第零章：数学预备知识

1、矢量代数的基本运算

$$\vec{A} = A_x\hat{e}_x + A_y\hat{e}_y + A_z\hat{e}_z$$

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{e}_x + (A_y + B_y)\hat{e}_y + (A_z + B_z)\hat{e}_z$$

$$\vec{A} \cdot \vec{B} = (A_x\hat{e}_x + A_y\hat{e}_y + A_z\hat{e}_z) \cdot (B_x\hat{e}_x + B_y\hat{e}_y + B_z\hat{e}_z) = A_xB_x + A_yB_y + A_zB_z$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\hat{e}_x + (A_zB_x - A_xB_z)\hat{e}_y + (A_xB_y - A_yB_x)\hat{e}_z$$

三重标积(混合积)

$$\vec{A}\cdot(\vec{B}\times\vec{C})=\begin{vmatrix} A_x & A_y & A_z \\ B_x & C_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B})$$ 轮换性

$$\vec{A}\cdot(\vec{B}\times\vec{C})=(\vec{A}\times\vec{B})\cdot\vec{C}$$

只要保持三矢量位置不变，可以交换点积和叉积的位置(先叉积后点积)

三重矢积

$$\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}$$

$$(\vec{C}\times\vec{B})\times\vec{A}=(\vec{C}\cdot\vec{A})\vec{B}-(\vec{B}\cdot\vec{A})\vec{C}$$

先远后近原则

2、矢量代数的三度运算

∇算符（nabla、del） 兼具矢量和微分的性质 $\nabla = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}$

梯度 ∇算符作用在标量上 $\nabla \varphi = \frac{\partial \varphi}{\partial x}\hat{e}_x + \frac{\partial \varphi}{\partial y}\hat{e}_y + \frac{\partial \varphi}{\partial z}\hat{e}_z$

散度 ∇算符与矢量的点积 $\nabla \cdot \vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$

旋度 ∇算符与矢量的叉积 $\nabla \times \vec{f} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = (\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z})\hat{e}_x + (\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x})\hat{e}_y + (\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y})\hat{e}_z$

常见的梯度运算 $\nabla r = \frac{\vec{r}}{r} = \vec{e}_r$ $\nabla(\vec{k}\cdot\vec{r}) = \vec{k}$ $\nabla\frac{1}{r} = -\frac{\vec{r}}{r^3}$

常见的散度运算 $\nabla\cdot\vec{r} = 3$ $\nabla\cdot\frac{\vec{r}}{r^3}(r\neq 0) = 0$

常见的旋度运算 $\nabla\times\vec{r} = 0$ $\nabla\times\frac{\vec{r}}{r^3}(r\neq 0) = 0$

对场点的三度运算与对源点的三度运算结果相反 $\nabla = -\nabla'$

复合函数的散度运算(习题**1.2**)

$$\nabla\varphi(u) = \nabla u\frac{d\varphi}{du} \qquad \nabla\cdot\vec{f}(u) = \nabla u\cdot\frac{d\vec{f}(u)}{du} \qquad \nabla\times\vec{f}(u) = \nabla u\times\frac{d\vec{f}(u)}{du}$$

3、加和与乘积的三度运算

加和的三度

$$\nabla(\varphi+\psi)=\nabla\varphi+\nabla\psi$$

$$\nabla\cdot(\vec{f}+\vec{g})=\nabla\cdot\vec{f}+\nabla\cdot\vec{g}$$

$$\nabla\times(\vec{f}+\vec{g})=\nabla\times\vec{f}+\nabla\times\vec{g}$$

乘积的三度（常数与标量或矢量乘积）

$$\nabla(k\varphi)=k\nabla\varphi$$

$$\nabla\cdot(k\vec{f})=k\nabla\cdot\vec{f}$$

$$\nabla\times(k\vec{f})=k\nabla\times\vec{f}$$

乘积的三度(标量矢量之间乘积) $\varphi\psi,\quad \varphi\vec{f},\quad \vec{f}\cdot\vec{g},\quad \vec{f}\times\vec{g}$

$$\nabla(\varphi\psi)=\varphi\nabla\psi+\psi\nabla\varphi$$

$$\nabla(\vec{f}\cdot\vec{g})=\vec{g}\times(\nabla\times\vec{f})+(\vec{g}\cdot\nabla)\vec{f}+\vec{f}\times(\nabla\times\vec{g})+(\vec{f}\cdot\nabla)\vec{g}$$

