Estimating means of bounded random variables by betting

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Carnegie Mellon University

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JRSS-B 2023 (w/ discussion)

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The downside? Not very sharp,
especially for small variance $\sigma^2 := \mathrm{Var}(X_i)$.

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and **make C_n as sharp as possible.**

Our bound:

$$C_n := \left\{ m : \prod_{i=1}^{n} (1+\lambda_i(X_i - m)) < \frac{1}{\alpha} \right\}. \qquad \text{(design } \lambda_i \text{ later)}$$

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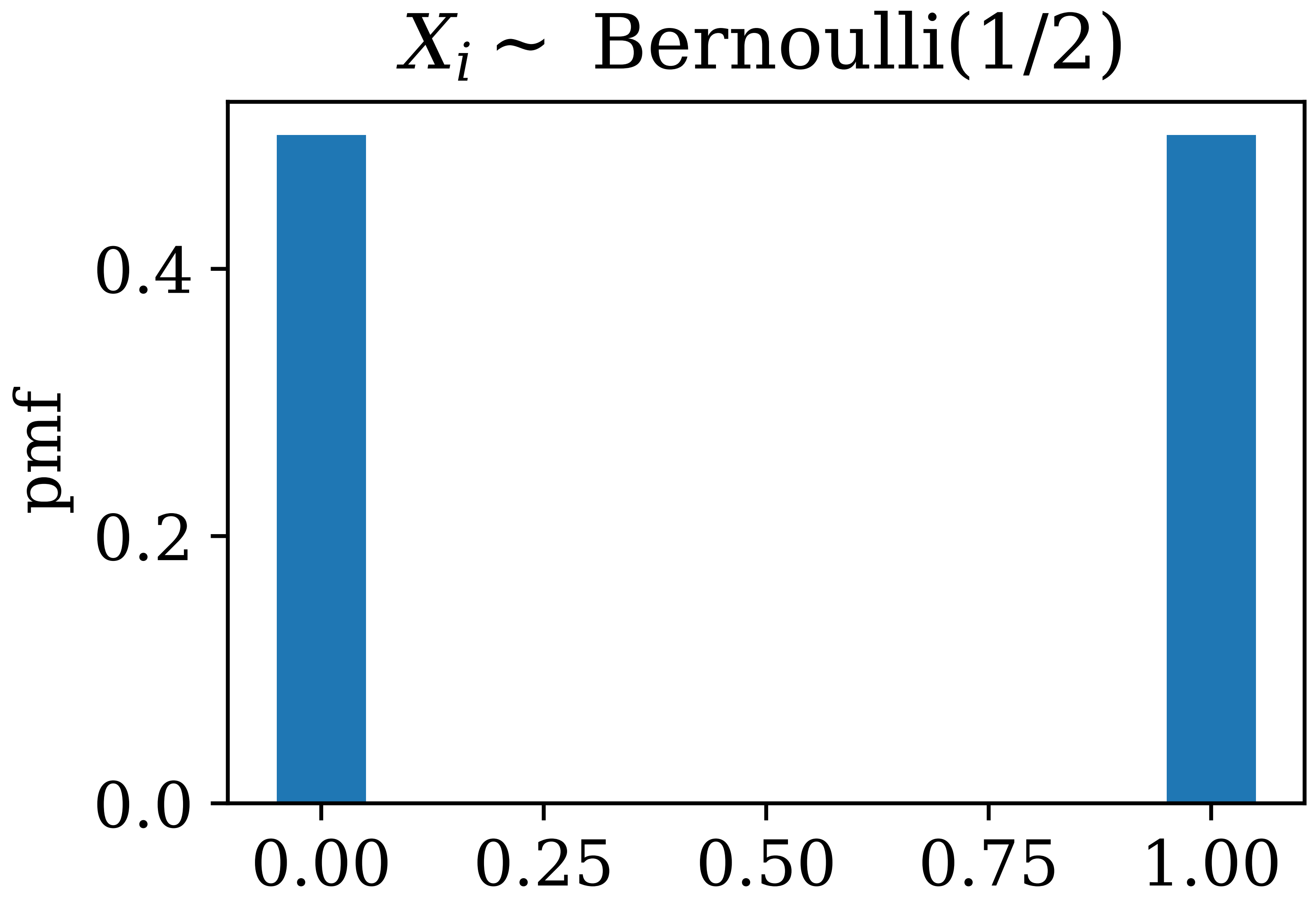
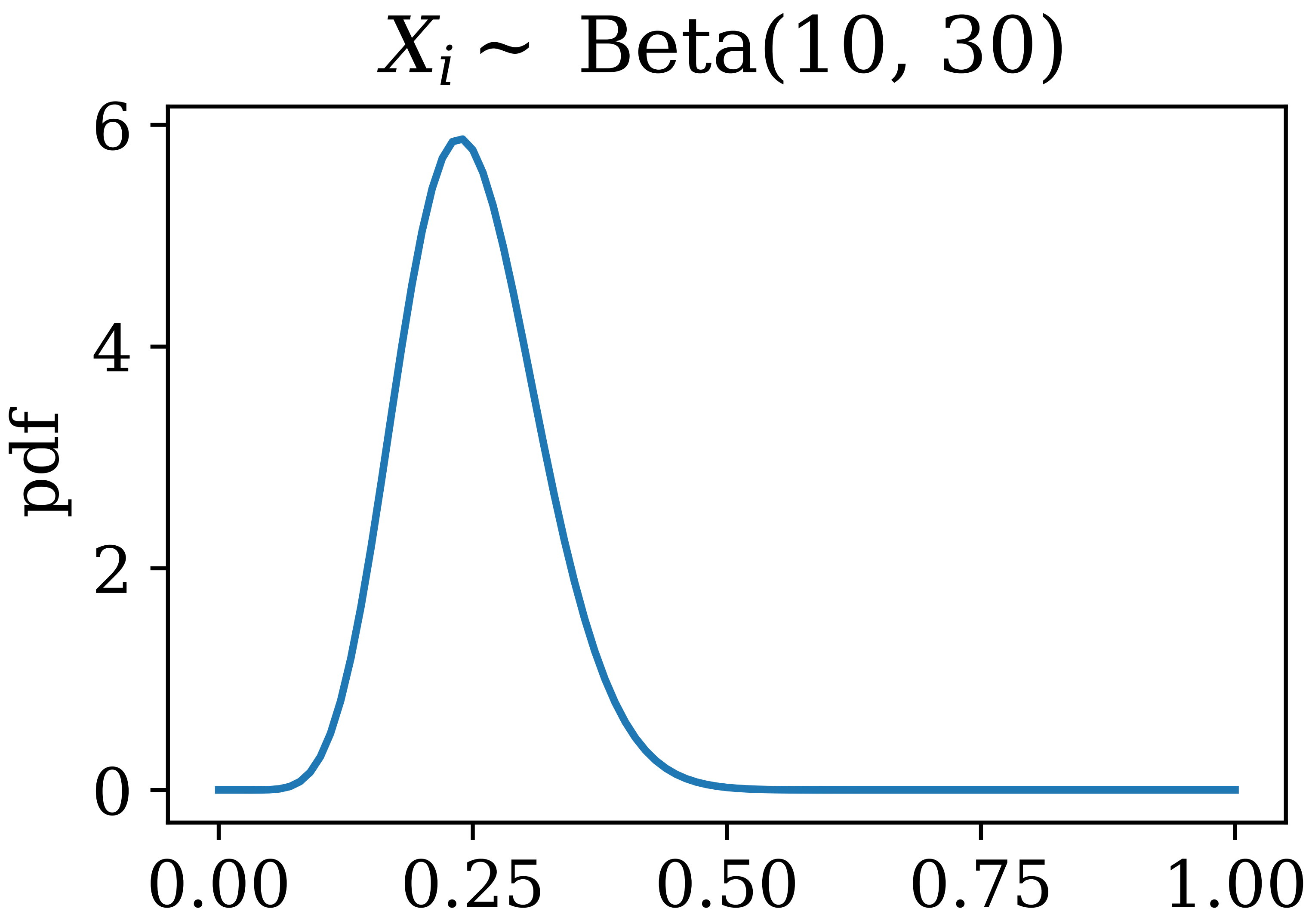
$$C_n^H := \left\{ m : \prod_{i=1}^{n} \exp\left\{\lambda(X_i - m) - \lambda^2/8\right\} < \frac{1}{\alpha} \right\}.$$

$\sigma^2 = 1/4$



$\sigma^2 \approx 0.0046$



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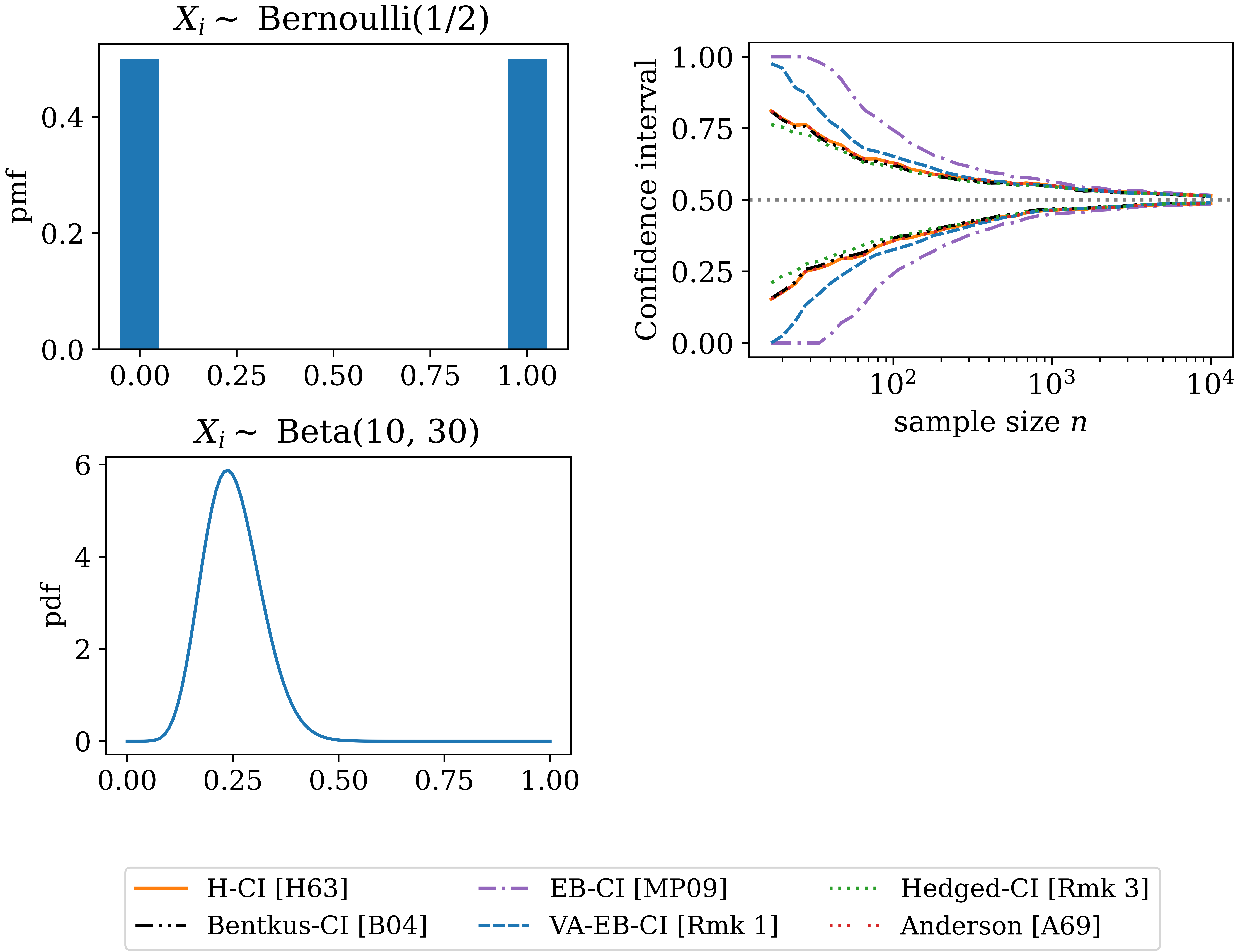

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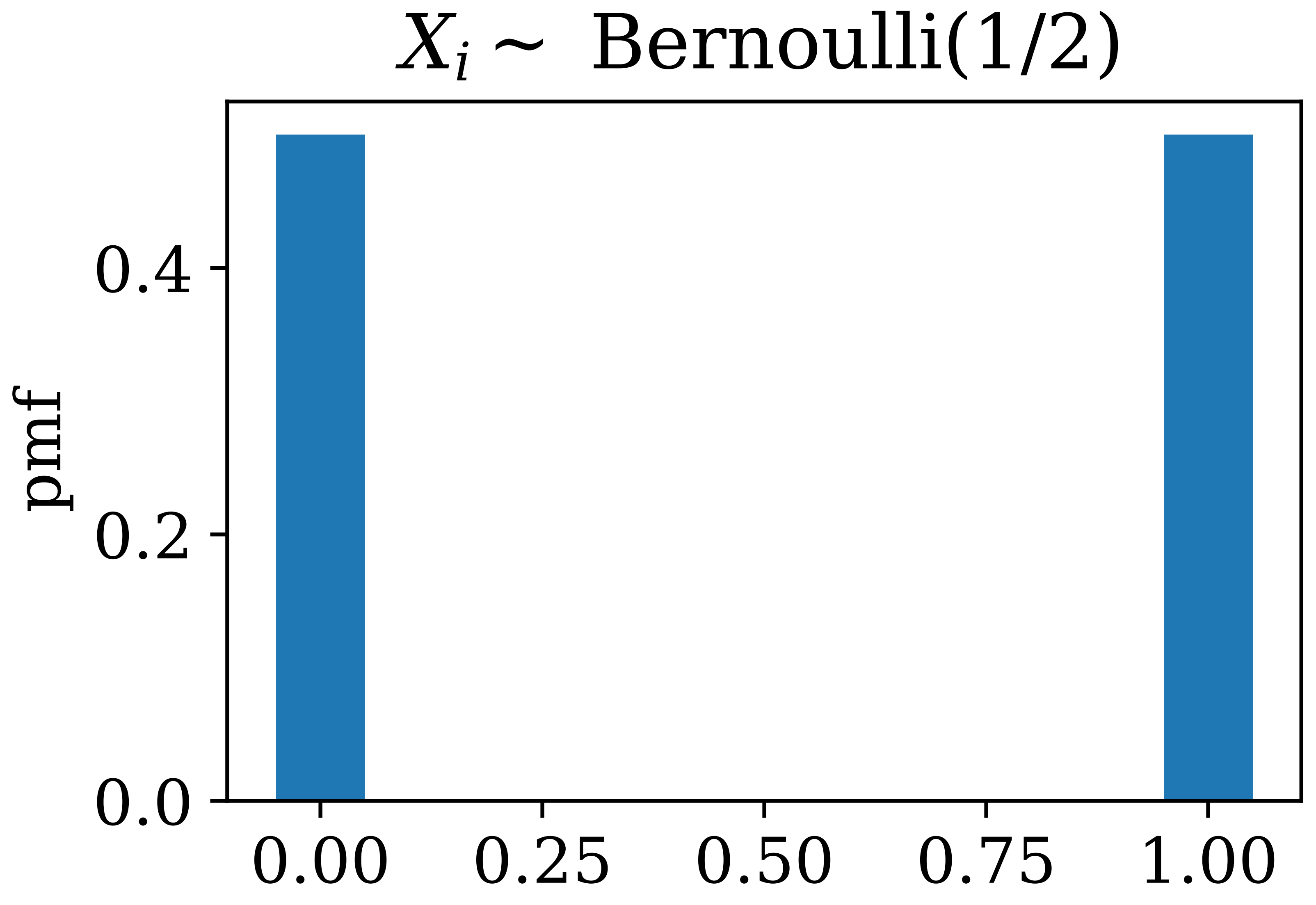

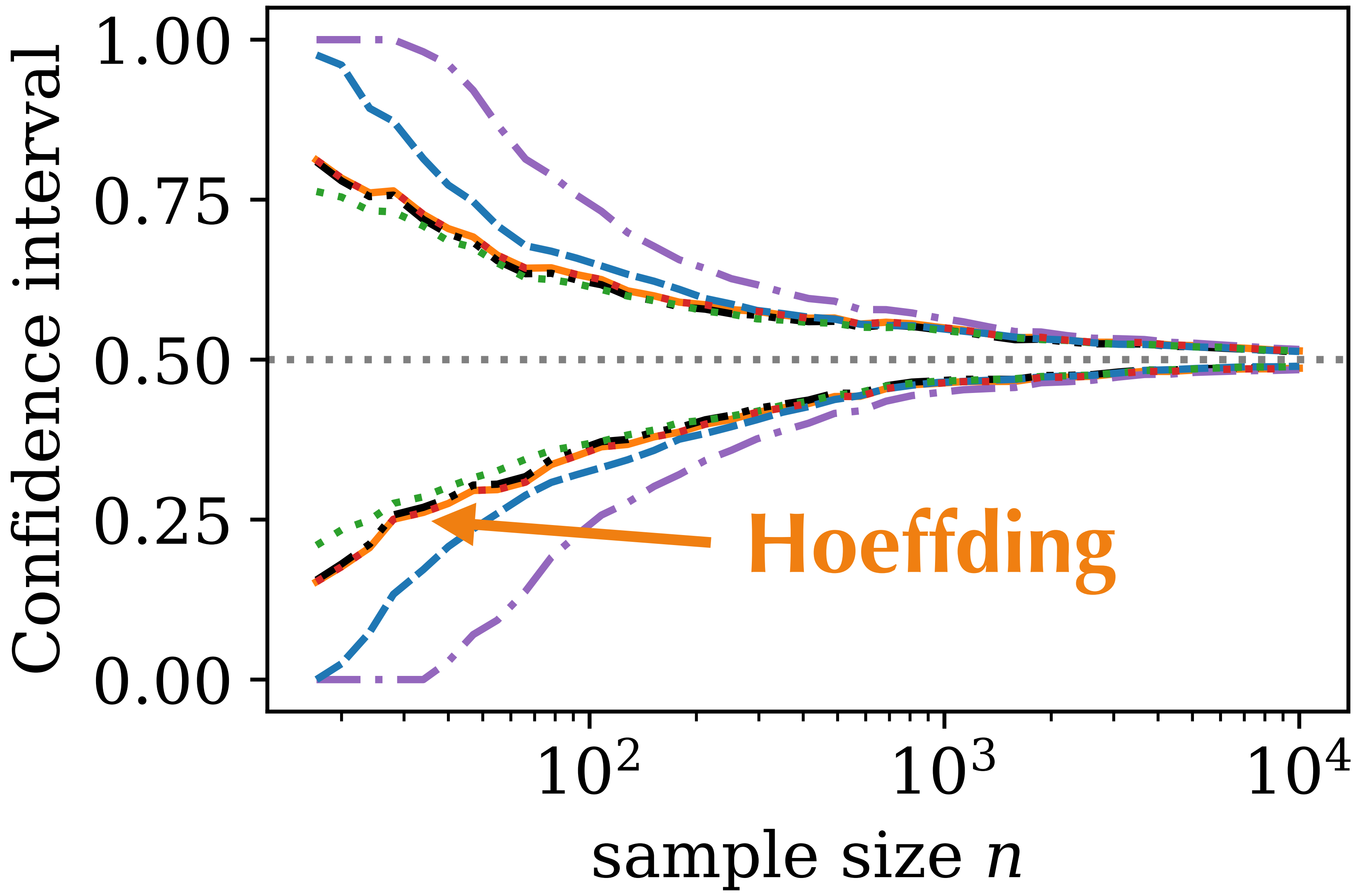

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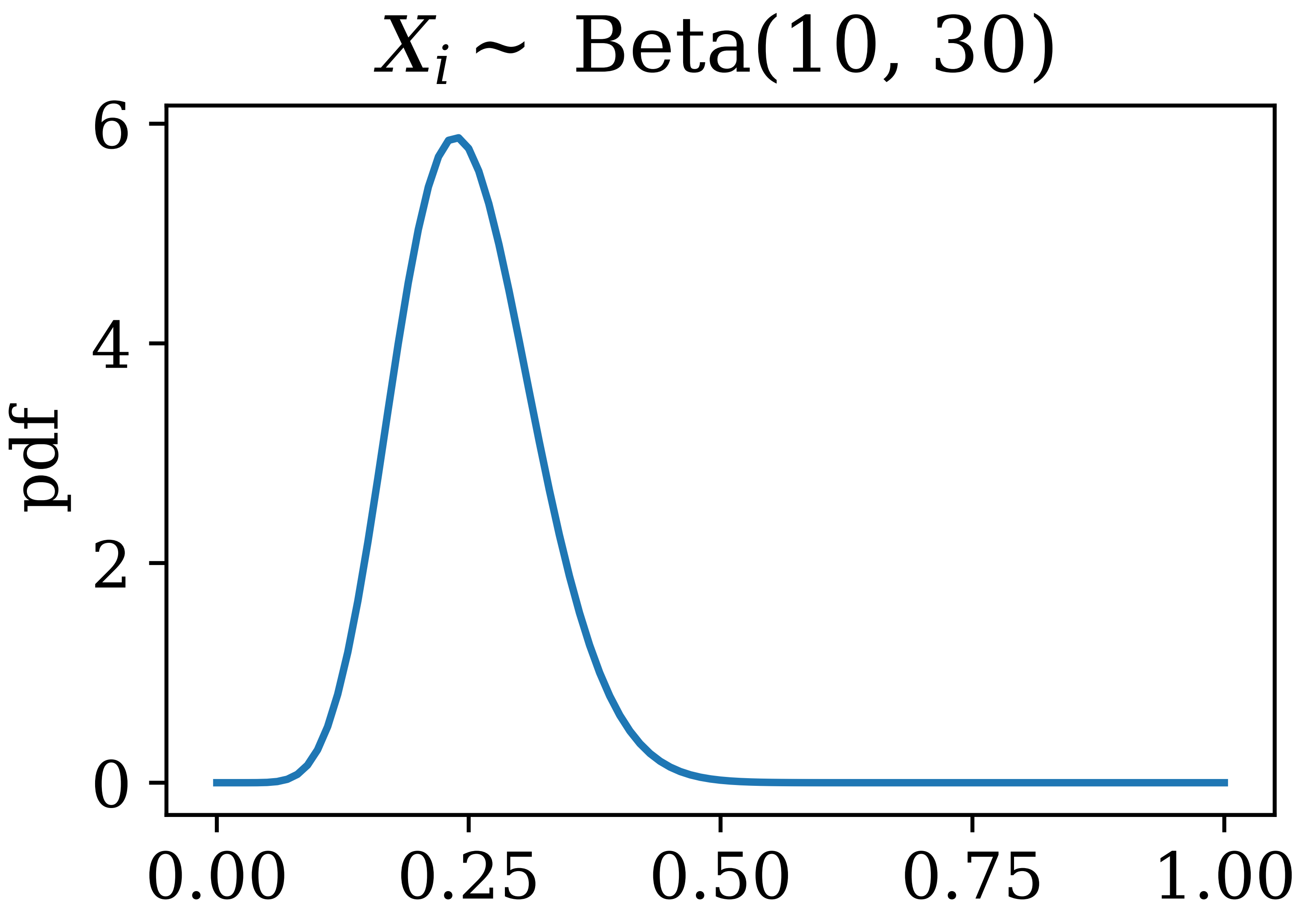

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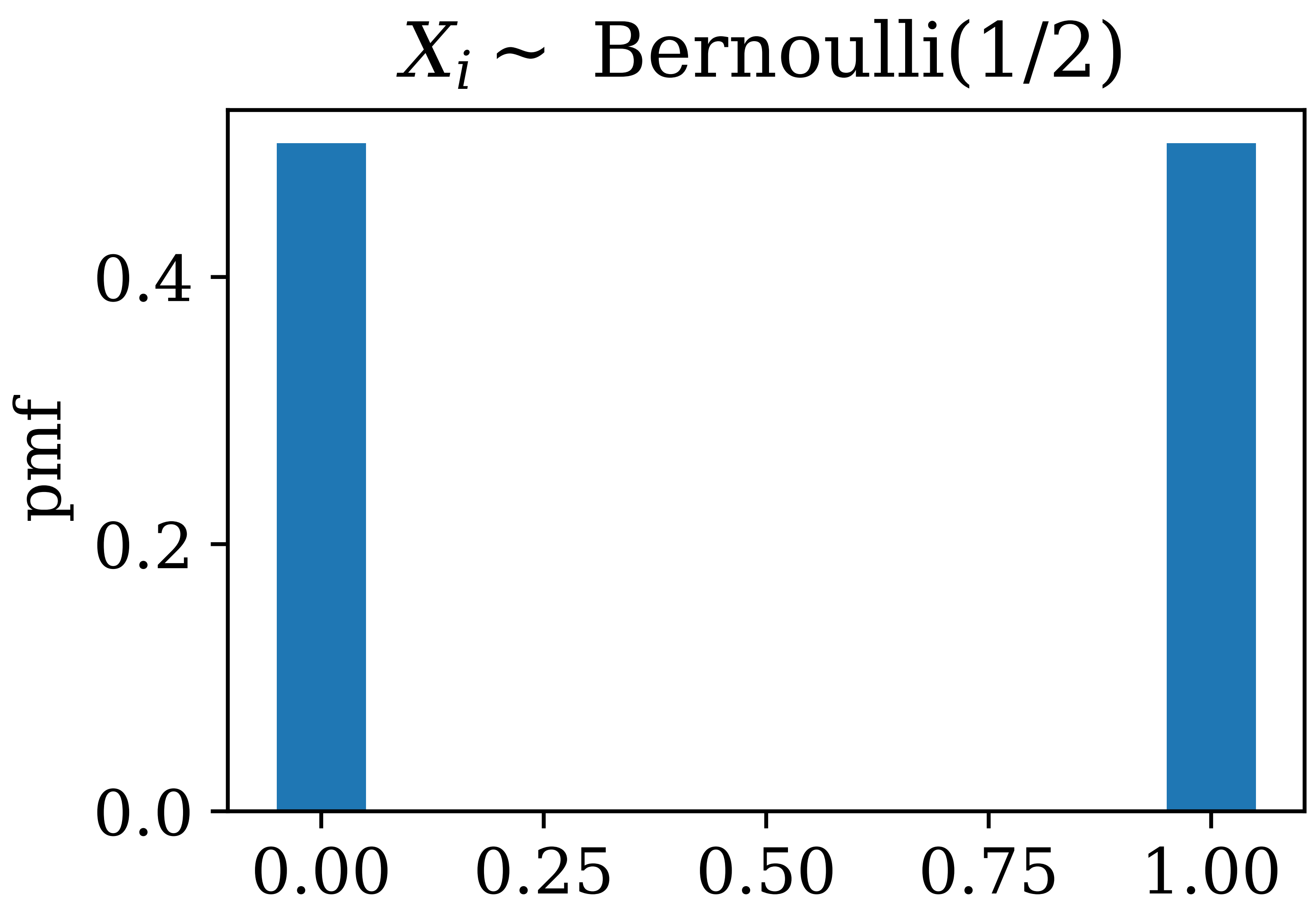

