1962–12/31/1984	36	5651
2.	NYSE (N)	01/01/1985–06/30/2010	23	6431
3.	SP500	01/02/1988–01/31/2003	25	1276
4.	DJIA	01/14/2001–01/14/2003	30	507

5.1. Simulation experiment

In this experiment, we trace the paths of several optimization algorithms and report the losses. The simulation environment is based on the Bohachevsky Function, which is a bowl-shaped, 3-dimensional function with many local minima and one global minimum (Bohachevsky et al., 1986). Specifically, the environment space is calculated as:

$$z = -(x^2 + 2y^2 - A\cos 3\pi x - B\cos 4\pi y + C) \tag{20}$$

with $A = 0.3, B = 0.4, C = 0.7$. Fig. 1(a) shows the environment space.

The loss is defined by Euclidean distance between global maximum and present point. The simulation experiment is performed for several sets with random starting points and same parameter setting including learning rate, maximal iteration round, etc. The optimization results vary in terms of starting points. we report two of the most representative results in this paper. As it is shown in Fig. 1(b), with a starting point far from global maximum, EG and strategies in EGM clearly outperform GD with a dramatic first step and little fluctuation in following rounds. This phenomenon becomes more pronounced with a further starting point. On the contrary, Fig. 1(c) illustrates that GD can achieve a better final result with a near-by starting point. Although EGR reaches a plateau similar to GD, the other two strategies in EGM suffer from early stopping.

In reality, the financial markets are generally considered as an non-stationary environment which is difficult to simulate. However, the markets can be regarded as stationary in each trading period, and the traders only have one updating round without knowing the explicit distribution of the markets. In this case, strategies in EGM significantly outperform GD in the simulation environment with respect to the first updating step by an average of 720%.

5.2. Data

In order to enhance the availability of data sources and improve the reproducibility of the experiments, we conduct comparison studies on four real-world datasets.

NYSE (O). The first one is NYSE (O) dataset with 5651 daily price relatives of 36 stocks in New York Stock Exchange for 22 years from Jul. 3rd 1962 to Dec. 31st 1984. This is the benchmark dataset first proposed by Cover (1991), followed by a certain number of researchers (Agarwal et al., 2006; Borodin et al., 2004; Györfi et al., 2006; Helmbold et al., 1998; Li et al., 2015, 2013; Singer, 1997).

Table 3
Summary of the evaluation criteria.

Criteria	Performance metrics	
Return	Cumulative Wealth (CW)	Annualized Percentage Yield (APY)
Risk	Maximum DrawDown (MDD)	Volatility (VO)
Risk-adjusted return	Annualized Sharp ratio (SR)	Calmar ratio (CR)

NYSE (N). The second one is NYSE (N), which is the updated version of the first dataset. For data consistency, Li et al. (2012) collected data in NYSE from Jan. 1st 1985 to Jun. 30th 2010 which covers a period of another 25 years. Note that due to mergers and bankruptcies, this new dataset consists of 23 stocks instead of previous 36 stocks.

SP500. The following dataset is SP500 which was collected by Borodin et al. (2004). This dataset contains the top 25 stocks in Standard & Poor's 500 in terms of market capitalization, which ranges from Jan. 2nd, 1998 to Jan. 31st 2003.

DJIA. The last dataset, DJIA, is collected by Borodin et al. (2004), which consists of 30 constituent stock of Dow Jones Industrial Average. This dataset contains price relatives of 507 trading days, ranging from Jan. 14th 2001 to Jan. 14th 2003.

Note that all four datasets contain daily prices. Table 2 summaries these four datasets. Datasets # 1, 3, 4 have been integrated by Borodin et al. (2004), and dataset # 2 was collected by Li et al. (2013).

5.3. Competing portfolio strategies

We refer to UBAH strategy as *market* strategy which indicates the comprehensive behavior of the market (Li et al., 2013). Moreover, we compare strategies in EGM with EG (Helmbold et al., 1998), ONS (Agarwal et al., 2006) and UP (Cover, 1991). We implement UP by random walk (Kalai & Vempala, 2002) with 50 independent trials, and take the average values for evaluation.