$$\nabla\cdot(\varphi\vec{f})=(\nabla\varphi)\cdot\vec{f}+\varphi\nabla\cdot\vec{f}$$

$$\nabla\cdot(\vec{f}\times\vec{g})=\vec{g}\cdot(\nabla\times\vec{f})-\vec{f}\cdot(\nabla\times\vec{g})$$

$$\nabla\times(\varphi\vec{f})=(\nabla\varphi)\times\vec{f}+\varphi\nabla\times\vec{f}$$

$$\nabla\times(\vec{f}\times\vec{g})=(\vec{g}\cdot\nabla)\vec{f}-(\nabla\cdot\vec{f})\vec{g}+(\nabla\cdot\vec{g})\vec{f}-(\vec{f}\cdot\nabla)\vec{g}$$

利用∇算符的微分性和矢量性进行运算

$\vec{f}\cdot\vec{g}$的梯度(习题1.1)

微分性

$$\nabla(\vec{f}\cdot\vec{g})=\nabla_f(\vec{f}\cdot\vec{g})+\nabla_g(\vec{f}\cdot\vec{g}) \qquad \vec{C}\Leftrightarrow\nabla;\vec{A}\Leftrightarrow\vec{f};\vec{B}\Leftrightarrow\vec{g}$$

$$\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}\Rightarrow\vec{C}(\vec{A}\cdot\vec{B})=(\vec{A}\cdot\vec{C})\vec{B}-\vec{A}\times(\vec{B}\times\vec{C})$$

$$\vec{C}(\vec{A}\cdot\vec{B})=(\vec{A}\cdot\vec{C})\vec{B}+\vec{A}\times(\vec{C}\times\vec{B})$$

矢量性

$$\vec{C}(\vec{A}\cdot\vec{B})=\vec{C}(\vec{B}\cdot\vec{A})=(\vec{B}\cdot\vec{C})\vec{A}+\vec{B}\times(\vec{C}\times\vec{A})$$

$$\nabla(\vec{f}\cdot\vec{g})=\nabla_f(\vec{f}\cdot\vec{g})+\nabla_g(\vec{f}\cdot\vec{g})$$

$$=\vec{g}\times(\nabla\times\vec{f})+(\vec{g}\cdot\nabla)\vec{f}+\vec{f}\times(\nabla\times\vec{g})+(\vec{f}\cdot\nabla)\vec{g}$$

4、二重∇算符运算

(1)梯度的散度 $\nabla\cdot(\nabla\varphi)=\nabla^2\varphi$ $\quad \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\nabla\cdot\nabla$

(2)梯度的旋度恒为0 $\nabla\times(\nabla\varphi)=0$

(3)旋度的散度恒为0 $\nabla\cdot(\nabla\times\vec{f})=0$

(4)旋度的旋度 $\nabla\times(\nabla\times\vec{f})=\nabla(\nabla\cdot\vec{f})-\nabla^2\vec{f}$

$$\nabla\times(\nabla\times\vec{f})=\nabla_{\nabla}\times(\nabla\times\vec{f})+\nabla_f\times(\nabla\times\vec{f})=\nabla_f\times(\nabla\times\vec{f})$$

$$=(\nabla_f\cdot\vec{f})\nabla-(\nabla_f\cdot\nabla)\vec{f}=(\nabla_f\cdot\vec{f})\nabla-(\nabla\cdot\nabla_f)\vec{f}$$

