Optimization Methods **Fall 2020**

Homework 1

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Notice

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Problem 1: Norms

A function $f : \mathbb{R}^n \to \mathbb{R}$ with $\mathrm{dom} f = \mathbb{R}^n$ is called a *norm* if

- f is nonnegative: $f(x) \geq 0$ for all $x \in \mathbb{R}^n$
- f is definite: $f(x) = 0$ only if $x = 0$
- f is homogeneous: $f(tx) = |t|f(x)$, for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$
- f satisfies the triangle inequality: $f(x + y) \leq f(x) + f(y)$, for all $x, y \in \mathbb{R}^n$

We use the notation $f(x) = \|x\|$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. The associated dual norm, denoted $\|\cdot\|_*$, is defined as

$$\|z\|_* = \sup\{z^{\mathrm{T}}x | \|x\| \leq 1\}$$

a) Prove that $\|\cdot\|_*$ is a valid norm.

b) Prove that the dual of Euclidean norm (ℓ_2-*norm*) is the Euclidean norm, *i.e.*, prove that

$$\|z\|_{2*} = \sup\{z^{\mathrm{T}}x | \|x\|_2 \leq 1\} = \|z\|_2$$

(*Hint*: Use Cauchy-Schwarz inequality.)

Solution. **a)**

1)

$\forall$ z if $z = 0$,then $z^{\mathrm{T}}x = \sum_{i=0}^{n} z_i x_i = 0$

if $z \neq 0$ for all $z_i > 0$ or $z_j < 0$, we can find a x_0 with $x_i > 0$ and $x_j > 0$. so

$$\sup\{z^{\mathrm{T}}x | \|x\| \leq 1\} \geq z^{\mathrm{T}}x_0 = \sum_{i=0}^{n} z_i x_i > 0.$$

thus,$\|z\|_* \geq 0$. it satisfies the first property.

2)

$z = 0 \Longrightarrow z^{\mathrm{T}}x = 0 \Longrightarrow \|z\|_* = 0$, which is obvious.

Let's prove the reverse is true. Suppose there is a z_0 and let $\|z\|_* = 0$ and $z_0 \neq 0$.

Therefore, for each $z_i > 0$ or $z_j < 0$, we can find a x_0 with $x_i > 0$ and $x_j < 0$ then $0 = \|z\|_* \geq z_0^{\mathrm{T}}x_0 > 0$, which is self-contradictory.

Thus $\|z\|_* = 0 \Longrightarrow z = 0$.

3)

if $t = 0$, it's obvious that $f(0) = 0f(x) = 0$, which means $\|0\|_* = \sup 0 | \|x\| \leq 1 = 0 = 0\|x\|_* = 0$.

if $t \neq 0$, since $\|z\|_*$ is the supremum of the set, we have $z^T x \leq \|z\|_*$. And we konw that $\|z\|_* \geq 0$,thus $tz^T x \leq |t|\|z\|_*, \forall x, \|x\| \leq 1$. Therefore , $|t|\|z\|_*$ is a upper bound of the set, which means $|t|\|z\|_* \geq \|tz\|_*$.

for any $|t|b$, if $|t|b$ is the upper bound of the $\{tz^T x | \|x\| \leq 1\}$ and $t \neq 0$,then b is the upper bound of $\{z^T x | \|x\| \leq 1\}$. then we have $\|x\|_* \leq b$. And thus, $|t|\|x\|_* \leq |t|b$. Since $|t|\|x\|_*$ is a upper bound and is no more than any upper bound of the set $\{tz^T x | \|x\| \leq 1\}$,it is a supermum of the set. Because uniqueness of the supremum $|t|\|x\|_* = \|tx\|_*$. It satisfies the third property.

4) It's obvious that for all x and z_1, z_2, we have $z_1^{\mathrm{T}} x \leq \|z_1\|_*$ and $z_2^{\mathrm{T}} x \leq \|z_2\|_*$.
Therefore,$(z_1^{\mathrm{T}} + z_2^{\mathrm{T}})x \leq \|z_1\|_* + \|z_2\|_*$.
Therefore, $\|z_1 + z_2\|_* \leq \|z_1\|_* + \|z_2\|_*$.
It satisfies the forth property.

