Causal Inference

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Effect of Training in JTPA

```
> d <- read.dta("jtpa.dta")
> summary(lm(earnings~training,data=d))
Call:
lm(formula = earnings ~ training, data = d)

Residuals:
   Min     1Q Median     3Q    Max
-17396 -13587  -4955   8776 141155

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  14605.1      209.8  69.624   <2e-16 ***
training      2791.1      318.6   8.761   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1

Residual standard error: 16710 on 11202 degrees of freedom
```

Motivation for Instrumental Variables: Non-Compliance

Problem

- *Often we cannot force subjects to take specific treatments*
- *Units choosing to take the treatment may differ in unobserved characteristics from units that refrain from doing so*

Example: Non-compliance in JTPA Experiment

	Not Enrolled in Training	Enrolled in Training	Total
Assigned to Control	3,663	54	3,717
Assigned to Training	2,683	4,804	7,487
Total	6,346	4,858	11,204

Two Views on Instrumental Variables

1. Traditional Econometric Framework
   - Constant treatment effects
   - Linearity in case of a multivalued treatment
2. Potential Outcome Model of IV
   - Heterogeneous treatment effects
   - Focus in Local Average Treatment Effect (LATE)

Recall the Omitted variable bias

- Long Regression model: $Y = \alpha_0 + \alpha_1 D + \alpha_3 A + \epsilon$
  - D is the treatment variable (e.g. training)
  - A is unobserved ability or motivation

- Structural Equation: $Y = \alpha_0 + \alpha_1 D + u_{SE}$ with $u_{SE} = \alpha_3 A + \epsilon$
  - D is endogenous so that $Cov[D, u_{SE}] \neq 0$ and therefore $\hat{\alpha}_1$ will suffer from omitted variable bias

- Recall that the OLS estimator for α_1 is given by:

$$\hat{\alpha}_{1,OLS} = \frac{Cov[Y, D]}{V[D]} = \frac{Cov[\alpha_0 + \alpha_1 D + u_2, D]}{Cov[D, D]}$$

$$\hat{\alpha}_{1,OLS} = \frac{\alpha_1 Cov[D, D] + Cov[D, u_{SE}]}{Cov[D, D]} = \alpha_1 + \frac{Cov[D, u_{SE}]}{Cov[D, D]}$$

$$E[\hat{\alpha}_{1,OLS}] = \alpha_1 + E[\frac{Cov[D, u_{SE}]}{Cov[D, D]}]$$

so bias depends on correlation between u_{SE} and D

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E[\hat{\alpha}_{1,OLS}] &= \alpha_1 + E[\frac{Cov[D, u_{SE}]}{Cov[D, D]}]
\end{aligned}$$

so bias depends on correlation between u_{SE} and D

Instrumental Variable Estimator Assumptions

Long Regression model: $Y = \alpha_0 + \alpha_1 D + \alpha_3 A + \epsilon$

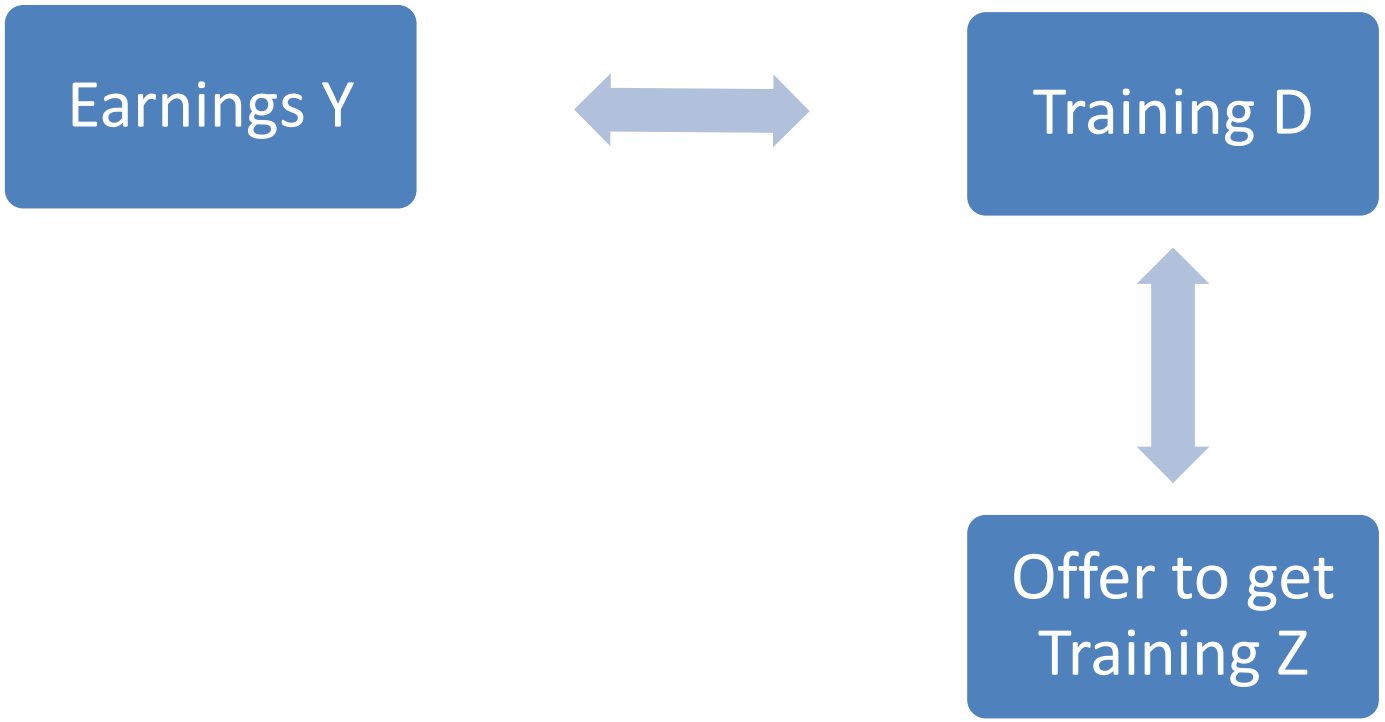
- IV solution is to isolate variation in D which is unrelated to A. The variable which does the isolating is the instrumental variable, call it Z.

A valid instrument Z needs to satisfy three assumption:

1. Z is as good as randomly assigned
2. Z affects the endogenous regressor D (called first stage or relevance)
3. Z satisfies the exclusion restriction, i.e. Z has no effect on Y other than through D. In other words, Z does not appear in the long regression we like to run

Of these only condition 2 can be tested. Conditions 1. and 2. have to be argued based on knowledge from outside the data we have.

Instrumental Variable Estimator Assumptions



Instrumental Variable Estimator Assumptions

Based on these IV assumptions we can identify three effects:

1. The first stage effect: Effect of Z on D.
2. Reduced form or intent-to-treat effect: Effect of Z on Y.
3. The instrumental variable treatment effect: Effect of D on Y, using only the exogenous variation in D that is induced by Z.