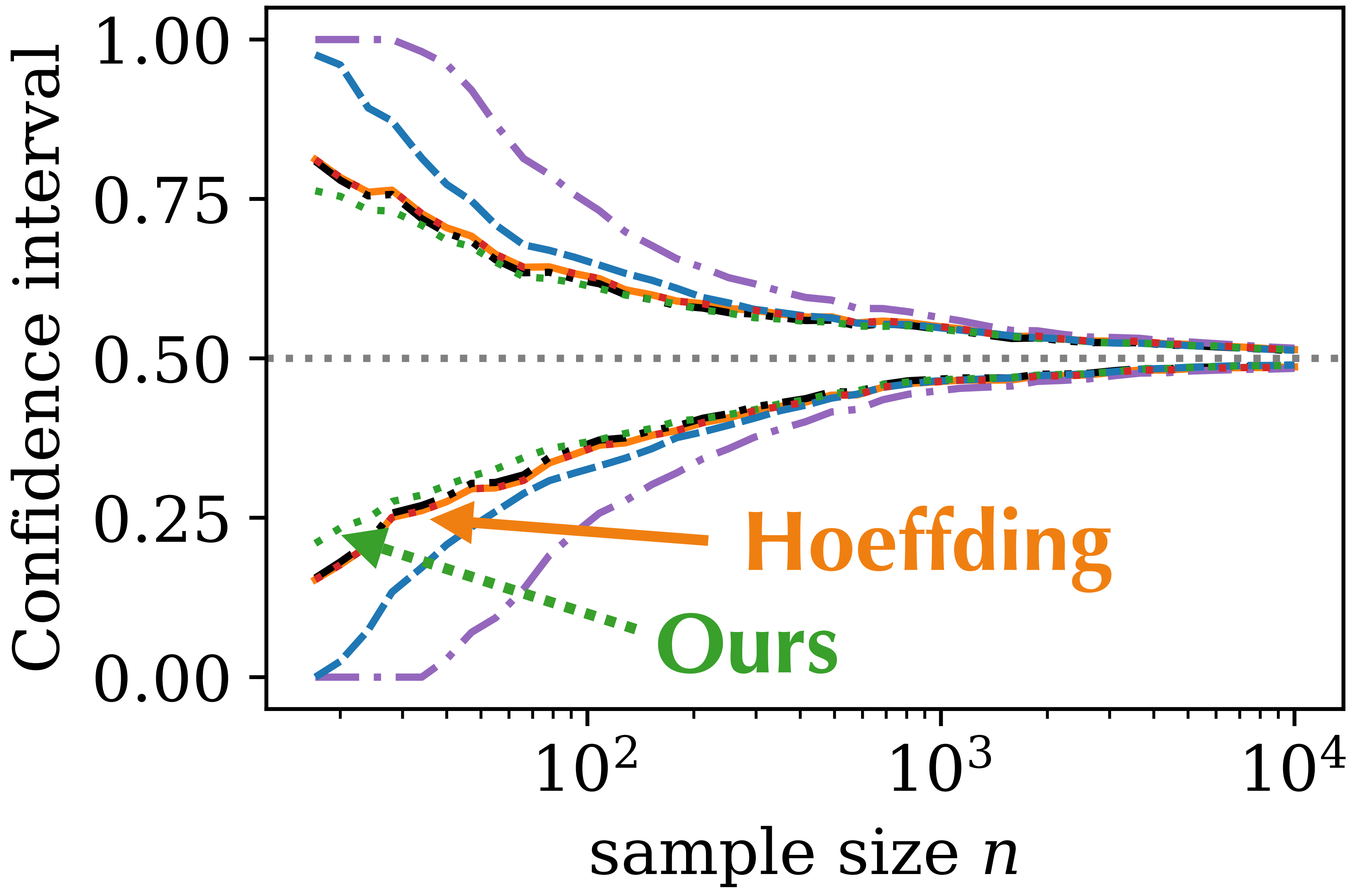

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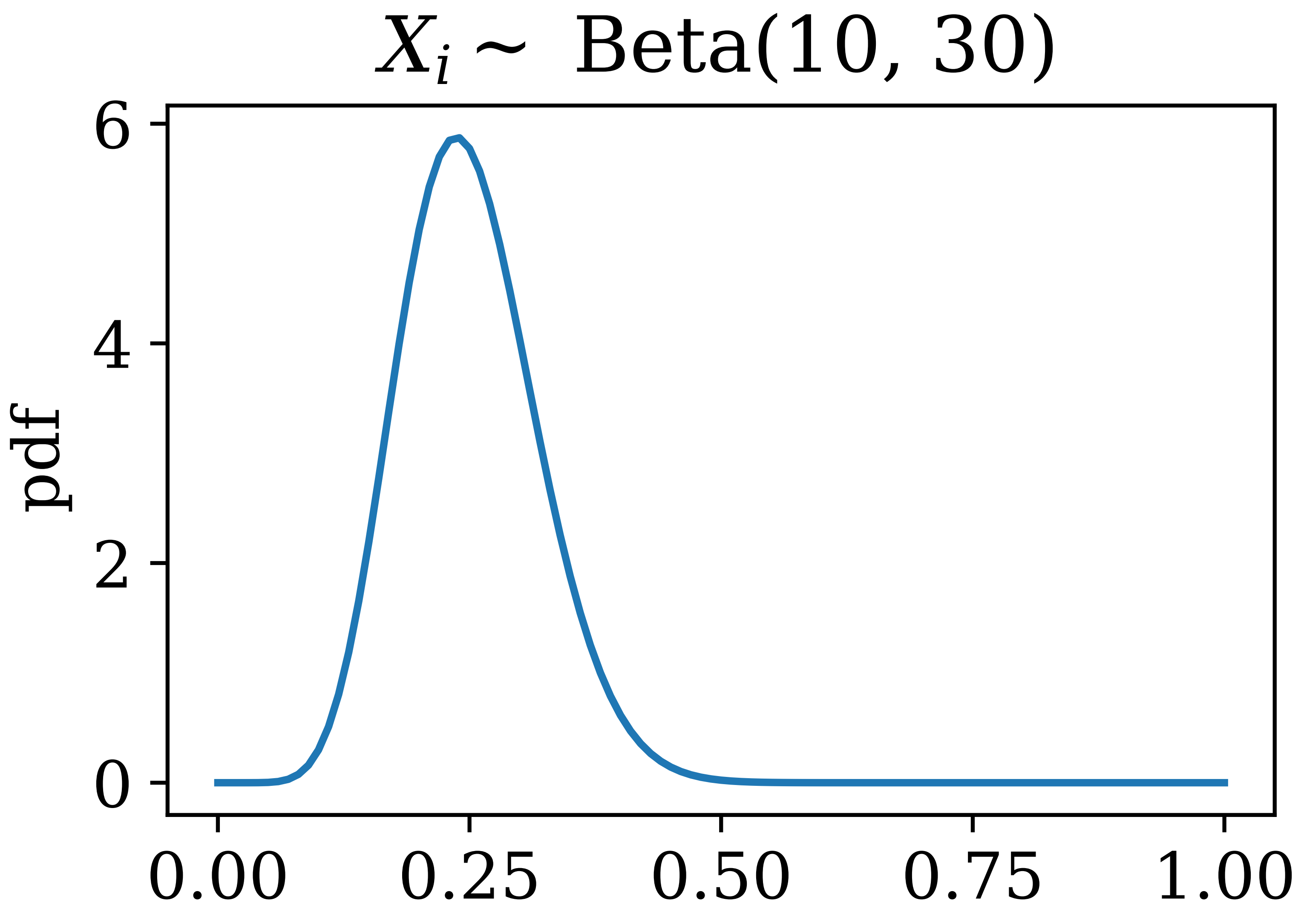

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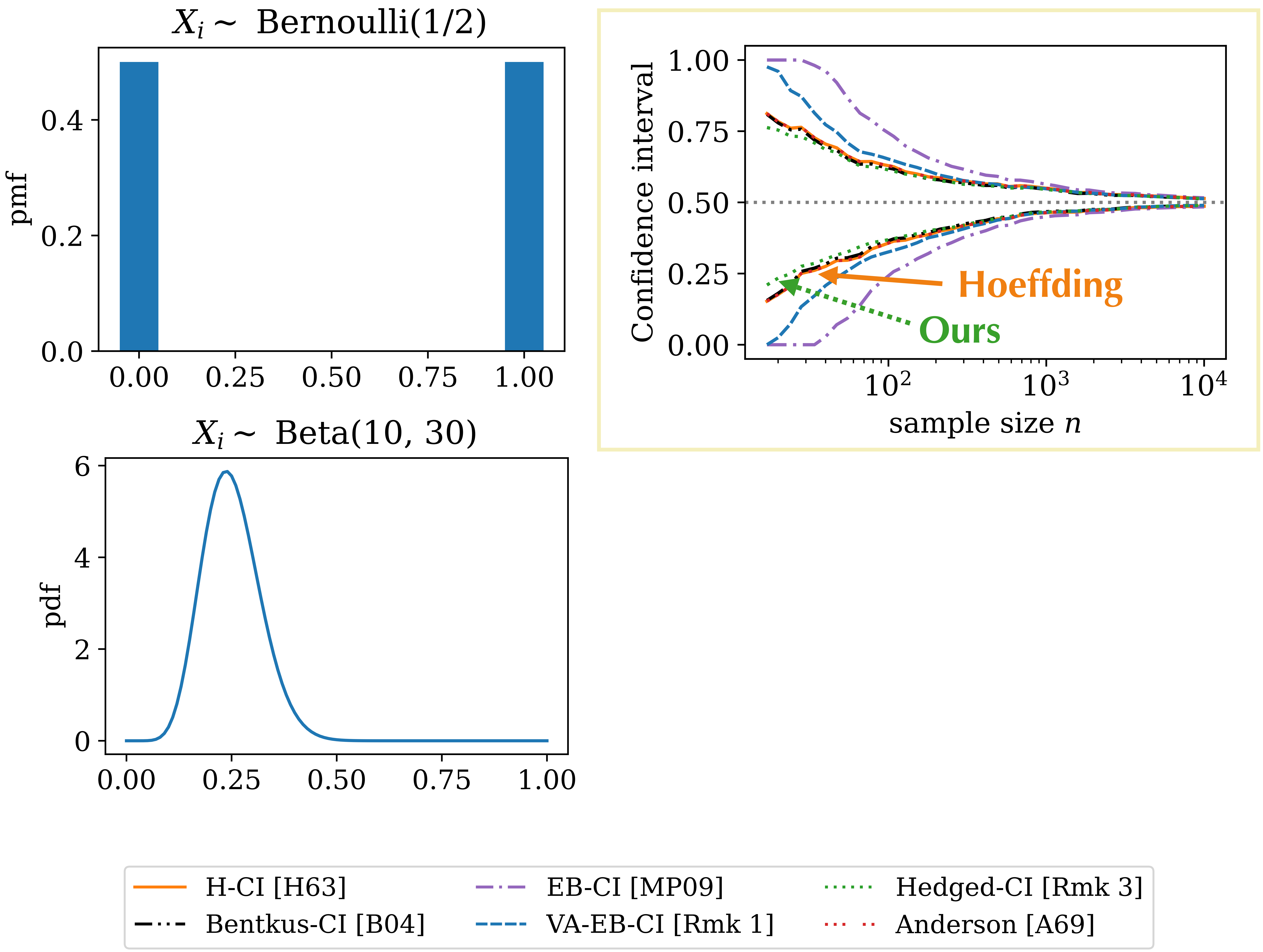

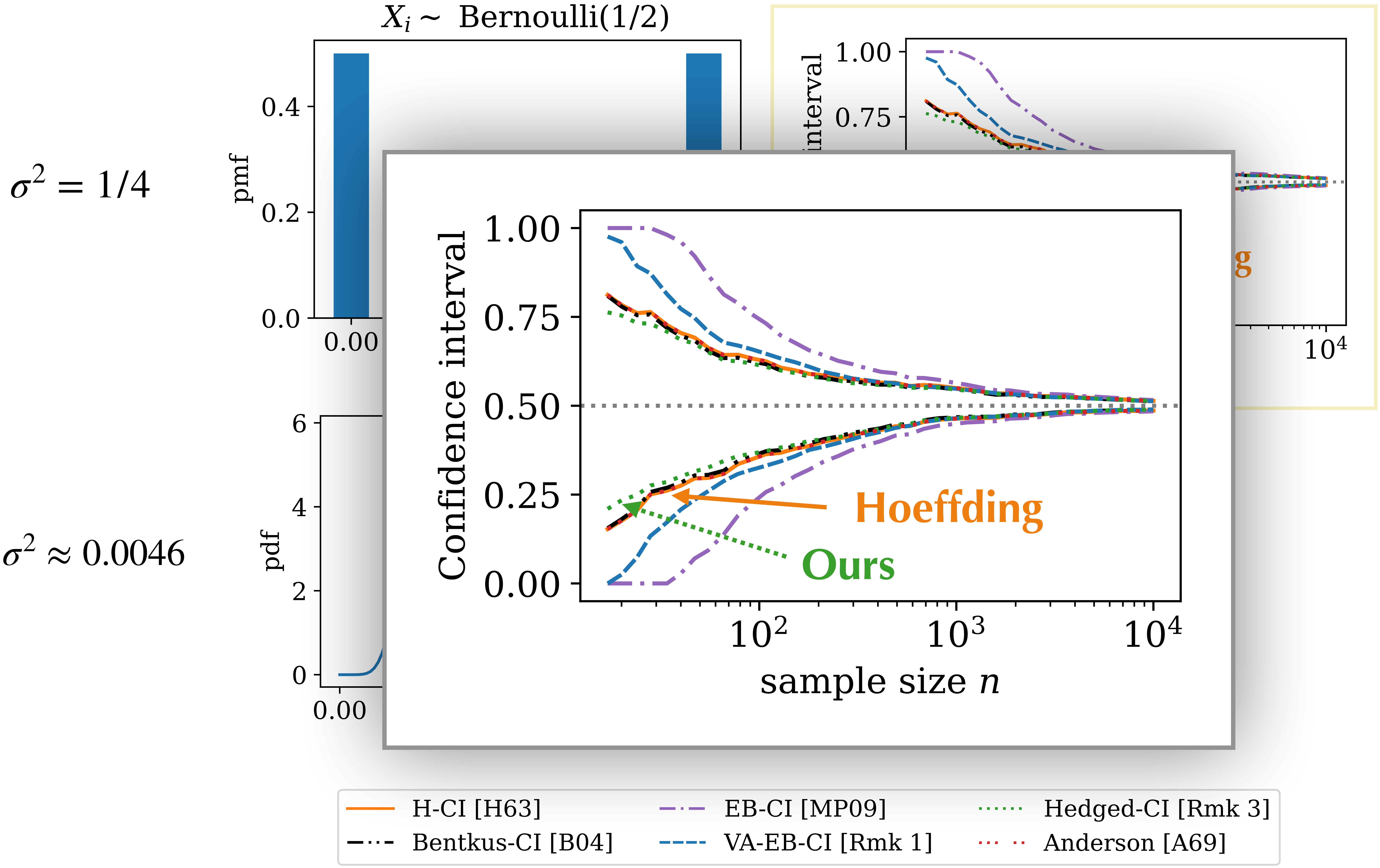

$\sigma^2 = 1/4$
$X_i \sim$ Bernoulli(1/2)
pmf
$\sigma^2 \approx 0.0046$
pdf
Confidence interval
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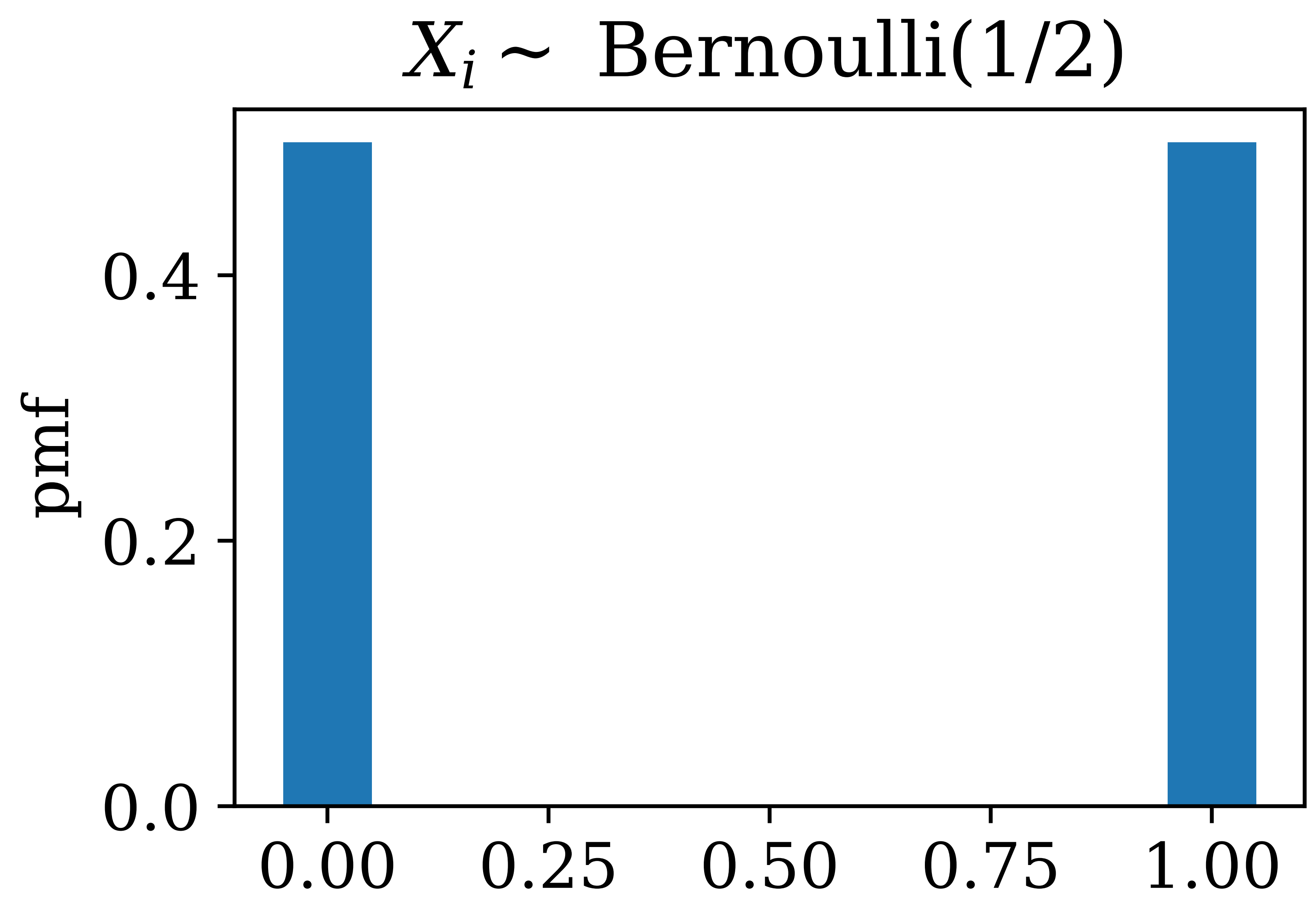

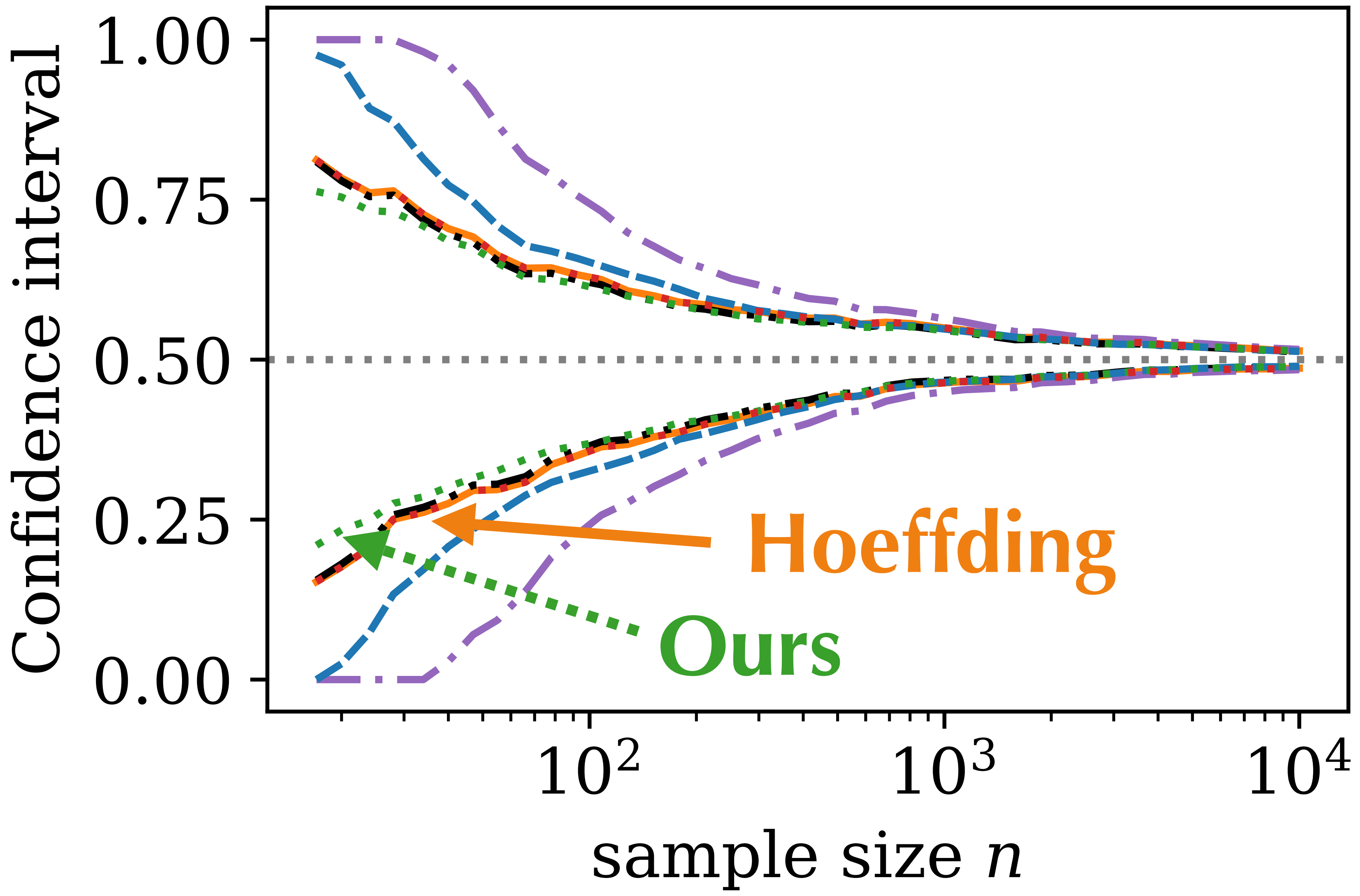

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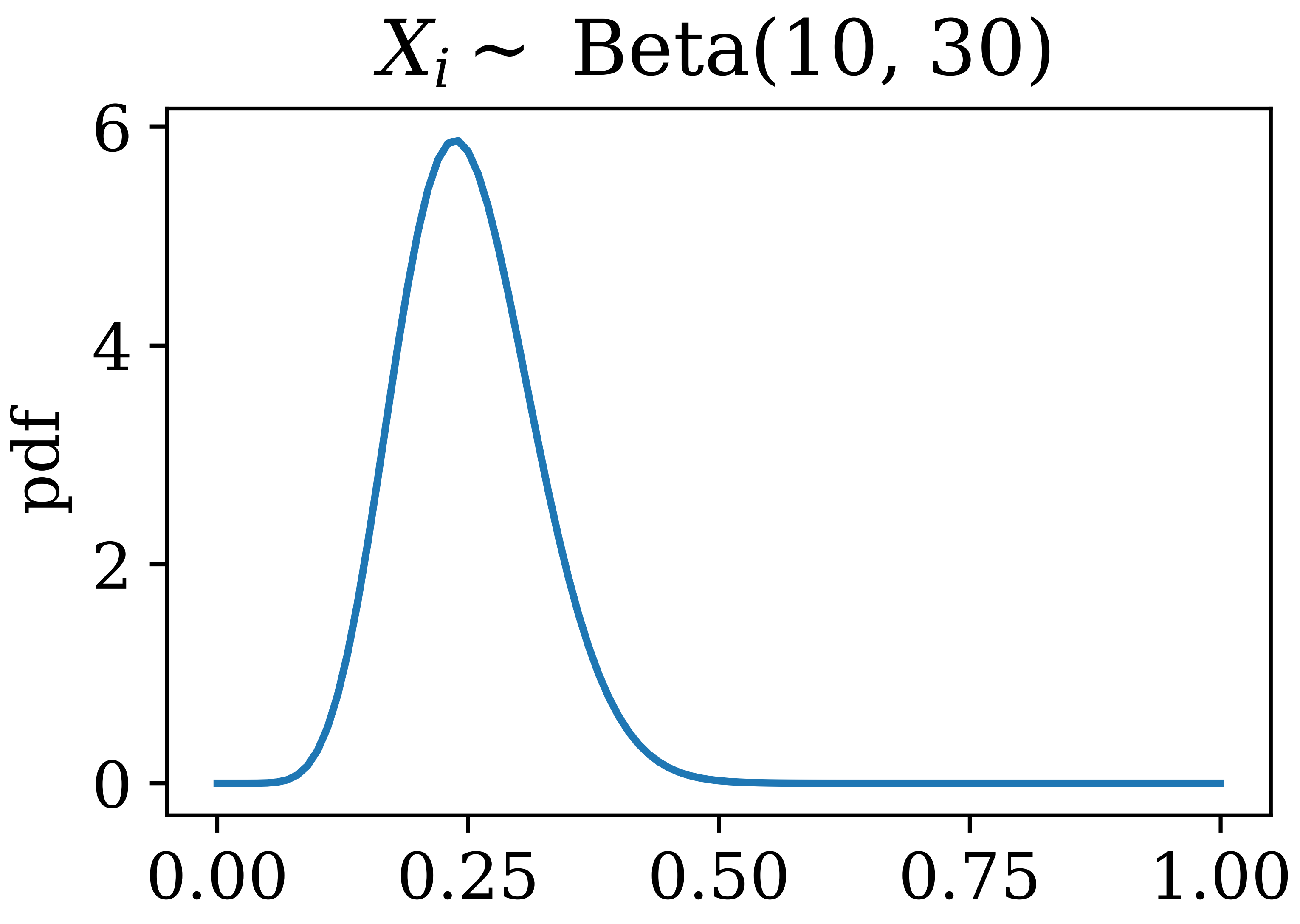

