Ocean Surface Wave Spectrum

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Abstract

In this lecture, the basic concepts of ocean wave spectrum will be introduced. In the first part, the concepts of wind generated waves are presented, showing the main characteristics and processes of its generation. Then, the concept of presenting waves as a probability density spectrum is shown. In the end, the Pierson-Moskowitz and JONSWAP models are presented, showing their features and relationships between them and wind waves.

1 Ocean Waves

Ocean and atmosphere, by their nature, form a thermodynamic coupled system, continuously exchanging heat, mass, and momentum. This means that any forcing that is imposed from one fluid to the other, leads to a change in the boundary conditions between them, sometimes called "inter-facial conditions" (Massel, 2013).

Due to the strong coupling between those two fluids, movements on the atmosphere will create movements on the ocean surface. These waves created by the movements of the atmosphere are called "Wind Waves". The wind blowing across the surface of the sea is the main generator of ocean waves (Shearman, 1983).

Generally, ocean waves are classified by their restoring forces. There are four main types of ocean waves, sometimes cited five, when gravity and internal waves are considered separately (Massel, 2013). They are:

- **Sound waves**: These waves are created due to the compressibility of the water. Since the water is almost incompressible, sound waves are really small.

- **Capillary waves**: Capillary waves are usually generated by pressure turbulence over the atmosphere. Their restoring force is the surface tension (capillarity) of the water.

- **Gravity waves**: These are the waves that rose to a size that the capillary forces are not strong enough to act as a restoring force. At this stage, the gravity is the one that restores the waves. Gravity waves can be divided in two different types:
  - *Surface gravity waves*: These are gravity waves observed at the surface. These waves are normally generated by the wind.
  - *Internal gravity waves*: These waves are observed at the interfaces of stratified fluids.

- **Planetary waves**: These waves are generated by the equilibrium of the potential vorticity due to changes in depth or latitude.

These waves, however, does not occur as independent events. They may be all occurring at the same time in the observed region. Waves generated by distant storms mix with waves generated at the observed region, all of them mixing with other kinds of waves, such as planetary waves. With this multiplicity of sources, it is not possible to deal with the sea with a deterministic approach. This leads us to a stochastic approach. One of the tools of stochastic variable analysis is the study of the *energy spectrum* of the referred random variable. In the study of ocean waves, this energy spectrum is called *ocean wave spectrum*.

2 Wind Wave Generation

Before exploring the ocean wave spectrum models, it is important to understand the generation of such waves. Even though there are other kinds of waves, we are going to focus on the wind generated waves. The generation of wind waves generally follow the given process:

1. At the initial stage, pressure variations on the atmosphere impose a certain roughness over the ocean surface. These are capillary waves, with very short period and wavelength.

2. When wind velocity increases, these waves grow in size and energy, making, changing the restoring force from surface tension to gravity. These are called gravity waves. At this stage, the stronger the wind, the greater are the waves.

3. A nonlinear energy transfer occur between the waves on the ocean surface, with the energy flowing from shorter waves to waves with slightly lower frequency than the peak waves (Hasselmann et al., 1973). This process eventually make the waves travel faster than the cross-wind speed (Pierson Jr and Moskowitz, 1963).

4. However, these waves do not grow indefinitely. At some point, for example, when a wave reaches a steepness of 1/7 in deep water (Massel, 2013), the flow in the crest and the bottom are not even, forcing the wave to break. Breaking waves is a field of study itself, so they will not be covered here.