□

Solution. b

For any x and z ,we have:

$$z^{\mathrm{T}} x = \sum_{i=1}^{n} z_i x_i \leq \sqrt{\sum_{i=1}^{n} z_i^2} \sqrt{\sum_{i=1}^{n} x_i^2} \leq 1 \times \|z\|_2 = \|z\|_2$$

. And the eual sign if and only if $\sum_{i=1}^{n} x_i = 0$ and $x_i = kz_i$, $i = 1, 2, 3 \ldots, n$.let $x = \frac{z}{\|z\|_2}$,we can make the euquation hold.Therefore, $\|z\|_2$ is the supremum of the set $\{z^T x | \|x\|_2 \leq 1\}$. Because the uniqueness of supremum, $\|z\|_{2*} = \|z\|_2$.

□

Problem 2: Convex sets

Convex C_c sets are the sets satisfying the constraints below:

$$\theta x_1 + (1 - \theta) x_2 \in C_c$$
$$\text{for all, } x_1, x_2 \in C_c, 0 \leq \theta \leq 1$$

a). Show that a set is convex if and only if its intersection with any line is convex.

b). Determine if each set below is convex.

1) $\{(x, y) \in \mathbf{R}^2_{++} | x/y \leq 1\}$

2) $\{(x, y) \in \mathbf{R}^2_{++} | x/y \geq 1\}$

3) $\{(x, y) \in \mathbf{R}^2_{++} | xy \leq 1\}$

4) $\{(x, y) \in \mathbf{R}^2_{++} | xy \geq 1\}$

5) $\{(x, y) \in \mathbf{R}^2_{++} | y = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}\}$

Solution. a)

$\Longrightarrow$:

Given an arbitrary convex set C, C is convex if and only if for any x_1 and $x_2 \in C$ and with any θ, $0 \leq \theta 1$, we have $\theta x_1 + (1 - 0) x_2 \in C$.
Since C is convex and an arbitrary line is convex and intersection preserve convexity.
Therefore we have the intersection of C and any line is convex.

$\Longleftarrow$: suppose the intersection of C and any line is convex.Take any distinct points x_1 and $x_2 \in C$. The intersection of C and the line through x_1, x_2 is convex. Therefore, for any convex combinations of x_1 and x_2 belong to the intersection, hence also belong to the set C.Therefore, for any two points in C, their convex combinations belong to C.Thus, C is convex.

□

Solution. b)

1) yes

2) yes

3) no

4) yes

5) no

□

Problem 3: Examples Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbb{R}^n | x^T Ax + b^T x + c \leq 0\}$$

with $A \in \mathbb{S}^n, b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

1) Show that C is convex if $A \succeq 0$.

2) Is the following statement true? The intersection of C and the hyperplane defined by $g^T x + h = 0$ is convex if $A + \lambda gg^T \succeq 0$ for some $\lambda \in \mathbb{R}$.

Solution. A set is convex iff for any arbitray line, their intersection is convex. Suppose line:$x+ty$,x is a given point in C ,so their intersection is $x+ty|(x+ty)^T A(x+ty)+b^T(x+ty)+c \leq 0$.

$(x+ty)^T A(x+ty)+b^T(x+ty)+c = (y^T Ay)t^2 + (x^T Ay + y^T Ax + b^T y)t + x^T Ax + b^T x + C = \alpha t^2 + \beta t + \gamma$. we can easliy find that $\alpha \geq 0$ and $\gamma \leq 0$ since $A \succeq$ and x is in set C.

*1.*when $\alpha = 0$

if $\beta = 0$, then $y = \gamma \leq 0$ is always true.

if $\beta \neq 0$, then $\beta t + \gamma$ is a linear function. for any t_1, t_2 and θ with $0 \leq \theta \leq 1$, we have $\beta(\theta t_1 + (1-\theta)t_2) + \gamma \leq -\theta\gamma - (1-\theta)\gamma + \gamma = 0$.