- Structural equation:
  $Y = \alpha_0 + \alpha_1 D + u_{SE}$ with $u_{SE} = \alpha_3 A + \epsilon$
- First Stage:
  $D = \pi_0 + \pi_1 Z + u_{FS}$
- Reduced form:
  $Y = \gamma_0 + \gamma_1 Z + u_{RF}$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

First Stage Effect

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

First stage effect: Z on D

$$\hat{\pi}_1 = \frac{Cov[D, Z]}{V[Z]} = \frac{Cov[\pi_0 + \pi_1 Z + u_1, Z]}{Cov[Z, Z]}$$

$$\hat{\pi}_1 = \frac{\pi_1 Cov[Z, Z] + Cov[Z, u_1]}{Cov[Z, Z]}$$

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$$E[\hat{\pi}_1] = \pi_1 + E[\frac{Cov[Z, u_1]}{Cov[Z, Z]}] = \pi_1$$

$\hat{\pi}_1$ is consistent since $Cov[u_1, Z] = 0$

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E[\hat{\pi}_1] &= \pi_1 + E[\frac{Cov[Z, u_1]}{Cov[Z, Z]}] = \pi_1
\end{aligned}$$

$\hat{\pi}_1$ is consistent since $Cov[u_1, Z] = 0$

First Stage Effect in JTPA

First stage effect: Z on D: $\hat{\pi}_1 = \frac{Cov[D,Z]}{V[Z]}$

```
> cov(d[,c("earnings","training","assignmt")])
              earnings    training    assignmt
earnings 2.811338e+08 685.5254685 257.0625061
training 6.855255e+02   0.2456123   0.1390407
assignmt 2.570625e+02   0.1390407   0.221713

> 0.1390407/0.2217139
[1] 0.6271177
```

First Stage Effect in JTPA

```
> summary(lm(training~assignmt,data=d))

Call:
lm(formula = training ~ assignmt, data = d)

Residuals:
     Min       1Q   Median       3Q      Max
-0.64165 -0.01453 -0.01453  0.35835  0.98547

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.014528   0.006529   2.225   0.0261 *
assignmt    0.627118   0.007987  78.522   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1

Residual standard error: 0.398 on 11202 degrees of freedom
Multiple R-squared: 0.355,  Adjusted R-squared: 0.355
F-statistic:  6166 on 1 and 11202 DF,  p-value: < 2.2e-1
```

Reduced Form/Intent-to-treat Effect

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

Reduced Form/Intent-to-treat Effect: Z on Y: Plug first into second stage:

$$\begin{aligned} Y &= \alpha_0 + \alpha_1(\pi_0 + \pi_1 Z + u_1) + u_2 \\ Y &= (\alpha_0 + \alpha_1\pi_0) + (\alpha_1\pi_1)Z + (\alpha_1 u_1 + u_2) \\ Y &= \gamma_0 + \gamma_1 Z + u_3 \end{aligned}$$

where $\gamma_0 = \alpha_0 + \alpha_1\pi_0$, $\gamma_1 = \alpha_1\pi_1$, and $u_3 = \alpha_1 u_1 + u_2$. Note that

$$\begin{aligned} \hat{\gamma}_1 &= \frac{Cov[Y, Z]}{Cov[Z, Z]} = \frac{Cov[\gamma_0 + \gamma_1 Z + u_3, Z]}{Cov[Z, Z]} \\ E[\hat{\gamma}_1] &= \gamma_1 + E[\frac{Cov[Z, u_3]}{Cov[Z, Z]}] = \gamma_1 \end{aligned}$$

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Reduced Form/Intent-to-treat Effect

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> summary(lm(earnings~assignmt,data=d))

Call:
lm(formula = earnings ~ assignmt, data = d)

Residuals:
   Min     1Q Median     3Q    Max
-16200 -13803  -4817   8950 139560

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  15040.5      274.9  54.716  < 2e-16 ***
assignmt      1159.4      336.3   3.448 0.000567 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1

Residual standard error: 16760 on 11202 degrees of freedom
Multiple R-squared: 0.00106,  Adjusted R-squared: 0.000971
F-statistic: 11.89 on 1 and 11202 DF,  p-value: 0.000566
```

Instrumental Variable Effect: Wald Estimator

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

IV Effect: X on Y using exogenous variation in D that is induced by Z. Recall

$$\begin{aligned} Y &= (\alpha_0 + \alpha_1\pi_0) + (\alpha_1\pi_1)Z + (\alpha_1 u_1 + u_2) \\ Y &= \gamma_0 + \gamma_1 Z + u_3 \end{aligned}$$

where $\gamma_0 = \alpha_0 + \alpha_1\pi_0$, $\gamma_1 = \alpha_1\pi_1$, and $u_3 = \alpha_1 u_1 + u_2$. Given this, we can identify α_1:

$$\begin{aligned} \alpha_1 &= \frac{\gamma_1}{\pi_1} = \frac{\text{Effect of Z on Y}}{\text{Effect of Z on D}} = \frac{Cov[Y,Z]/Cov[Z,Z]}{Cov[D,Z]/Cov[Z,Z]} = \frac{Cov[Y,Z]}{Cov[D,Z]} \\ \hat{\alpha}_1 &= \frac{Cov[\alpha_0 + \alpha_1 D + u_2, Z]}{Cov[D,Z]} = \frac{\alpha_1 Cov[D,Z] + Cov[u_2,Z]}{Cov[D,Z]} = \alpha_1 + \frac{Cov[u_2,Z]}{Cov[D,Z]} \\ E[\hat{\alpha}_1] &= \alpha_1 + E[\frac{Cov[u_2,Z]}{Cov[D,Z]}] = \alpha_1 \end{aligned}$$

$\hat{\alpha}_1$ is consistent if $Cov[u_2, Z] = 0$. What if $\pi_1 = 0$?

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- IV assumptions: $Cov[u_1, Z] = 0$, $\pi_1 \neq 0$, and $Cov[u_2, Z] = 0$

IV Effect: X on Y using exogenous variation in D that is induced by Z. Recall

$$\begin{aligned} Y &= (\alpha_0 + \alpha_1\pi_0) + (\alpha_1\pi_1)Z + (\alpha_1 u_1 + u_2) \\ Y &= \gamma_0 + \gamma_1 Z + u_3 \end{aligned}$$

where $\gamma_0 = \alpha_0 + \alpha_1\pi_0$, $\gamma_1 = \alpha_1\pi_1$, and $u_3 = \alpha_1 u_1 + u_2$. Given this, we can identify α_1:

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$$\hat{\alpha}_1 = \frac{Cov[\alpha_0 + \alpha_1 D + u_2, Z]}{Cov[D,Z]} = \frac{\alpha_1 Cov[D,Z] + Cov[u_2,Z]}{Cov[D,Z]} = \alpha_1 + \frac{Cov[u_2,Z]}{Cov[D,Z]}$$

$$E[\hat{\alpha}_1] = \alpha_1 + E[\frac{Cov[u_2,Z]}{Cov[D,Z]}] = \alpha_1$$

$\hat{\alpha}_1$ is consistent if $Cov[u_2, Z] = 0$. What if $\pi_1 = 0$?

Instrumental Variable Effect: Wald Estimator

- Second Stage: $Y = \alpha_0 + \alpha_1 D + u_2$
- First Stage: $D = \pi_0 + \pi_1 Z + u_1$
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$\hat{\alpha}_1$ is consistent if $Cov[u_2, Z] = 0$. What if $\pi_1 = 0$?

Instrumental Variable Effect: Wald Estimator

Instrumental Variable Effect: $\alpha_1 = \frac{\text{Effect of Z on Y}}{\text{Effect of Z on D}} = \frac{Cov[Y,Z]}{Cov[D,Z]}$

```
> cov(d[,c("earnings","training","assignmt")])
             earnings    training    assignmt
earnings 2.811338e+08 685.5254685 257.0625061
training 6.855255e+02   0.2456123   0.1390407
assignmt 2.570625e+02   0.1390407   0.221713