The first two parts of the described process are known as the *Phillips-Miles Theory*. The first part, regarding turbulent winds and capillary waves generation, is described in Phillips's 1957 paper (Phillips, 1957), while the second part was described in Miles's same year paper (Miles, 1957). As it can be seen, these are two different works, however, their theories form a continuum, as can be seen in Figure 1

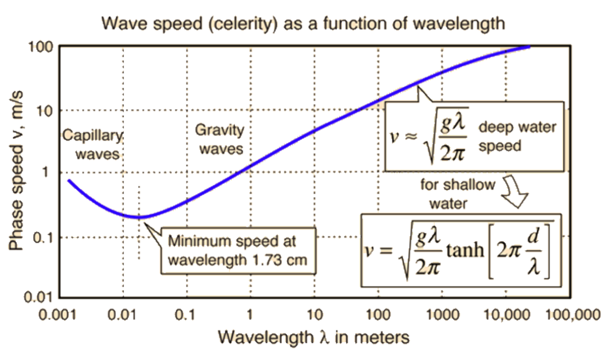

Figure 1: Wave Velocity *versus* Wavelength (Nave, 2015)

3 Spectral Analysis of Ocean Waves

3.1 General Concepts

While observing ocean surface waves, their randomness is readily noticeable, as well as the quasi-regular overall behaviour. The wave profile can be seen as a random variable, in a way that the wave properties are not readily available to the observer in a wave-by-wave fashion. Therefore, the prediction of waves parameters can only be achieved by stochastic analysis of the sea data. One of the most common approaches for the stochastic analysis is the analysis of the frequency spectrum of the variable (Massel, 2013).

In time domain, a given random time series $\zeta(t)$, its autocorrelation function can be written as in Equation 3.1. The autocorrelation of a signal can be interpreted as the similarity between observations with a time lag τ between them.

$$K(\tau) = \int_{-\infty}^{\infty} \zeta(t)\zeta(t-\tau)dt = E[\zeta(t)\zeta(t-\tau)] \quad (3.1)$$

There are two ways of analyzing the spectrum of a random variable: by applying the Fast Fourier Transform (FFT) on the random signal $\zeta(t)$, or applying the Fourier Transform over the autocorrelation function. The latter method results in the spectral density function $S(\omega)$, as shown in Equation 3.2.

$$S(\omega) = \int_{-\infty}^{\infty} K(\tau)e^{j\omega\tau}d\tau \quad (3.2)$$

In spectral analysis, the spectral moments play an important role. Generally, the r-moment of a spectrum is given my the Equation 3.3.

$$m_r = \int_{0}^{\infty} \omega^r S(\omega)d\omega \quad (3.3)$$

By definition, the zero-moment is the variance of the random variable, in this case, the surface elevation $zeta$, represented by σ_ζ^2. (Massel, 2013). Some important spectral characteristics can be derived from the moments, most notably the mean and average wave frequency and period, as shown in the Equation 3.4 and Equation 3.5 respectively.

$$\begin{aligned} \bar{\omega} &= \frac{m_1}{m_0} \\ \bar{T} &= \frac{2\pi}{\bar{\omega}} = 2\pi\frac{m_0}{m_1} \end{aligned} \quad (3.4)$$

$$\begin{aligned} \bar{\omega}_0 &= \sqrt{\frac{m_2}{m_0}} \\ \bar{T}_0 &= \frac{2\pi}{\bar{\omega}} = 2\pi\sqrt{\frac{m_0}{m_2}} \end{aligned} \quad (3.5)$$

Another important measurement in wind-driven seas is the significant wave height $H_{1/3}$. The concept of significant wave height evolved in time, from World War II, where it corresponded to the average of the upper 30% of the wave height observations, to a more modern concept, based on the standard deviation of the ocean surface. The accepted method to calculate the significant wave-height is then presented in Equation 3.6 (Stewart, 2004).

$$H_{1/3} = 4\sqrt{\sigma_\zeta^2} \tag{3.6}$$

Due to the coupling between ocean and atmosphere, the frequency spectrum of the ocean waves depend on various external wave generation conditions, such as wind speed, wind fetch and duration, water depth, presence of swells, and storm stace, as well as internal mechanisms, such as the nonlinear interaction between the wave components, energy dissipation (breaking waves and bottom friction). These various factors, however, does not contribute for an arbitrary wave spectrum, allowing to analyze some fundamental properties for all ocean wave spectra.

