$\mathcal{P}$-uniform anytime-valid inference

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where $\alpha \in (0, 1)$ and

$$\dot{C}_n := \frac{1}{n} \sum_{i=1}^{n} X_i \pm \frac{\widehat{\sigma}_n \cdot \Phi^{-1}(1 - \alpha/2)}{\sqrt{n}}.$$

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Only holds for a *fixed* and *prespecified* sample size n.

(Asymptotic) confidence sequences enable statistical inference at stopping times.

For example, continuously monitoring the CIs of a sequential experiment.

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Confidence interval	**Confidence sequence**
$\lim_{n \to \infty} \mathbb{P}_P \left(\mu_P \notin \dot{C}_n \right) = \alpha$	$\lim_{m \to \infty} \mathbb{P}_P \left(\exists k \geqslant m : \mu_P \notin \bar{C}_k \right) = \alpha$

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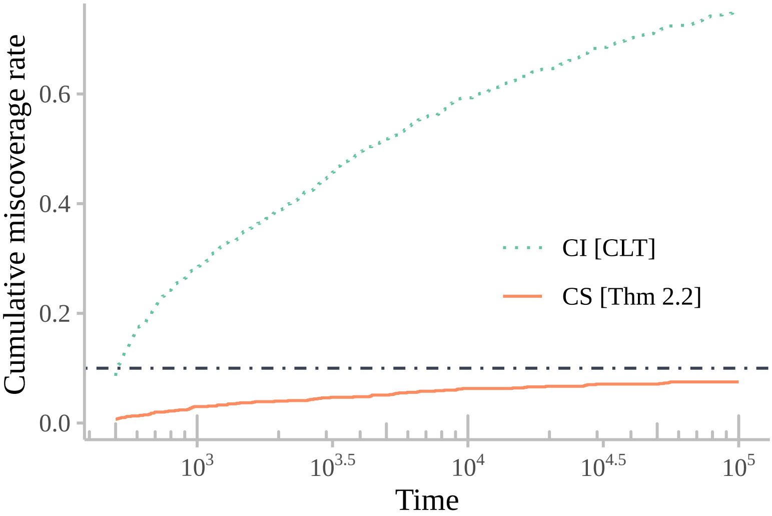

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$$\sup_{P \in \mathcal{P}} \mathbb{E}_P |X - \mu_P|^{2+\delta} < \infty,$$

then the guarantees of $\dot{C}_n$ hold *uniformly* in $\mathcal{P}$:

$$\lim_{n \to \infty} \sup_{P \in \mathcal{P}} \mathbb{P}_P \left(\mu_P \notin \dot{C}_n \right) = \alpha.$$

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*Can similar uniform statements be made for confidence **sequences**?*

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Can similar uniform statements be made for confidence ***sequences****?*

	Confidence interval	**Confidence sequence**
P-pointwise	$\lim_{n\to\infty} \mathbb{P}_P\left(\mu_P \notin \dot{C}_n\right) = \alpha$	$\lim_{m\to\infty} \mathbb{P}_P\left(\exists k \geqslant m : \mu_P \notin \bar{C}_k\right) = \alpha$
$\mathcal{P}$-uniform	$\lim_{n\to\infty} \sup_{P\in\mathcal{P}} \mathbb{P}_P\left(\mu_P \notin \dot{C}_n\right) = \alpha$	$\boxed{\lim_{m\to\infty} \sup_{P\in\mathcal{P}} \mathbb{P}_P\left(\exists k \geqslant m : \mu_P \notin \bar{C}_k\right) = \alpha}$

The answer is **"Yes"**, but this required first filling a gap in the probability literature (i.e. strong invariance principles).

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Informal: Need to show that for iid $(X_n)_{n=1}^{\infty}$, there exist standard Gaussians $(Z_n)_{n=1}^{\infty}$ so that

$$\sum_{i=1}^{n} \frac{X_i - \mu_P}{\sigma_P} - \sum_{i=1}^{n} Z_i = o\left(\sqrt{n \log \log n}\right),$$

P-almost surely uniformly in $P \in \mathcal{P}$.

Theorem: $\mathcal{P}$-uniform asymptotic confidence sequences

Suppose $(X_n)_{n=1}^{\infty}$ have $\mathcal{P}$-uniformly bounded $(2+\delta)^{\text{th}}$ moments. Letting $\bar{C}_k^{(m)}$ be given by

$$\bar{C}_k^{(m)} := \widehat{\sigma}_k \sqrt{\frac{\Psi^{-1}(1-\alpha) + \log(k/m)}{k}}.$$

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$$\bar{C}_k^{(m)} := \widehat{\sigma}_k \sqrt{\frac{\Psi^{-1}(1-\alpha) + \log(k/m)}{k}}.$$

Then the asymptotic guarantees of $(\bar{C}_k^{(m)})_{k=m}^{\infty}$ hold *uniformly* in $\mathcal{P}$:

$$\lim_{m \to \infty} \sup_{P \in \mathcal{P}} \mathbb{P}_P \left(\exists k \geqslant m : \mu_P \notin \bar{C}_k^{(m)} \right) = \alpha.$$

Thank you!

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