> 257.0625061/0.1390407
[1] 1848.829
```

Instrumental Variable Effect: Two Stage Least Squares

The instrumental variable estimator:

$$\alpha_1 = \frac{\gamma_1}{\pi_1} = \frac{Cov[Y, Z]}{Cov[D, Z]}$$

is numerically equivalent to the following two step procedure:

1. Fit first stage and obtain fitted values $\hat{D} = \hat{\pi}_0 + \hat{\pi}_1 Z$
2. Plug into second stage:

$$\begin{aligned} Y &= \alpha_0 + \alpha_1 \hat{D} + u_2 \\ Y &= \alpha_0 + \alpha_1(\hat{\pi}_0 + \hat{\pi}_1 Z) + u_2 \\ Y &= (\alpha_0 + \alpha_1 \hat{\pi}_0) + \alpha_1(\hat{\pi}_1 Z) + u_2 \end{aligned}$$

- α_1 is solely identified based on variation in D that comes from Z
- Point estimates from second regression are equivalent to IV estimator, the standard errors are not quite correct since they ignore the estimation uncertainty in $\hat{\pi}_0$ and $\hat{\pi}_1$.

Instrumental Variable Effect: Two Stage Least Squares

```
> training_hat <- lm(training~assignmt,data=d)$fitted
> summary(lm(earnings~training_hat,data=d))

Call:
lm(formula = earnings ~ training_hat, data = d)

Residuals:
   Min     1Q Median     3Q    Max
-16200 -13803  -4817   8950 139560

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   15013.6      281.3  53.375  < 2e-16 ***
training_hat   1848.8      536.2   3.448 0.000567 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1

Residual standard error: 16760 on 11202 degrees of freedom
Multiple R-squared: 0.00106,  Adjusted R-squared: 0.000971
```

Instrumental Variable Effect: Two Stage Least Squares

```
> library(AER)
> summary(ivreg(earnings ~ training | assignmt,data = d))
Call:
ivreg(formula = earnings ~ training | assignmt, data = d)
Residuals:
   Min     1Q Median     3Q    Max
-16862 -13716  -4943   8834 140746
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  15013.6      280.6  53.508  < 2e-16 ***
training      1848.8      534.9   3.457 0.000549 ***
---
Residual standard error: 16720 on 11202 degrees of freedom
Multiple R-Squared: 0.00603,  Adjusted R-squared: 0.005941
Wald test: 11.95 on 1 and 11202 DF,  p-value: 0.0005491
```

IV Estimator: Multivariate Case

- Let $\mathbf{X} = [1, X_1, ..., X_K, D]$ and $\mathbf{Z} = [1, X_1, ..., X_K, Z]$.
- Second Stage: $Y = \mathbf{X}\alpha + u_2$ with $\alpha = [\alpha_0, \alpha_1, ..., \alpha_K, \alpha_D]$
- First Stage: $D = \mathbf{Z}\pi + u_1$ with with $\pi = [\pi_0, \pi_1, ..., \pi_K, \pi_Z]$
- Identification: $Cov[\mathbf{Z}, u_1] = 0$, $Cov[\mathbf{Z}, u_2] = 0$, and $\pi_Z \neq 0$ (non-zero partial effect of Z on D)

The multivariate IV estimator is consistent:

$$\begin{aligned}
\hat{\alpha}_{IV} &= (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'Y \\
\hat{\alpha}_{IV} &= (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'(\mathbf{X}\alpha + u_2) \\
\hat{\alpha}_{IV} &= (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'\mathbf{X}\alpha + (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'u_2 \\
\hat{\alpha}_{IV} &= \alpha + (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'u_2 \\
E[\hat{\alpha}_{IV}] &= \alpha + E[(\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'u_2] = \alpha
\end{aligned}$$

2SLS Estimator: Multivariate Case

1. First stage regression to get fitted values

$$\begin{aligned} D &= \mathbf{Z}\pi + u_1 \Rightarrow \hat{\pi} = (\mathbf{Z}\mathbf{Z}')^{-1}\mathbf{Z}'D \\ \hat{D} &= \mathbf{Z}\hat{\pi} = \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'D = \mathbf{P}_z D \end{aligned}$$

2. Regress fitted values on Y

$$Y = \hat{D}\alpha_{2SLS} + u_3$$

We can show that:

$$\begin{aligned} \alpha_{2SLS} &= (\hat{D}'\hat{D})^{-1}\hat{D}'Y \\ &= (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'Y = \alpha_{IV} \end{aligned}$$

For reference, the variance under homoscedasticity is given by:

$$V[\hat{\alpha}_{2SLS|\mathbf{X},\mathbf{Z}}] = \sigma^2(\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'\mathbf{Z}(\mathbf{X}'\mathbf{Z})^{-1}$$

A consistent estimator for the error variance is given by: $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - x_i'\hat{\beta}_{2SLS})^2$ where x_i corresponds to the original variables (endogenous treatment and controls) and $\hat{\beta}_{2SLS}$ is the vector of 2SLS parameter estimates (so this is not the residual from the second stage)

2SLS Estimator: Multivariate Case

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Instrumental Variable Effect: Two Stage Least Squares

```
> summary(ivreg(earnings ~ training + prevearn + sex + age + marrie
+               | prevearn + sex + age + married +assignmt,data = d))
Call:
ivreg(formula = earnings ~ training + prevearn + sex + age +
    married | prevearn + sex + age + married + assignmt, data = d)
Residuals:
   Min     1Q Median     3Q    Max
-58052 -10916  -4050   8316 117239

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.162e+04  6.042e+02  19.238  < 2e-16 ***
training     1.927e+03  4.998e+02   3.855 0.000116 ***
prevearn     1.270e+00  3.885e-02  32.675  < 2e-16 ***
sex          3.760e+03  3.053e+02  12.316  < 2e-16 ***
age         -9.592e+01  1.543e+01  -6.215  5.3e-10 ***
married      2.707e+03  3.488e+02   7.760  9.2e-15 ***
---
Residual standard error: 15600 on 11198 degrees of freedom
```

Multiple Instruments

- 2SLS estimator can be used to combine multiple instruments for the same endogeneous variable. Strong assumptions needed:
  - Each instrument captures the same effect
  - Exogeneity holds for all instruments

$$D = X\beta + Z_1\pi_1 + Z_2\pi_2 + ... + Z_k\pi_k + u_1$$

where $Cov(Z_j, u_1) = 0$ and $Cov(Z_j, u_2) = 0$ for all $j = 1, .., k$.

- Need at least as many instruments as endogenous regressors:
  - Let k be number of endogenous regressors and m number of instruments
  - Exactly or just identified case: $m = k$
  - Overidentified case: $m > k$
  - Underidentified case: $m < k$

Judging the Credibility of IV Estimates

- The probability limit of the IV estimator is given by:

$$plim\, \alpha_{D,IV} = \alpha_D + \frac{Corr(Z, u_2)}{Corr(Z, D)} \frac{\sigma^{u_2}}{\sigma^D}$$

  so to obtain consistent estimates the instrument Z must be:

- Relevant: $Cov(Z, D) \neq 0$ (testable)
  - If $Cov(Z, D)$ is small, the instrument is weak. We get consistency in asymptotia, but in small (finite) samples we can get strong bias even if instrument is perfectly exogenous

- Exogenous: $Cov(Z, u_2) = 0$ (untestable)
  - If Z has an independent effect on Y other than through D we have $Cov(Z, u_2) \neq 0$ and estimates are inconsistent
  - Even small violations can lead to significant large sample bias unless instruments are strong

- Failure of either condition is a problem! But both conditions can be hard to satisfy at the same time. There often is a tradeoff.

Instrumental Variable Examples

Study	**Outcome**	**Treatment**	**Instrument**
Angrist and Evans (1998)	Earnings	More than 2 Children	Multiple Second Birth (Twins)
Angrist and Evans (1998)	Earnings	More than 2 Children	First Two Children are Same Sex
Levitt (1997)	Crime Rates	Number of Policemen	Mayoral Elections
Angrist and Krueger (1991)	Earnings	Years of Schooling	Quarter of Birth
Angrist (1990)	Earnings	Veteran Status	Vietnam Draft Lottery
Miguel, Satyanath and Sergenti (2004)	Civil War Onset	GDP per capita	Lagged Rainfall
Acemoglu, Johnson and Robinson (2001)	Economic performance	Current Institutions	Settler Mortality in Colonial Times
Cleary and Barro (2006)	Religiosity	GDP per capita	Distance from Equator

Exogenous, but weak Instruments

- In contrast to OLS, the IV estimator is not unbiased in small (finite) samples even when instrument is perfectly exogenous
- Because of sampling variability in first stage estimation of fitted values, some part of the correlation between errors in first and second stage seeps into 2SLS estimates (correlation disappears in large samples)
- Finite sample bias can be considerable (e.g., 20 - 30%), even when the sample size is over 100,000 if the instrument is weak
- Relative bias of $\alpha_{D,IV}$ versus $\alpha_{D,OLS}$ is approximately $1/F$ where F is the F-statistic for testing H_0: $\pi_Z = 0$, i.e. partial effect of Z on D is zero (or against joint zero for multiple instruments)

Testing For Relevance