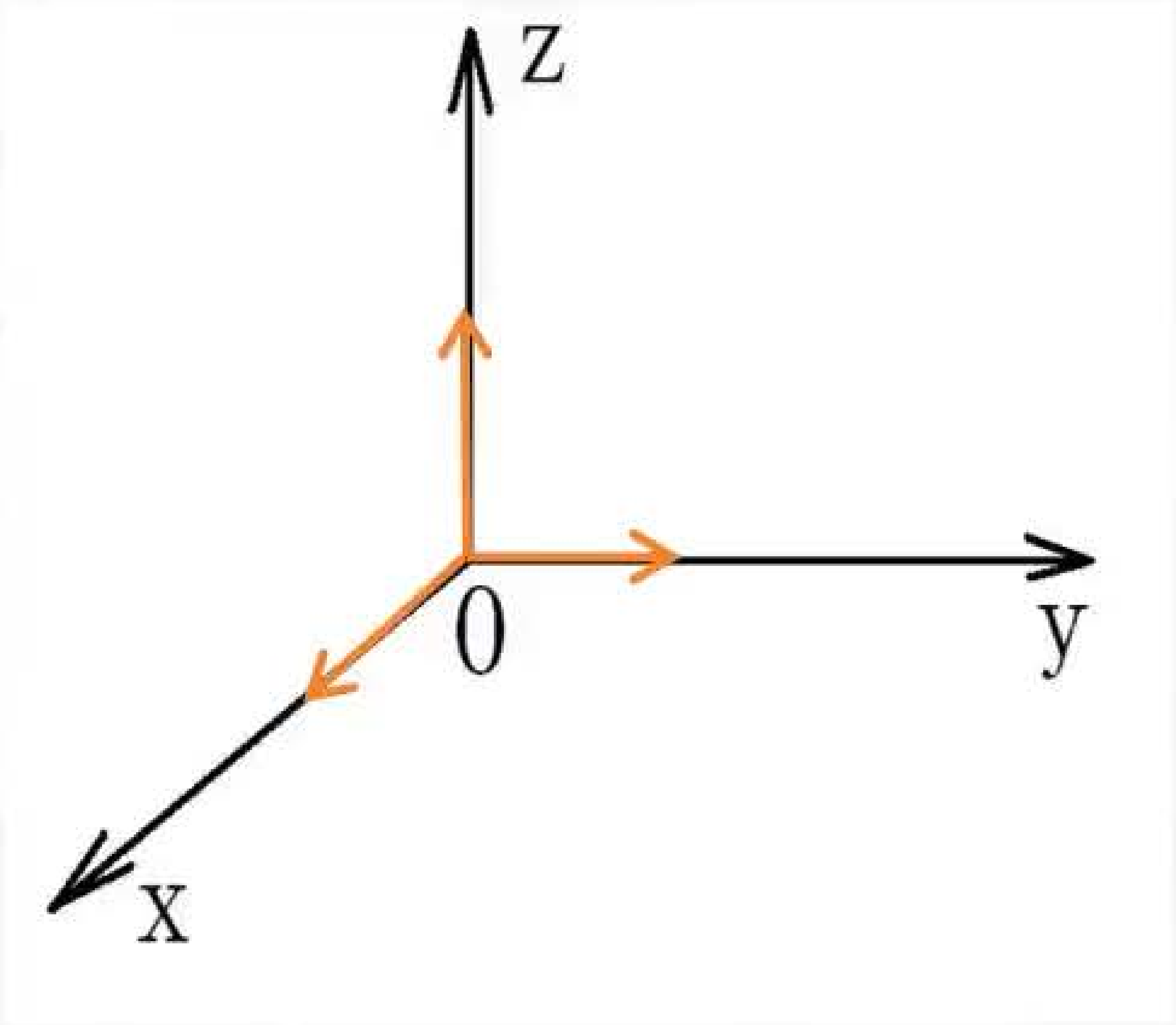
$$=\nabla(\nabla\cdot\vec{f})-\nabla^2\vec{f}$$