The wave spectral density function reaches a peak at $\omega = \omega_p$, decreasing for frequency values lower and greater than ω_p, with a faster decrease for lower than for higher frequencies, resembling a Rayleigh distribution, with a small delay in the increasing. The lowest frequency of a wind wave is approximately 0.03 Hz, with waves with lower frequency being normally generated by Earth movement and gravity field variations. The highest frequency for a wind wave is 13.6 Hz, while waves with higher frequencies are capillary waves (Massel, 2013). The presented limits are theoretical, with real values being smaller than the presented ones.

The wave growth, as pointed out before, is not infinite. If the amount of energy transferred to the waves is greater than a given limit, the waves will break, generating a different kind of breaking wave for each situation. The presence of such dissipation mechanism suggests that there is a saturation of the wave components, that impedes the further growth of wind waves. Since the breaking of the wave happens in different conditions depending on the local physical parameters, the saturation range should also be limited by these local parameters. These local parameters are the angular frequency ω $(2\pi f)$, the friction velocity u_*, the gravitational acceleration g. By dimensional analysis, the spectral density function considering this saturation argument can be described as in Equation 3.7 (Phillips, 1958).

$$S(\omega) = f\left(\frac{\omega u_*}{g}\right) g^2 \omega^{-5} \tag{3.7}$$

3.2 Wave Spectrum Models

In his 1962 paper, Kitaigorodskii, used a similarity analysis to derive the general form of a wave spectrum. This form is presented in Equation 3.8, where X is the fetch length, which is the length of water over the wind has blown.

$$S(\omega) = F(\omega, g, u_*, X) \tag{3.8}$$

However, the friction velocity u_* is not a directly measured variable. To solve this problem, the friction velocity is then substituted by the wind measured in an anemometer placed in a ship. These instruments are placed at a height h above the sea level, being this variable then called U_h. The spectral density is then a function of the variables shown in Equation 3.9 (Pierson Jr and Moskowitz, 1963).

$$S(\omega) = F(\omega, g, U_h, X) \tag{3.9}$$

Based on the proposed similarity model, two of the most important wave models in oceanography studies were derived.

3.2.1 Pierson-Moskowitz Model

In 1963, Pierson Jr and Moskowitz, by request of the US Naval Oceanic Office, wrote the technical report entitled "A proposed spectral form for fully developed wind seas based on the similarity theory of SA Kitaigorodskii". In this technical report was described one of the most popular models for ocean wave spectra, known as the *Pierson-Moskowitz model*.

Based on the propositions of Kitaigorodskii's 1962 paper, and in the limitations imposed by the saturation equilibrium range described by Phillips's 1958, a spectral density function was derived, based on samples collected by Moskowitz. This function is presented in Equation 3.10, where $\alpha = 8.1 \cdot 10^{-3}$ and $\beta = 0.74$ Massel (2013). In the original text, $\frac{g}{U_{19.5}} = \omega_0$. Figure 2 shows the Pierson-Moskowitz model for different wind speeds.

$$S(\omega) = \alpha g^2 \omega^{-5} \exp\left[-\beta\left(\frac{g}{\omega U_{19.5}}\right)^4\right] \tag{3.10}$$

Equation 3.10 yields the following (Massel, 2013):

$$\frac{\omega_p U_{19.5}}{g} = \text{const.} = 0.879 \tag{3.11}$$

Substituting Equation 3.11 in Equation 3.10, the following result is achieved

$$S(\omega) = \alpha g^2 \omega^{-5} \exp\left[-\frac{5}{4}\left(\frac{\omega}{\omega_p}\right)^{-4}\right] \tag{3.12}$$