*2.*when $\alpha > 0$

the function is a quadractic function open upward with images like Figure 1.for t_1, t_2 and θ with $0 \leq \theta \leq 1$, we have $\min(t1, t2) \leq \theta t_1 + (1-\theta)t_2 \leq \max(t1, t2)$. therefore, the points with $\theta t_1 + (1-\theta)t2$,satisfies, $\alpha(\theta t_1 + (1-\theta)t_2)^2 + \beta(\theta t_1 + (1-\theta)t_2) + \gamma \leq \max(\alpha t_1^2 + \beta t_1 + \gamma, \alpha t_2^2 + \beta t_2 + \gamma) \leq 0$.

□

Solution. Supoose $x \in C \cap \{x | g^T x + h = 0\}$ and an arbitrary line x+ty. The intersection of the line with the set is $\{x + ty | g^T(x+ty) + h = 0, \alpha t^2 + \beta t + \gamma \leq 0\}$.

since $g^T(x+ty) + h = 0$ and $g^T x + h = 0$, $g^T(x+ty) + h = g^T x + h + g^T y = tg^T y = 0$.

if $g^T y \neq 0$, then t=0. Thus, the set is $\emptyset$ or x. No matter which it is ,the set is convex.

if $g^T y = 0$, then the set becomes $\{x + ty | \alpha t^2 + \beta t + \gamma \leq 0\}$.

$\because \alpha = y^T Ay$ and $g^T y = 0$

$\therefore \alpha = y^T Ay + \lambda y^T gg^T y = y^T(A + \lambda gg^T)y$.

$\because A + \lambda gg^T \succeq 0$

$\therefore A + \lambda$ is a positive semi-definite matrix.

$\therefore, \alpha = y^T(A + \lambda gg^T)y \geq 0$.

so we get the same condition as the question one.Therefore, the set is convex.

Therefore, the intersection of C and the hyperplane defined by $g^T x + h = 0$ is convex if $A + \lambda gg^T \succeq 0$ for some $\lambda \in \mathbb{R}$.

□

Problem 4: Operations That Preserve Convexity

Suppose $\phi : \mathbb{R}^n \to \mathbb{R}^m$ and $\psi : \mathbb{R}^m \to \mathbb{R}^p$ are the linear-fractional functions