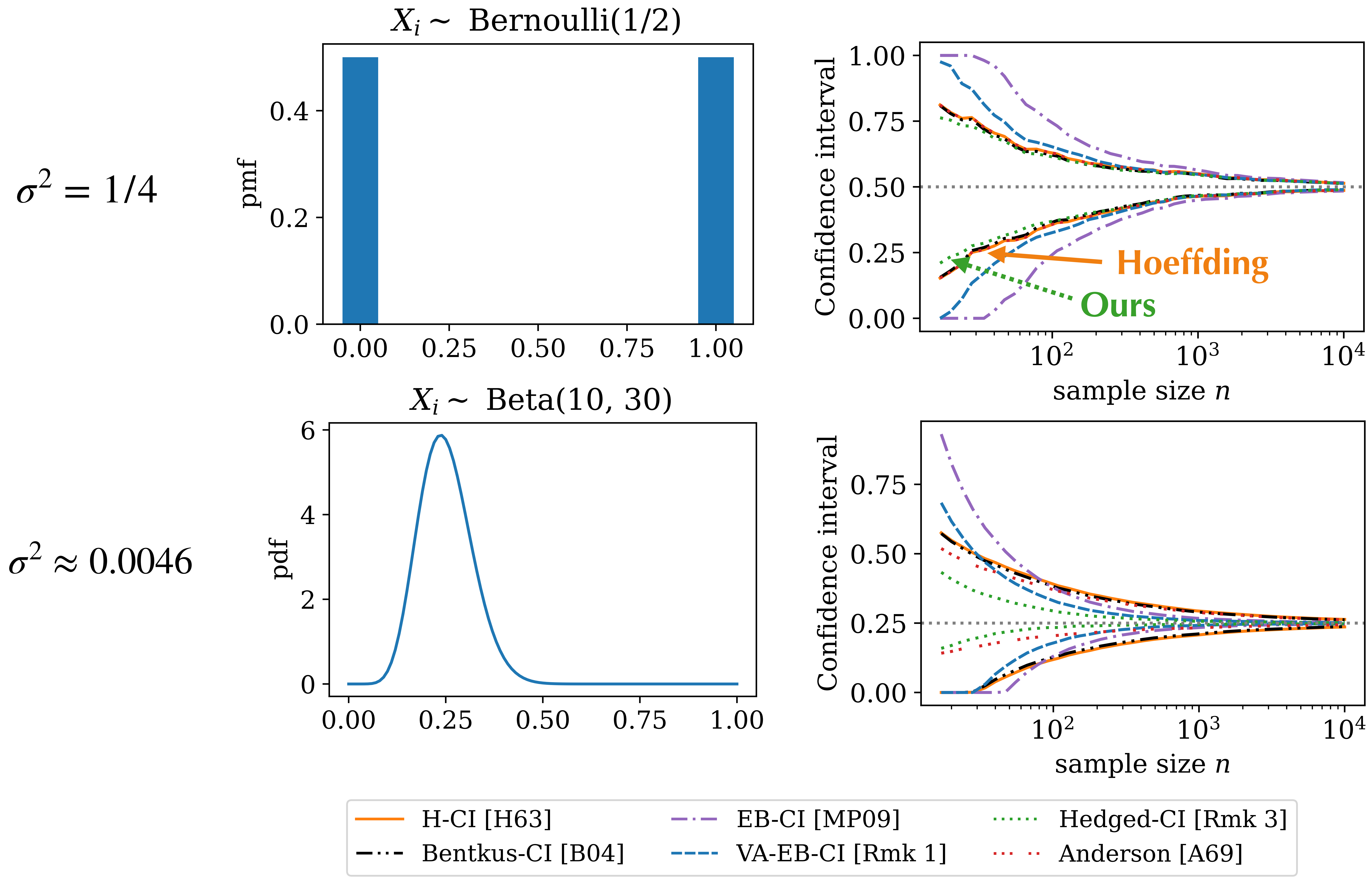
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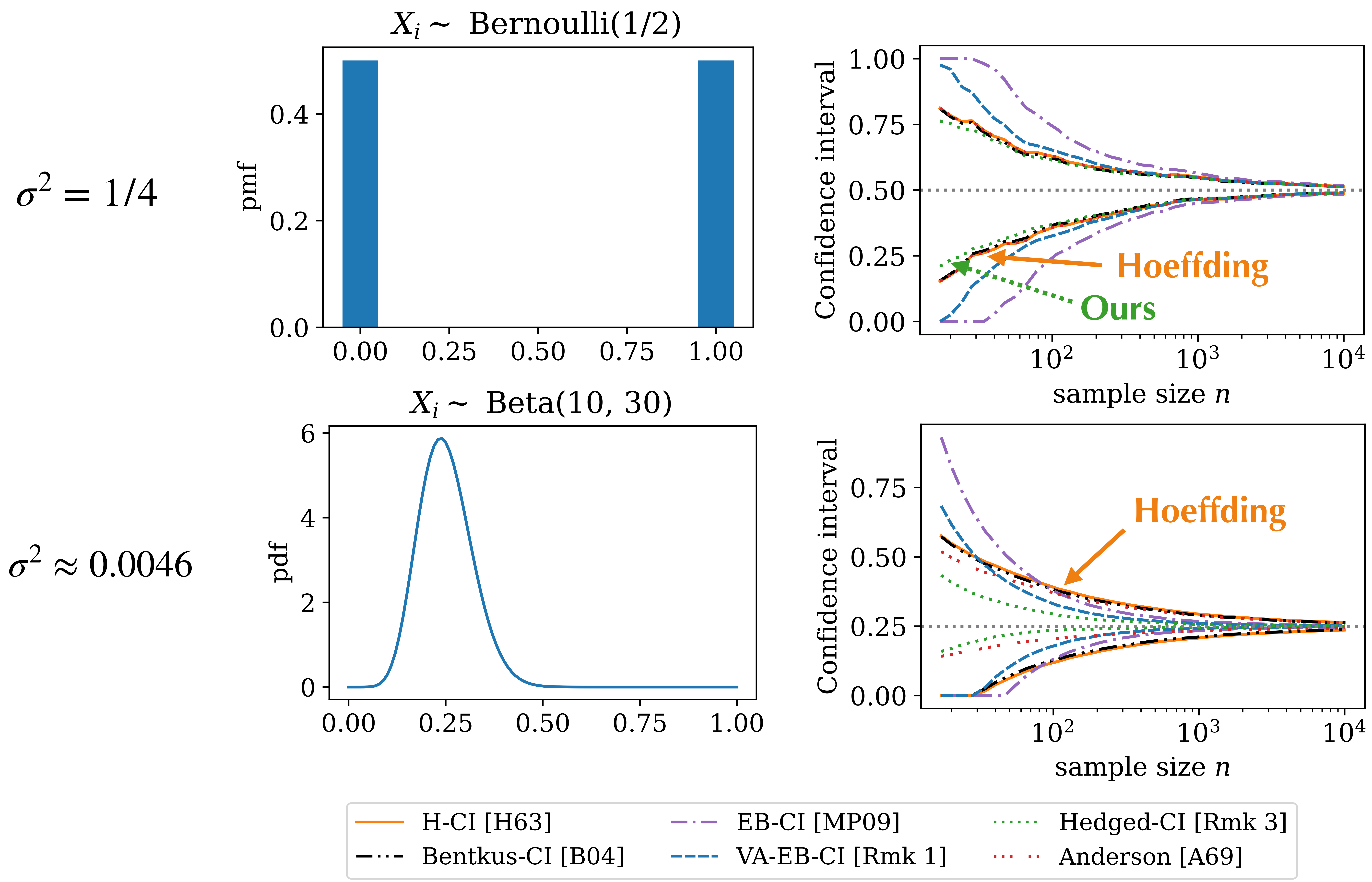

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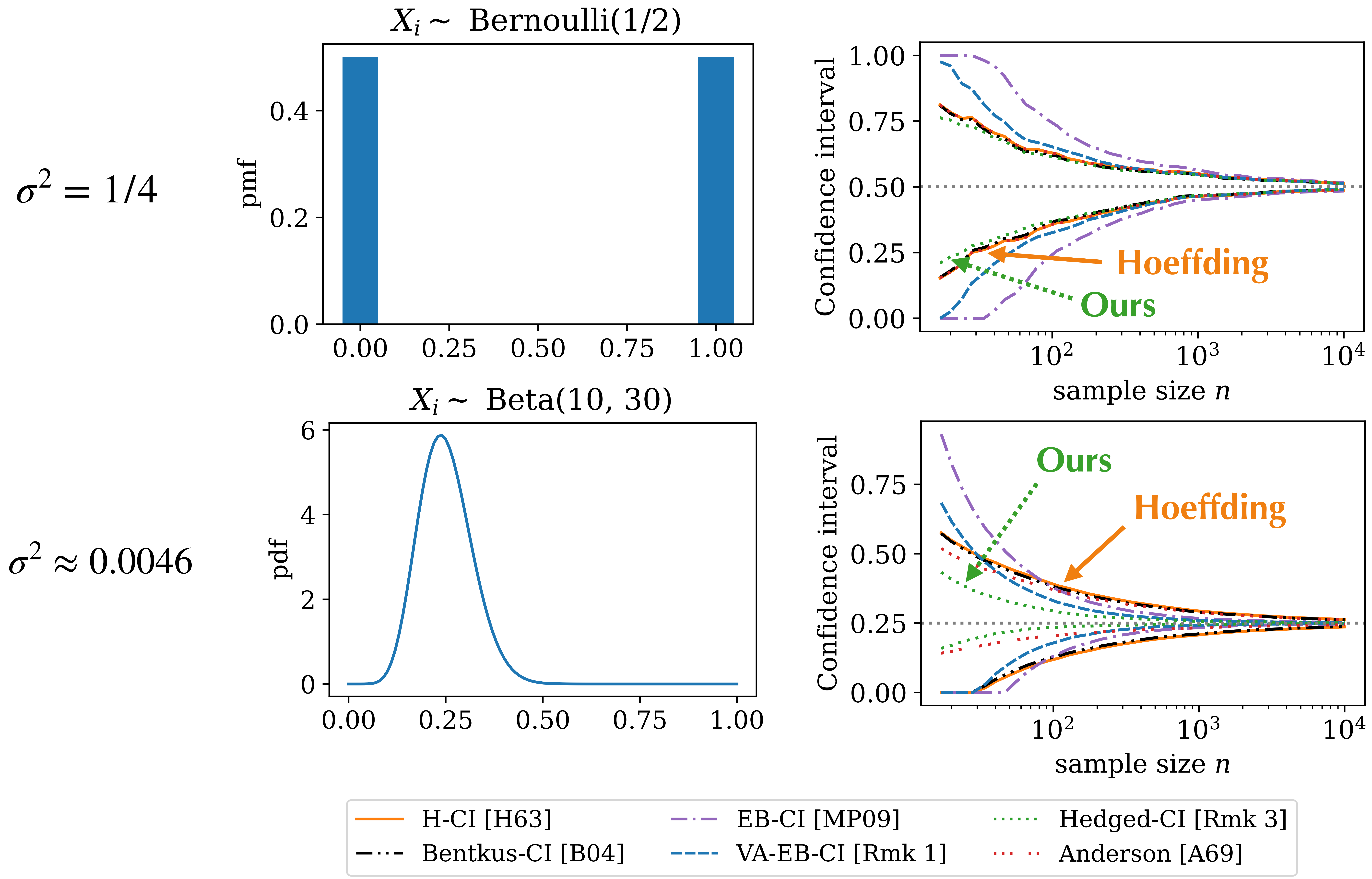
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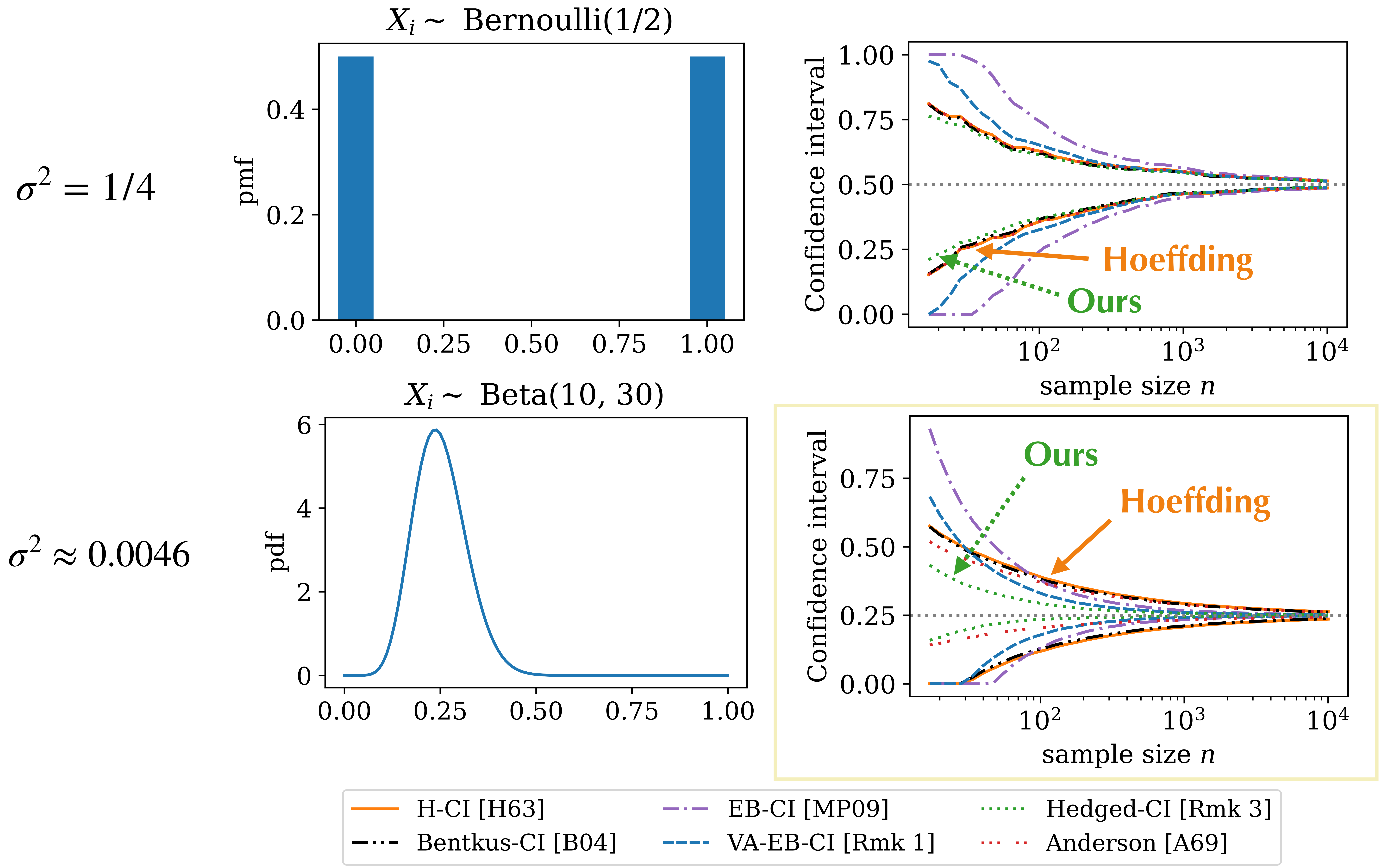

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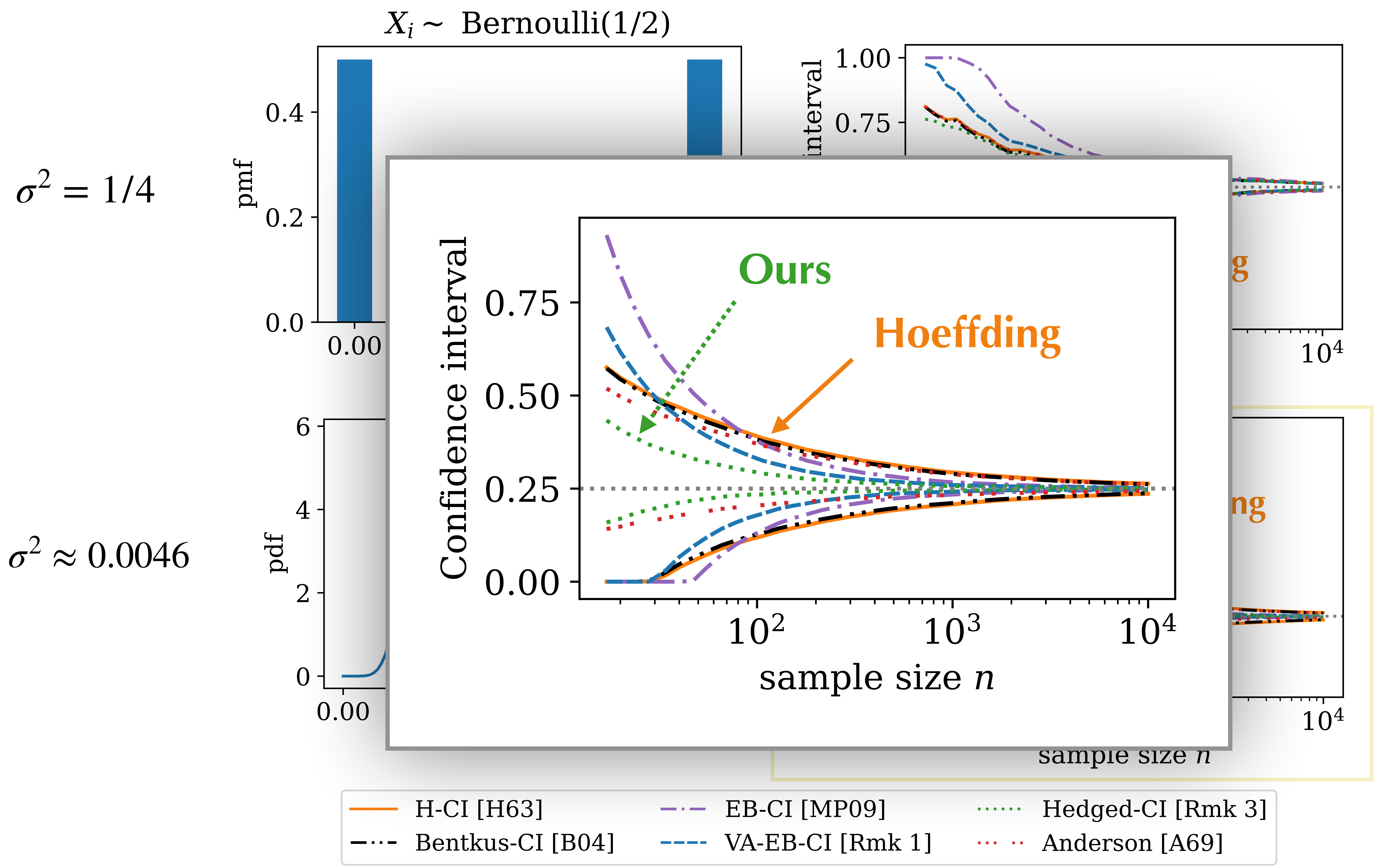
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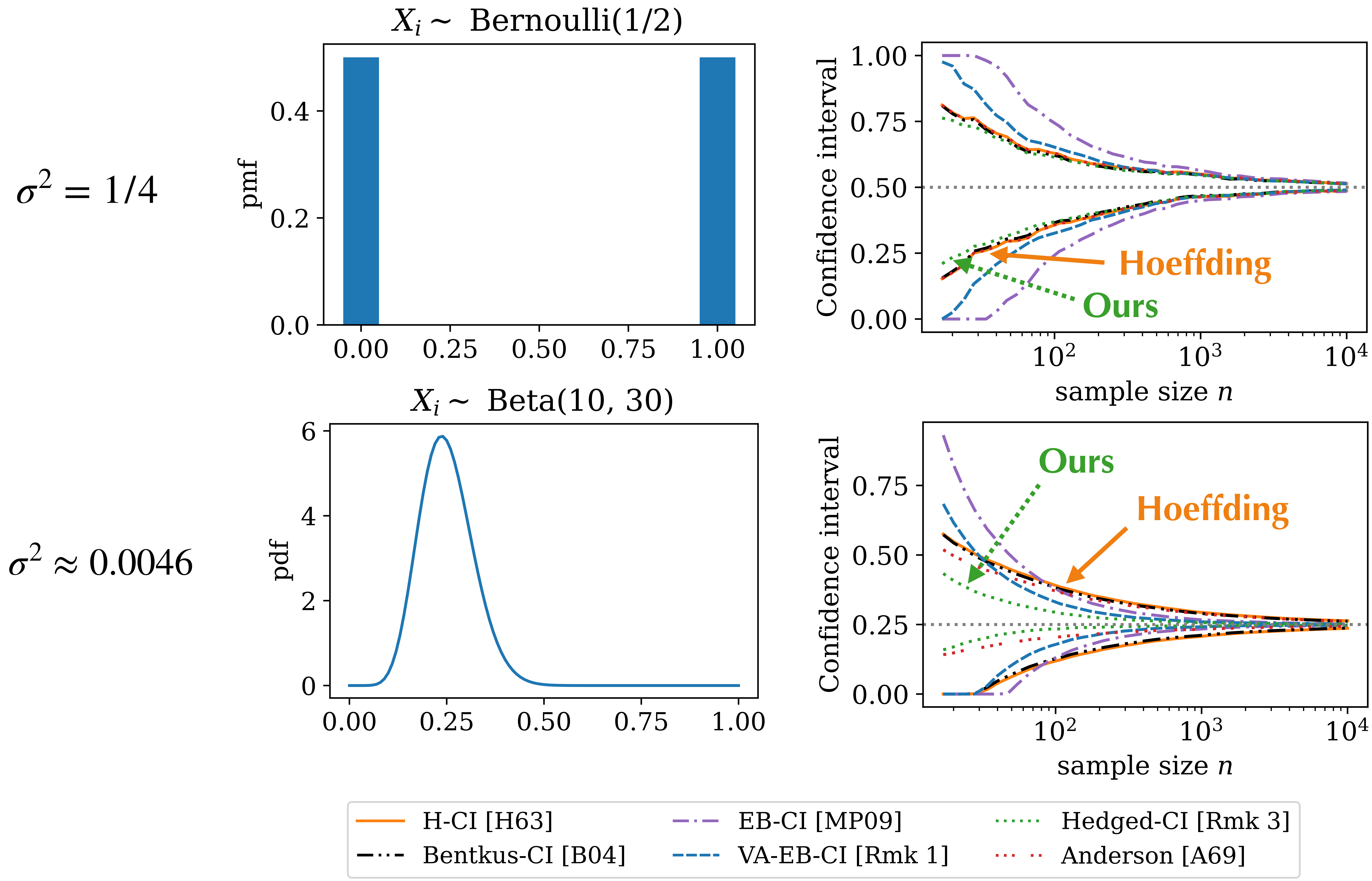

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Familiar special case: $X_1, X_2, \ldots \overset{iid}{\sim} \mathbb{P}$, with $\mathbb{E}_{\mathbb{P}}(X_1) = \mu$.

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What if

$\mathbb{E}(X_t) \equiv \mu \gg m$?

or $\mu \ll m$?

<u>or $\mu = m$?</u>

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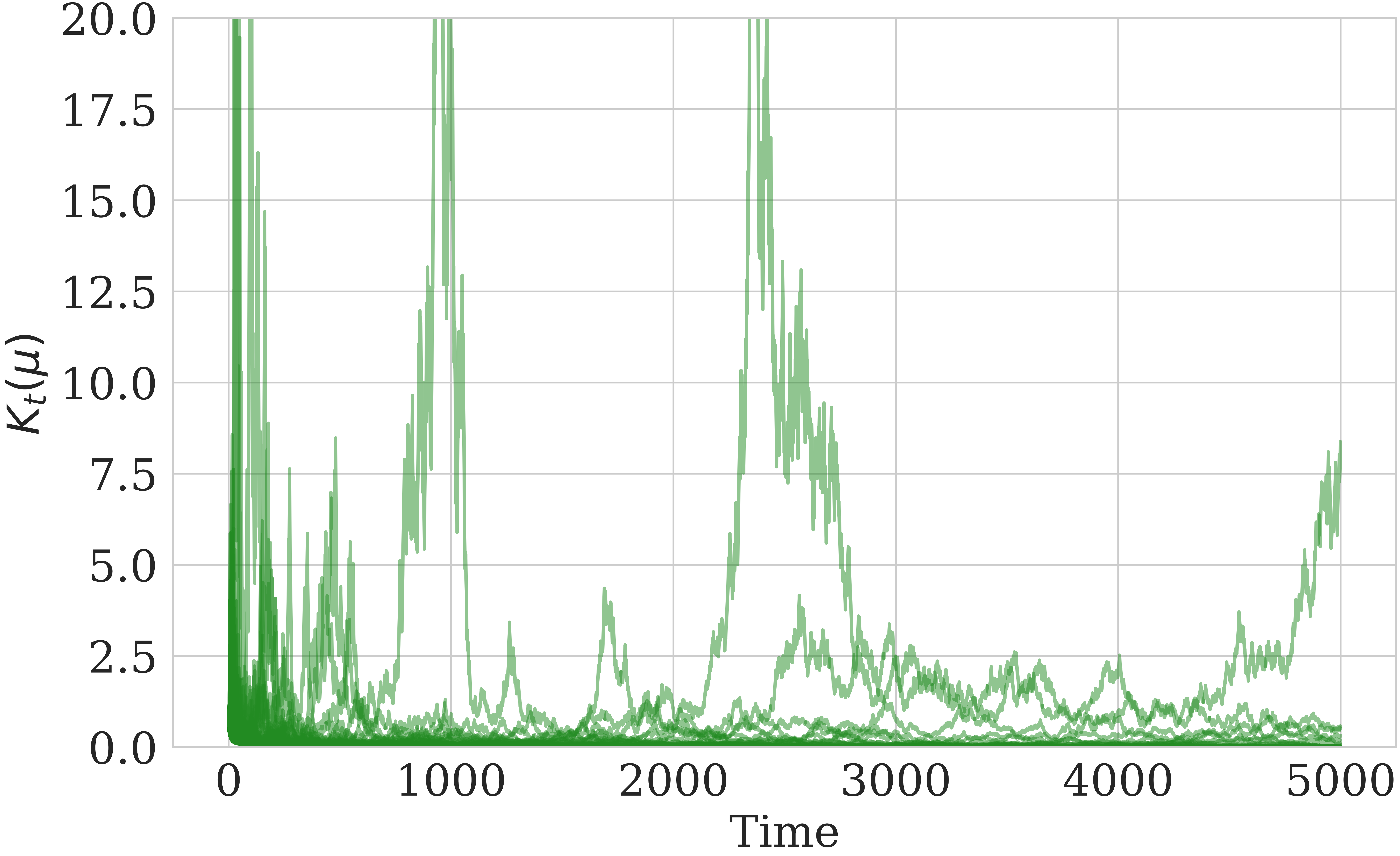
Similar ideas from Shafer & Vovk, Hendriks, Jun & Orabona, etc.