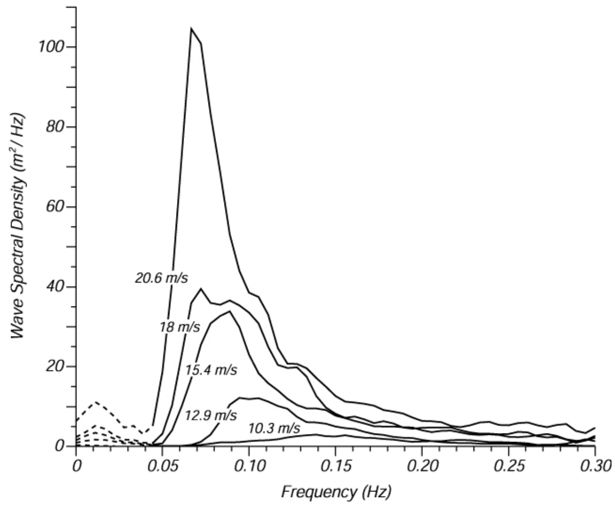

Figure 2: Wave Spectra for Different Wind Speeds according to the Pierson-Moskowitz Model (Stewart, 2004)

In deep water, the peak waves group velocity can be determined by the Equation 3.13 (Stewart, 2004). It can be observed that the peak wave velocity is 14% greater that the wind speed at 19.5 m. This is due to the nonlinear energy transfer between surface waves, explained by Hasselmann et al., which makes the waves travel faster than the blowing wind at 19.5 m (Hasselmann et al., 1973).

$$c_p = \frac{g}{\omega_p} = 1.14 U_{19.5} \tag{3.13}$$

Calculating the zero-moment of the Equation 3.10, the variance of the surface wave elevation can be obtained, as expressed in the Equation 3.14 (Stewart, 2004)

$$\sigma_\zeta^2 = \int_0^\infty S(\omega) d\omega = 2.74 \cdot 10^{-3} \frac{{U_{19.5}}^4}{g^2} \tag{3.14}$$

An interesting relationship rises from the value of the variance of surface elevation, expressed in the Equation 3.14. From the mathematical derivation for the probability distribution for wave period described by Krylov (1966), the Pierson-Moskowitz model can be obtained again, however, with a value for α that depends on the physical and spectral quantities of the system, as expressed in Equation 3.15 (Massel, 2013).

$$\alpha = 5 \left(\frac{\omega_p^2 \sigma_\zeta}{g} \right) \tag{3.15}$$

Returning to the result obtained in Equation 3.14, the significant wave height can be obtained by using Equation 3.6. The significant wave height for the Pierson-Moskowitz wave model is shown in Equation 3.16 (Stewart, 2004).

$$H_{1/3} = 0.21 \frac{{U_{19.5}}^2}{g} \tag{3.16}$$

When presented in the report, the Pierson-Moskowitz (PM) Model was then proposed for a fully-developed sea, where phase speed is equal to the wind speed, which happens when the wind blows steadily for a long time over a large area ($X \to \infty$). However, after careful examination of the PM spectra, Hasselmann et al. found that only part of these spectra corresponds to the fully-developed sea assumption.

3.2.2 JONSWAP Model

Using the Kitaigorodskii approach and the Phillips saturation range, same approach used by Pierson Jr and Moskowitz, Hasselmann et al. derived a new wave spectrum model, based on results retrieved by the Joint North Sea Wave Project (JONSWAP). Instead of considering an infinitely large fetch, a limited fetch was considered, as well as the number of coefficients were increased from the PM Model to JONSWAP.

The JONSWAP model was one of the most thoroughly studied wave spectrum models, scrutinized by various specialists. Some of these studies resulted in modifications on the original JONSWAP spectrum, sometimes to fit to different data structures, other times to correct some discrepancies. Massel (2013) contains a detailed explanation of the changes on JONSWAP, as well as the explanation for other wave spectrum models. This text, however, will present the original JONSWAP model, the corner stone for further modifications. The original model, presented in Hasselmann et al. (1973), is then expressed in the Equation 3.17, in which δ is defined by the Equation 3.18.

$$S(\omega) = \alpha g^2 \omega^{-5} \exp\left[-\frac{5}{4} \left(\frac{\omega}{\omega_p} \right)^{-4} \right] \gamma^\delta \tag{3.17}$$

$$\delta = \exp\left[-\frac{(\omega - \omega_p)^2}{2\sigma_0^2 \omega_p^2} \right] \tag{3.18}$$

