

Concentration inequalities for strong laws of large numbers

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Setup

$$X_1, X_2, X_3, \dots \stackrel{\text{i.i.d.}}{\sim} \mathsf{P}.$$

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We will study the P -almost sure behavior of

$$\bar{X}_n := \frac{1}{n} \sum_{i=1}^n X_i.$$

Kolmogorov's strong law of large numbers



$$\mathbb{E}_P[|X_1|] < \infty$$

↓

$$P\left[\lim_{n \rightarrow \infty} \bar{X}_n = \mathbb{E}_P[X_1]\right] = 1.$$

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$$\Downarrow$$

$$P\left[\lim_{n \rightarrow \infty} \bar{X}_n = \mathbb{E}_P[X_1]\right] = 1.$$

This is a qualitative statement.

For “quantitative” analogues, the literature tends to study the probability

$$P_m \equiv P_m^{(\mathsf{P})} := \mathsf{P} \left[\sup_{k \geq m} |\bar{X}_{\textcolor{blue}{k}} - \mathbb{E}_{\mathsf{P}}[X_1]| \geq \varepsilon \right]$$

for any fixed $\varepsilon > 0$.

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Justification: Kolmogorov’s SLLN is equivalent to the statement

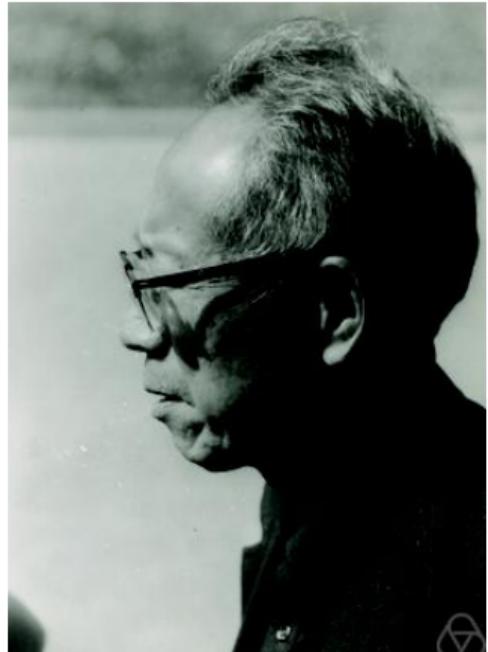
$$\mathbb{E}_{\mathbb{P}}[|X_1|] < \infty \implies \lim_{m \rightarrow \infty} P_m^{(\mathbb{P})} = 0.$$

Chung's strong law of large numbers [1951]

$$\lim_{m \rightarrow \infty} \sup_{P \in \mathcal{P}} \mathbb{E}_P [|X_1| \mathbf{1}\{|X| > m\}] = 0$$

↓

$$\lim_{m \rightarrow \infty} \sup_{P \in \mathcal{P}} P_m^{(P)} = 0.$$



Baum & Katz' strong law of large numbers [1961]

$$\mathbb{E}_P [|X_1| \log(|X_1| + 1)] < \infty$$

↓

$$\sum_{m=1}^{\infty} \frac{P_m}{m} < \infty$$

Baum & Katz' strong law of large numbers [1961]

$$\mathbb{E}_{\mathsf{P}} [|X_1| \log(|X_1| + 1)] < \infty$$

↓

$$\sum_{m=1}^{\infty} \frac{\textcolor{violet}{P}_m}{m} < \infty$$

$$\text{e.g. } \textcolor{violet}{P}_m = o\left(\frac{1}{\log m}\right).$$

There are a few others...

- (i) Kolmogorov [1930]
- (ii) Marcinkiewicz & Zygmund [1937]
- (iii) Chung [1951]
- (iv) Baum & Katz [1961]
- (v) Ruf, Larsson, Koolen, and Ramdas [2023]
- (vi) **W-S**, Larsson, and Ramdas [2024]

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However, none* of these results imply each other.

*Actually, (iii) \implies (i) and (vi) \implies (ii).

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None provide exact upper bounds on $P_{\textcolor{red}{m}}$.

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Theorem 1: Concentration inequalities for strong laws

For any $\varepsilon > 0$ and any $q \in [1, 2)$,

$$\mathsf{P} \left[\sup_{k \geq m} \left| \frac{1}{k^{1/q}} \sum_{i=1}^k X_i \right| \geq \varepsilon \right] \lesssim \exp \{-\sqrt{m}\} + \frac{1}{\varepsilon^2} \mathbb{E}_{\mathsf{P}} \left[|X_1|^q \mathbb{1}\{|X_1|^q \geq \varepsilon^q \sqrt{m}\} \right].$$

Theorem 1: Concentration inequalities for strong laws

For any $\varepsilon > 0$ and any $q \in [1, 2)$,

$$\begin{aligned} \mathsf{P} \left[\sup_{k \geq m} \left| \frac{1}{k^{1/q}} \sum_{i=1}^k X_i \right| \geq \varepsilon \right] &\lesssim \exp \{-\sqrt{m}\} + \\ &\quad \frac{1}{\varepsilon^2} \mathbb{E}_{\mathsf{P}} \left[|X_1|^q \mathbf{1}\{|X_1|^q \geq \varepsilon^q \sqrt{m}\} \right]. \end{aligned}$$

Theorem 1 \implies (i)–(vi) from the previous slide.

Note: All hidden constants are universal and known.

Other stuff in the paper:

- (a) Moment-dependent rates of convergence (a la Marcinkiewicz-Zygmund).
- (b) Iterated logarithm inequalities (“finite laws of the iterated logarithm”).
- (c) Pathwise (“Game-Theoretic”) strong laws.
- (d) Non-asymptotic / uniform generalizations of the Baum-Katz strong laws.

Thank you

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