Wealth processes / nonnegative martingales



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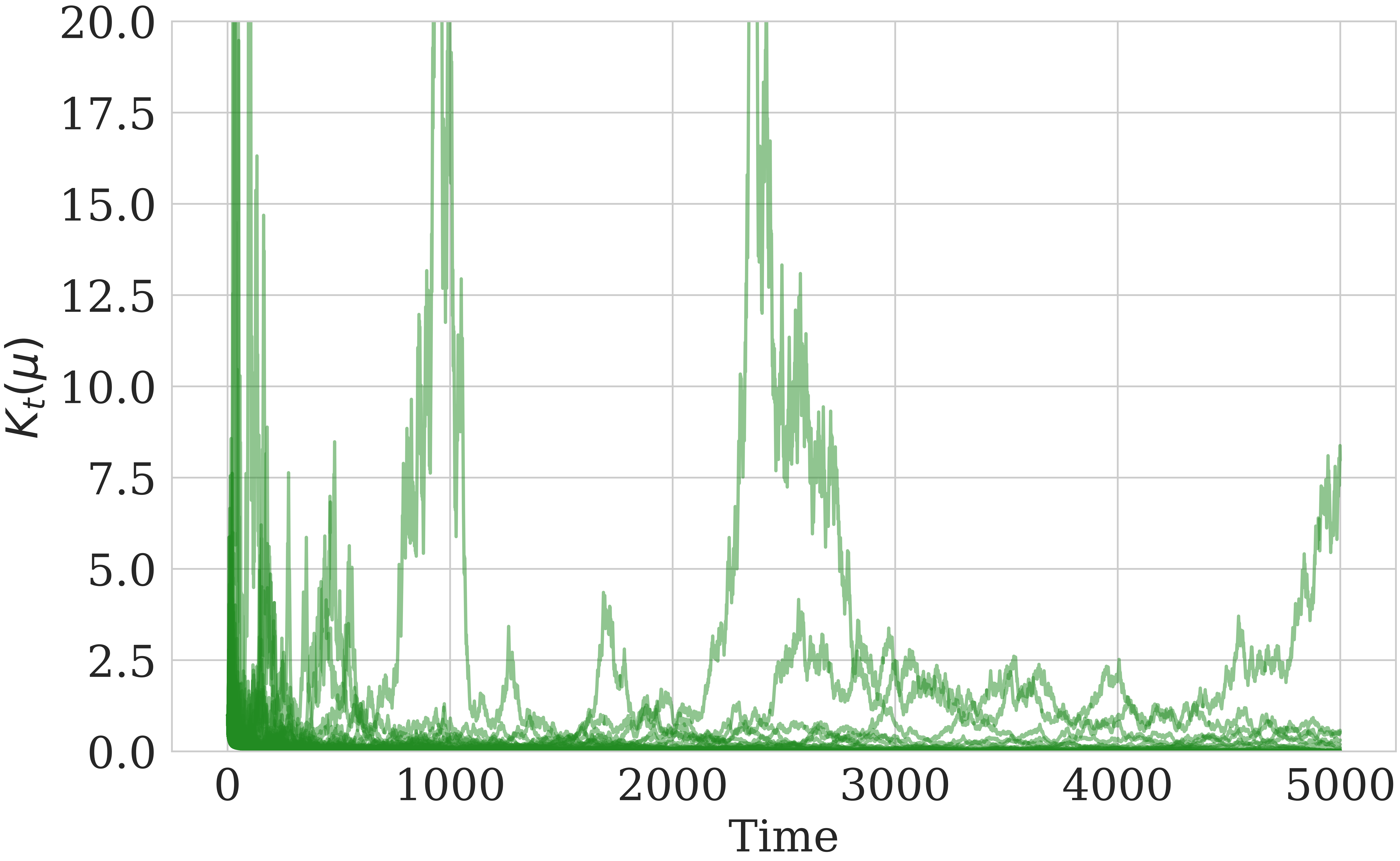

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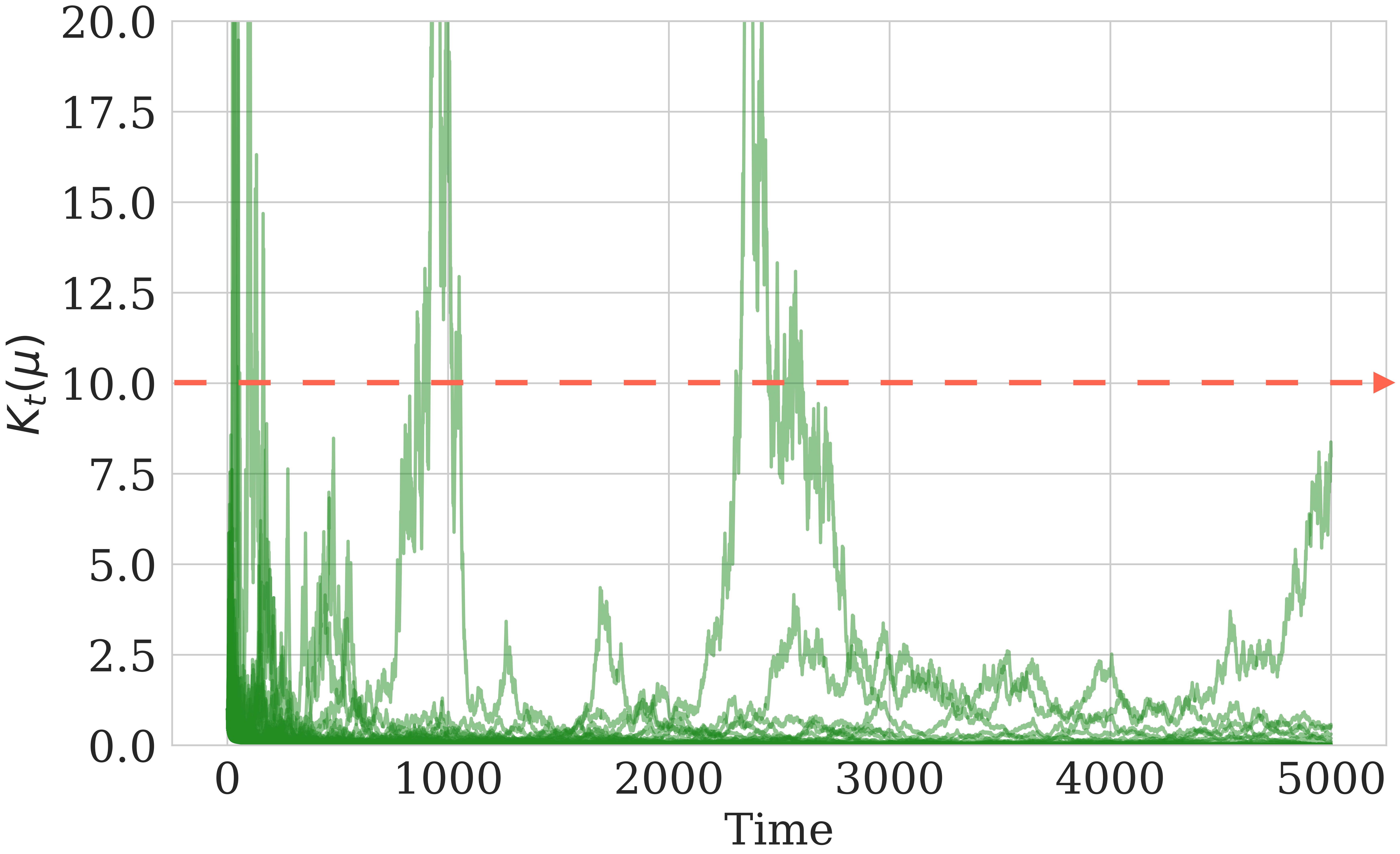

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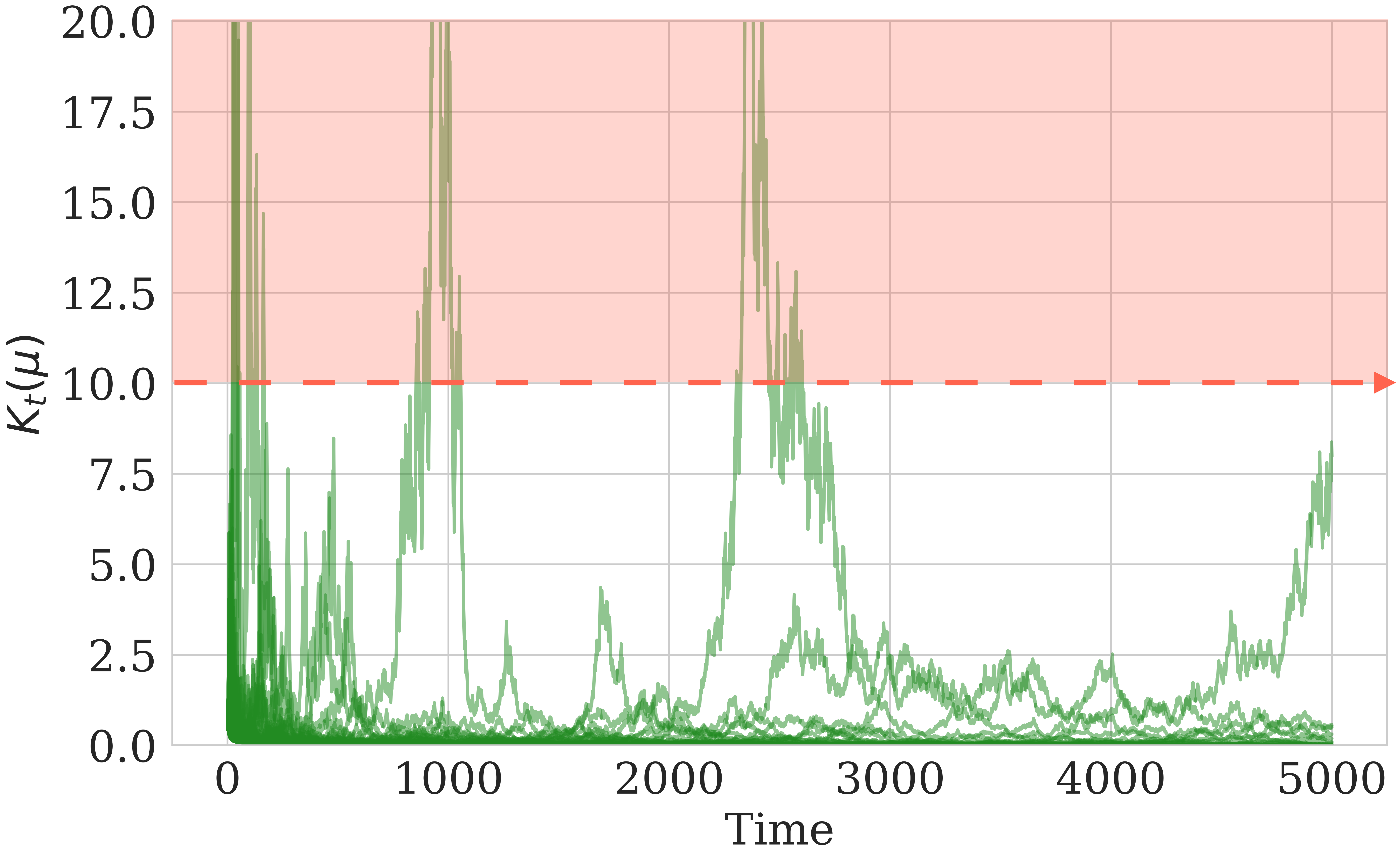

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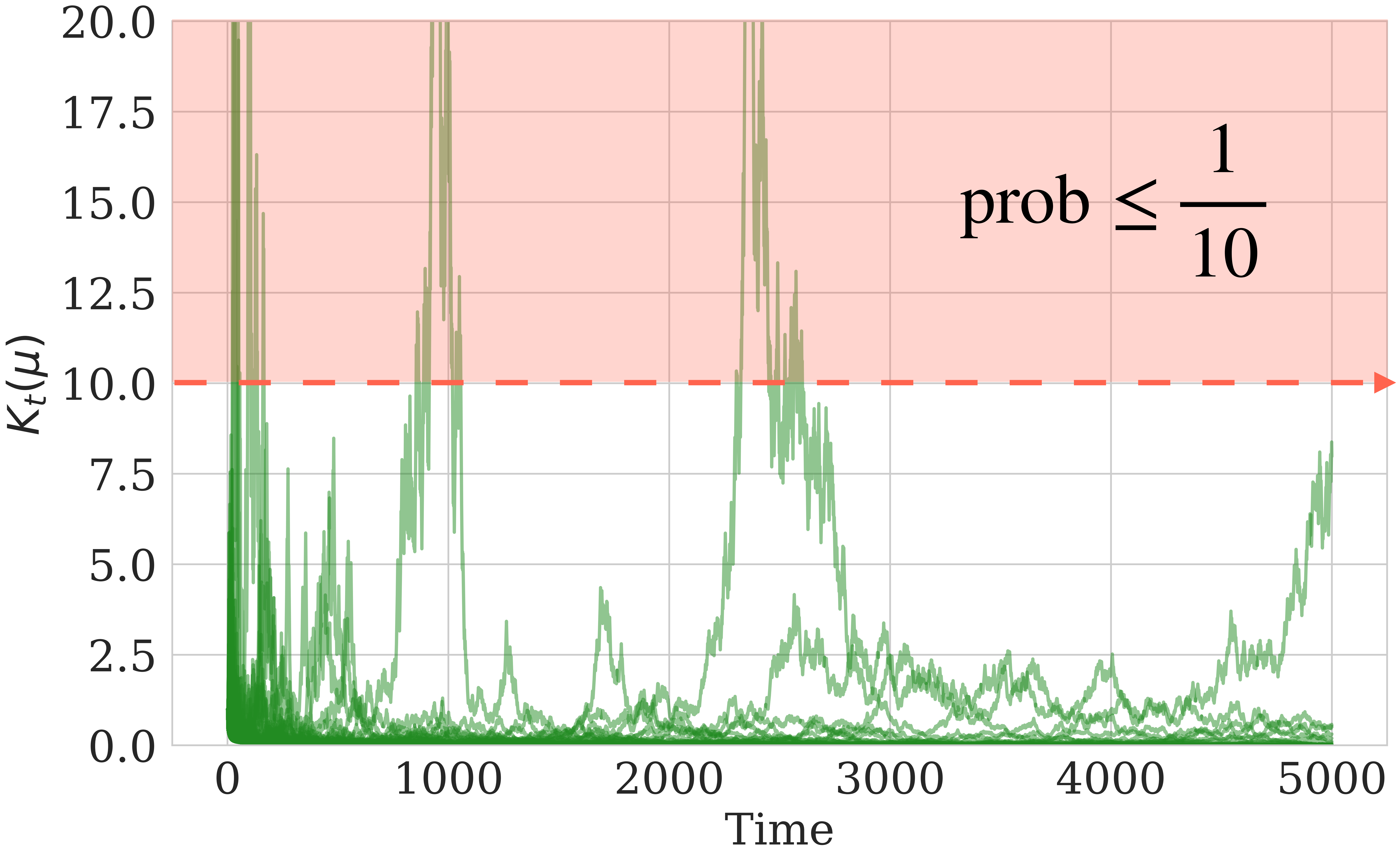

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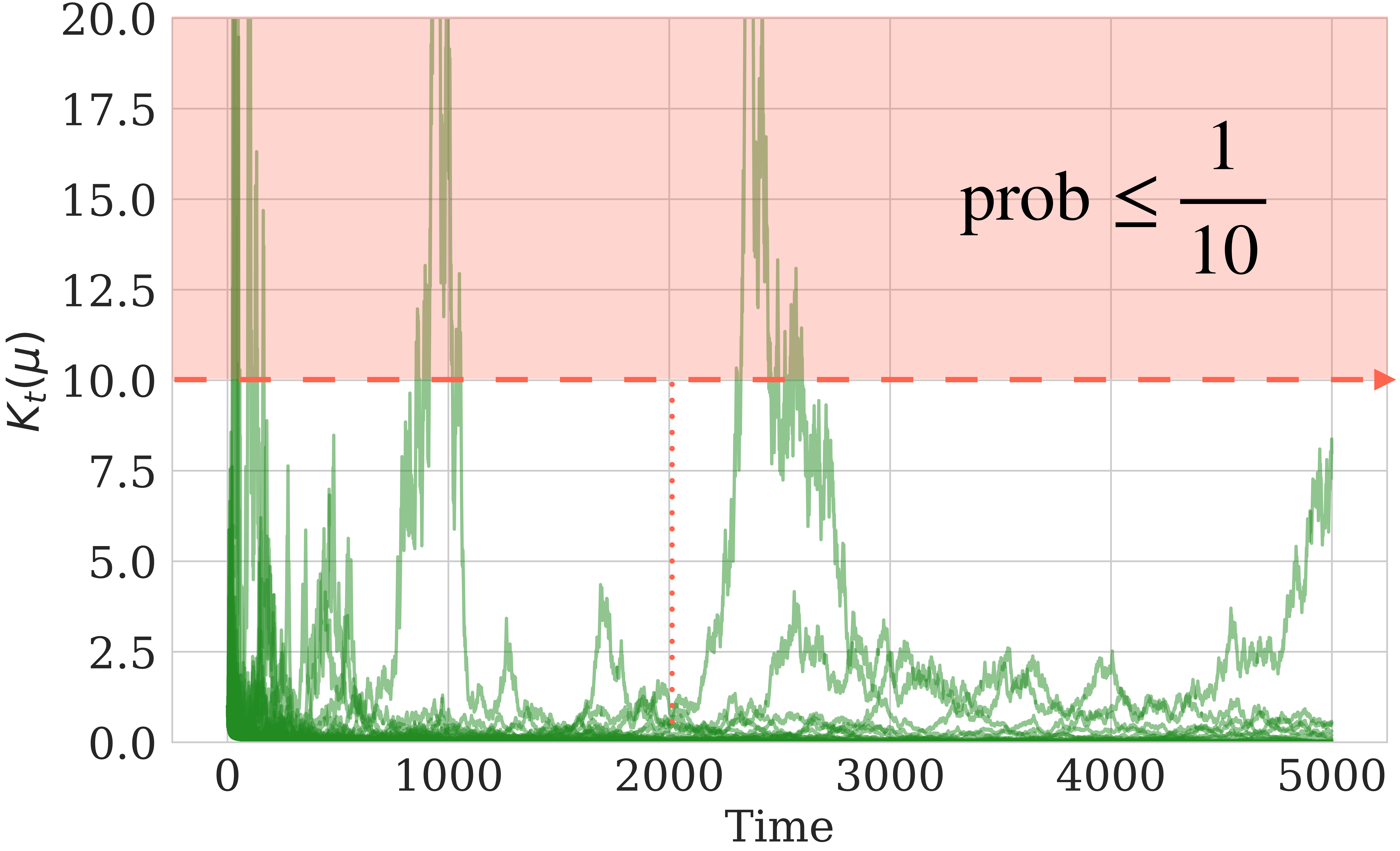

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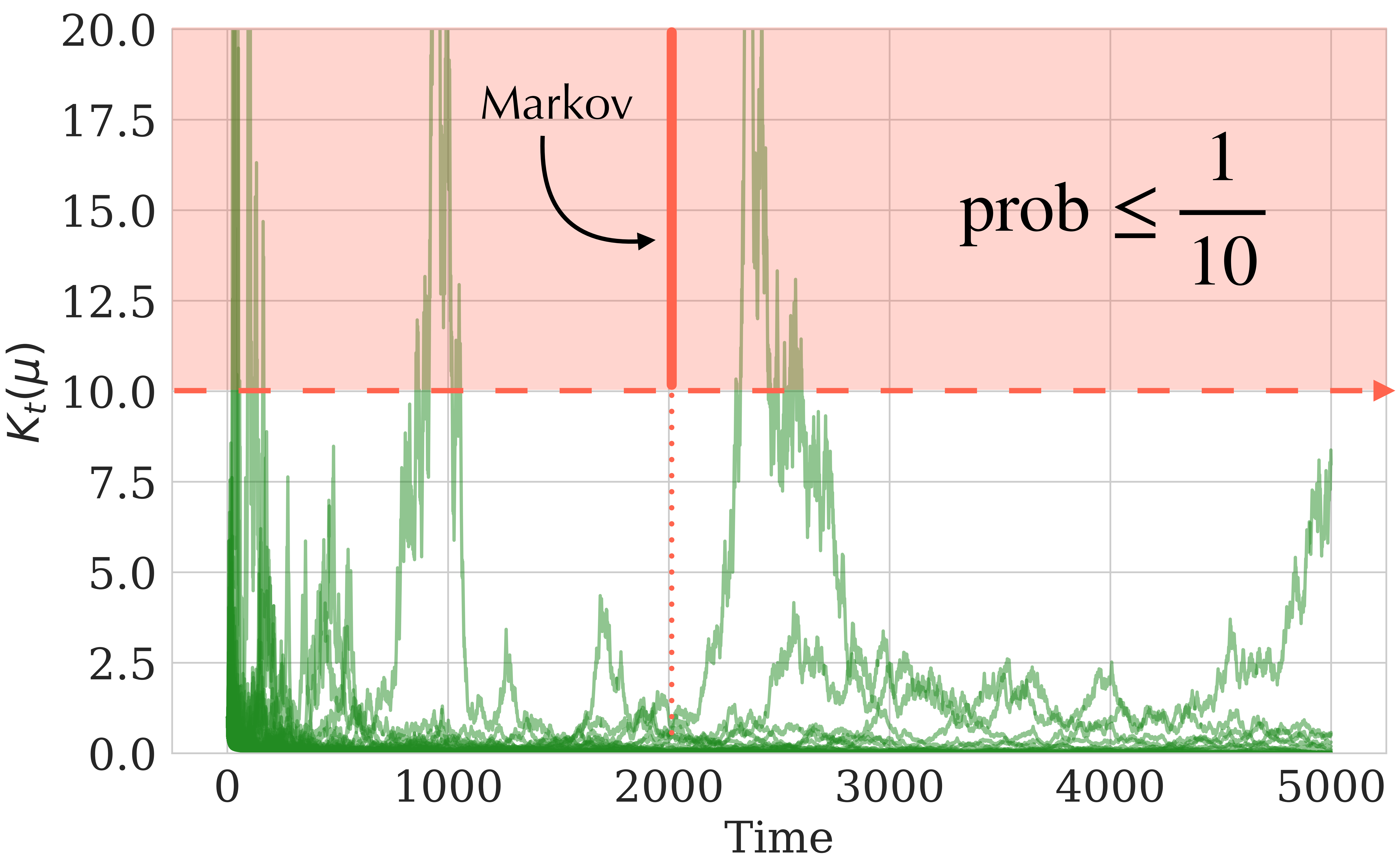

“Invert” Ville’s inequality:

$$C_t := \left\{ m \in [0,1] : K_t(m) < \frac{1}{\alpha} \right\}.$$

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$(C_t)_{t=1}^{\infty}$ forms a $(1-\alpha)$-***confidence sequence.***

Detour: confidence sequences

Detour: confidence sequences

Herbert Robbins, 1960s/70s

+ Siegmund, Darling, & Lai

Confidence sequence

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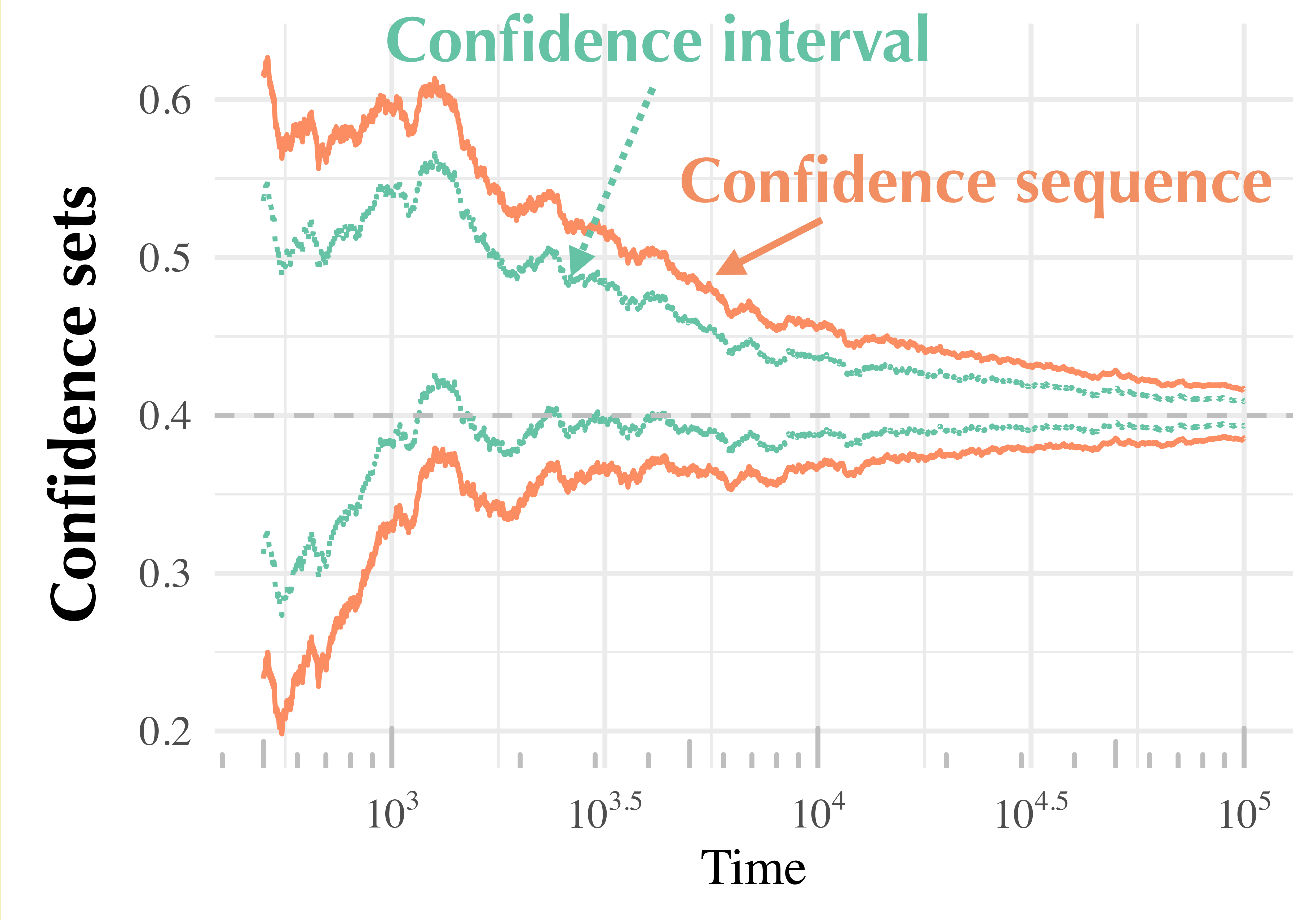
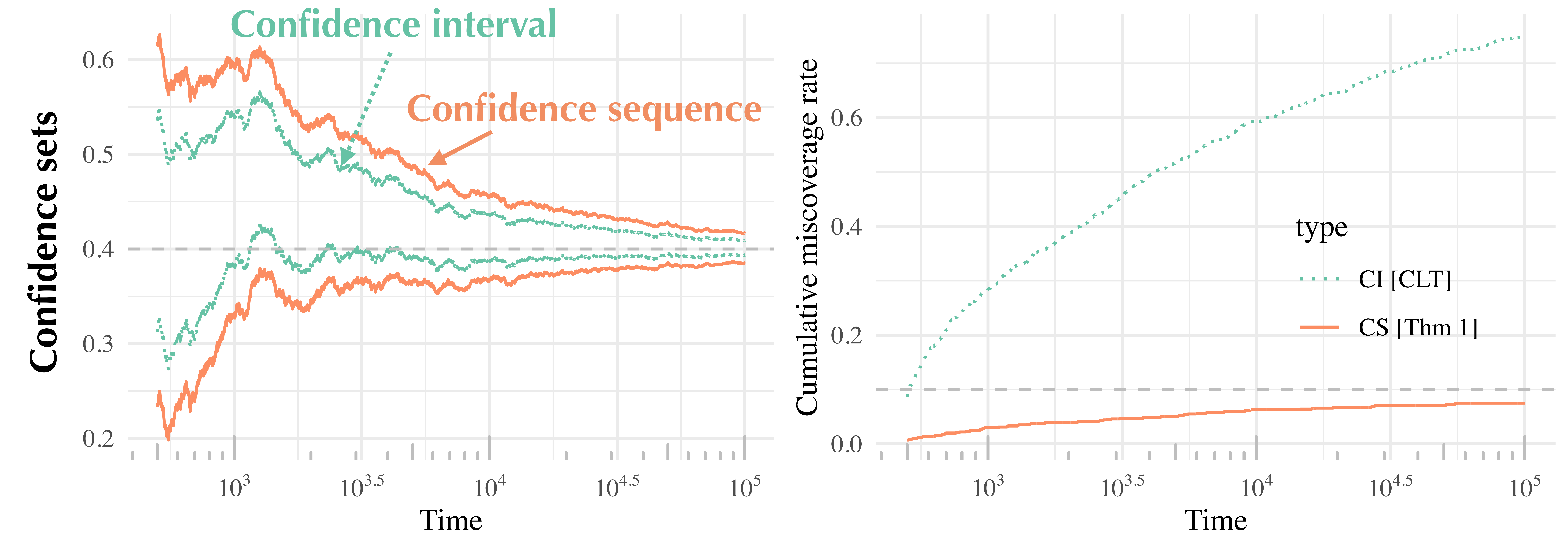