Comparing the presented models, JONSWAP has far more information regarding the spectrum for different sea states. In growing seas, the sea spectrum is more peaked than the fully-developed sea spectrum. To allow the representation of a non fully-developed sea, Hasselmann et al. added the *peak enhancement factor* γ^δ, where σ_0 is the width of the peak region. For a peak enhancement factor $\gamma^\delta = 1$, the spectrum

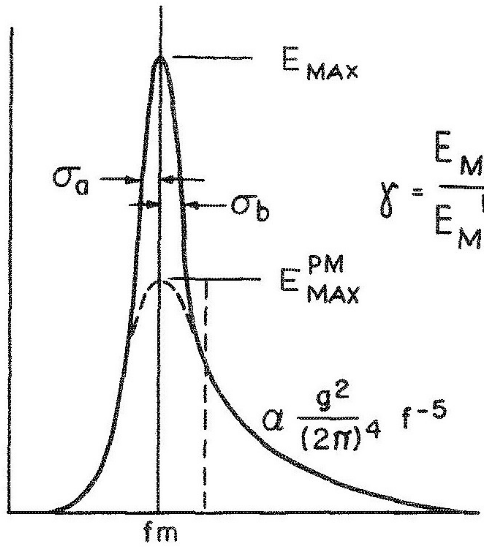

Figure 3: Comparison between the Pierson-Moskowitz and JONSWAP spectra (Hasselmann et al., 1973)

behaves like a Pierson-Moskowitz model. The difference between the two wave spectra can be observed in Figure 3. The parameter σ_0 changes $\omega = \omega_p$, being described by the Equation 3.19.

$$\sigma_0 = \begin{cases} \sigma_a(\text{or } \sigma_0'), \text{ if } \omega \leq \omega_p \\ \sigma_b(\text{or } \sigma_0''), \text{ if } \omega > \omega_p \end{cases} \tag{3.19}$$

The local parameter dependence is shown in the definition of the model parameters, expressed in the Equations 3.20 to 3.23 (Massel, 2013). The spectra that carries these model parameters is also called *mean JONSWAP spectrum.*

$$\alpha = 0.076 \left(\frac{gX}{U_{10}^2} \right)^{-0.22} \tag{3.20}$$

$$\omega_p = 7\pi \left(\frac{g}{U_{10}} \right) \left(\frac{gX}{U_{10}^2} \right)^{-1/3} \tag{3.21}$$

$$\gamma = 3.3 \tag{3.22}$$

$$\sigma_0 = \begin{cases} 0.07, \text{ if } \omega \leq \omega_p \\ 0.09, \text{ if } \omega > \omega_p \end{cases} \tag{3.23}$$

Observing the parameter determination, it can be seen that JONSWAP spectra are, as described initially, dependent of the fetch X. This dependence can strongly affect the behaviour of the spectrum. Figure 4 shows the fetch dependence of the JONSWAP spectrum, under the same wind conditions.

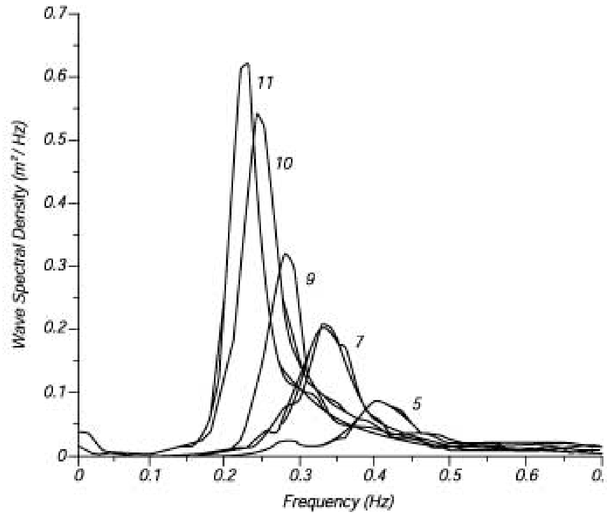

Figure 4: Fetch dependence of the JONSWAP spectrum (Stewart, 2004)

The equation Equation 3.24 shows the sea surface variance for the mean JONSWAP spectrum Stewart (2004).

$$\sigma_\zeta^2 = 1.67 \cdot 10^{-7} \frac{U_{10}^2}{g} X \tag{3.24}$$

Analyzing the Equation 3.24, it can be seen that the sea surface variance σ_ζ^2 is proportional to the fetch X. The zero-moment is also the energy measurement for the system, showing that a greater fetch directly affects the energy of the system. This can also be observed from the spectra with different fetch shown in Figure 4.