5.4. Evaluation criteria

For the convenience of calculations, we assume that the initial wealth is 1$ (Li et al., 2018). Under such a circumstance, the single-period growth rate s_t^c is equal to portfolio's gross return at tth period, and the cumulative growth rate S_t is equal to cumulative wealth. Then we evaluate the characteristics of portfolio strategies above by three criteria and six performance metrics in Table 3. The three criteria are return, risk and risk-adjusted return. There are two metrics in each criterion. Basically, the higher the value of metrics in return and risk-adjusted return criteria, the better the portfolio strategy performs. On the contrary, the lower the value of metrics in risk criterion, the more performance favorable the strategy is. These metrics are commonly used and considered standard in finance (Brandt, 2010).

Cumulative Wealth. CW is the most common and elementary metric for measuring the performance of a investment strategy, which is computed by Eq. (1).

Annualized Percentage Yield. APY is an another common metric in return-based evaluation, which describes the average return of the strategy in a year. APY is computed as follow:

$$\mathrm{APY} = \mathrm{CW}^{1/y} - 1, \tag{21}$$

where y is the number of year according to T trading days. In this work, all four datasets contain daily prices. Thus, the number of year is calculated by $y = T/252$ where 252 is the average number of annual trading days.

Maximum DrawDown. MDD is a typical measure for downside risk. It is defined as the maximum drop percentage of CW from its running maximum over all periods, which looks for the most considerable movement from a peak point to a trough point. Generally speaking, the smaller the MDD, the more downside risk tolerable the trading strategy is. MDD is computed as follows:

$$\mathrm{MDD} = \max_{t\in[1,T]} \frac{M_t - S_t}{M_t}, M_t = \max_{k\in[1,t]} S_k, \tag{22}$$

where M_t stands for the running maximum of CW.

Volatility. VO is a quantitative risk measure in finance. MDD only focuses on the size of the greatest declination, without thinking about the frequency of declination. Thus, we include VO to allow for a comprehensive measure of risk. Simply, it is computed by $\sqrt{H}\sigma$ where H is 252 since all datasets consist of daily prices and σ denotes the standard deviation of return in each period.

Sharpe Ratio. SR is one of the most accepted portfolio strategy criterion that takes both expected return and risk into consideration (Reed & Sharp, 1987). To be specified, it is calculated as $\mathrm{SR} = (\mathrm{APY} - R_f)/\sigma$ with R_f denotes the risk-free annual return. As following the work of Li et al. (2012), we fixed R_f at 4%, which is the typical return of Treasury bills.

Calmar Ratio. CR is a comparison of the average annual compound return and the MDD risk (Young, 1991), which is widely adopted in fund management. The calculation formula is CR = APY/MDD.

In addition, we introduce t-statistic method into the measurement of the proposed strategies. In fund management operations, managers often use resemble statistical method to test whether the strategy achieves its returns just by luck (Grinold & Kahn, 1999). To be specific, we establish a regression model of portfolio excess returns and market excess returns as follows:

$$s_t - s_t^F = \alpha + \beta(s_t^M - s_t^F), \tag{23}$$

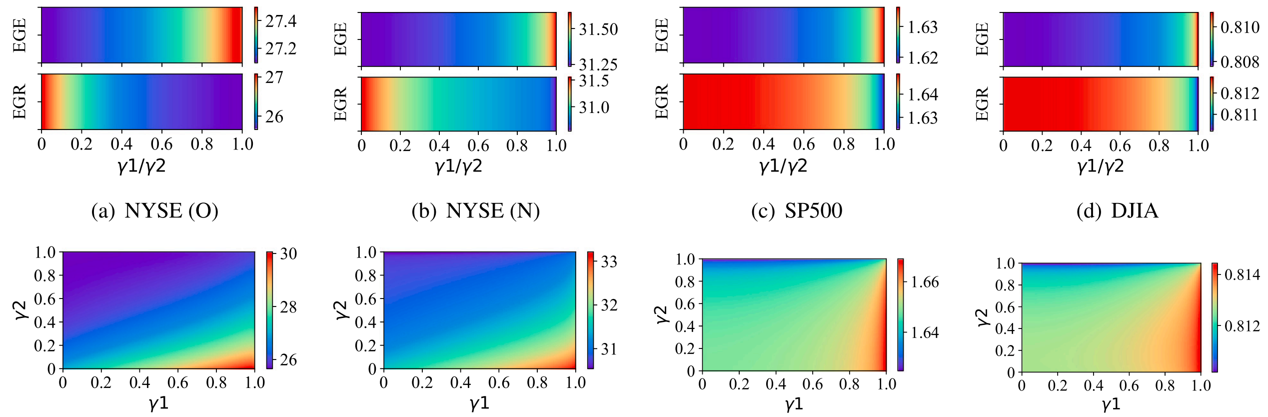
where s_t denotes single-period return as we introduced before, s_t^F and s_t^M stand for single-period return of risk-free asset and that of the market respectively. In this work, we follow the setting above (Li et al., 2012) and set $s_t^F = 1.000156$ which means an annual interest of 4%. By assuming that parameter α follows an normal distribution, we conduct a statistic t-test on α and derive the probability of achieving the excess return by luck.

5.5. Results

5.5.1. Parameter sensitivity

In order to analyze parameter sensitivity and robustness of our proposed framework, we fine tune hyper parameters in EGM. There are three hyper parameters in EGM: learning rate η, trade-off parameters γ_1 and γ_2.

The learning rate η is the only one out of three hyper parameters that we set into a fixed value. EG is able to bound the regret by $O(T \log N)$ with $\eta = 2r\sqrt{2 \log N/T}$ where r is referred to a transformed lower bound of price relatives (Helmbold et al., 1998). Incorporating momentum with EG can be regarded as applying certain transformations to its updating gradients. EWMA, for example, applies linear transformation to the original gradients. Therefore, we maintain the theoretical optimal value of η in EG for parameter setting of EGM framework. Note that the value of T can be determined by doubling



(a) NYSE (O) (b) NYSE (N) (c) SP500 (d) DJIA

(e) NYSE (O) (f) NYSE (N) (g) SP500 (h) DJIA

Fig. 2. Parameter sensitivity of EGM strategies' parameters in terms of CW. In the first row, (a), (b), (c) and (d) are results of EGE's γ_1 and EGR's γ_2 on four datasets. In the second row, there are results of EGA's γ_1 and γ_2.

Table 4
Optimal results of fine tuning. FR denotes fluctuation rate which is the difference between maximum CW and minimum CW divided by maximum CW.

Dataset	EGE		EGR		EGA		
	γ_1	FR (%)	γ_2	FR (%)	γ_1	γ_2	FR (%)
NYSE (O)	0.97	1.45	0.00	5.24	0.97	0.00	14.65
NYSE (N)	0.99	1.19	0.00	3.20	0.99	0.00	8.03
SP500	0.99	1.10	0.00	1.46	0.99	0.00	2.66
DJIA	0.99	0.38	0.09	0.31	0.99	0.14	0.53

trick in real-world cases where T is unknown in advance (Helmbold et al., 1998).