Confidence interval
Confidence sequence
Confidence sets
0.6
0.5
0.4
0.3
0.2
10^3
$10^{3.5}$
10^4
$10^{4.5}$
10^5
Time



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Time
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CI [CLT]
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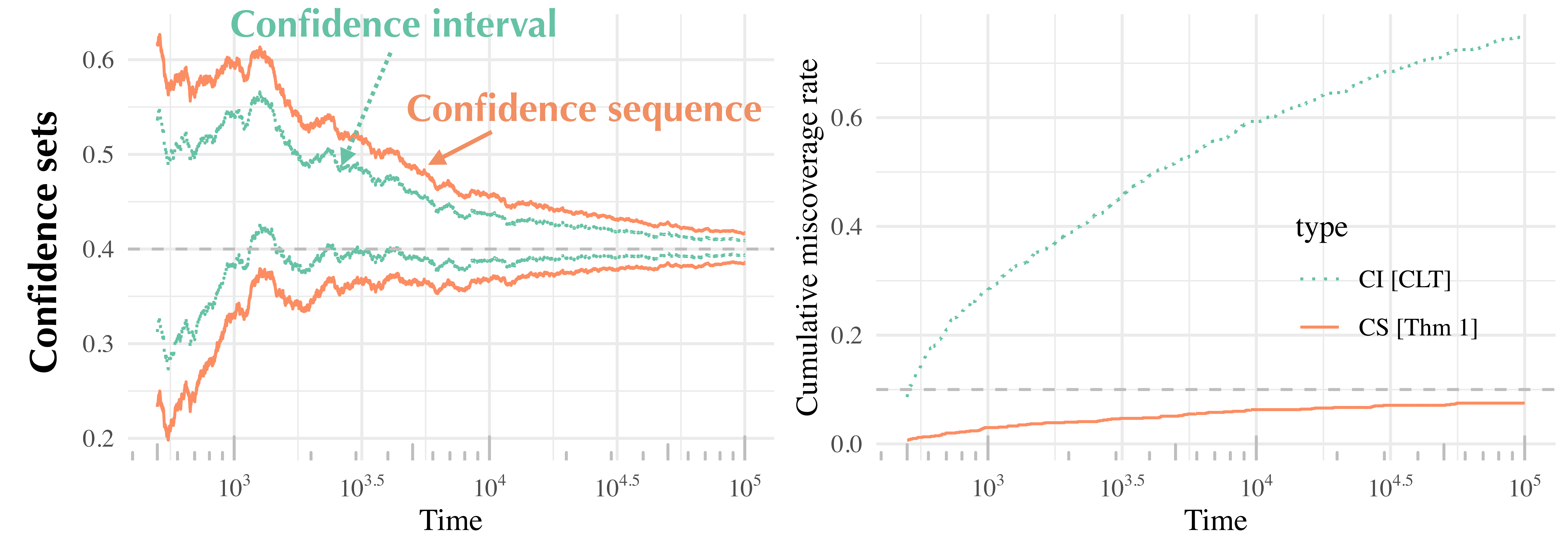

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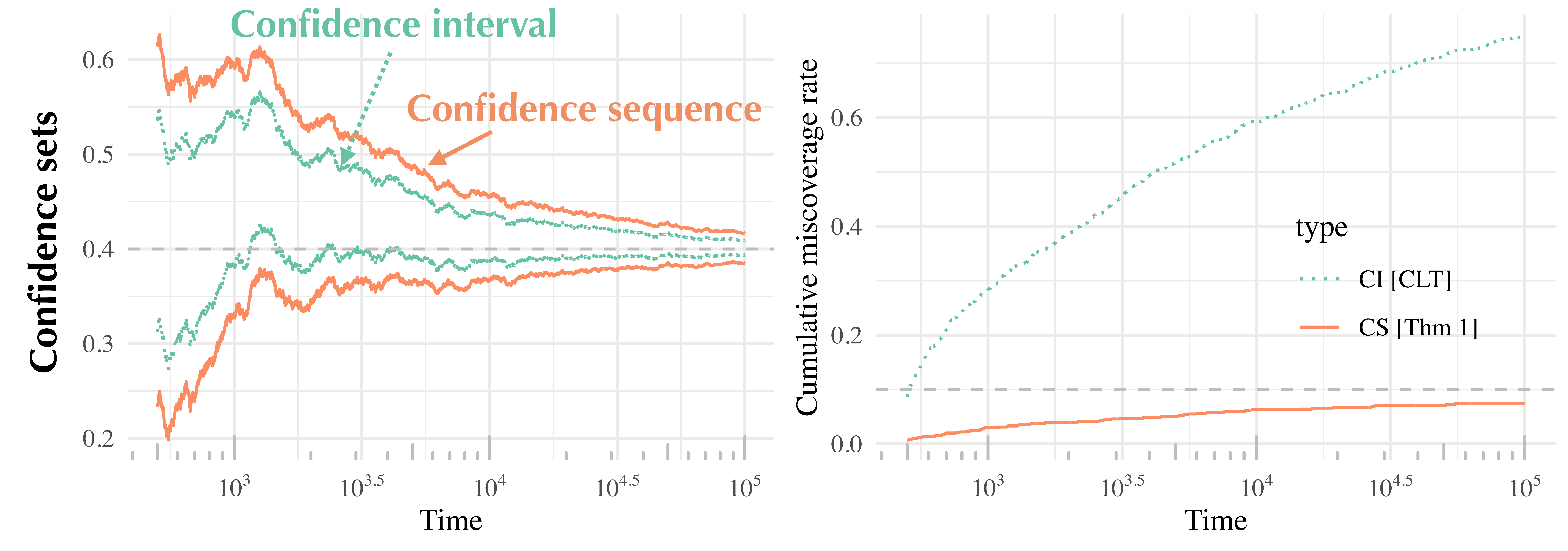

Confidence intervals are valid *at a single sample size*.
Confidence sequences are valid *at all sample sizes simultaneously*.

Back to confidence sequences for means of bounded random variables

Our confidence sequence:

$$C_t := \left\{ m \in [0,1] : \prod_{i=1}^{t} (1+\lambda_i(m) \cdot (X_i - m)) < \frac{1}{\alpha} \right\}$$

is valid for any λ_i but what is a smart choice?

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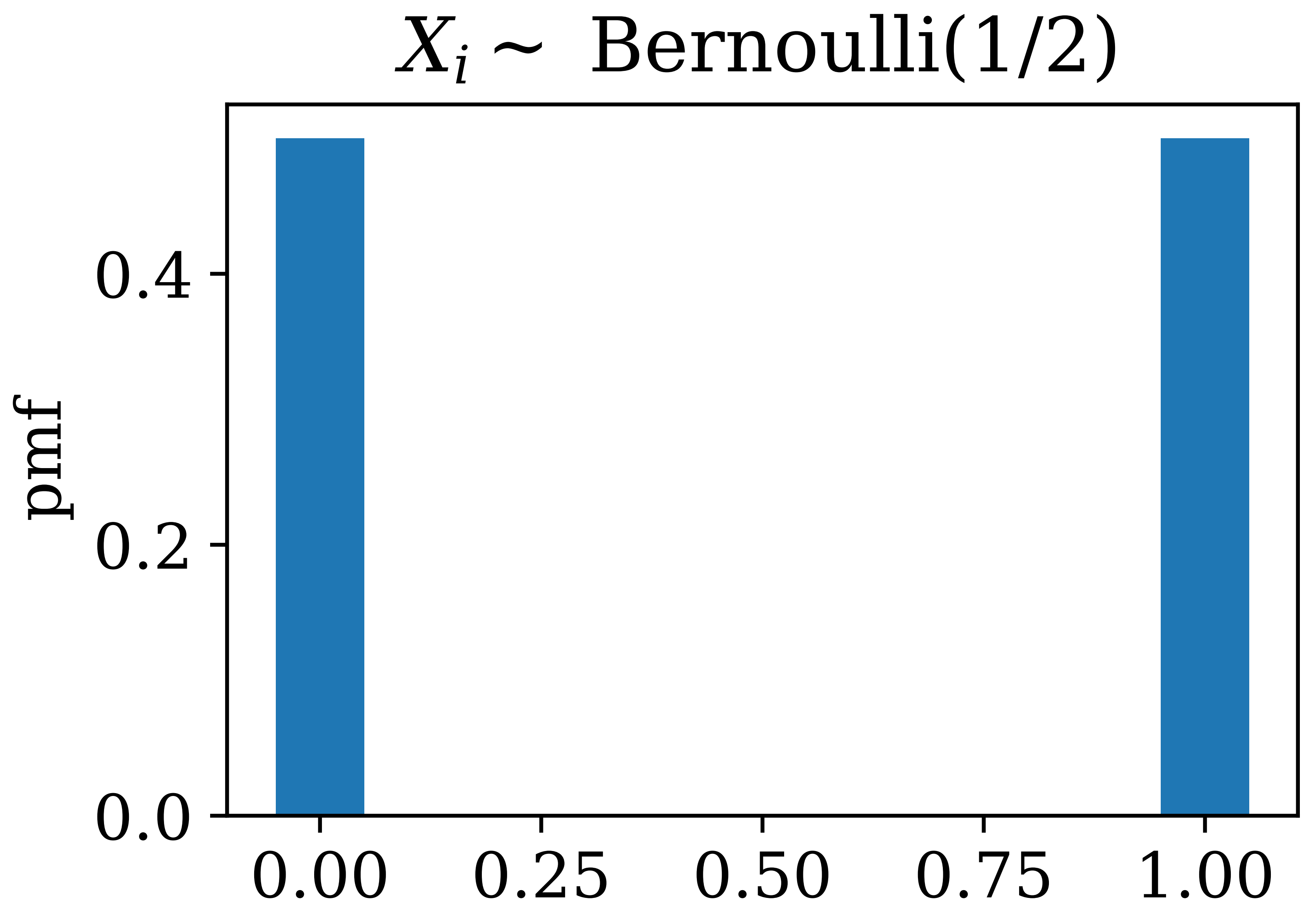
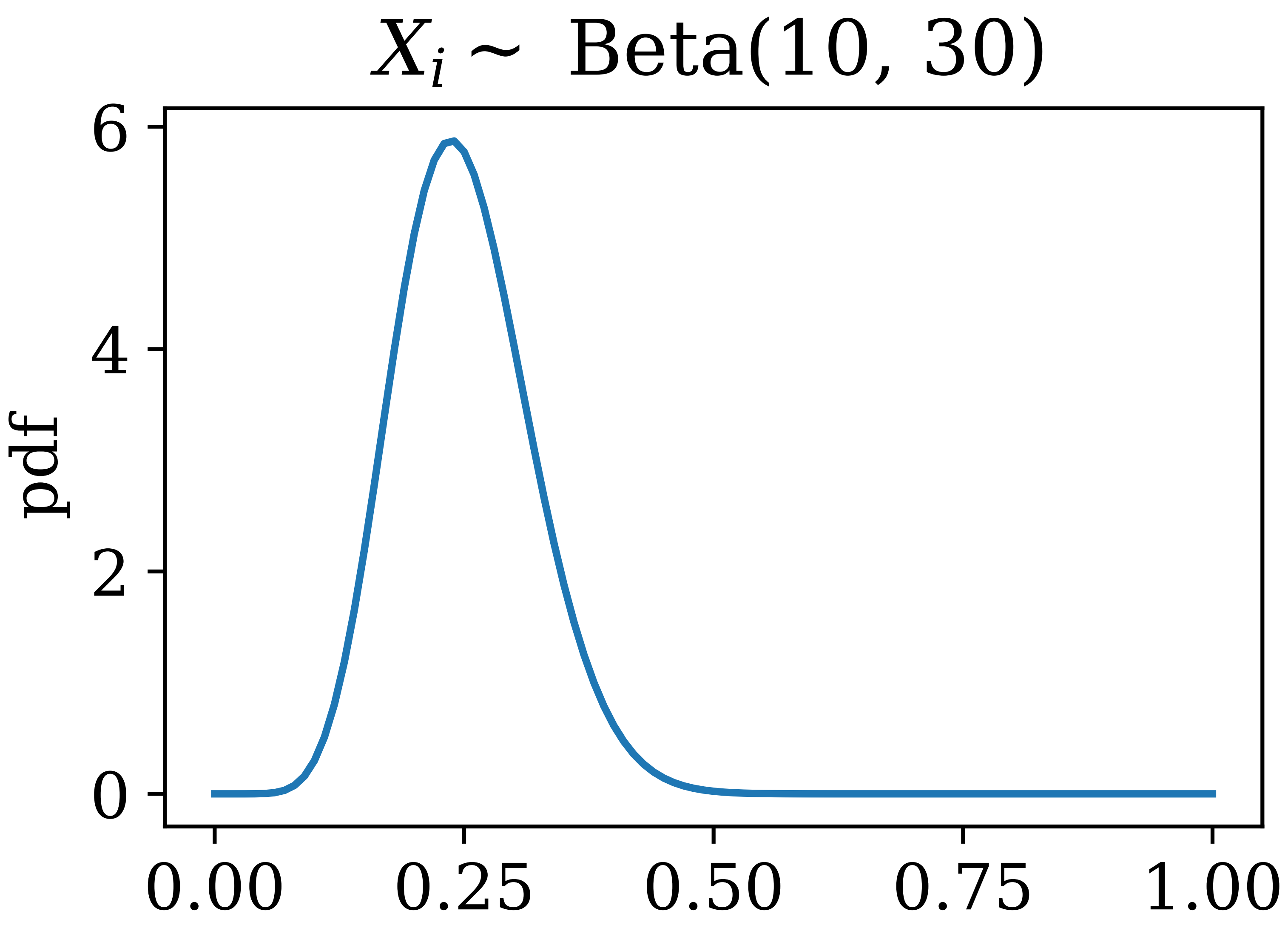


$\sigma^2 \approx 0.0046$



PM-H [Prop 1]
PM-EB [Thm 2]
Hedged [Thm 3]
hgKelly [Prop 4]
BANCO [JO19]
A-Bentkus [KZ21]
Bernoulli [HRMS20]

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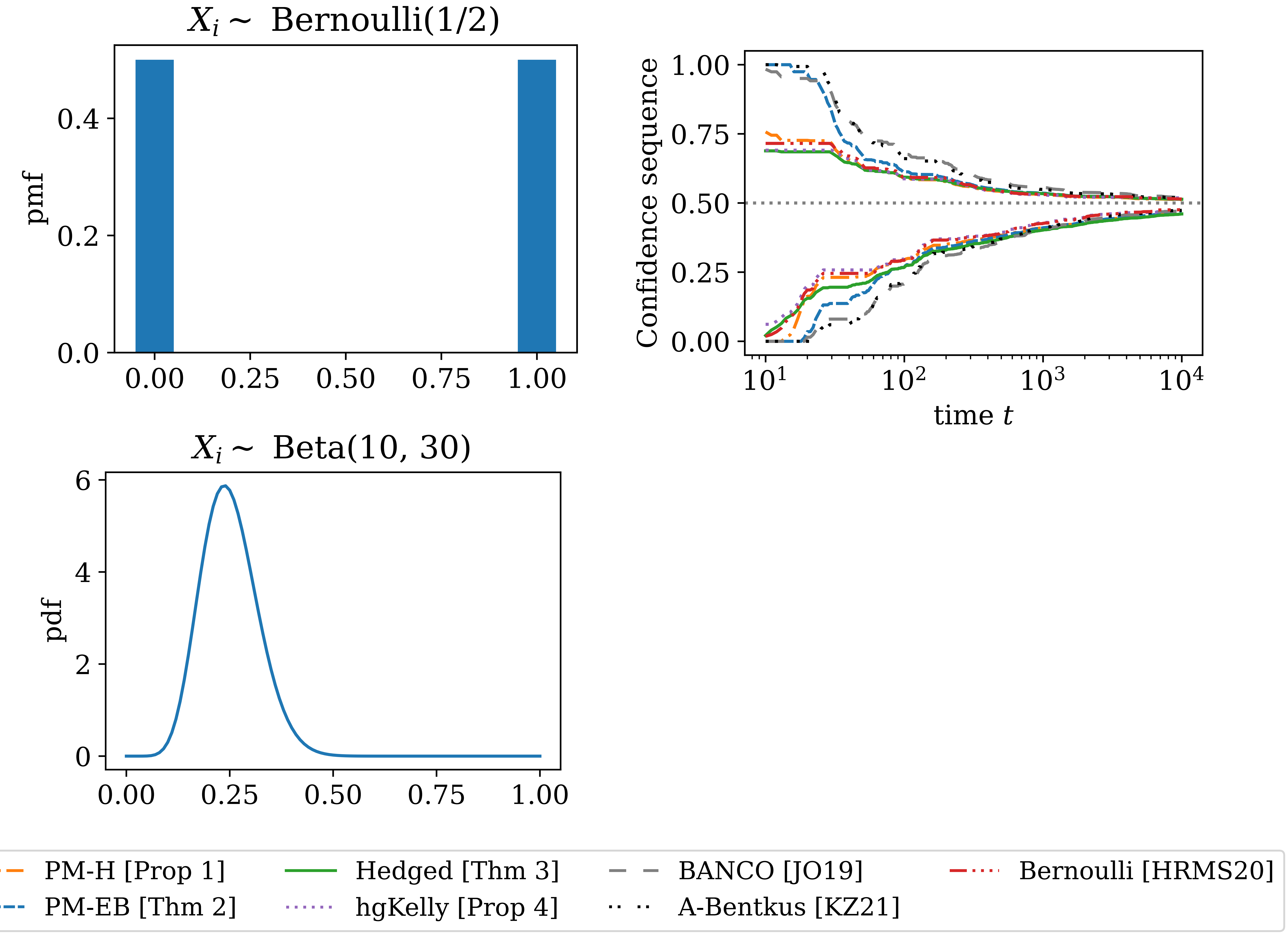

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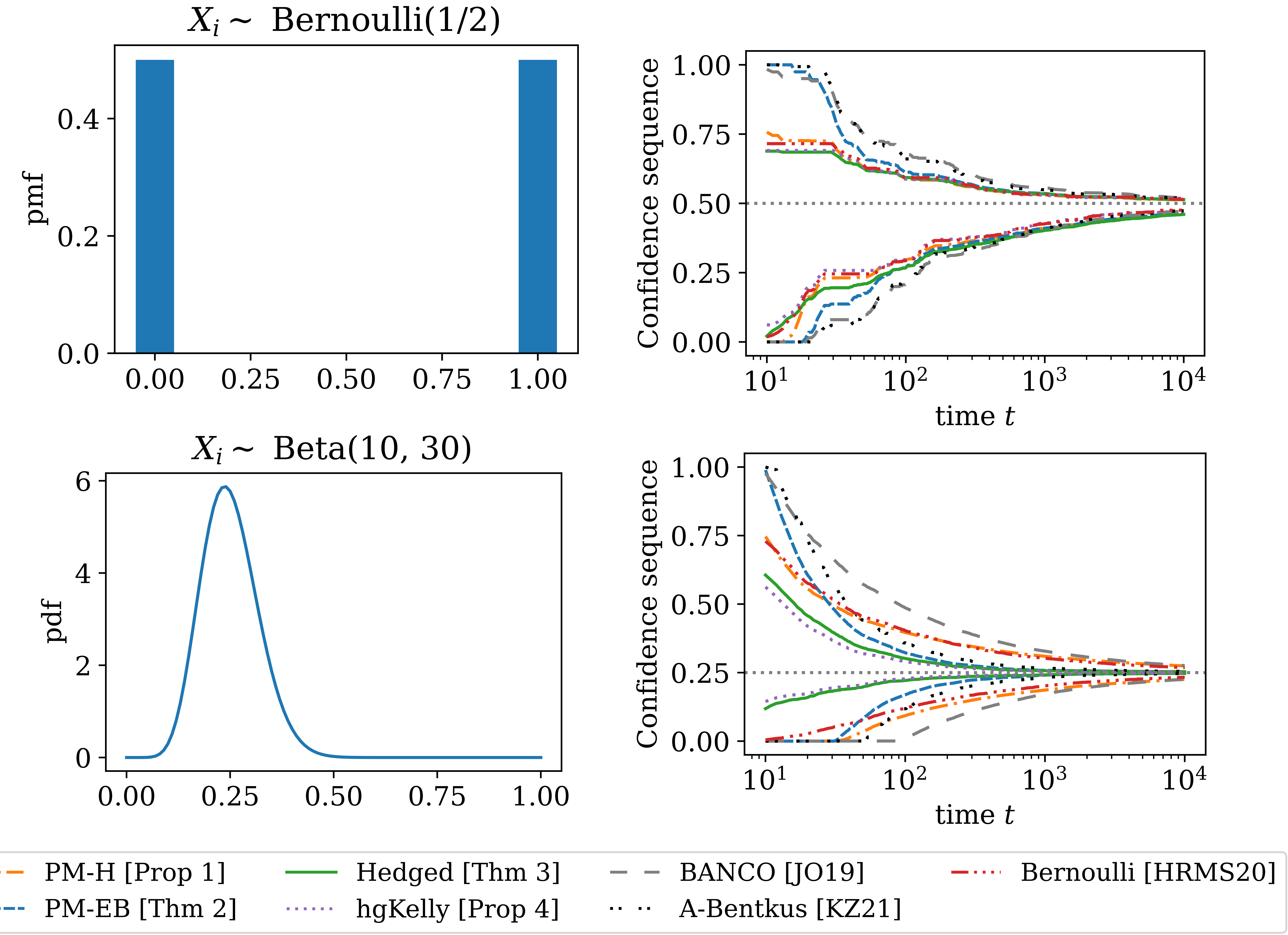

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+ Closed-form empirical Bernstein confidence sets*
+ Extensions to sampling without replacement*

Thank you.

ian.waudbysmith.com

Fixed-time confidence intervals

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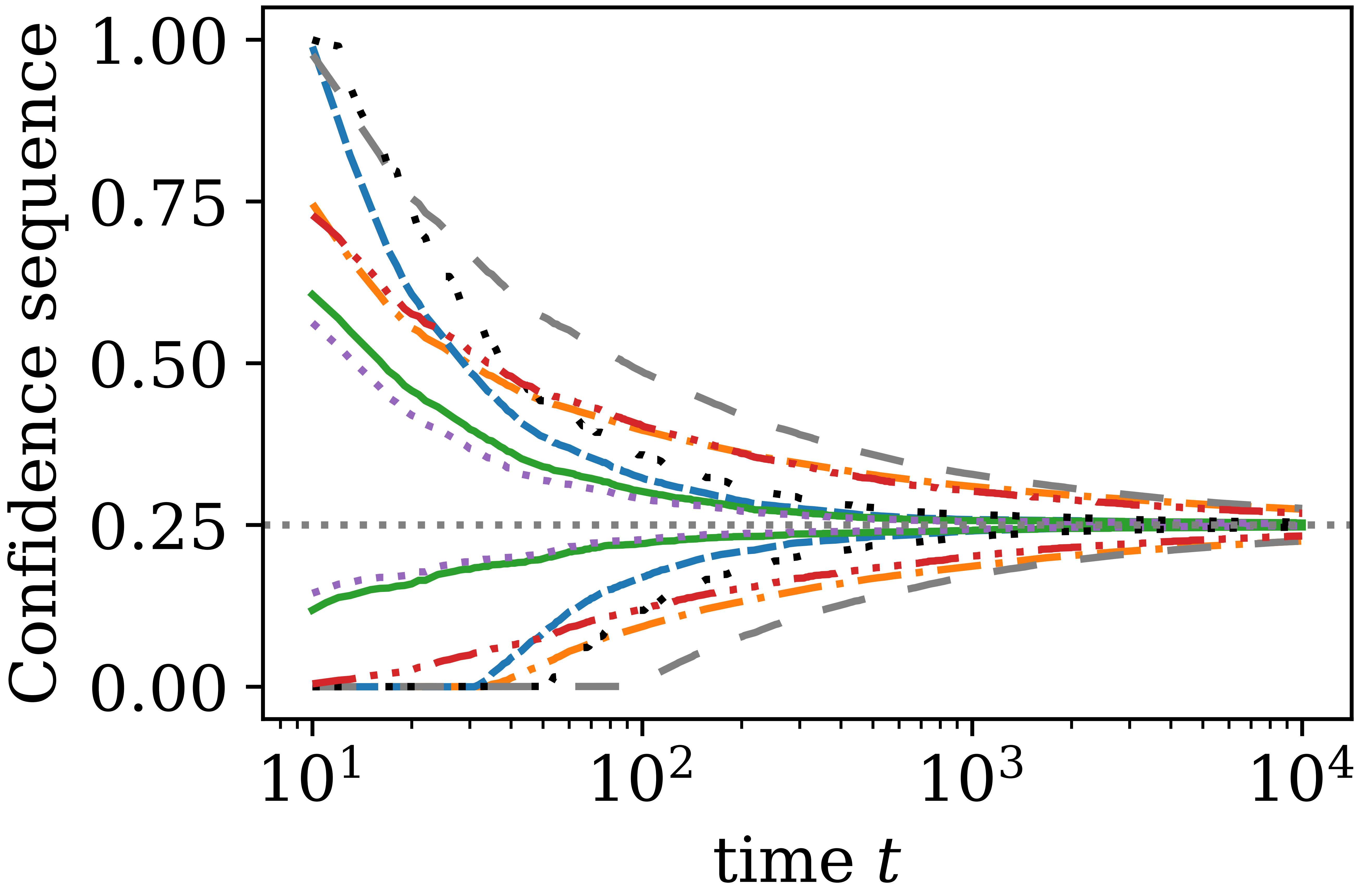