```
> library(lmtest)
> fs1 <- lm(training~prevearn + sex + age + married +assignmt,data=d
> fs2 <- lm(training~prevearn + sex + age + married,data=d)
> waldtest(fs1, fs2)
Wald test

Model 1: training ~ prevearn + sex + age + married + assignmt
Model 2: training ~ prevearn + sex + age + married
  Res.Df Df      F    Pr(>F)
1  11198
2  11199 -1 6158.8 < 2.2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1
```

Exogenous, but weak Instruments

- Adding instruments increases the relevance of the instrument set (increases the first stage F)
- But too many instruments increases small sample bias (compared to few instruments) and also call into doubt the exclusion restrictions
- Best to have single, strong instrument
- There are more complex competitors to 2SLS:
  - Limited Information Maximum Likelihood (LIML) estimation
  - Jackknife instrumental variables
  - Imbens and Rosenbaum (2005) robust IV.
- Small sample studies suggest that LIML and robust IV may be superior to 2SLS in small samples (but remains open area of research)

Failure of Exogeneity

- Recall the probability limit:

$$plim\, \alpha_{D,IV} = \alpha_D + \frac{Corr(Z, u_2)}{Corr(Z, D)} \frac{\sigma_{u_2}}{\sigma_D}$$

- In general we get inconsistent estimates if $Corr(Z, u_2) \neq 0$. This large sample bias can often be considerable but is hard to quantify precisely because it depends on unobservables
- If the instrument is stronger, large sample bias can be attenuated, but often magnitude of $Corr(Z, u_2)$ dominates
- The best we can often do is often to be skeptical and to make sure exogeneity is highly plausible in the setting to which we apply IV
- Sensitivity analysis:
  - Is the instrument plausibly exogenous or can it be easily predicted from covariates?
  - Formal sensitivity tests
    - E.g. Stata code from "Plausibly Exogenous" (Hanson et. al, 2009)
    - R code from Wand (2002)

Failure of Exogeneity

- Does a randomly assigned instrument Z always satisfy $Cov(Z, u_2) = 0$?
- No! Encouragement may still have independent effect on outcome other than through the treatment
- When designing an encouragement experiment we need to be careful to design encouragements so that they are relevant, but also narrowly targeted to only create variation in treatment intake
- SUTVA may be a concern as well

Failure of Exogeneity

- Does a randomly assigned instrument Z always satisfy $Cov(Z, u_2) = 0$?
- No! Encouragement may still have independent effect on outcome other than through the treatment
- When designing an encouragement experiment we need to be careful to design encouragements so that they are relevant, but also narrowly targeted to only create variation in treatment intake
- SUTVA may be a concern as well

Conclusion

- IV works only under very specific circumstances (e.g. well designed encouragement design experiments)
- Often, it will be difficult to find instruments that are both relevant (strong enough) and exogenous
- Violations of assumptions can lead to large biases and estimation theory is complicated
- So far, we have assumed constant treatment effects α_D which seems unrealistic in most settings. Often an instrument affects only a subpopulation of interest and we have little information about treatment effects for other units that may not be affected by the instrument at all.
- Next we'll discuss modern IV with heterogeneous potential outcomes

Two Views on Instrumental Variables

1. Traditional Econometric Framework
   - Constant treatment effects
   - Linearity in the case of a multivalued treatment
2. Potential Outcome Model of IV
   - Heterogeneous treatment effects
   - Focus in Local Average Treatment Effect (LATE)

Identification with Traditional Instrumental Variables

Definition

Two equations:

- $Y = \gamma + \alpha D + \varepsilon$ (Second Stage)
- $D = \tau + \rho Z + \eta$ (First Stage)

Identification Assumption

1. *Exogeneity and Exclusion:* $Cov(Z, \eta) = 0$ *and* $Cov(Z, \varepsilon) = 0$
2. *First Stage:* $\rho \neq 0$
3. $\alpha = Y_{1,i} - Y_{0,i}$ *constant for all units* i.
   Or in the case of a multivalued treatment with s *levels we assume* $\alpha = Y_{s,i} - Y_{s-1,i}$.

Instrumental Variable Estimator

- True model: $Y = D\alpha + X\beta + \varepsilon$
- Given the IV assumptions, we could regress: $Y = Z\rho + \omega$ and obtain an unbiased effect of $\hat{\rho}$, the effect of Z on Y
- But we can also obtain an unbiased effect of the estimate of β_a, the effect of D on Y by using only the exogenous variation in D that is induced by Z

Assume $Cov[\nu = \varepsilon + X\beta, Z] = 0$. And consider the IV estimator:

Potential Outcome Model for Instrumental Variables

Definition (Instrument)

Z_i: Binary instrument for *unit i*.

$$Z_i = \begin{cases} 1 & \text{if unit } i \text{ "encouraged" to receive treatment} \\ 0 & \text{if unit } i \text{ "encouraged" to receive control} \end{cases}$$

Definition (Potential Treatments)

D_z indicates *potential* treatment status given $Z = z$

- $D_1 = 1$ encouraged to take treatment and takes treatment

Assumption

Observed treatments are realized as

$$D = Z \cdot D_1 + (1 - Z) \cdot D_0 \text{ so } D_i = \begin{cases} D_{1i} & \text{if } Z_i = 1 \\ D_{0i} & \text{if } Z_i = 0 \end{cases}$$

Potential Outcome Model for Instrumental Variables

Following Angrist, Imbens, and Rubin (1996), we can define:

Definition

- *Compliers*: $D_1 > D_0$ ($D_0 = 0$ and $D_1 = 1$).
- *Always-takers*: $D_1 = D_0 = 1$.
- *Never-takers*: $D_1 = D_0 = 0$.
- *Defiers*: $D_1 < D_0$ ($D_0 = 1$ and $D_1 = 0$).

Problem

Only one of the potential treatment indicators (D_0, D_1) *is observed, so we cannot identify which group any particular individual belongs to*

Who are the Compliers?

Study	Outcome	Treatment	Instrument
Angrist and Evans (1998)	Earnings	More than 2 Children	Multiple Second Birth (Twins)
Angrist and Evans (1998)	Earnings	More than 2 Children	First Two Children are Same Sex
Levitt (1997)	Crime Rates	Number of Policemen	Mayoral Elections
Angrist and Krueger (1991)	Earnings	Years of Schooling	Quarter of Birth
Angrist (1990)	Earnings	Veteran Status	Vietnam Draft Lottery
Miguel, Satyanath and Sergenti (2004)	Civil War Onset	GDP per capita	Lagged Rainfall
Acemoglu, Johnson and Robinson (2001)	Economic performance	Current Institutions	Settler Mortality in Colonial Times
Cleary and Barro (2006)	Religiosity	GDP per capita	Distance from Equator

Potential Outcome Model for Instrumental Variables

Definition (Potential Outcomes)

Given the binary instrument $Z_i \in (1, 0)$ and the binary treatment $D_i \in (1, 0)$ every unit now has four potential outcomes $Y_i(D, Z)$:

- $Y(D = 1, Z = 1)$; $Y(D = 1, Z = 0)$; $Y(D = 0, Z = 1)$; $Y(D = 0, Z = 0)$

e.g. the causal effect of the treatment given the unit's realized encouragement status is given by $Y(D = 1, Z_i) - Y(D = 0, Z_i)$.

Assumption (Ignorability)

Ignorability of the Instrument: $(Y_0, Y_1, D_0, D_1) \perp\!\!\!\perp Z$

- *Independence:* $(Y(D, Z), D_1, D_0) \perp\!\!\!\perp Z$ *which implies that causal effects of* Z *on* Y *and* Z *on* D *are identified.*
- *Exclusion:* $Y(D, 0) = Y(D, 1)$ *for* $D = 0, 1$ *so we can simply define potential outcomes indexed solely by treatment status:* (Y_1, Y_0)