The trade-off parameters γ_1 and γ_2 are constrained in the range of $[0,1]$. These parameters adjust the proportion of past gradients and present gradient for the next updating round. We fine tune both parameters in terms of cumulative wealth with the step of 0.01 in four real-world datasets, and report the results in Fig. 2. Roughly speaking, CW of proposed strategies in EGM framework appears to change smoothly with the adjustment of trade-off parameters. Fig. 2(c), 2(d), 2(g) and 2(h) even show that EGM's CW reaches a plateau in SP500 and DJIA datasets. However, Fig. 2 also illustrates that the trade-off parameters have significant margin effect, which means the parameters are sensitive around the edge of range. Table 4 lists the optimal fine tuning values and fluctuation rate of EGM. We observe that the fluctuation rates are low in most cases. For all strategies in EGM, γ_1 tends to be 0.99 while γ_2 tends to be 0.00 in all four datasets, which indicates that it seems better for EWMA to focus on past information and RMSprop to focus on the present. In consideration of generalization and the fact that most datasets appear to be parameter-insensitive with low fluctuation rates, we recommend to choose $\gamma_1 = 0.99$ and $\gamma_2 = 0.00$ for our proposed EGM, and it is also the parameter setting for following experiments.

In general, our proposed EGM framework is robust and stable in most testing datasets.

5.5.2. Comparison studies

Table 5 shows the performance of strategies in EGM and competing strategies on four real-world datasets. We evaluate the performance of portfolio strategies comprehensively with respect to the criteria including return, risk, and risk-adjusted return. We observe that EGM not only enhance the performance of the original EG across all criteria, but also are capable of outperforming the market and acknowledged benchmark in most testing cases except DJIA. Although ONS is the leading strategy with respect to CW and SR in DJIA dataset, EGA is still the second promising strategy. Moreover, ONS appears to be unstable with high MDD and VO in three other datasets. ONS and EGM both take all historical information into consideration. However, they leverage the historical data in different ways. Eq. (3) indicates that ONS focuses on the logarithmic return which can be regarded as allocating equal weights on historical data. The EGM framework we proposed allocates a greater weigh on more recent data which seems more reasonable in real markets. We suppose the difference of mechanism makes ONS outperform others on a short time-range dataset (DJIA, 507 periods) while performing poorly on other long time-range datasets (over 1000 periods), especially on two datasets with over 5000 periods (NYSE(O), NYSE(N)). In summary, strategies in the proposed EGM framework, especially EGA, appear to establish a comprehensive advantage over competing strategies.

In order to display the evaluation result in a intuitionistic way and compare the difference of strategies in EGM framework, we present the radar charts of EGM with respect to all six metrics in four datasets(See Fig. 3). There are six directions in the radar charts. Each direction represents a performance metric. Note that we reverse the values in two risk metrics (MDD and VO), which means the larger the space covered, the comprehensively better the strategy is. It is obvious that all proposed strategies in EGM outperform EG in terms of return and risk-adjusted return, especially EGA. However, risk management appears to be a weakness of EGA. With overall consideration, we still deem that EGA outperforms other strategies, because EGA achieves the best performance in risk-adjusted return criterion among all strategies. Additionally, it is usually acceptable that high return is associated with high risk.

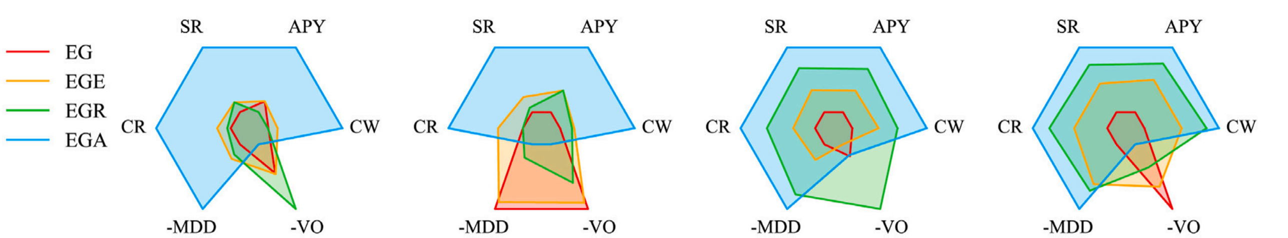
Besides the evaluation result, Fig. 4 illustrates EGM's trend of relative cumulative wealth through all trading periods. In most testing cases, three proposed strategies in EGM framework are able to outperform the original EG. To be specific, EGE improves the performance of EG with stable enhancement in all datasets. We also observe that EGR outperforms EGE at a large part of periods while is surpassed by EG at some others. It is noteworthy that EGA significantly outperforms EG and other two strategies in terms of cumulative wealth in all datasets.