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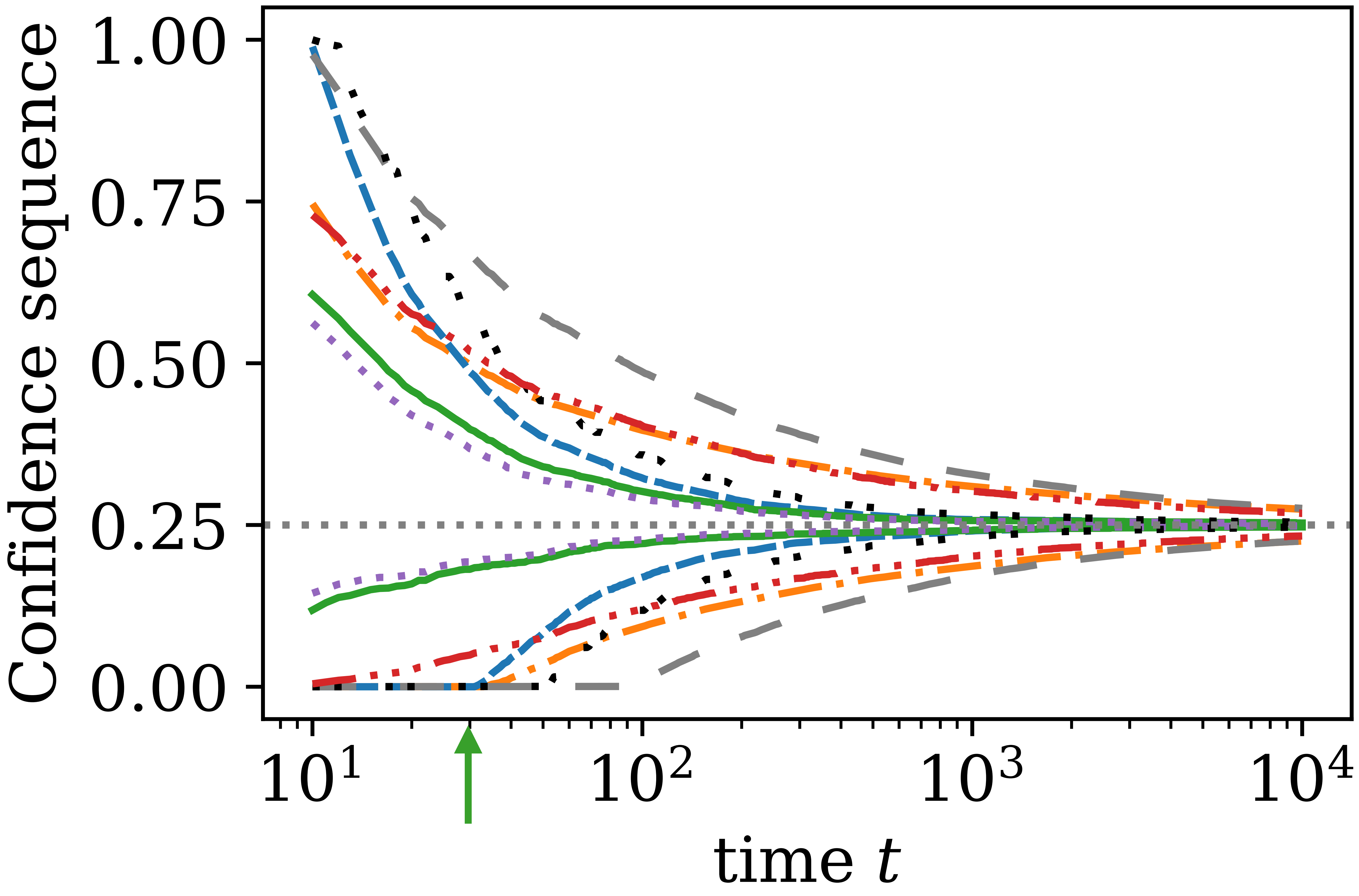

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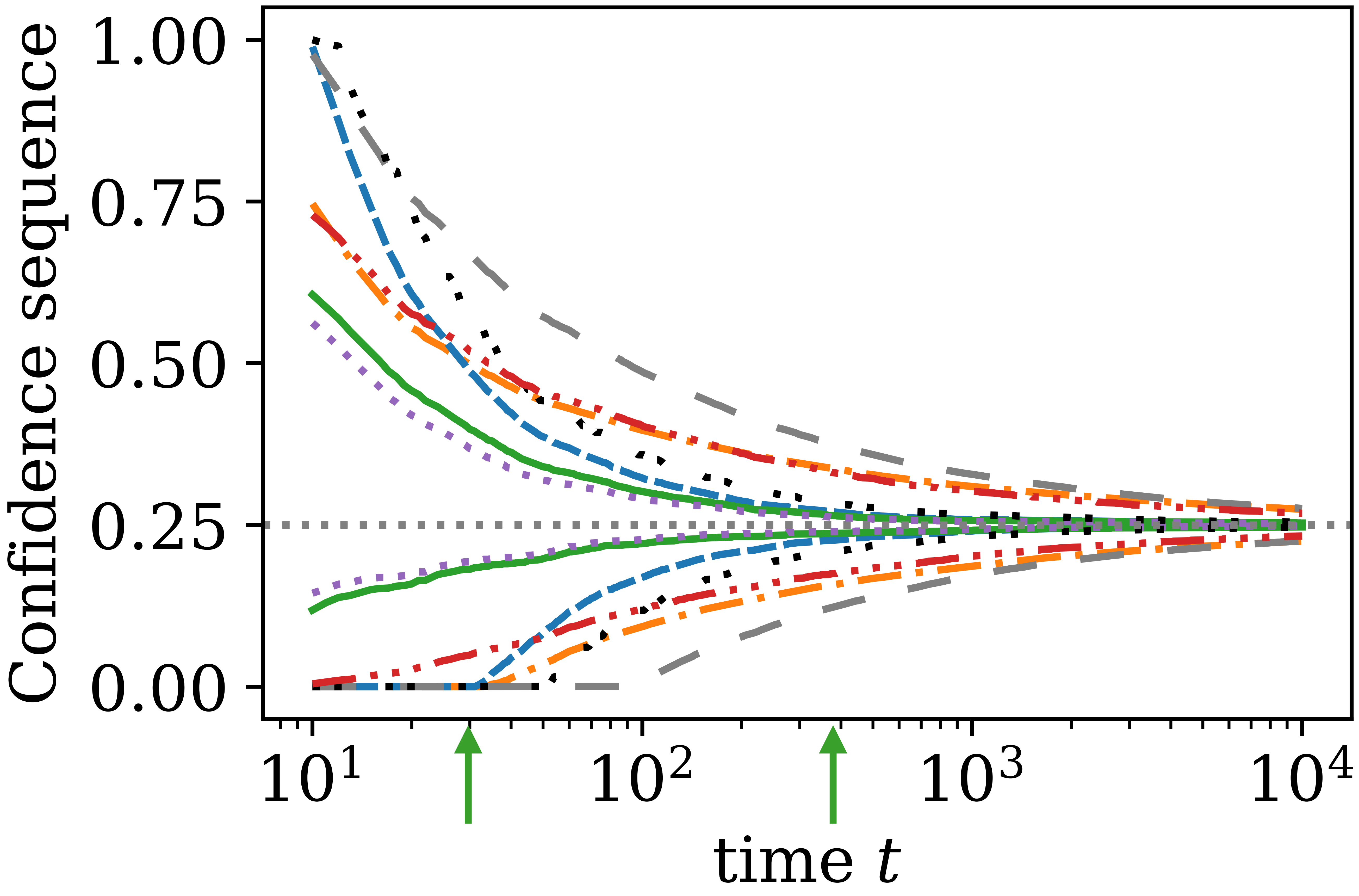

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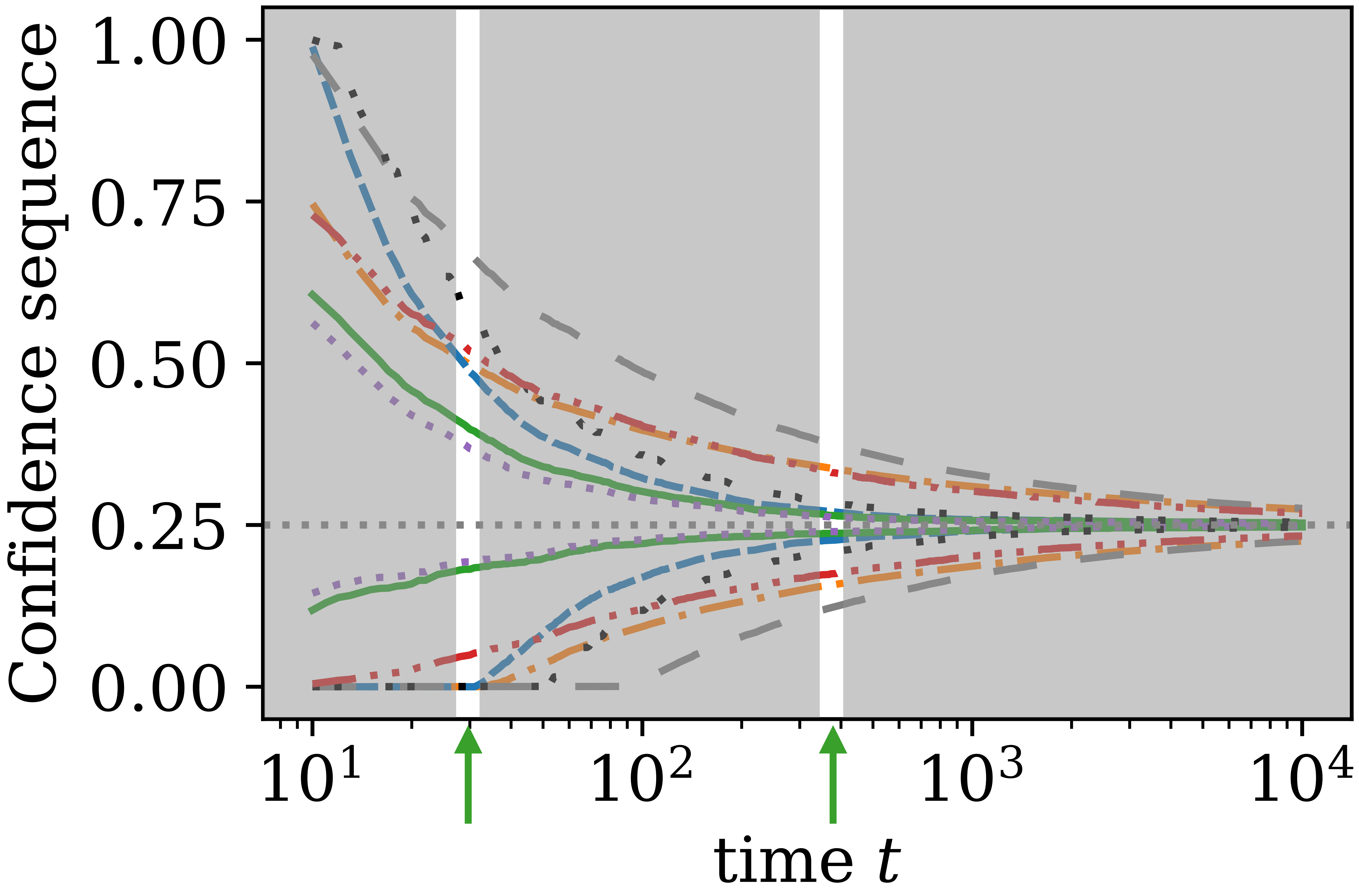

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$$K_n^{\pm}(m) := \max\left\{ \frac{1}{2}\prod_{i=1}^{n}(1 + \lambda_i^{+} \cdot (X_i - m)), \frac{1}{2}\prod_{i=1}^{n}(1 - \lambda_i^{-} \cdot (X_i - m)) \right\}$$

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data-dependent tuning parameters without sample splitting!

So, given $X_1, \ldots, X_n$ bounded,

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There's one more modification we can make to get **strictly** tighter confidence intervals!

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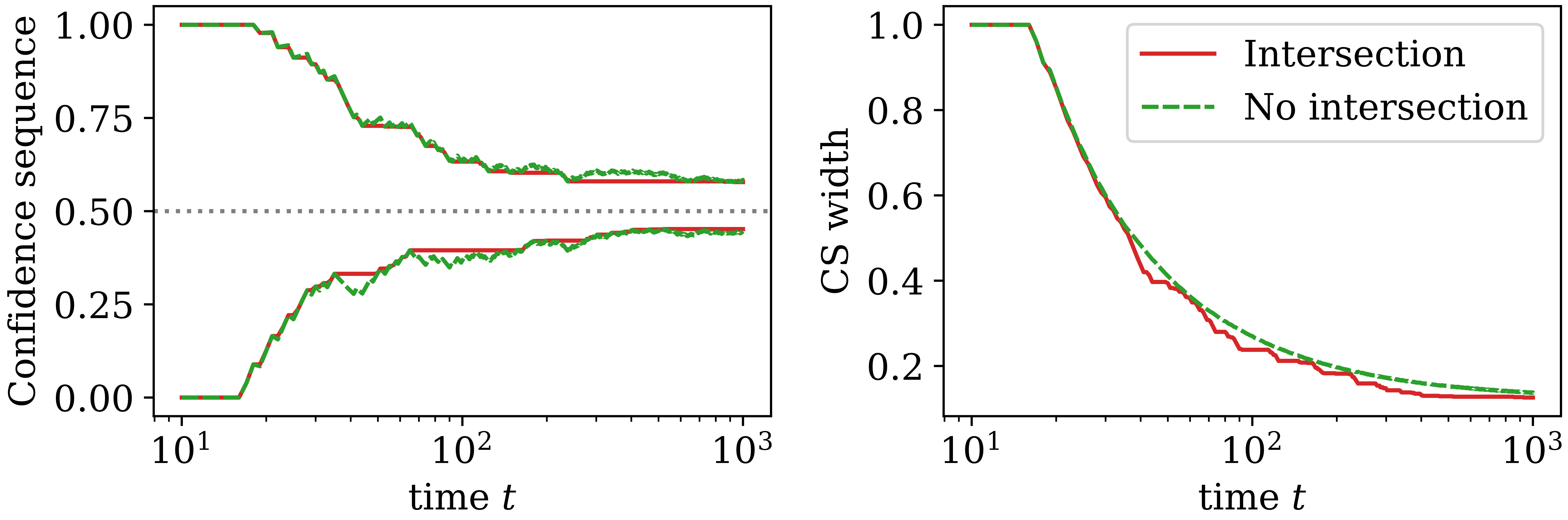

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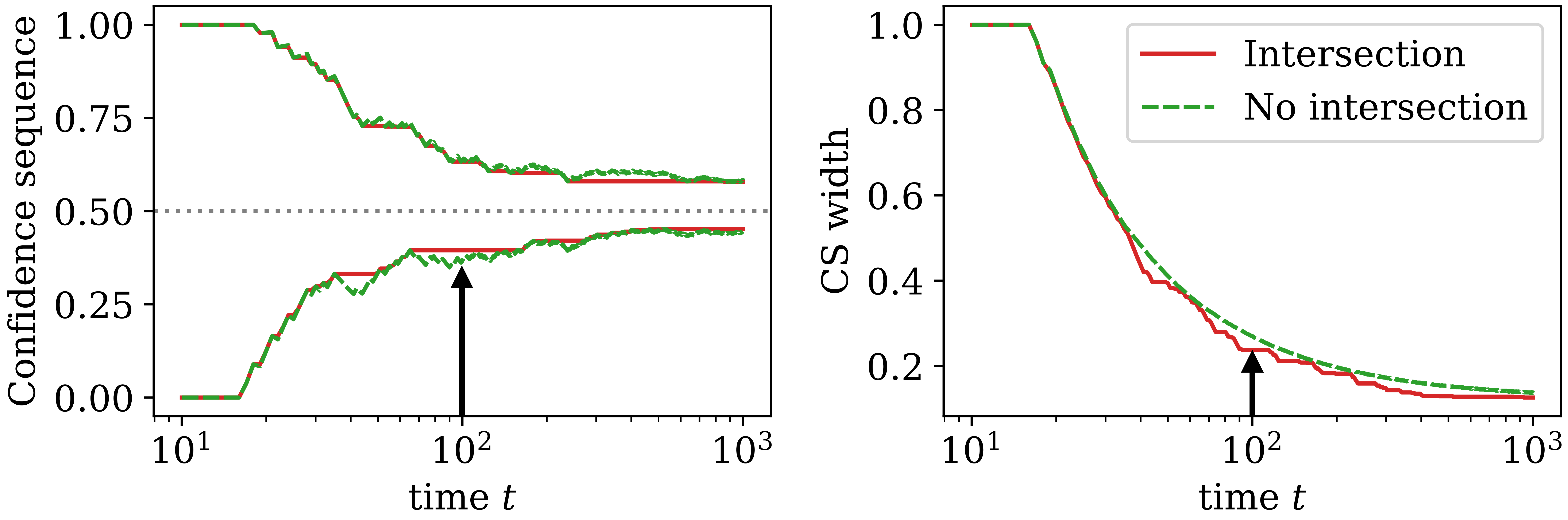

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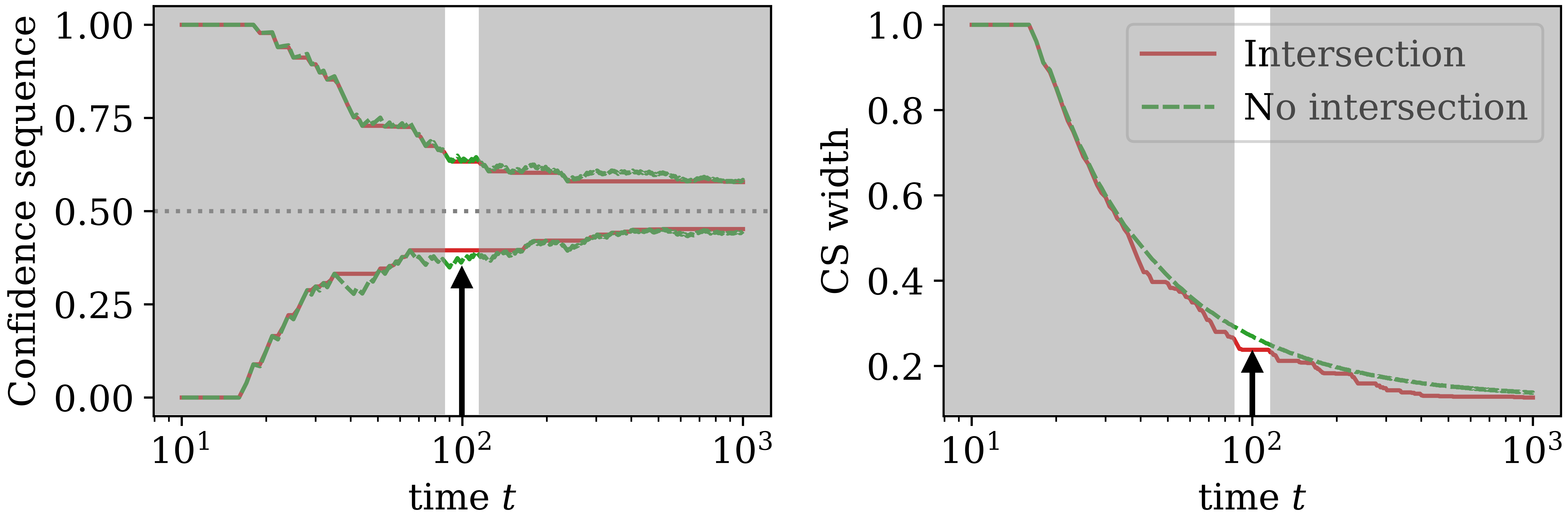

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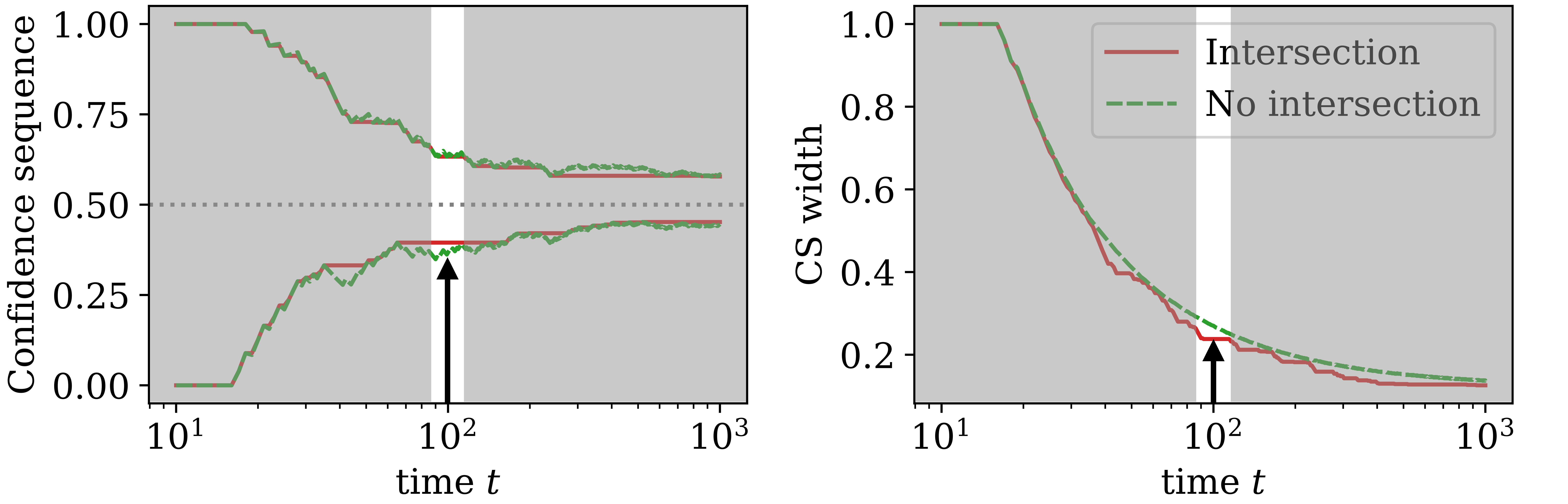

So, $\bigcap_{i \le n} C_i^{\pm}$ is a strict improvement over $C_n^{\pm}$ **for free!**

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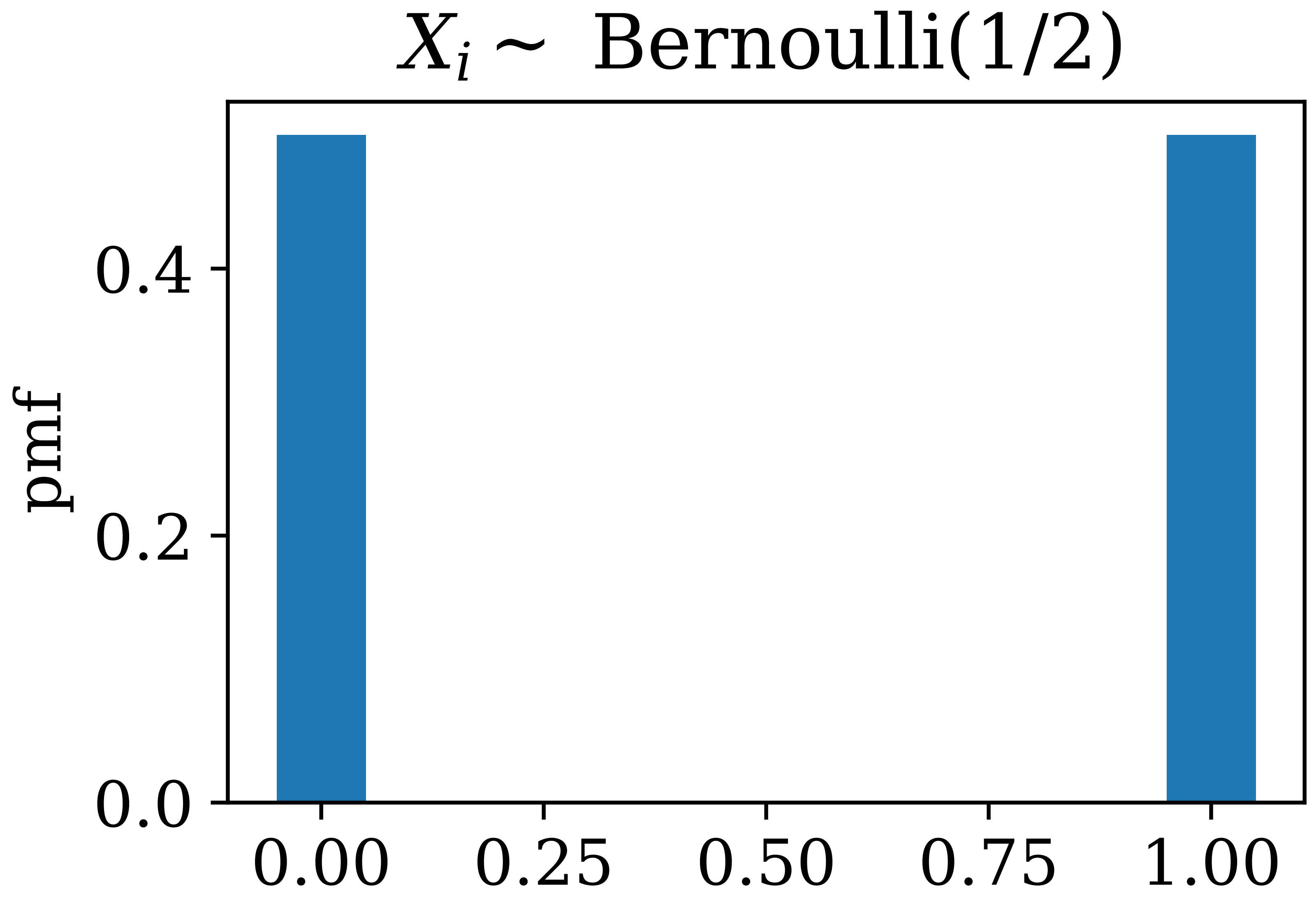

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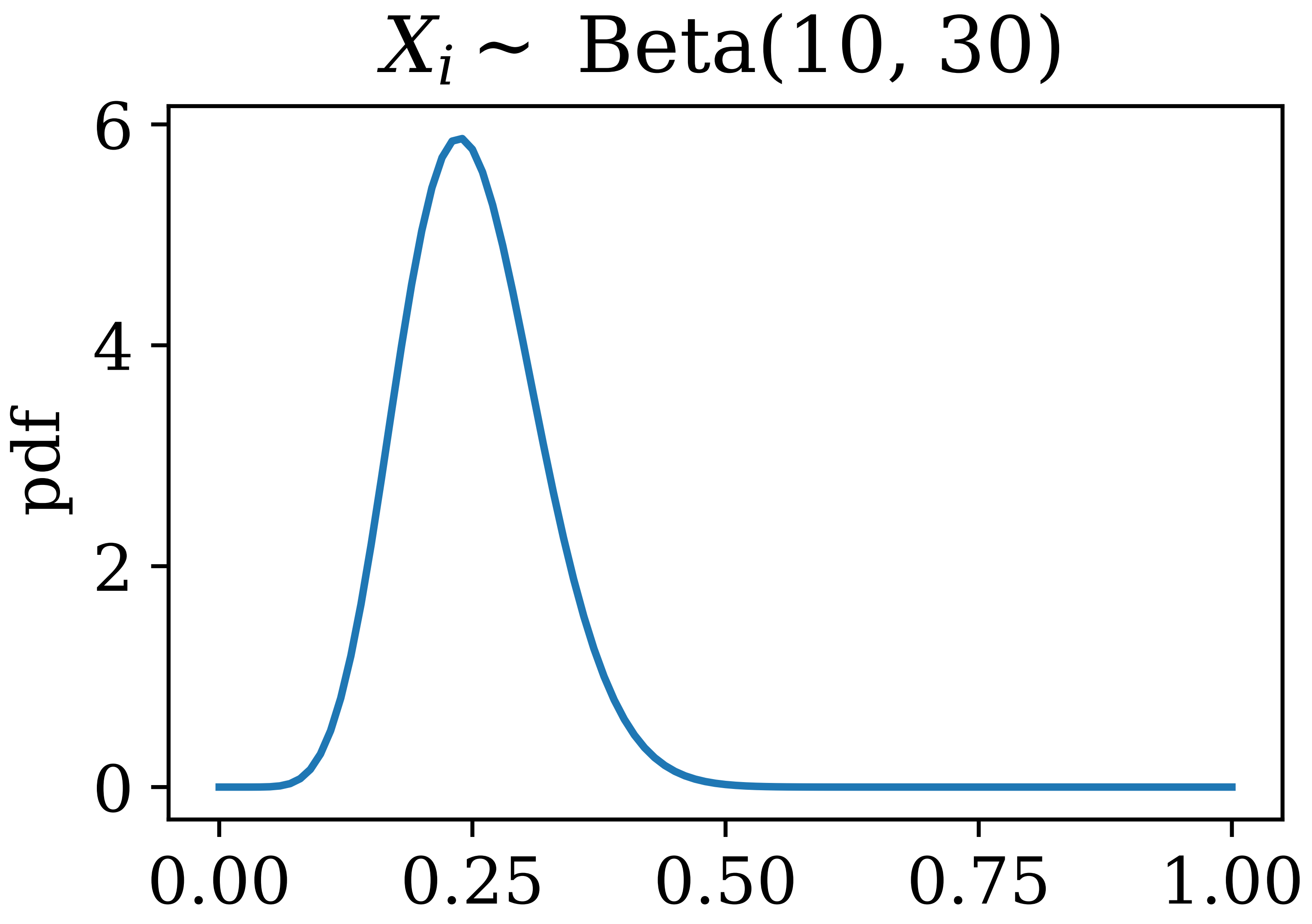