Potential Outcome Model for Instrumental Variables

Estimand (LATE)

$\alpha_{LATE} = E[Y_1 - Y_0 | D_1 > D_0]$ *is defined as the Local Average Treatment Effect for Compliers*

- *This estimand varies with the particular instrument Z*

Proposition (Special Cases)

- *When the treatment intake, D, is itself randomized, then $Z = D$ and every individual is a complier*
- *Given one-sided noncompliance, $D_0 = 0$:*

$$E[Y_1 | D_1 > D_0] = E[Y_1 | D_1 = 1] = E[Y_1 | Z = 1, D_1 = 1] = E[Y_1 | D = 1]$$

, and

$$E[Y_0 | D_1 > D_0] = E[Y_0 | D = 1]$$

so $\alpha_{LATE} = E[Y_1 - Y_0 | D_1 > D_0] = E[Y_1 - Y_0 | D = 1] = \alpha_{ATET}$

Potential Outcome Model for Instrumental Variables

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$$E[Y_0 | D_1 > D_0] = E[Y_0 | D = 1]$$

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Potential Outcome Model for Instrumental Variables

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, and

$$E[Y_0 | D_1 > D_0] = E[Y_0 | D = 1]$$

so $\alpha_{LATE} = E[Y_1 - Y_0 | D_1 > D_0] = E[Y_1 - Y_0 | D = 1] = \alpha_{ATET}$

Identification with Instrumental Variables

Identification Assumption

1. *Ignorability of the Instrument:* $(Y_0, Y_1, D_0, D_1) \perp\!\!\!\perp Z$
2. *First Stage:* $0 < P(Z = 1) < 1$ *and* $P(D_1 = 1) \neq P(D_0 = 1)$
3. *Monotonicity:* $D_1 \geq D_0$

Identification Result

$$
\begin{aligned}
E[Y_1 - Y_0 | D_1 > D_0] &= \frac{E[Y|Z=1] - E[Y|Z=0]}{E[D|Z=1] - E[D|Z=0]} \\
&= \frac{\textit{Intent to Treat Effect of } Z \textit{ on } Y}{\textit{First Stage Effect of } Z \textit{ on } D} \\
&= \frac{\textit{Intent to Treat Effect}}{\textit{Proportion of Compliers}}
\end{aligned}
$$

Identification with Instrumental Variables

Identification Assumption

1. *Ignorability of the Instrument:* $(Y_0, Y_1, D_0, D_1) \perp\!\!\!\perp Z$
2. *First Stage:* $0 < P(Z = 1) < 1$ *and* $P(D_1 = 1) \neq P(D_0 = 1)$
3. *Monotonicity:* $D_1 \geq D_0$

Proof.

$$\frac{E[Y|Z=1] - E[Y|Z=0]}{E[D|Z=1] - E[D|Z=0]} = \frac{E[Y_0 + (Y_1 - Y_0)D_1|Z=1] - E[Y_0 + (Y_1 - Y_0)D_0|Z=0]}{E[D_1|Z=1] - E[D_0|Z=0]}$$

$$= \frac{E[Y_0 + (Y_1 - Y_0)D_1] - E[Y_0 + (Y_1 - Y_0)D_0]}{E[D_1] - E[D_0]} = \frac{E[(Y_1 - Y_0)(D_1 - D_0)]}{E[D_1 - D_0]}$$

$$= \frac{E[Y_1 - Y_0|D_1 > D_0]P(D_1 > D_0) - E[Y_1 - Y_0|D_1 < D_0]P(D_1 < D_0)}{E[D_1 - D_0]} \text{ as } (D_1 - D_0)=(1, 0, -1)$$

$$= \frac{E[Y_1 - Y_0|D_1 > D_0]P[D_1 > D_0]}{P(D_1 > D_0)} = E[Y_1 - Y_0|D_1 > D_0]$$

□

Identification Assumptions

- Ignorability of the Instrument: $(Y_0, Y_1, D_0, D_1) \perp\!\!\!\perp Z$
  - Implies that Z is randomly assigned so that the intent to treat effect and first stage effect are causally identified
  - $Y(d, z)$ implies exclusion restriction so that $Y(d, 0) = Y(d, 1)$ for $d = (1, 0)$. Rules out independent effect of Z on Y
  - Allows to attribute correlation between Z and Y to the effect of D alone; assumption is not testable
    - Random assignment is a not a sufficient condition for exclusion.
- First Stage: $0 < P(Z = 1) < 1$ and $P(D_1 = 1) \neq P(D_0 = 1)$
  - Implies that the instrument Z induces variation in D
  - Testable by regressing D on Z
- Monotonicity: $D_1 \geq D_0$
  - Rules out defiers
  - Often easy to assess from institutional knowledge

Instrumental Variable: Estimators

Estimand (LATE)

$$E[Y_1 - Y_0 | D_1 > D_0] = \frac{E[Y|Z=1] - E[Y|Z=0]}{E[D|Z=1] - E[D|Z=0]} \left(= \frac{cov(Y,Z)}{cov(D,Z)} \right)$$

Estimator (Wald Estimator)

The sample analog estimator is:

$$\left(\frac{\sum_{i=1}^N Y_i Z_i}{\sum_{i=1}^N Z_i} - \frac{\sum_{i=1}^N Y_i (1-Z_i)}{\sum_{i=1}^N (1-Z_i)} \right) \Bigg/ \left(\frac{\sum_{i=1}^N D_i Z_i}{\sum_{i=1}^N Z_i} - \frac{\sum_{i=1}^N D_i (1-Z_i)}{\sum_{i=1}^N (1-Z_i)} \right)$$

Instrumental Variable: Estimators

Estimand (LATE)

$$E[Y_1 - Y_0 | D_1 > D_0] = \frac{E[Y|Z=1] - E[Y|Z=0]}{E[D|Z=1] - E[D|Z=0]} \left(= \frac{cov(Y,Z)}{cov(D,Z)} \right)$$

Estimator (Wald Estimator as IV Regression)

Can also implement Wald Estimator using an IV regression:

$$Y = \mu + \alpha D + \varepsilon$$

where $E[\varepsilon|Z] = 0$, *so* $\alpha = cov(Y,Z)/cov(D,Z)$

To estimate α *we run the simple IV regression of* Y *on a constant and* D *and instrument* D *with* Z.

Instrumental Variable: Estimators

Estimand (LATE)

$$E[Y_1 - Y_0 | D_1 > D_0] = \frac{E[Y|Z=1] - E[Y|Z=0]}{E[D|Z=1] - E[D|Z=0]} \left(= \frac{cov(Y,Z)}{cov(D,Z)} \right)$$

Estimator (Two Stage Least Squares)

If identification assumptions only hold after conditioning on X, covariates are often introduced using 2SLS regression:

$$Y = \mu + \alpha D + X'\beta + \varepsilon,$$

where $E[\varepsilon|X,Z] = 0$. Now α and β are computed regressing Y on D and X, and using Z and X as instruments.

In general, α estimated in this way does not necessarily have a clear causal interpretation (see Abadie (2003))

Example: The Vietnam Draft Lottery (Angrist (1990))

- Effect of military service on civilian earnings
- Simple comparison between Vietnam veterans and non-veterans are likely to be a biased measure
- Angrist (1990) used draft-eligibility, determined by the Vietnam era draft lottery, as an instrument for military service in Vietnam
- Draft eligibility is random and affected the probability of enrollment
- Estimate suggest a 15% effect of veteran status on earnings in the period 1981-1984 for white veterans born in 1950-51; although the estimators are quite imprecise

Wald Estimates for Vietnam Draft Lottery (Angrist (1990))

		Draft-Eligibility Effects in Current $				
Cohort	Year	FICA Earnings (1)	Adjusted FICA Earnings (2)	Total W-2 Earnings (3)	$\hat{p}^e - \hat{p}^n$ (4)	Service Effect in 1978 $ (5)
1950	1981	−435.8	−487.8	−589.6	0.159	−2,195.8
		(210.5)	(237.6)	(299.4)	(0.040)	(1,069.5)
	1982	−320.2	−396.1	−305.5		−1,678.3
		(235.8)	(281.7)	(345.4)		(1,193.6)
	1983	−349.5	−450.1	−512.9		−1,795.6
		(261.6)	(302.0)	(441.2)		(1,204.8)
	1984	−484.3	−638.7	−1,143.3		−2,517.7
		(286.8)	(336.5)	(492.2)		(1,326.5)
1951	1981	−358.3	−428.7	−71.6	0.136	−2,261.3
		(203.6)	(224.5)	(423.4)	(0.043)	(1,184.2)
	1982	−117.3	−278.5	−72.7		−1,386.6
		(229.1)	(264.1)	(372.1)		(1,312.1)
	1983	−314.0	−452.2	−896.5		-2,181.8
		(253.2)	(289.2)	(426.3)		(1,395.3)
	1984	−398.4	−573.3	−809.1		−2,647.9
		(279.2)	(331.1)	(380.9)		(1,529.2)
1952	1981	−342.8	−392.6	−440.5	0.105	−2,502.3
		(206.8)	(228.6)	(265.0)	(0.050)	(1,556.7)
	1982	−235.1	−255.2	−514.7		−1,626.5
		(232.3)	(264.5)	(296.5)		(1,685.8)
	1983	−437.7	−500.0	−915.7		−3,103.5
		(257.5)	(294.7)	(395.2)		(1,829.2)
	1984	−436.0	−560.0	−767.2		−3,323.8
		(281.9)	(330.1)	(376.0)		(1,959.3)

Example: Minneapolis Domestic Violence Experiment

- Minneapolis Domestic Violence Experiment was first field experiment to examine effectiveness of methods used by police to reduce domestic violence (Sherman and Berk 1984)
- **Sample**: 314 cases of male-on-female spousal assault in two high-density precincts, in which both parties present at scene. 51 patrol officers participated in the study.
- **Treatments**: Random assignment of cases to one of three approaches:
  - Send the abuser away for eight hours
  - Advice and mediation of disputes
  - Make an arrest
