With-replacement vs without-replacement notes

Suppose $\mu := \frac{1}{N}\sum_{i=1}^N X_i \gg m$. Write out the WoR capital process

$$K_t^{\text{WoR}}(m) := \prod_{i=1}^t (1 + \lambda_i(X_i - m_t^{\text{WoR}}))$$
$$\text{where } m_t := \frac{Nm - \sum_{i=1}^{t-1} X_i}{N - t + 1}.$$

Suppose for the sake of example that $t - 1 = N/2$. Then, consider the marginal mean $\mathbb{E}(m_t^{\text{WoR}})$ of m_t^{WoR},

$$\begin{aligned}\mathbb{E}(m_t^{\text{WoR}}) &= \frac{Nm - \sum_{i=1}^{N/2} \mathbb{E}(X_i)}{N/2} \\ &= \frac{Nm - \mu N/2}{N/2} \\ &= 2m - \mu.\end{aligned}$$

As such, we have that

$$\mathbb{E}(X_t - m_t^{\text{WoR}}) = \mu - 2m + \mu = 2(\mu - m).$$

Contrast this with what we would have in the with-replacement regime,

$$\mathbb{E}(X_t - m) = \mu - m,$$

which is half the size of $\mathbb{E}(X_t - m_t^{\text{WoR}})$.

More generally, suppose that $t - 1 = N\delta$ for some $\delta \in (0, 1)$ such that $N\delta$ is an integer (avoiding uninteresting technicalities). Then,

$$\mathbb{E}(m_t^{\text{WoR}}) = \frac{Nm - N\delta\mu}{(1 - \delta)N} = \frac{m - \delta\mu}{1 - \delta}.$$

It is easy to then show that

$$\mathbb{E}(X_t - m_t^{\text{WoR}}) = \frac{\mu - m}{1 - \delta},$$

or alternatively written,

$$\mathbb{E}(X_t - m_t^{\text{WoR}}) = \frac{\mathbb{E}(X_t - m)}{(1 - \delta)}.$$

Therefore, for small t (i.e. close to 1), we have that $\mathbb{E}(X_t - m_t^{\text{WoR}}) \approx \mathbb{E}(X_t - m)$, while if t is large (i.e. approaching N), we have

$$\mathbb{E}(X_t - m_t^{\text{WoR}}) \gg \mathbb{E}(X_t - m)$$

Rewriting $\delta = \frac{t-1}{N}$, we have

$$\mathbb{E}(X_t - m_t^{\text{WoR}}) = \frac{N - t + 1}{N}$$

Fix $\lambda \in (0, 1)$, and consider the expected WoR wealth,

$$\mathbb{E}\left(\prod_{i=1}^t (1 + \lambda(X_i - m_t^{\text{WoR}}))\right) = \prod_{i=1}^t \left(1 + \lambda\frac{N(\mu - m)}{N - i + 1}\right)$$

versus the expected WR wealth

$$\prod_{i=1}^t (1 + \lambda(\mu - m))$$