$$\phi(x) = \frac{Ax + b}{c^\top x + d}, \psi(y) = \frac{Ey + f}{g^\top y + h}$$

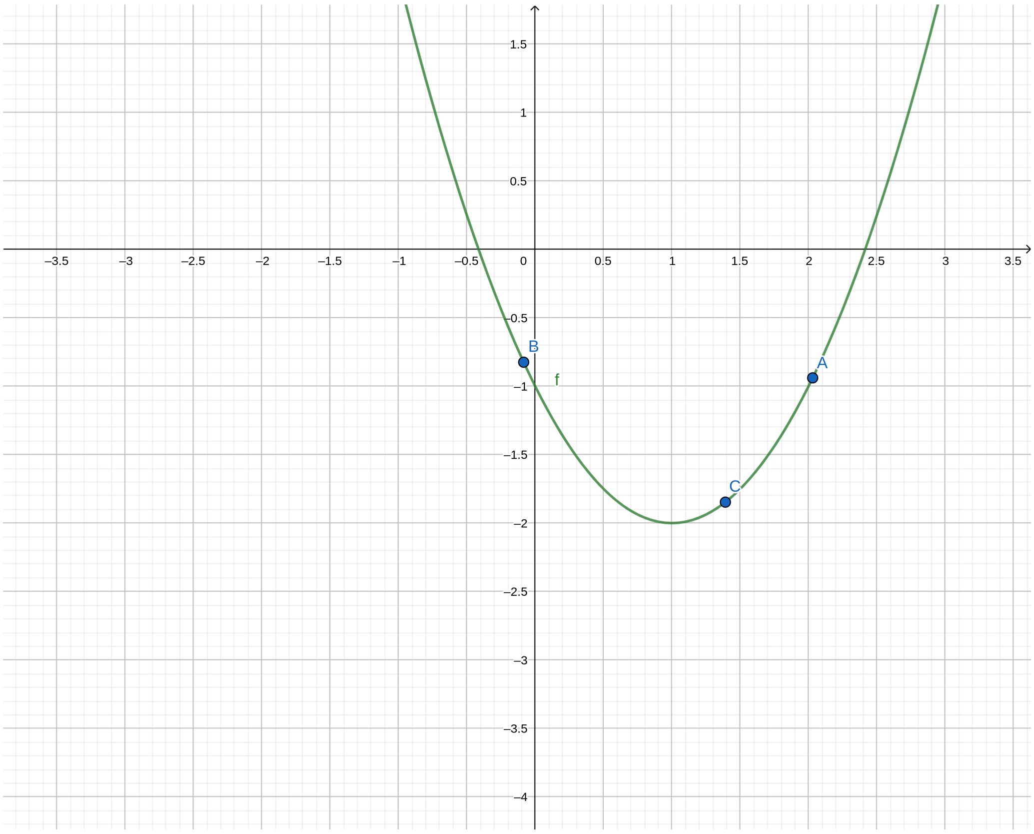

Figure 1:

with domains **dom** $\phi = \left\{x \mid c^\top x + d > 0\right\}$, **dom** $\psi = \left\{y \mid g^\top y + h > 0\right\}$. We associate with ϕ and ψ the matrices respectively.

$$\begin{bmatrix} A & b \\ c^\top & d \end{bmatrix}, \begin{bmatrix} E & f \\ g^\top & h \end{bmatrix}$$

Now, consider the composition Γ of ϕ and ψ, i.e., $\Gamma(x) = \psi(\phi(x))$, with domain

$$\mathbf{dom}\Gamma = \{x \in \mathbf{dom}\phi \mid \phi(x) \in \mathbf{dom}\psi\}$$

Show that Γ is linear-fractional, and that the matrix associate with it is the product

$$\begin{bmatrix} E & f \\ g^\top & h \end{bmatrix} \begin{bmatrix} A & b \\ c^\top & d \end{bmatrix}$$

Solution.

$$\Gamma(x) = \frac{E\phi(x) + F}{g\phi(x) + h} = \frac{EAx + Eb + f(c^T x + d)}{(g^T a + hc^T)x + g^T b + hd} = \frac{(EA + fc^T)x + Eb + fd}{(g^T A + hc^T)x + g^T b + hd}.$$

Therefore,Γ is a linear-fractional.

The martix:

$$\begin{bmatrix} E & f \\ g^\top & h \end{bmatrix} \begin{bmatrix} A & b \\ c^\top & d \end{bmatrix} = \begin{bmatrix} EA + fc^T & Eb + fd \\ g^T A + hc^T & g^T b + hd \end{bmatrix}$$

Thus, the matrix of associate with Γ is the product $\begin{bmatrix} E & f \\ g^\top & h \end{bmatrix} \begin{bmatrix} A & b \\ c^\top & d \end{bmatrix}$

$\square$

Problem 5: Generalized Inequalities

Let K^* be the dual cone of a convex cone K. Prove the following

1) K^* is indeed a convex cone.

2) $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$

Solution. 1)
$K^* = \{y|x^Ty \geq 0 \forall x \in K\}$. For any $y_1, \quad y_2 \in K^*$ and $\theta_1, \quad \theta_2 \geq 0$ and for all $x \in K$,we have $x^T(\theta_1 y_1 + \theta_2 y_2) = \theta_1 x^T y_1 + \theta_2 x^T y_2 \geq 0 + 0 = 0$. Therefore, $\theta_1 y_1 + \theta_2 y_2 \in K^*$. Thus, k^* is a convex cone.

□

Solution. 2)
$\forall y \in K_2^*, \forall x \in K_2$,we have $x^Ty \geq 0$. And $K_1 \subseteq K_2$. Therefore, for all x in K_1, it's also in K_2. Thus, for all x in K_1 and for all y in K_2^*,we have $x^Ty \geq 0$. Since K_1^* includes all y that makes for all x in k_1, $x^Ty \geq 0$, for any y in K_2^*, it must be in K_1^*. Therefore,$K_2^* \subseteq K_1^*$.

□