- **Outcome**: 6-month follow-up period, with both victims and offenders, as well as official records consulted to determine whether or not re-offending had occurred

Non-Compliance In Minneapolis Experiment

Table 1: Assigned and Delivered Treatments in Spousal Assault Cases

Assigned Treatment	Delivered Treatment			
		Coddled		
	Arrest	Advise	Separate	Total
Arrest	98.9 (91)	0.0 (0)	1.1 (1)	29.3 (92)
Advise	17.6 (19)	77.8 (84)	4.6 (5)	34.4 (108)
Separate	22.8 (26)	4.4 (5)	72.8 (83)	36.3 (114)
Total	43.4 (136)	28.3 (89)	28.3 (89)	100.0(314)

Notes: The table shows statistics from Sherman and Berk (1984), Table 1.

ITT Effect in Minneapolis Experiment

Table 2. First stage and reduced forms for Model 1.

	Endogenous variable is coddled			
	First stage		*Reduced form (ITT)*	
	(1)	*(2)**	*(3)*	*(4)**
Coddled-assigned	0.786 (0.043)	0.773 (0.043)	0.114 (0.047)	0.108 (0.041)
Weapon		−0.064 (0.045)		−0.004 (0.042)
Chem. influence		−0.088 (0.040)		0.052 (0.038)
Dep. var. mean	0.567 (Coddled–delivered)		0.178 (V Failed)	

The table reports OLS estimates of the first-stage and reduced form for Model 1 in the text.
*Other covariates include year and quarter dummies, and dummies for non-white and mixed race.

Treatment Effect in Minneapolis Experiment

Table 3. OLS and 2SLS estimates for Model 1.

	Endogenous variable is coddled			
	OLS		*IV/2SLS*	
	(1)	*(2)**	*(3)*	*(4)**
Coddled–delivered	0.087 (0.044)	0.070 (0.038)	0.145 (0.060)	0.140 (0.053)
Weapon		0.010 (0.043)		0.005 (0.043)
Chem. influence		0.057 (0.039)		0.064 (0.039)

The Table reports OLS and 2SLS estimates of the structural equation in Model 1.
*Other covariates include year and quarter dummies, and dummies for non-white and mixed race.

Estimating the Size of the Complier Group

- Since we never observe both potential treatment assignments for the same unit, we cannot identify individual units as compliers
- However, we can easily identify the proportion of compliers in the population using the first stage effect:

$$\begin{aligned} P(D_1 > D_0) &= E[D_1 - D_0] = E[D_1] - E[D_0] \\ &= E[D|Z = 1] - E[D|Z = 0] \end{aligned}$$

- Using a similar logic we can identify the proportion of compliers among the treated or controls only. For example:

$$P(D_1 > D_0|D = 1) = \frac{P(Z = 1)(E[D|Z = 1] - E[D|Z = 0])}{P(D = 1)}$$

- We can also estimate characteristics of the compliers using kappa weighting

Size of Complier Group

TABLE 4.4.2
Probabilities of compliance in instrumental variables studies

Source (1)	Endogenous Variable (D) (2)	Instrument (Z) (3)	Sample (4)	$P[D=1]$ (5)	First Stage, $P[D_1 > D_0]$ (6)	$P[Z=1]$ (7)	Compliance Probabilities $P[D_1 > D_0 \mid D=1]$ (8)	Compliance Probabilities $P[D_1 > D_0 \mid D=0]$ (9)
Angrist (1990)	Veteran status	Draft eligibility	White men born in 1950	.267	.159	.534	.318	.101
			Non-white men born in 1950	.163	.060	.534	.197	.033
Angrist and Evans (1998)	More than two children	Twins at second birth	Married women aged 21–35 with two or more children in 1980	.381	.603	.008	.013	.966
		First two children are same sex		.381	.060	.506	.080	.048
Angrist and Krueger (1991)	High school graduate	Third- or fourth-quarter birth	Men born between 1930 and 1939	.770	.016	.509	.011	.034
Acemoglu and Angrist (2000)	High school graduate	State requires 11 or more years of school attendance	White men aged 40–49	.617	.037	.300	.018	.068

Notes: The table computes the absolute and relative size of the complier population for a number of instrumental variables. The first stage, reported in column 6, gives the absolute size of the complier group. Columns 8 and 9 show the size of the complier population relative to the treated and untreated populations.

Precision for LATE Estimation

- When N is large the standard error on the instrumental variable estimator of the LATE is approximately

$$SE_{\widehat{LATE}} \approx \frac{SE_{\widehat{ITT}}}{Compliance\ Ratio}$$

  - In JTPA data we get $330/.62 = 532$ which is close to the standard error estimate from the instrumental variable regression of 526.
- Two estimates converge if there is perfect compliance
- Otherwise, all else equal, the standard error on the LATE decreases linearly with the compliance!
  - If compliance ratio drops from 100% to 10%, the LATE standard error increases by a factor of 10
- Always wise to conduct a pilot to test the encouragement
- Design it to boost compliance, but do not violate exclusion restriction

Instrumental Variables with Covariates

- Sometimes IV identification assumptions may only hold once we condition on a set of pre-treatment characteristics X
- In traditional econometric view of IV this poses no additional problems. Since treatment effects are assumed to be constant we can just include X in the 2SLS model
- Potential Outcome Model of IV allows for heterogenous treatment effects and 2SLS does not formally identify LATE conditional on X

Instrumental Variables with Covariates

Two ways to get at LATE conditional on X:

1. Add covariates to 2SLS and interpret it as a linear approximation to a weighted average of covariate-specific LATE estimates.

2. Use the local average response function (LARF) estimator that relies on kappa weighting and formally identifies the LATE conditional on X.
   - If $P(Z = 1|X)$ is linear, the results are often similar
   - Kappa weighting is always useful, because it allows us to estimate any characteristic or fit any model of interest to the subpopulation of compliers (even though we cannot identify individual compliers)

IV Identification with Covariates

Identification Assumption

1. *Conditional Independence of the Instrument:* $(Y_0, Y_1, D_0, D_1) \perp\!\!\!\perp Z|X$
2. *First Stage:* $0 < P(Z = 1|X) < 1$ *and* $P(D_1 = 1|X) > P(D_0 = 1|X)$
3. *Monotonicity:* $P(D_1 \geq D_0|X) = 1$

Proposition

Given our identification assumptions:

$$P(D_1 > D_0|X) = E[D|Z = 1, X] - E[D|Z = 0, X] > 0$$

IV Identification with Covariates

Identification Assumption

1. *Conditional Independence of the Instrument:* $(Y_0, Y_1, D_0, D_1) \perp\!\!\!\perp Z|X$
2. *First Stage:* $0 < P(Z = 1|X) < 1$ *and* $P(D_1 = 1|X) > P(D_0 = 1|X)$
3. *Monotonicity:* $P(D_1 \geq D_0|X) = 1$

Theorem (Magic Kappa)

Let $g(\cdot)$ *be any measurable real function of* (Y, D, X) *such that* $E|g(Y, D, X)| < \infty$. *Define*

$$\kappa = 1 - \frac{D \cdot (1 - Z)}{P(Z = 0|X)} - \frac{(1 - D) \cdot Z}{P(Z = 1|X)}.$$

Given our identification assumptions we have that:

$$E[g(Y, D, X)|D_1 > D_0] = \frac{1}{P(D_1 > D_0)} E[\kappa \cdot g(Y, D, X)]$$

Using Magic Kappa to Characterize Compliers

Problem

Let X be a pre-treatment covariate (e.g. age). We like to estimate the average X for compliers, but we cannot identify compilers individually.

Identification Result (Characterize Compliers)

We can use kappa theorem to identify any characteristic for compliers:

$$E[g(Y,D,X)|D_1 > D_0] = \frac{1}{P(D_1 > D_0)} E[\kappa \cdot g(Y,D,X)]$$

so for example: $E[X|D_1 > D_0] = \frac{E[\kappa \cdot X]}{P(D_1 > D_0)} = \frac{E[\kappa \cdot X]}{E[\kappa]}$

Suggests a 2 step procedure:

1. *Estimate* $\kappa = 1 - \frac{D \cdot (1-Z)}{P(Z=0|X)} - \frac{(1-D) \cdot Z}{P(Z=1|X)}$
2. *Compute weighted average of X using κ weights*

Estimate $\kappa = 1 - \frac{D \cdot (1-Z)}{P(Z=0|X)} - \frac{(1-D) \cdot Z}{P(Z=1|X)}$