Last but not least, we list the results of t-test in Table 6 to justify the effectiveness of proposed EGM framework. Table 6 shows that in the datasets of NYSE (O) and DJIA, it is almost impossible for EGM to produce corresponding returns simply by luck. There are also more than 90% of confidence level for EGM not to produce their returns by luck in the other two datasets.

In general, EGM shows noticeable advantages over other competing strategies in terms of comprehensive performance.

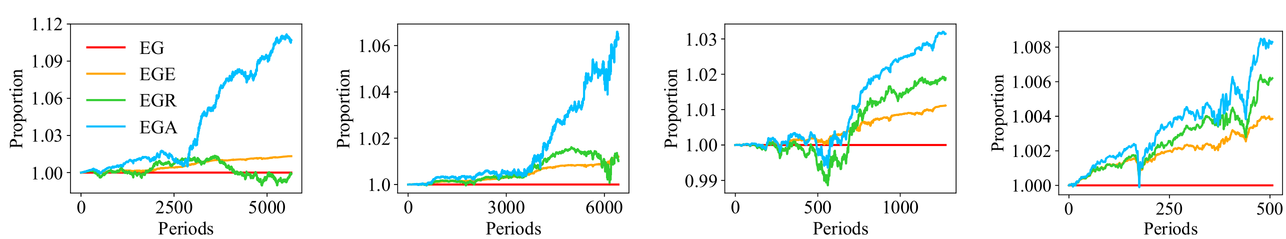
Table 5
Performance of strategies. I and c of all strategies are set to 1$ and 0% respectively. The best score is highlighted in each row.

Dataset	Metrics	Market	UP	ONS	EG	EGE	EGR	EGA
NYSE (O)	CW	14.497	27.080	10.299	27.095	27.454	27.075	**29.988**
	APY	0.127	0.158	0.110	0.159	0.159	0.158	**0.164**
	MDD(%)	41.635	36.592	92.190	36.698	36.551	36.596	**36.042**
	VO(%)	14.866	13.438	72.337	13.520	13.518	**13.437**	13.587
	SR	9.252	13.996	1.527	13.914	13.997	13.996	**14.459**
	CR	0.304	0.433	0.119	0.432	0.436	0.433	**0.454**
NYSE (N)	CW	18.057	31.602	6.666	31.236	31.612	31.552	**33.212**
	APY	0.120	0.145	0.077	0.144	0.145	0.145	**0.147**
	MDD(%)	**53.532**	64.382	86.666	64.061	64.429	64.429	64.524
	VO(%)	**17.855**	19.133	34.980	19.074	19.088	19.132	19.217
	SR	7.118	8.703	1.687	8.687	8.725	8.698	**8.850**
	CR	0.224	0.225	0.089	0.225	0.226	0.225	**0.228**
SP500	CW	1.342	1.647	1.163	1.618	1.636	1.649	**1.669**
	APY	0.060	0.104	0.030	0.100	0.102	0.104	**0.106**
	MDD(%)	45.863	**31.128**	52.686	32.087	31.794	31.144	**30.870**
	VO(%)	24.212	22.022	34.941	22.197	22.237	**22.022**	22.201
	SR	1.296	4.600	−0.440	4.271	4.435	4.598	**4.752**
	CR	0.130	0.333	0.058	0.311	0.321	0.333	**0.345**
DJIA	CW	0.764	0.813	**1.007**	0.808	0.811	0.813	0.814
	APY	−0.125	−0.098	**0.003**	−0.101	−0.099	−0.098	−0.097
	MDD(%)	38.546	37.765	41.596	37.887	37.802	37.788	**37.749**
	VO(%)	**24.333**	25.430	39.132	25.334	25.387	25.433	25.488
	SR	−10.766	−8.600	**−1.486**	−8.816	−8.691	−8.609	−8.533
	CR	−0.324	−0.259	**0.008**	−0.266	−0.262	−0.259	−0.257