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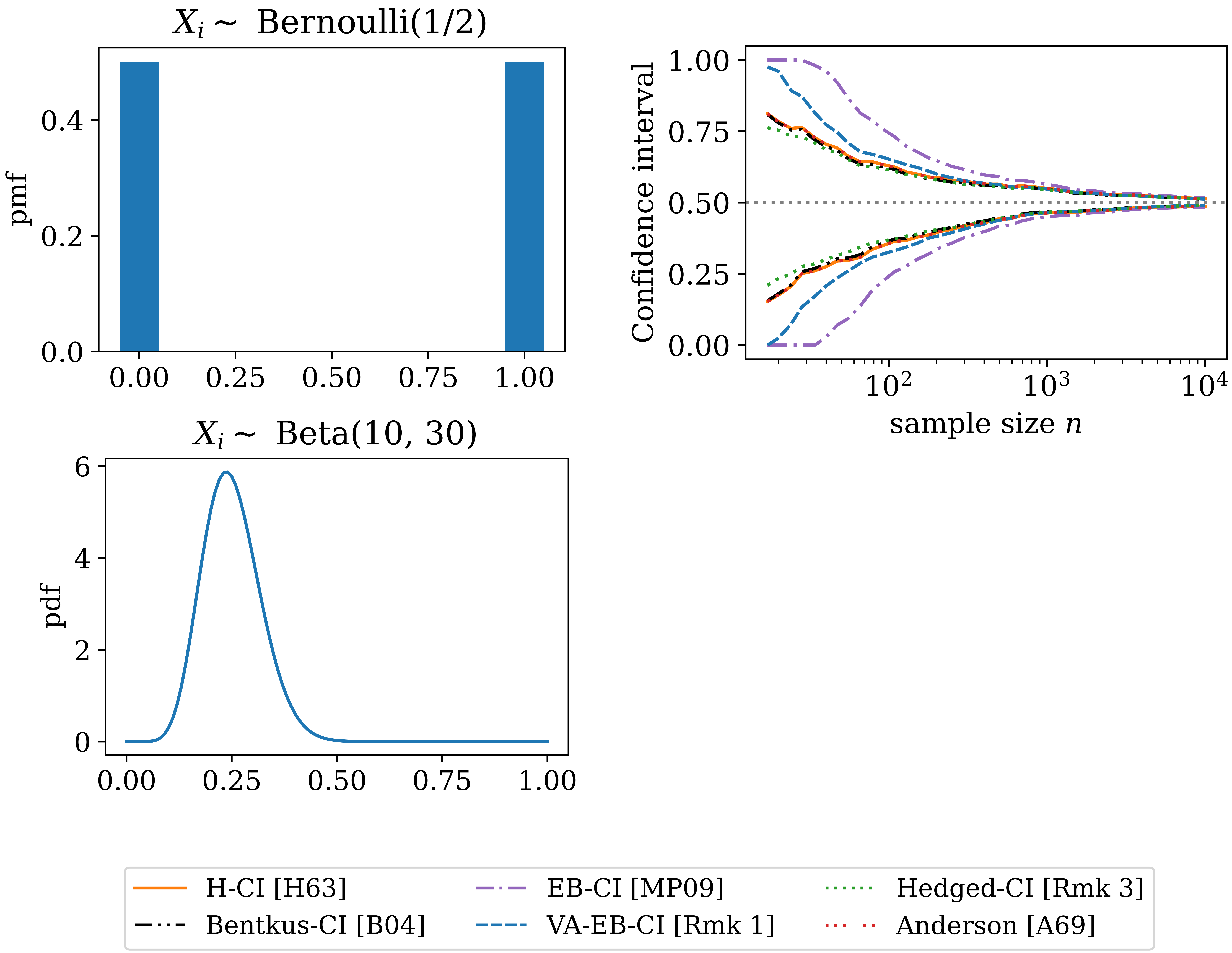

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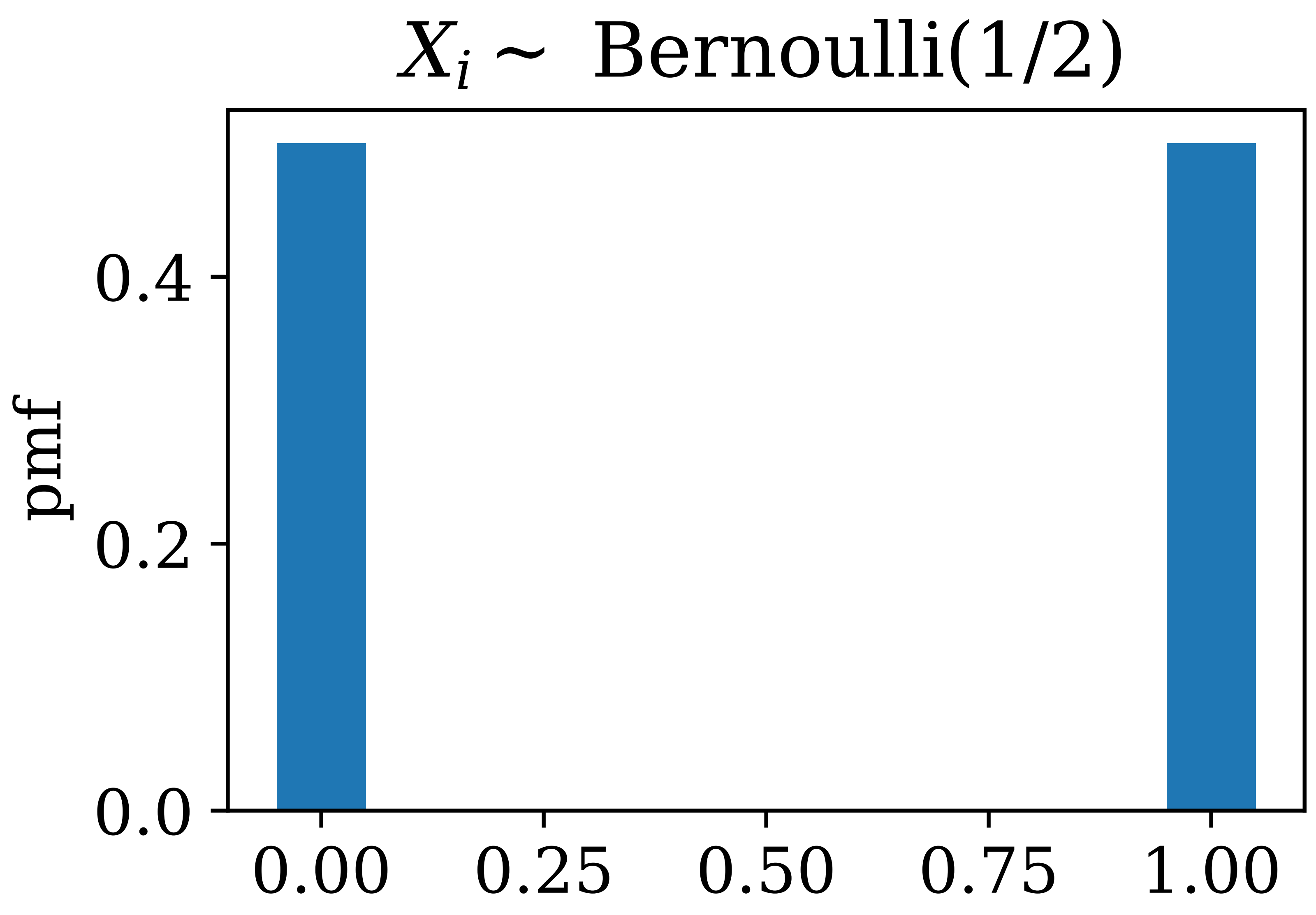

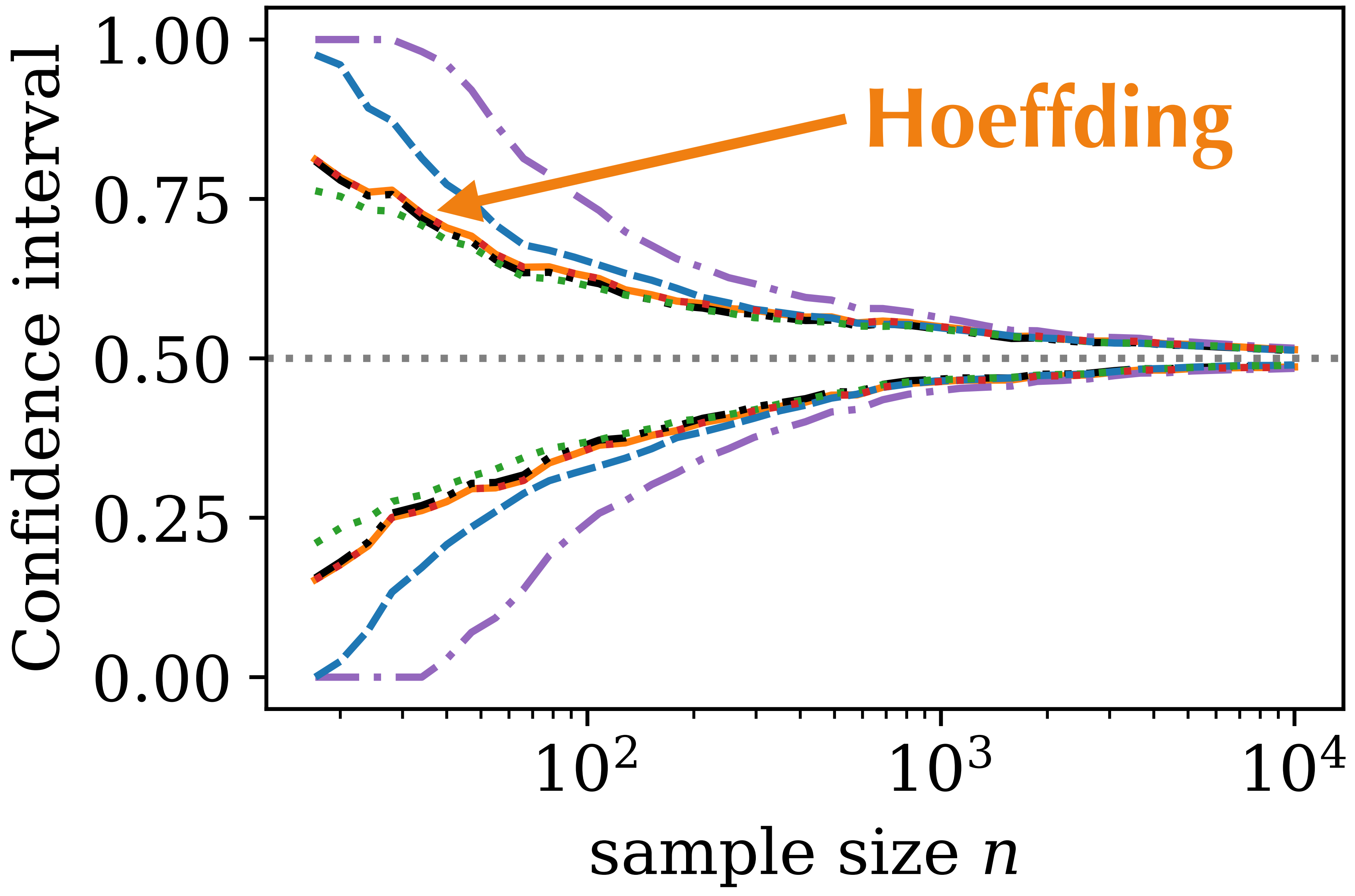

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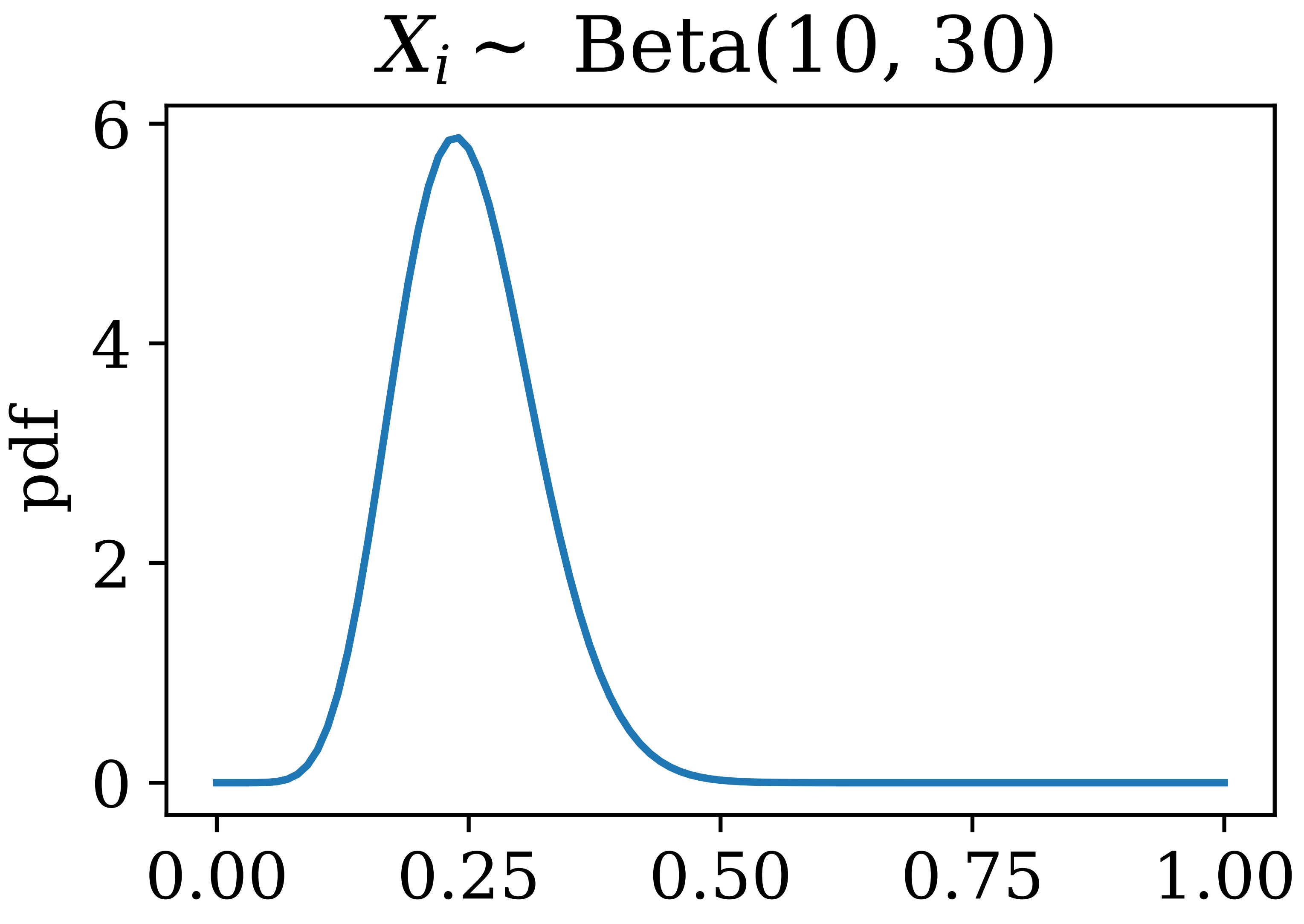

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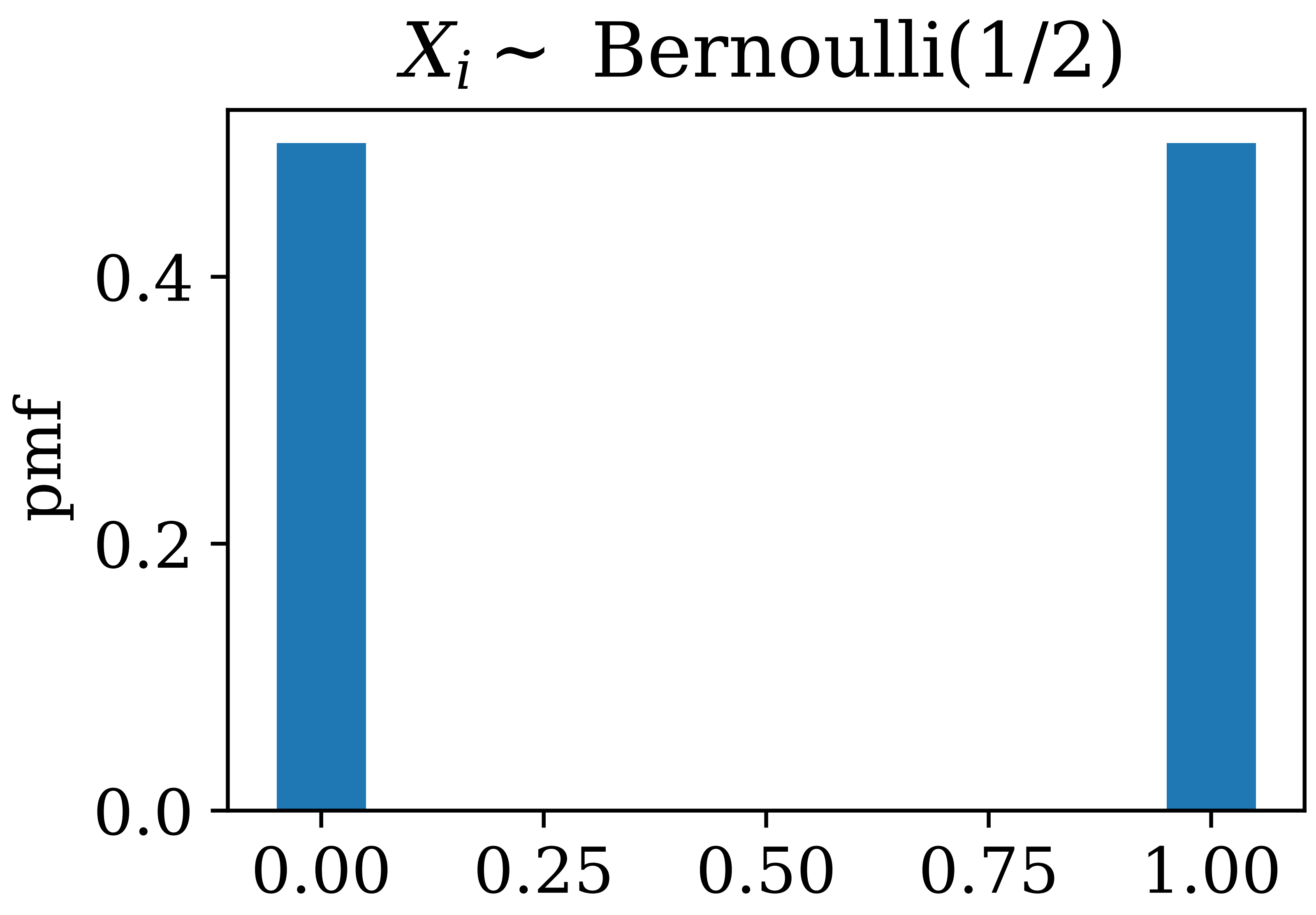

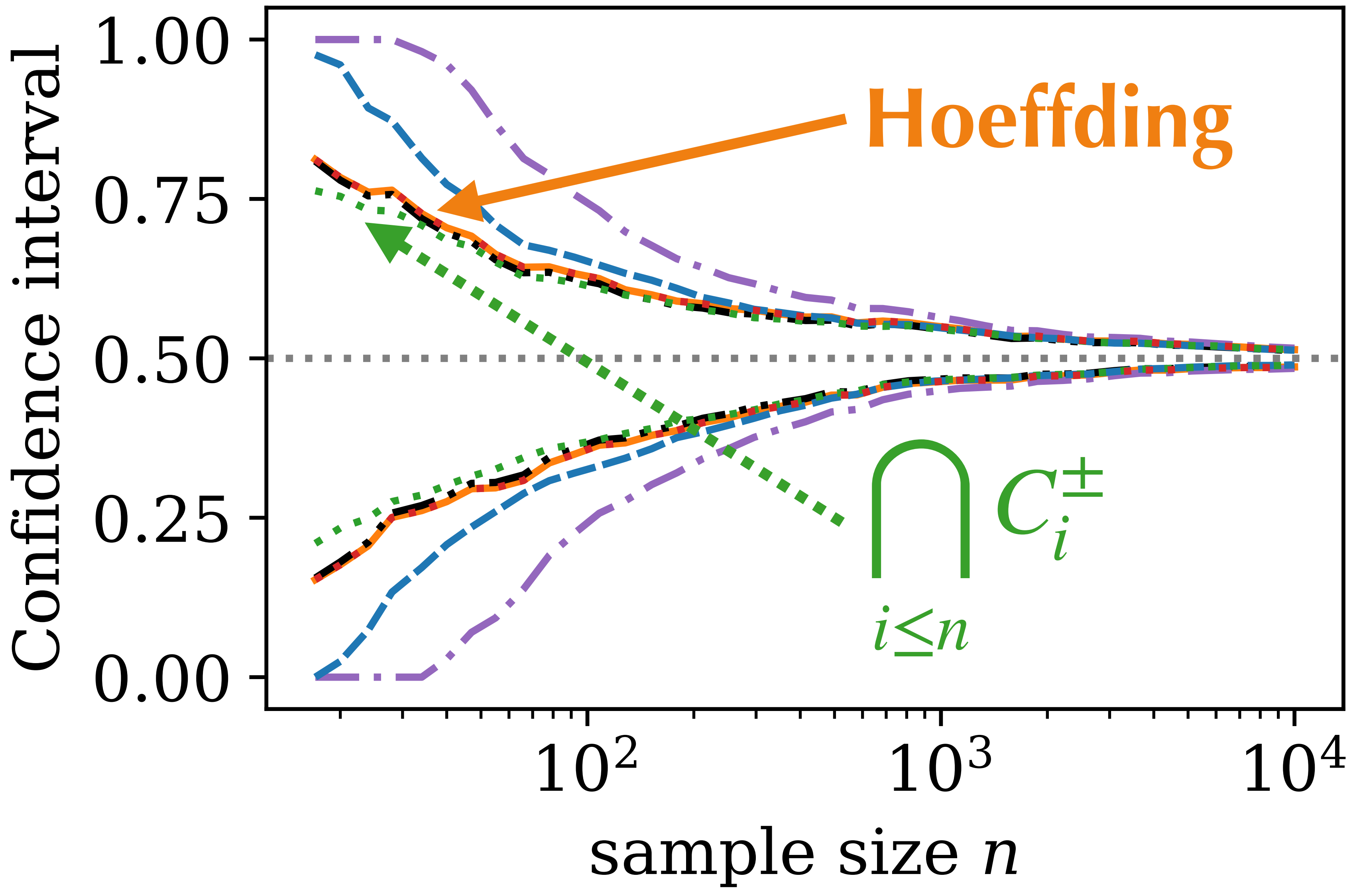

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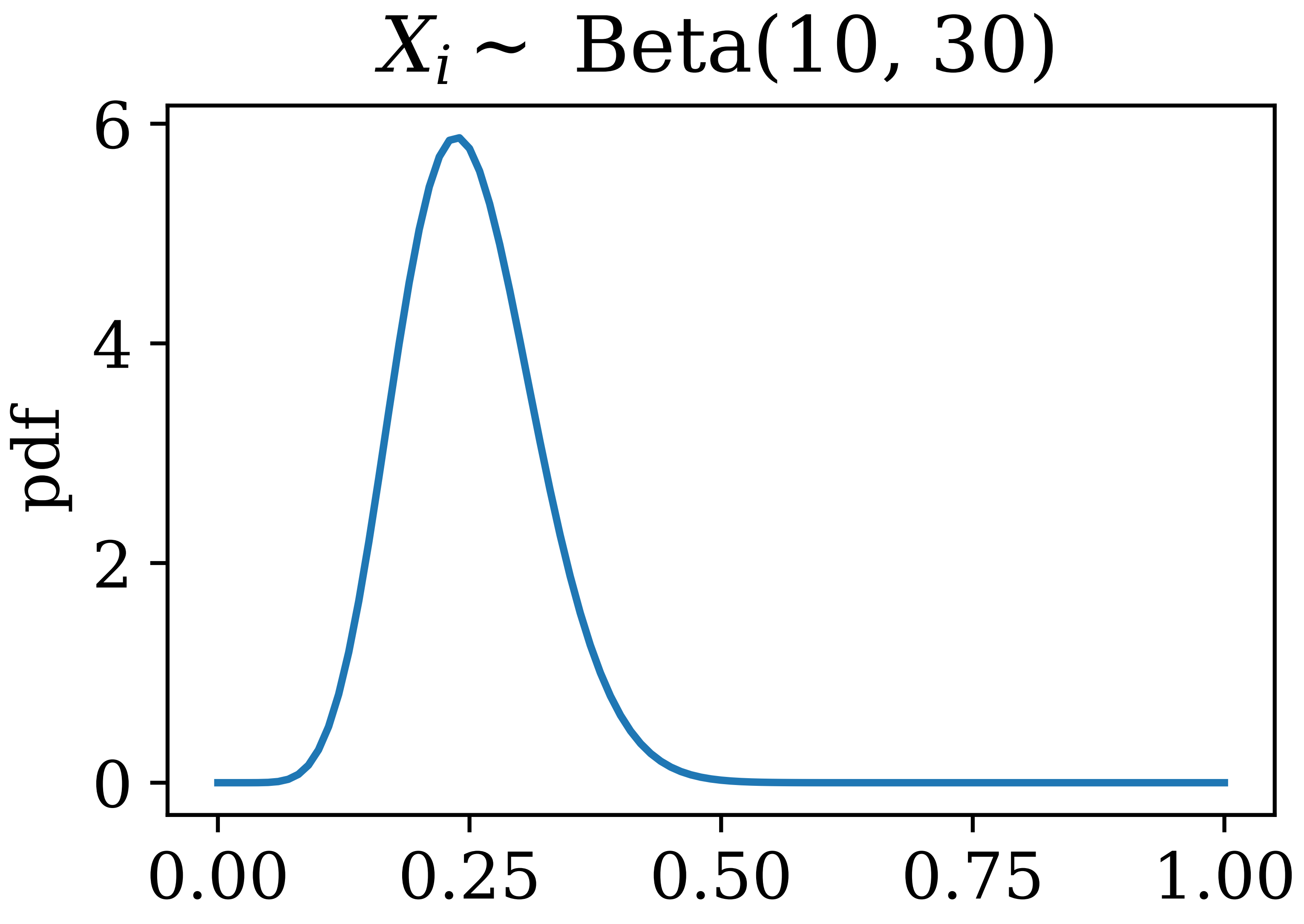