```
. probit  assignmt   married

Iteration 0:   log likelihood = -7119.1964
Iteration 1:   log likelihood = -7117.1398
Iteration 2:   log likelihood = -7117.1398

Probit regression                                 Number of obs   =      11204
                                                  LR chi2(1)      =       4.11
                                                  Prob > chi2     =     0.0425
Log likelihood = -7117.1398                       Pseudo R2       =     0.0003
```

assignmt	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
married	.0529502	.0261262	2.03	0.043	.0017438	.1041566
_cons	.4176759	.0149452	27.95	0.000	.3883838	.446968

```
. predict PrZ_X
(option pr assumed; Pr(assignmt))

. tab  PrZ_X  married
```

Pr(assignmt)	RECODE of married (Married, living with spouse) 0	1	Total
.6619079	7,495	0	7,495
.6810461	0	3,709	3,709
Total	7,495	3,709	11,204

```
. gen kappa=1-((training*(1-assignmt))/(1- PrZ_X))-(((1-training)*assignmt)/ PrZ_X)
```

Estimate $E[X|D_1 > D_0] = \frac{E[\kappa \cdot X]}{P(D_1 > D_0)} = \frac{E[\kappa \cdot X]}{E[\kappa]}$

```
. tab kappa
```

kappa	Freq.	Percent	Cum.
-2.13525	12	0.11	0.11
-1.957774	42	0.37	0.48
-.5107841	1,817	16.22	16.70
-.4683293	866	7.73	24.43
1	8,467	75.57	100.00
Total	11,204	100.00	

```
. egen numerator = mean( kappa* married)

. egen denominator = mean(kappa)

. sum  denominator
```

Variable	Obs	Mean	Std. Dev.	Min	Max
denominator	11204	.6270513	0	.6270513	.6270513