5、矢量的积分变换定理

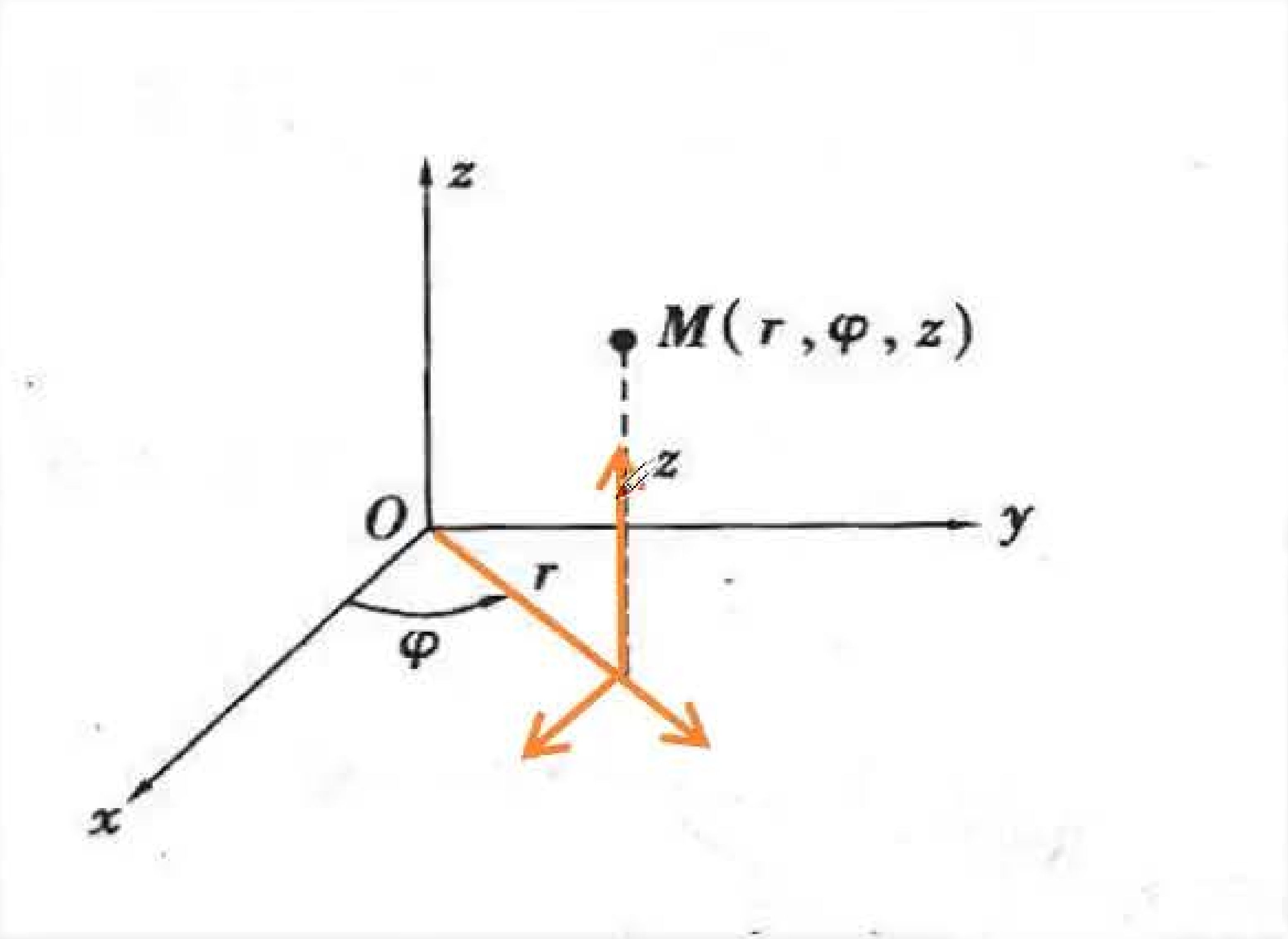
高斯定理： $\oint_s \vec{f} \cdot d\vec{S} = \int_V (\nabla \cdot \vec{f}) dV$ 　　面积分 $\Leftrightarrow$ 体积分

斯托克斯定理： $\oint_l \vec{f} \cdot d\vec{l} = \int_S (\nabla \times \vec{f}) \cdot d\vec{S}$ 　　线积分 $\Leftrightarrow$ 面积分

