

Concentration inequalities for strong laws of large numbers

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Setup

$$X_1, X_2, X_3, \dots \stackrel{\text{i.i.d.}}{\sim} \mathbf{P}.$$

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We will study the \mathbf{P} -almost sure behavior of

$$\bar{X}_n := \frac{1}{n} \sum_{i=1}^n X_i.$$

Kolmogorov's strong law of large numbers



$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[|X_1|] < \infty \\ \Downarrow \\ \mathbb{P} \left[\lim_{n \rightarrow \infty} \bar{X}_n = \mathbb{E}_{\mathbb{P}}[X_1] \right] = 1. \end{aligned}$$

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This is a qualitative statement.

For “quantitative” analogues, the literature tends to study the probability

$$P_{\textcolor{red}{m}} \equiv P_{\textcolor{red}{m}}^{(\text{P})} := \mathbf{P} \left[\sup_{\textcolor{blue}{k} \geq \textcolor{red}{m}} |\bar{X}_{\textcolor{blue}{k}} - \mathbb{E}_{\text{P}}[X_1]| \geq \varepsilon \right]$$

for any fixed $\varepsilon > 0$.

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$$P_m \equiv P_m^{(P)} := \mathbb{P} \left[\sup_{k \geq m} |\bar{X}_k - \mathbb{E}_P[X_1]| \geq \varepsilon \right]$$

for any fixed $\varepsilon > 0$.

Justification: Kolmogorov’s SLLN is equivalent to the statement

$$\mathbb{E}_P[|X_1|] < \infty \implies \lim_{m \rightarrow \infty} P_m^{(P)} = 0.$$

Chung's strong law of large numbers [1951]

$$\lim_{m \rightarrow \infty} \sup_{P \in \mathcal{P}} \mathbb{E}_P [|X_1| \mathbb{1}\{|X| > m\}] = 0$$

\Downarrow

$$\lim_{m \rightarrow \infty} \sup_{P \in \mathcal{P}} P_m^{(P)} = 0.$$



Baum & Katz' strong law of large numbers [1961]

$$\mathbb{E}_{\mathbb{P}} [|X_1| \log(|X_1| + 1)] < \infty$$



$$\sum_{\textcolor{red}{m}=1}^{\infty} \frac{\textcolor{violet}{P}_{\textcolor{red}{m}}}{\textcolor{red}{m}} < \infty$$

Baum & Katz' strong law of large numbers [1961]

$$\mathbb{E}_{\mathbf{P}} [|X_1| \log(|X_1| + 1)] < \infty$$



$$\sum_{m=1}^{\infty} \frac{P_m}{m} < \infty$$

$$\text{e.g. } P_m = o\left(\frac{1}{\log m}\right).$$

There are a few others...

- (i) Kolmogorov [1930]
- (ii) Marcinkiewicz & Zygmund [1937]
- (iii) Chung [1951]
- (iv) Baum & Katz [1961]
- (v) Ruf, Larsson, Koolen, and Ramdas [2023]
- (vi) **W-S**, Larsson, and Ramdas [2024]

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None provide exact upper bounds on P_m .

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Theorem 1: Concentration inequalities for strong laws

For any $\varepsilon > 0$ and any $q \in [1, 2)$,

$$\mathbb{P} \left[\sup_{k \geq m} \left| \frac{1}{k^{1/q}} \sum_{i=1}^k X_i \right| \geq \varepsilon \right] \lesssim \exp \{ -\sqrt{m} \} + \frac{1}{\varepsilon^2} \mathbb{E}_{\mathbb{P}} \left[|X_1|^q \mathbf{1} \{ |X_1|^q \geq \varepsilon^q \sqrt{m} \} \right].$$

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Theorem 1 \implies (i)–(vi) from the previous slide.

Note: All hidden constants are universal and known.

Other stuff in the paper:

- (a) Moment-dependent rates of convergence (a la Marcinkiewicz-Zygmund).
- (b) Iterated logarithm inequalities (“finite laws of the iterated logarithm”).
- (c) Pathwise (“Game-Theoretic”) strong laws.
- (d) Non-asymptotic / uniform generalizations of the Baum-Katz strong laws.

Thank you

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