(a) NYSE (O) (b) NYSE (N) (c) SP500 (d) DJIA

Fig. 3. Comprehensive performance of strategies in EGM. (a), (b), (c) and (d) share the same legend. Note that two risk metrics in the radar chart are minus MDD and minus VO.



(a) NYSE (O) (b) NYSE (N) (c) SP500 (d) DJIA

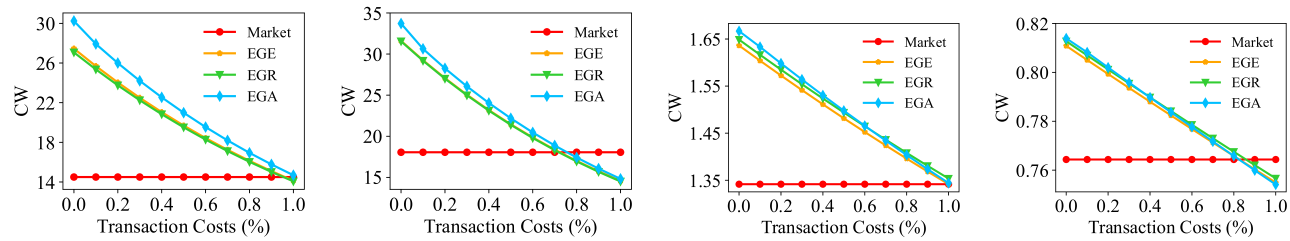
Fig. 4. Relative cumulative wealth. As the baseline, EG's CW is set to one through all periods. (a), (b), (c) and (d) share the same legend.

Table 6
Statistical t-test of performance of strategies in EGM. MER stands for mean excess return.

Statistics	NYSE (O)			NYSE(N)			SP500			DJIA		
	EGE	EGR	EGA	EGE	EGR	EGA	EGE	EGR	EGA	EGE	EGR	EGA
MER(EGM)	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005	0.0003	0.0003	0.0003	−0.0004	−0.0004	−0.0004
MER(Market)	0.0004	–	–	0.0004	–	–	0.0002	–	–	−0.0006	–	–
α	0.0002	0.0002	0.0002	0.0001	0.0001	0.0001	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
β	0.8677	0.8617	0.8630	1.0179	1.0174	1.0173	0.8948	0.8814	0.8899	1.0412	1.0429	1.0450
t-statistics	4.5197	4.4805	4.5563	1.9557	1.9062	1.9992	1.7782	1.7089	1.8356	3.3252	3.2971	3.3137
p-value	0.0000	0.0000	0.0000	0.0505	0.0567	0.0456	0.0756	0.0877	0.0667	0.0009	0.0010	0.0001