```
. gen married_CP =  numerator/ denominator

. sum  married_CP married
```

Variable	Obs	Mean	Std. Dev.	Min	Max
married_CP	11204	.3415856	0	.3415856	.3415856
married	11204	.3310425	.4706093	0	1

Characteristics of Complier Group

TABLE 4.4.3
Complier characteristics ratios for twins and sex composition instruments

		Twins at Second Birth		First Two Children Are Same Sex	
Variable	$P[x_{1i}=1]$ (1)	$P[x_{1i}=1 \mid D_{1i} > D_{0i}]$ (2)	$P[x_{1i}=1 \mid D_{1i} > D_{0i}]/P[x_{1i}=1]$ (3)	$P[x_{1i}=1 \mid D_{1i} > D_{0i}]$ (4)	$P[x_{1i}=1 \mid D_{1i} > D_{0i}]/P[x_{1i}=1]$ (5)
Age 30 or older at first birth	.0029	.004	1.39	.0023	.995
Black or hispanic	.125	.103	.822	.102	.814
High school graduate	.822	.861	1.048	.815	.998
College graduate	.132	.151	1.14	.0904	.704

Notes: The table reports an analysis of complier characteristics for twins and sex composition instruments. The ratios in columns 3 and 5 give the relative likelihood that compliers have the characteristic indicated at left. Data are from the 1980 census 5 percent sample, including married mothers aged 21–35 with at least two children, as in Angrist and Evans (1998). The sample size is 254,654 for all columns.

Local Average Response Function: Intuition

We want to estimate $E[g(Y,D,X)|D_1 > D_0]$ but

$$\begin{aligned} E[g(Y,D,X)] &= E[g(Y,D,X)|D_1 > D_0]\Pr(D_1 > D_0) \\ &+ E[g(Y,D,X)|D_1 = D_0 = 1]\Pr(D_1 = D_0 = 1) \\ &+ E[g(Y,D,X)|D_1 = D_0 = 0]\Pr(D_1 = D_0 = 0) \end{aligned}$$

Therefore,

$$\begin{aligned} &E[g(Y,D,X)|D_1 > D_0] \\ &= \frac{1}{\Pr(D_1 > D_0)}\Big\{E[g(Y,D,X)] \\ &- E[g(Y,D,X)|D_1 = D_0 = 1]\Pr(D_1 = D_0 = 1) \\ &- E[g(Y,D,X)|D_1 = D_0 = 0]\Pr(D_1 = D_0 = 0)\Big\} \end{aligned}$$

Kappa:

$$\kappa = 1 - \frac{D \cdot (1-Z)}{P(Z=0|X)} - \frac{(1-D)\cdot Z}{P(Z=1|X)}$$

Estimation of κ

- Estimate $\tau_0(x) = P(Z = 1|X = x)$ using a probit of Z on X
- Compute: $\kappa_i = 1 - \frac{D_i \cdot (1-Z_i)}{1-\tau_0(X_i)} - \frac{(1-D_i) \cdot Z_i}{\tau_0(X_i)}$
  - κ is 1 for $(D = Z)$ which includes "true" compliers but also "fake" compliers that may be always takers or never takers

```
. tab  kappa married if  training==assignmt
```

kappa	married 0	1	Total
1	5,636	2,831	8,467

Estimation of κ

- Estimate $\tau_0(x) = P(Z = 1|X = x)$ using a probit of Z on X
- Compute: $\kappa_i = 1 - \frac{D_i \cdot (1-Z_i)}{1-\tau_0(X_i)} - \frac{(1-D_i) \cdot Z_i}{\tau_0(X_i)}$
  - κ takes larger negative values for identifiable always takers $(Z = 0, D = 1)$ that, based on their X characteristics, look very likely to be assigned to treatment

```
. tab  kappa married if  (training==1 &  assignmt==0)
```

kappa	married 0	1	Total
-2.13525	0	12	12
-1.957774	42	0	42

Estimation of κ

- Estimate $\tau_0(x) = P(Z = 1|X = x)$ using a probit of Z on X
- Compute: $\kappa_i = 1 - \frac{D_i \cdot (1-Z_i)}{1-\tau_0(X_i)} - \frac{(1-D_i) \cdot Z_i}{\tau_0(X_i)}$
  - κ takes larger negative values for identifiable never takers $(Z = 1, D = 0)$ that, based on their X characteristics, look very unlikely to be assigned to treatment

```
. tab  kappa married if  (training==0 &  assignmt==1)
```

kappa	married 0	1	Total
-.5107841	1,817	0	1,817
-.4683293	0	866	866

Local Average Response Function

Definition (LARF)

We call the CEF of Y given X and D for the subpopulation of compliers the local average response function:

$$E[Y|X, D, D_1 > D_0]$$

Since $D \equiv Z$ for compliers and Z is ignorable given X, it follows that the LARF identifies the LATE conditional on X:

$$\begin{aligned} E[Y|X, D = 0, D_1 > D_0] &= E[Y_0|X, Z = 0, D_1 > D_0] \\ &= E[Y_0|X, D_1 > D_0] \text{ and similarly} \\ E[Y|X, D = 1, D_1 > D_0] &= E[Y_1|X, D_1 > D_0] \end{aligned}$$

Therefore:
$E[Y|X, D = 1, D_1 > D_0] - E[Y|X, D = 0, D_1 > D_0] = E[Y_1 - Y_0|X, D_1 > D_0]$

Local Average Response Function: Identification

Problem

The LARF conditions on being a complier, but we cannot identify compilers individually

Identification Result

We can use kappa theorem to identify the LARF:
$E[Y|X, D, D_1 > D_0] = 1/P(D_1 > D_0)E[\kappa Y|X, D]$

Suggests a 2 step procedure:

1. *Estimate* κ
2. *Approximate* $E[Y|X, D, D_1 > D_0]$ *by estimating* $E[Y|X, D]$ *in the whole population while weighting by* κ
   - *E.g.* $(\widehat{\alpha}, \widehat{\beta}) = \text{argmin}_{(\alpha,\beta)\in\Theta} \frac{1}{N} \sum_{i=1}^{N} \kappa_i \cdot (Y_i - \alpha D_i - X_i'\beta)^2$

Fitting a LARF

```
> d <- read.dta("jtpa.dta")
> fstage.probit <- glm(assignmt~sex+age+married+hsorged+prevea
+                      family=binomial(link="probit"),data=d)
> phi <- fstage.probit$fitted.values
> kappa <- 1 - ((d$training*(1-d$assignmt))/(1-phi))-(((1-d$tr
> Y <- matrix(d$earnings)
> X <- as.matrix(cbind(1,d[,c("sex","age","married","hsorged",
> larf <- solve(t(X) %*% diag(kappa) %*% X ) %*% t(X) %*% diag
> larf
                [,1]
1       8612.456173
sex     4857.976712
age      -69.550698
married 2719.630642
hsorged 3093.402183
prevearn    1.301121
training 1854.525498
```

Fitting a LARF

```
> d <- read.dta("jtpa.dta")
>
> out <- larf.ls(
+     fstage.eq          = training~sex+age+married+hsorged+pre
+     sstage.eq          = earnings~sex+age+married+hsorged+pre
+     datafr             = d,
+     cluster.id         = FALSE
+     )
> out$coefficients
                  beta       asy.se
training  1854.525498 489.79328922
Intercept 8612.456173 850.28299445
sex       4857.976712 460.16638082
age        -69.550698  19.80140360
married   2719.630642 531.72234361
hsorged   3093.402183 472.01737178
prevearn     1.301121   0.08658346
```

Example: Effect on Family Net Financial Assets (in $)

		Endogenous Treatment		
		Two Stage Least Squares		
	Ordinary Least Squares (1)	First Stage (2)	Second Stage (3)	Least Squares Treated (4)
Participation in 401(k)	13,527.05		9,418.83	10,800.25
	(1,810.27)		(2,152.89)	(2,261.55)
Constant	-23,549.00	-0.0306	-23,298.74	-27,133.56
	(2,178.08)	(0.0087)	(2,167.39)	(3,212.35)
Family Income (in thousand $)	976.93	0.0013	997.19	982.37
	(83.37)	(0.0001)	(83.86)	(106.65)
Age (minus 25)	-376.17	-0.0022	-345.95	312.30
	(236.98)	(0.0010)	(238.10)	(371.76)
Age (minus 25) square	38.70	0.0001	37.85	24.44
	(7.67)	(0.0000)	(7.70)	(11.40)
Married	-8,369.47	-0.0005	-8,355.87	-6,646.69
	(1,829.93)	(0.0079)	(1,829.67)	(2,742.77)
Family Size	-785.65	0.0001	-818.96	-1,234.25
	(410.78)	(0.0024)	(410.54)	(647.42)
Eligibility for 401(k)		0.6883		
		(0.0080)		