With a similar process that was used in the PM Model, Equation 3.25 shows the significant wave height for the mean JONSWAP spectrum. Since $H_{1/3}$ is dependent of σ_ζ, the significant wave height will also depend on the fetch. It means that, for a longer fetch, higher waves will be present in the field, which matches with the physical observations.

$$H_{1/3} = 1.63 \cdot 10^{-3} \sqrt{\frac{U_{10}^2 X}{g}} \tag{3.25}$$

Another interesting point to be observed is that the wind measurement height changes between JONSWAP and PM Models. To compare both models under the same conditions, an approximate relationship between wind speed values at the different heights is considered, as expressed in Equation 3.26 (Stewart, 2004).

$$U_{19.5} \approx 1.026 \cdot U_{10} \tag{3.26}$$

In principle, since the JONSWAP model considers a limited fetch and Pierson-Moskowitz consider a sufficiently large fetch, JONSWAP spectrum should asymptotically tend to Pierson-Moskowitz spectrum for a large fetch. However, the peak-enhancement factor shows no tendency to converge to 1 when $X \to \infty$ (Massel, 2013). After reanalyzing the data from Pierson and Moskowitz, Hasselmann et al. observed that more than half of the spectra contained multiple peaks, in other words, two or more peaks were fitted by a single function, which may result in ill-fitted functions. Excluding those multiple peak results, it was observed that the peak-enhancement factor should be $\gamma = 1.4$ for a fully developed sea, instead of $\gamma = 1$.

References

K. Hasselmann, T. Barnett, E. Bouws, H. Carlson, D. Cartwright, K. Enke, J. Ewing, H. Gienapp, D. Hasselmann, P. Kruseman, et al. Measurements of wind-wave growth and swell decay during the joint north sea wave project (jonswap). Technical report, Deutches Hydrographisches Institut, 1973.

K. Hasselmann, W. Sell, D. Ross, and P. Müller. A parametric wave prediction model. *Journal of Physical Oceanography*, 6(2):200–228, 1976.

S. A. Kitaigorodskii. Applications of the theory of similarity to the analysis of wind-generated wave motion as a stochastic process. *Izv. Geophys. Ser. Acad. Sci., USSR*, 1:105–117, 1962.

Y. M. Krylov. Spectral methods of studying and predicting of wind waves. *Gidrometeoizidat*, 1966. (in Russian).

S. R. Massel. *Ocean surface waves: their physics and prediction*, volume 36. World scientific, 2013.

J. W. Miles. On the generation of surface waves by shear flows. *Journal of Fluid Mechanics*, 3(02):185–204, 1957.

R. Nave. *Wave Motion.* Hyperphysics. Georgia State University, `http://hyperphysics.phy-astr.gsu.edu/hbase/waves/watwav2.html`, 2015.

O. M. Phillips. On the generation of waves by turbulent wind. *Journal of Fluid Mechanics*, 2(05): 417–445, 1957.

O. M. Phillips. The equilibrium range in the spectrum of wind-generated waves. *Journal of Fluid Mechanics*, 4(04):426–434, 1958.

W. J. Pierson Jr and L. Moskowitz. A proposed spectral form for fully developed wind seas based on the similarity theory of sa kitaigorodskii. Technical report, DTIC Document, 1963.

E. Shearman. Radio science and oceanography. *Radio science*, 18(3):299–320, 1983.

R. H. Stewart. *Introduction to physical oceanography.* Texas A & M University, 2004.