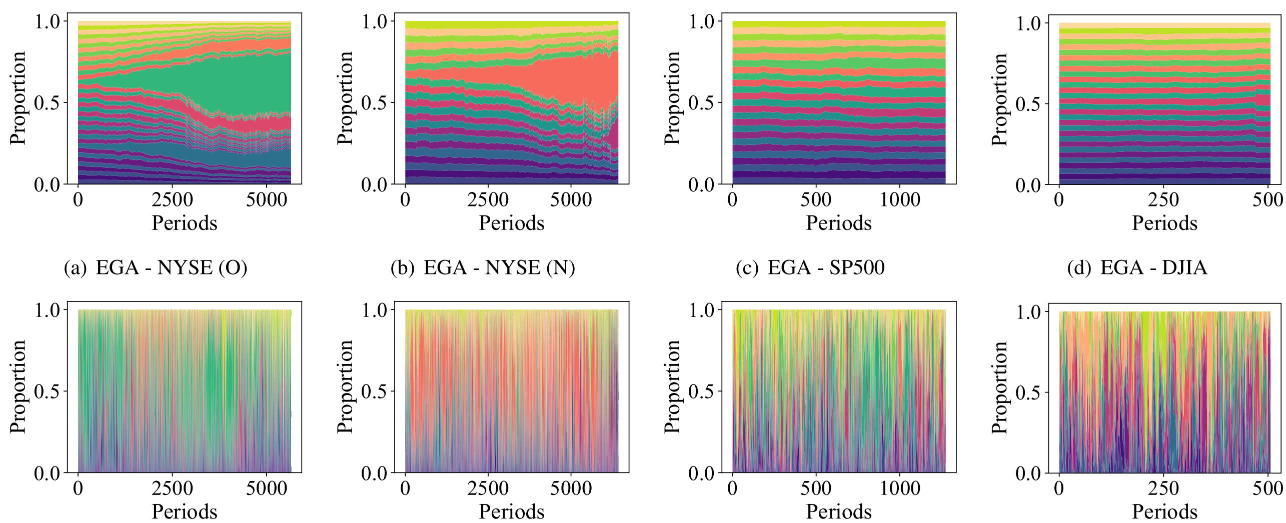
5.5.3. Transaction costs

Transaction costs is a notable and inevitable issue for HFT strategies, who are demanded to transact orders with high frequency. In order to evaluate the practical applicability for HFT, we evaluate the transaction cost scalability of the proposed EGM framework. There are mainly two ways of dealing with transaction costs. The first way is that the portfolio selection process go on regardless of transaction costs but the rebalancing process takes transaction costs into consideration. In



(a) NYSE (O) (b) NYSE (N) (c) SP500 (d) DJIA

Fig. 5. Scalability of transaction cost in terms of cumulative wealth.



(a) EGA - NYSE (O) (b) EGA - NYSE (N) (c) EGA - SP500 (d) EGA - DJIA

(e) TCO - NYSE (O) (f) TCO - NYSE (N) (g) TCO - SP500 (h) TCO - DJIA

Fig. 6. Distribution changes of portfolio weight over periods. In the first row, (a), (b), (c) and (d) are results of EGA on four datasets. In the second row, there are results of TCO. In each sub-figure, each color represents a different assets and the width of a color bar represents the proportion of a asset in the portfolio weight. In order to distinguish each asset, we use different color maps for adjacent color.

second way, transaction costs are straightly engaged in the portfolio selection process. As we formulated in 2, we choose the first way and assume symmetric proportional transaction costs. The computational details can be found in 2.

Fig. 5 illustrates the CW achieved by EGM and the market on four real-world datasets with considering about transaction costs in the range of $[0, 1]\%$. We observe that EGM can strongly withstand a range of reasonable transaction costs. In most testing case, EGM can still outperform the market. On the one hand, achieving more CW in no-transaction-cost condition, EGM has inherent advantage over the market. On the other hand, we think that momentum smooths the update gradients as well as the portfolio weights, which reduces the difference between portfolio weights. Therefore, the practical applicability of EGM framework is promising for HFT.

6. Discussions

In this section, we discuss about the main research contributions and implication of our works in practice.

6.1. Discussion on research contributions

Stochastic gradient-based optimization is of practical significance in many fields of science and engineering (Kingma & Ba, 2014). We introduce classical benchmarks and state-of-the-art stochastic gradient-based optimization methods into online portfolio selection problem to enhance the performance of EG.

To the best of our knowledge, this is the first work to incorporate exponential gradient optimization with momentum-oriented boosting methods. Although momentum is extremely widely used in gradient decent optimization of deep learning models, it is underestimated in other stochastic gradient-based optimization algorithms. As it is illustrated in the simulation experiment (Fig. 1), EG is way more competitive than GD for online portfolio selection problem. Consequently, we propose EGM framework to enhance the performance of EG with the help of momentum. In addition, EGM' update rules are incremental which maintain both the time and space complexities in $O(N)$. This high efficiency makes it possible for strategies in EGM to be HFT strategies.