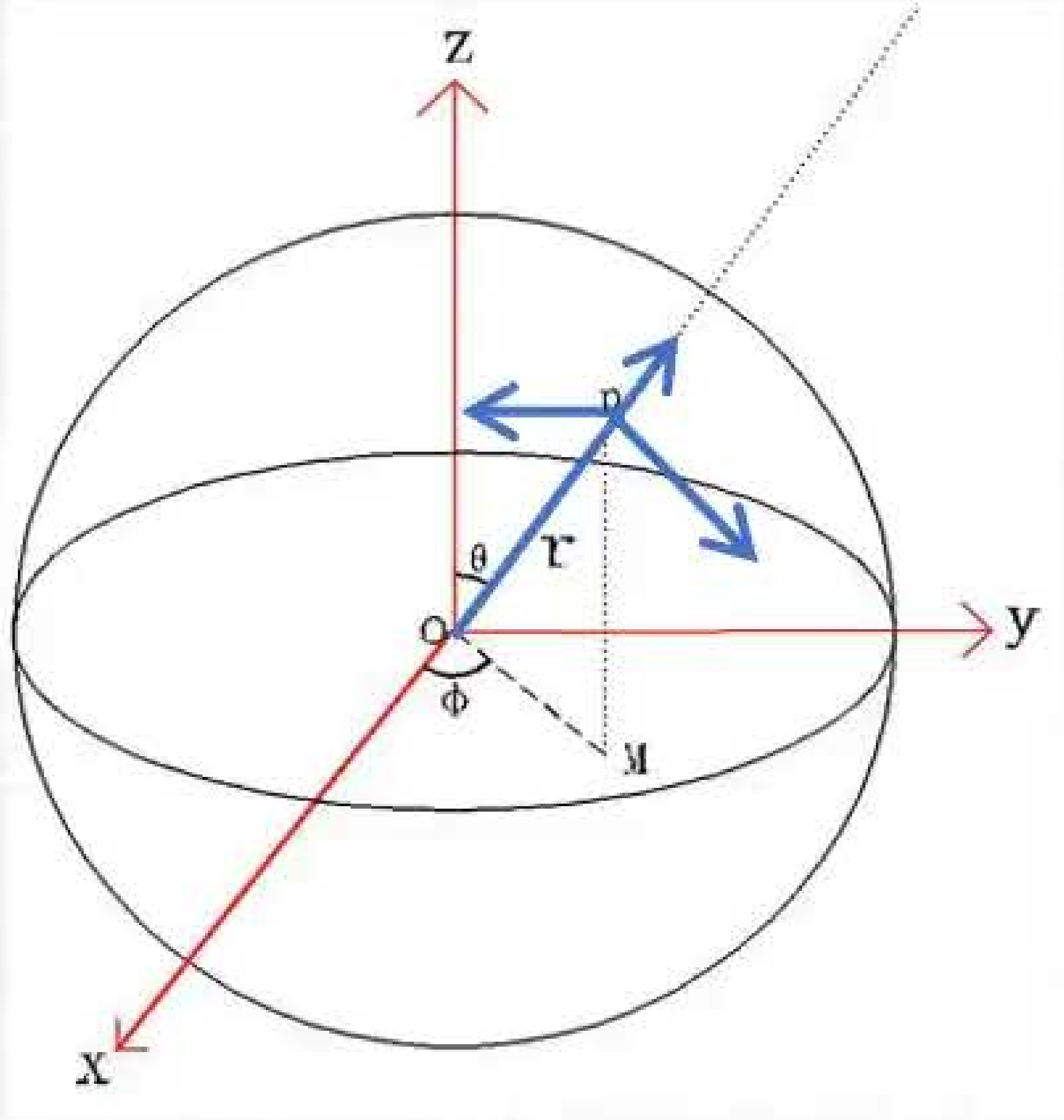
6、正交曲线坐标系



直角坐标系 (x,y,z)



柱坐标系 (r,φ,z)



球坐标系 (r,θ,ϕ)

不同坐标系下单位矢量运算

点乘：　对于任何一种坐标系，都有$\hat{e}_i \cdot \hat{e}_i = 1$

叉乘：　直角坐标系(x, y, z)

$$\hat{e}_x \times \hat{e}_y = \hat{e}_z \quad \hat{e}_y \times \hat{e}_z = \hat{e}_x \quad \hat{e}_z \times \hat{e}_x = \hat{e}_y$$

柱坐标系(r, ϕ, z)

$$\hat{e}_r \times \hat{e}_\phi = \hat{e}_z \quad \hat{e}_\phi \times \hat{e}_z = \hat{e}_r \quad \hat{e}_z \times \hat{e}_r = \hat{e}_\phi$$

球坐标系(r, θ, ϕ)

$$\hat{e}_r \times \hat{e}_\theta = \hat{e}_\phi \quad \hat{e}_\theta \times \hat{e}_\phi = \hat{e}_r \quad \hat{e}_\phi \times \hat{e}_r = \hat{e}_\theta$$

不同坐标系下三度微分运算

查表　附录**I.5**

直角坐标系积分　$$\int f(x, y, z) dV = \int f_1(x) dx \int f_2(y) dy \int f_3(z) dz$$

柱坐标系积分　$$\int f(r, \phi, z) dV = \int f_1(r) r dr \int f_2(\phi) d\phi \int f_3(z) dz$$

球坐标系积分　$$\int f(r, \theta, \phi) dV = \int f_1(r) r^2 dr \int f_2(\theta) \sin\theta d\theta \int f_3(\phi) d\phi$$