Furthermore, being incorporated with momentum enables EG to utilize historical information. The follow-up works of EG, including ONS and other FTRL approaches, take historical information into consideration and achieve smaller regret bounds with the cost of higher time and space complexities. On the contrast, our EGM framework utilizes historical information to elevate the effectiveness without significantly increasing time and space complexities.

6.2. Discussion on practical implication

As we introduced in Section 1, HFT has become popular in recent year. In order to trade with high frequency, HFT strategies have a high requirement for efficiency. As a result, our EGM framework is born for this HFT background.

On the one hand, compared with existing FTW approaches, the proposed EGM framework can not only keep the computational time and space costs at the minimal level, but also achieve greater performance in most testing cases. On the other hand, compared with the state-of-the-art FTL approach, i.e. VAWR (Cai, 2020), EGM framework has close form update rules, which is easy and efficient to implement, while VAWR has to deal with a quadratic programming problem in each period. Even if some FTL approaches, i.e., PAMR, OLMAR and RMR, also have close form update rules and can achieve better performance in terms of cumulative wealth, but they are not applicable in practice, especially for HFT background, for the following problems.

The first problem is the high transaction costs, which is the most fatal drawback of FTL approaches and also an unavoidable issue that should be addressed in practice. Because of the mean reversion assumption that FTL approaches rely on, FTL approaches regularly allocate all wealth into a single asset, producing a one-hot coding portfolio weight vector. This unique portfolio weight distribution bring FTL with high cumulative wealth in most cases. In the meanwhile, it causes a significant transaction costs problem. In order to address this problem, Li et al. (2018) introduced L1-norm into formulations and proposed Transaction Cost Optimization (TCO). However, TCO appears to have limited effectiveness in reducing the transaction costs. Fig. 6 shows the EGA's and TCO's distribution changes of portfolio weight over periods. We observe that in all four testing datasets, EGA's portfolio weight changes in a relatively smooth way while TCO's portfolio weight changes far more aggressively. This unique portfolio weight distribution also causes the second problem.

The second problem is the high risk. One of the key concepts of managing a portfolio is diversification, which means not to put all the eggs in one basket. However, FTL approaches regularly invest in a single asset and always achieve high risk. In fact, risk is an important metric, which is as significant as return. Although Li et al. (2012) explained that "high return is associated with high risk", a portfolio with high risk is also extremely hard to be accepted in practice.

In summary, the proposed EGM framework is effective and appropriate for HFT background in practice.

7. Conclusion and future work

In this paper, we propose an online portfolio selection framework named EGM for high-frequency trading, which enhances the performance of EG by incorporating momentum. Leveraging historical information helps EGM to outperform the benchmark and well-acknowledged strategies on the real-world datasets. In the meanwhile, EGM framework is simple to implement and execute with only constant memory and computing time per asset in each trading period, which meets the requirements of high-frequency trading in practice. The theoretical analysis reveals that EGM can bound the regret sublinearly with respect to number of both periods and assets. Additionally, momentum-oriented boosting methods can be widely adapted to other gradient-based stochastic learning problems that considerably concern about computational efficiency.

In the future, we would like to design other HFT strategies that fully leverage historical information and hopefully break through the regret bound of EG. Moreover, addressing the practical issues of Follow-The-Loser approaches, which are mentioned in Section 6.2, is also an interesting and meaningful research direction of online portfolio selection.

CRediT authorship contribution statement

Yuyuan Li: Conceptualization, Methodology, Software, Writing – original draft. **Xiaolin Zheng:** Resources, Supervision, Funding acquisition. **Chaochao Chen:** Supervision, Writing – review & editing. **Jiawei Wang:** Visualization, Writing – review & editing. **Shuai Xu:** Software, Investigation, Data curation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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