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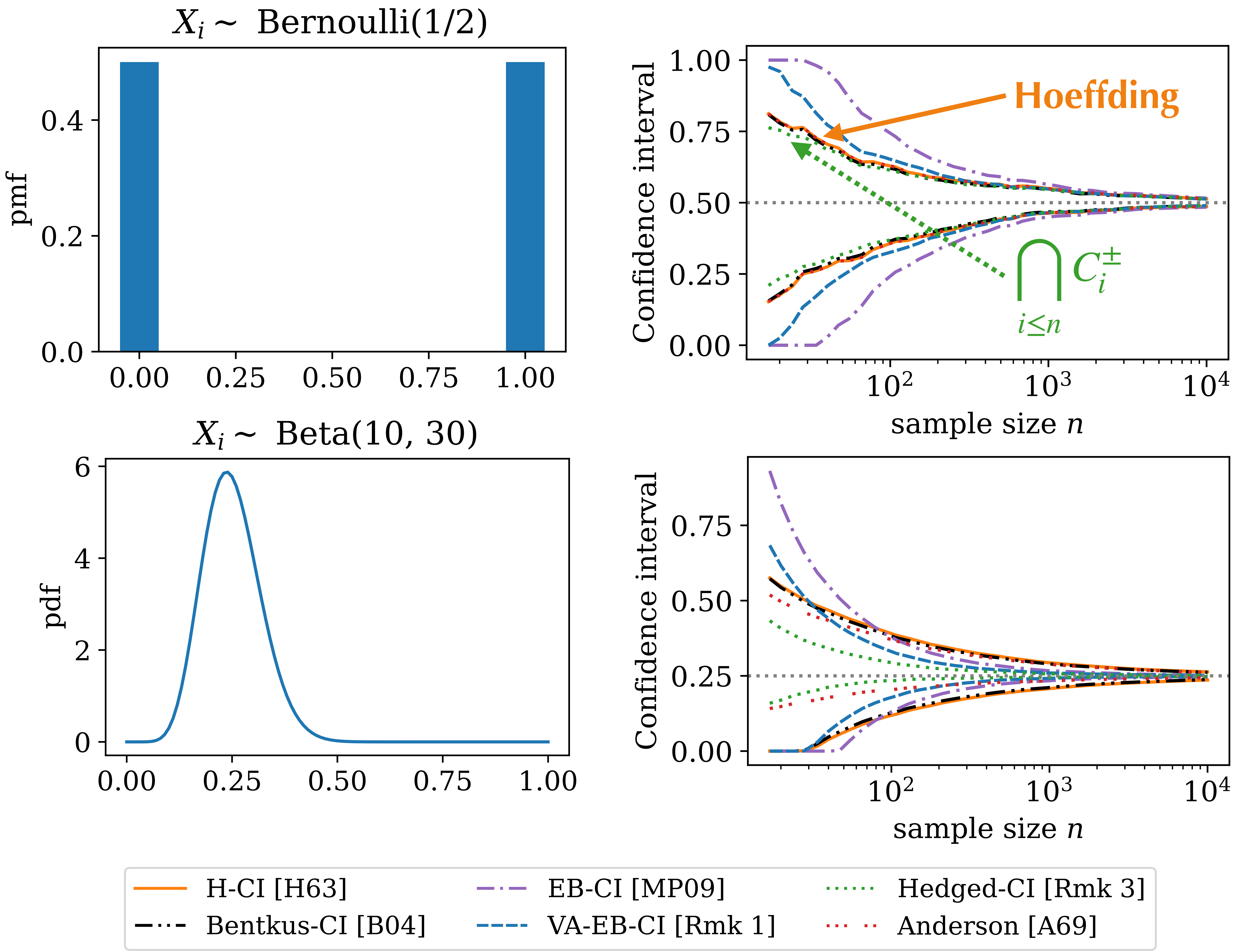

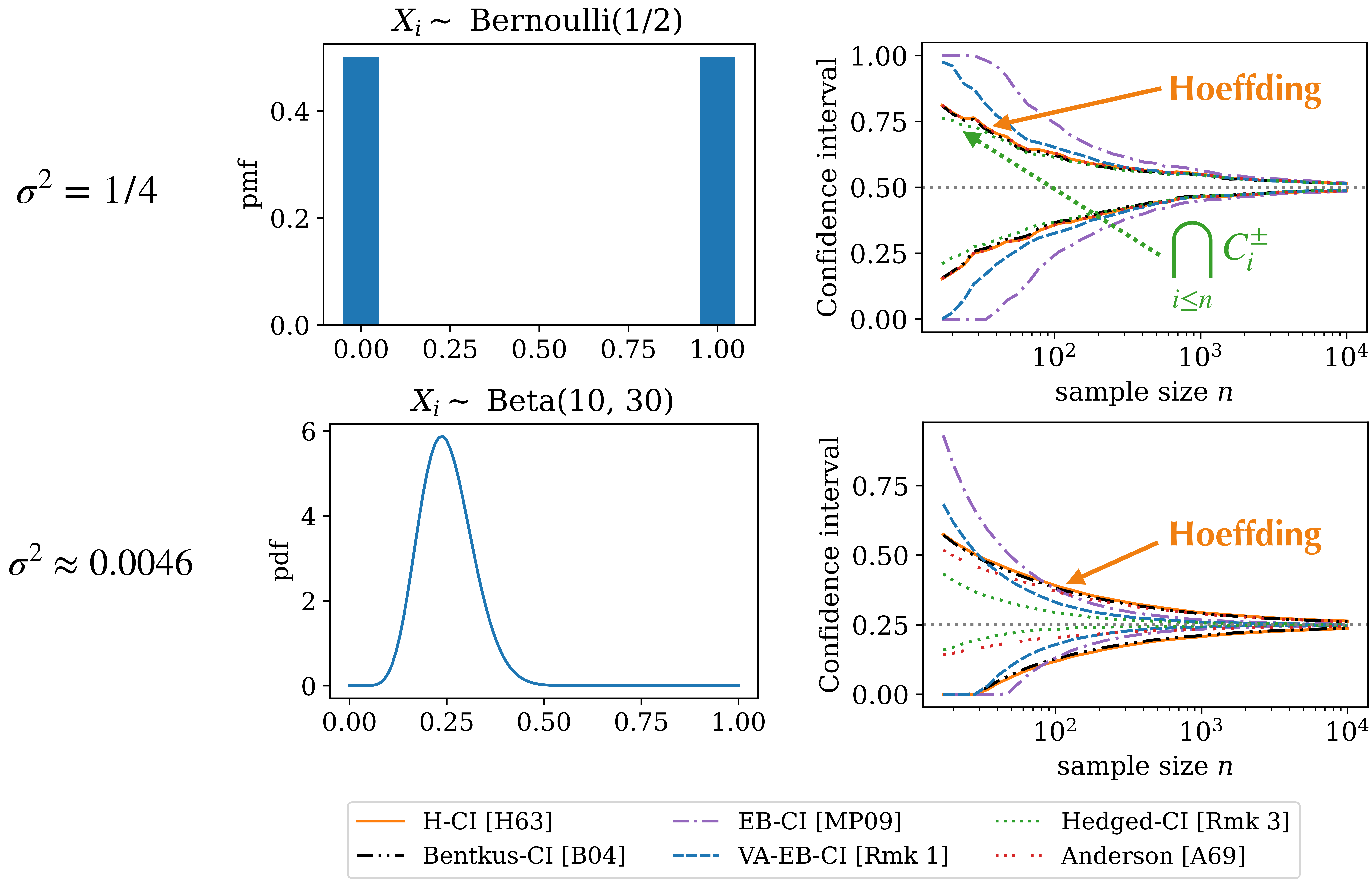

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pdf
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4
6
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EB-CI [MP09]
VA-EB-CI [Rmk 1]
Hedged-CI [Rmk 3]
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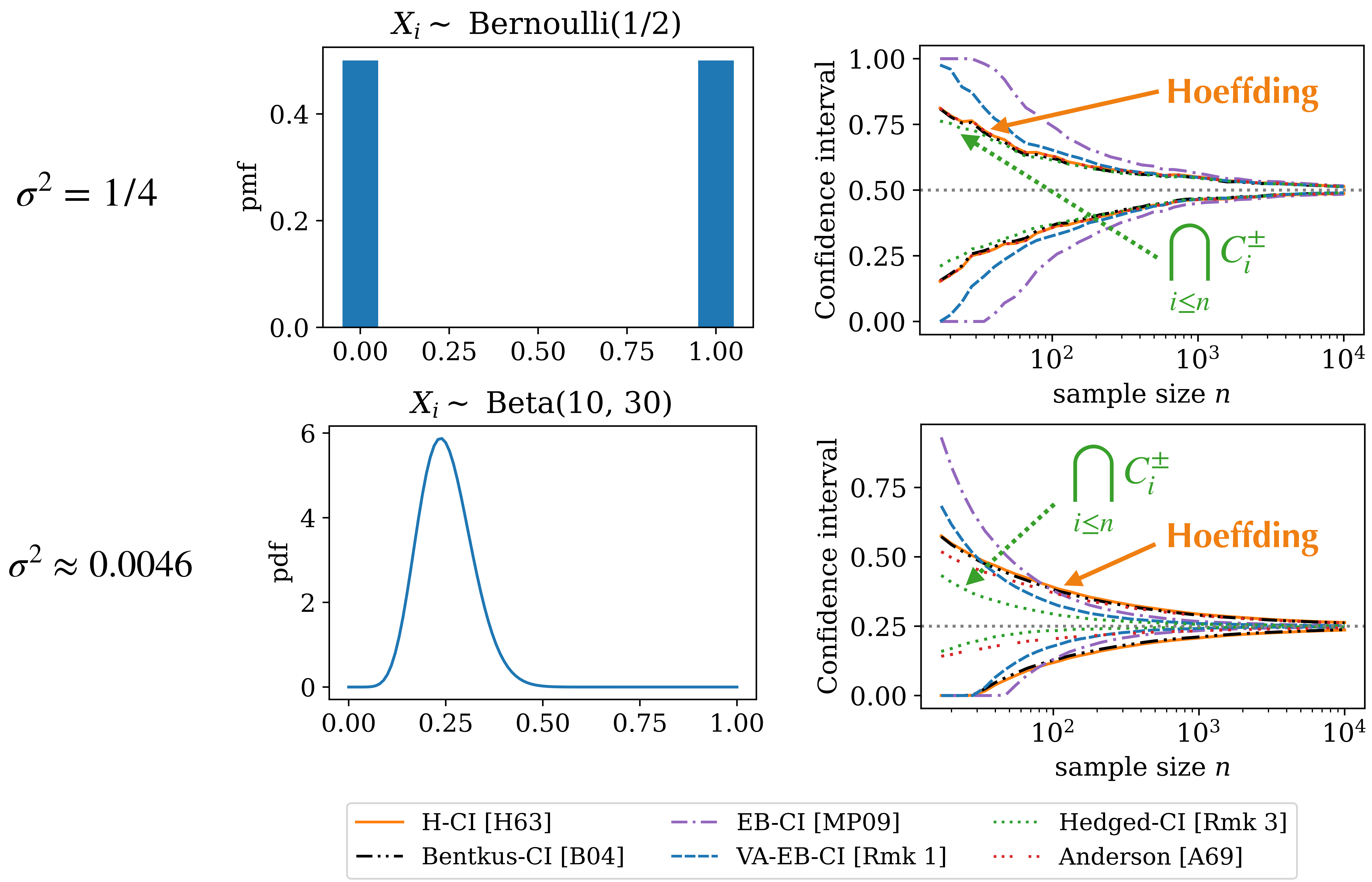

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Confidence sets for sampling without replacement

Without-replacement (WoR) sampling:

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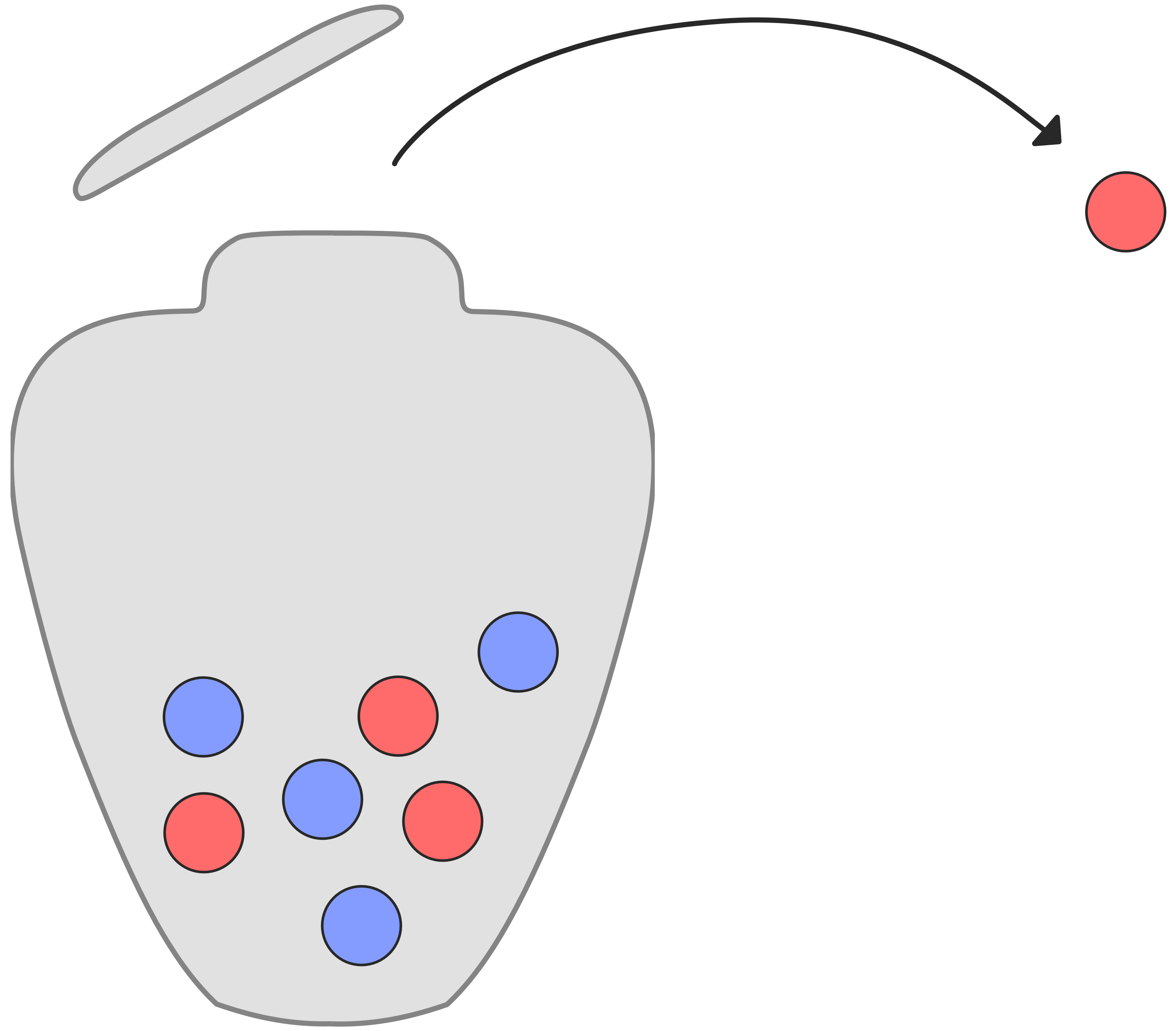

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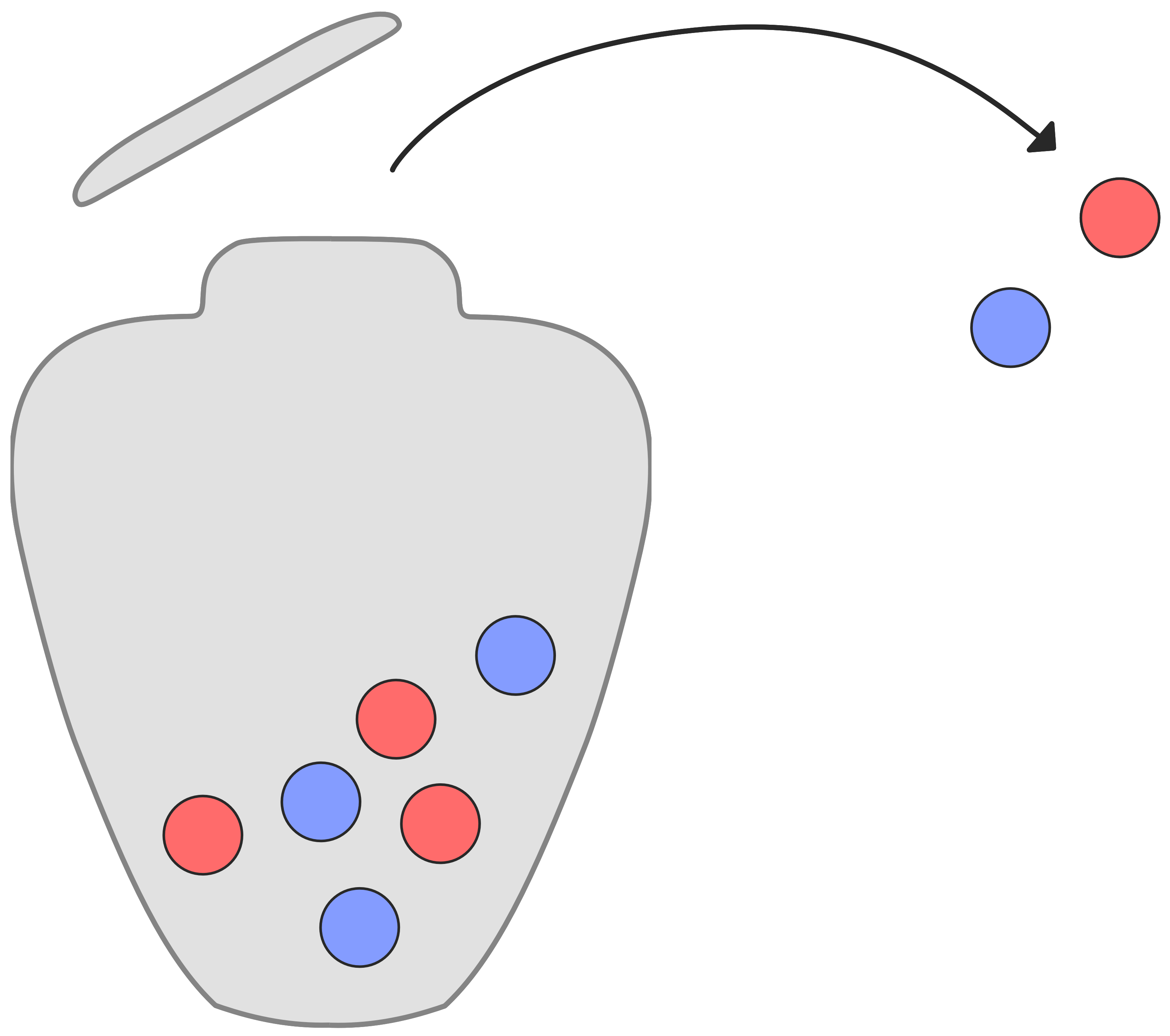

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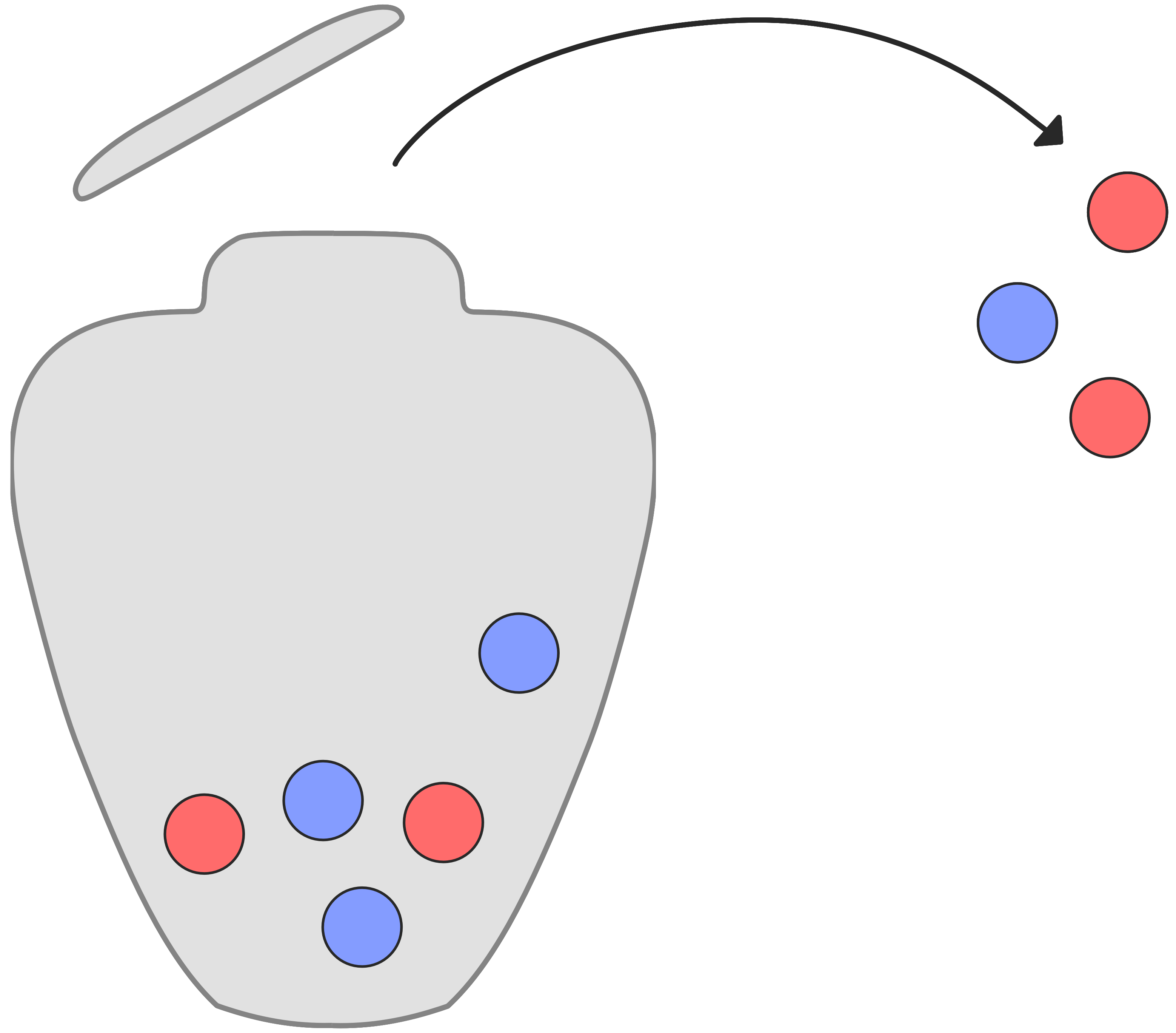

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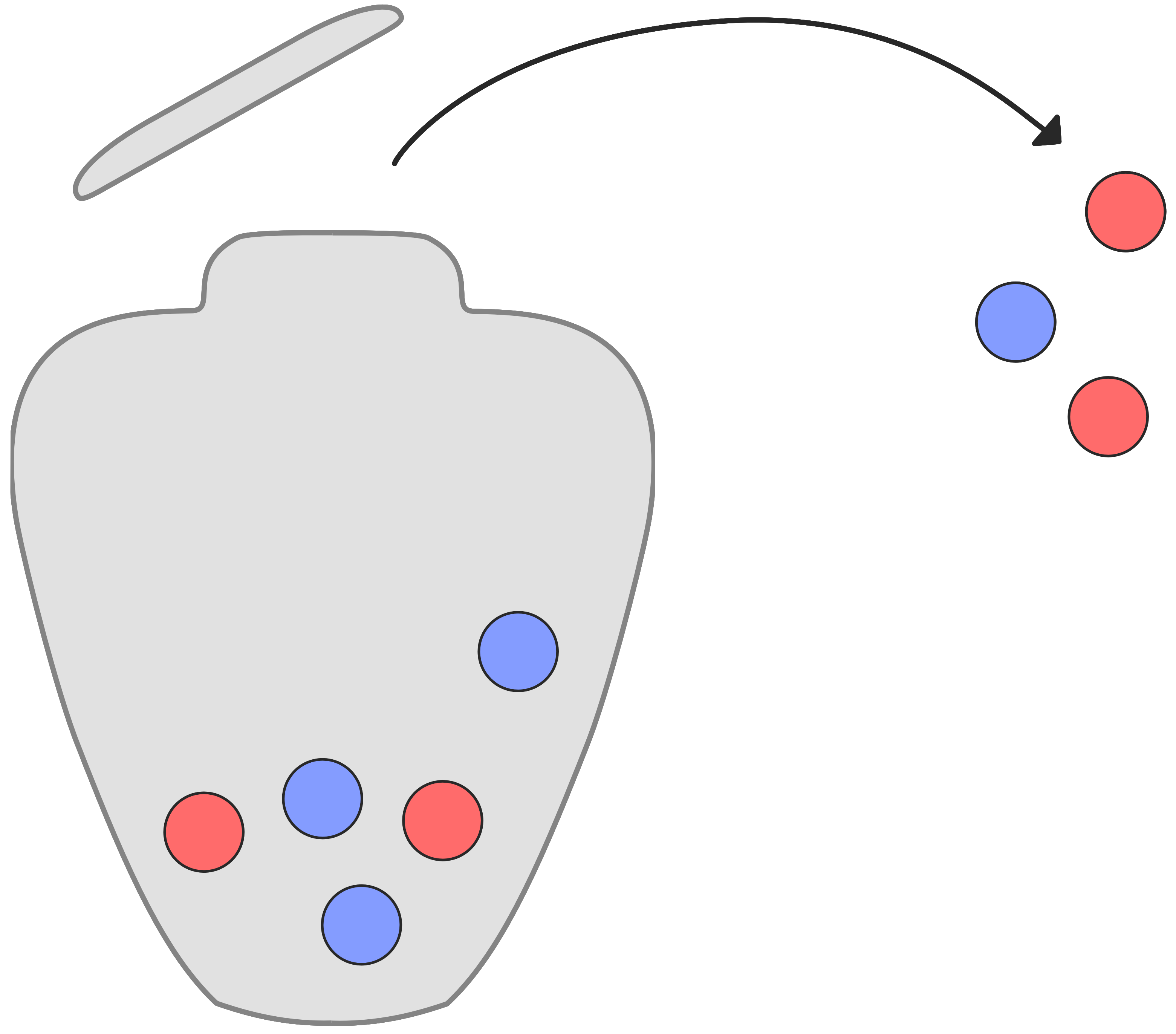
$$(x_1, \ldots, x_N) \in [0,1]^N, \ \ \mu := \frac{1}{N}\sum_{i=1}^{N} x_i$$



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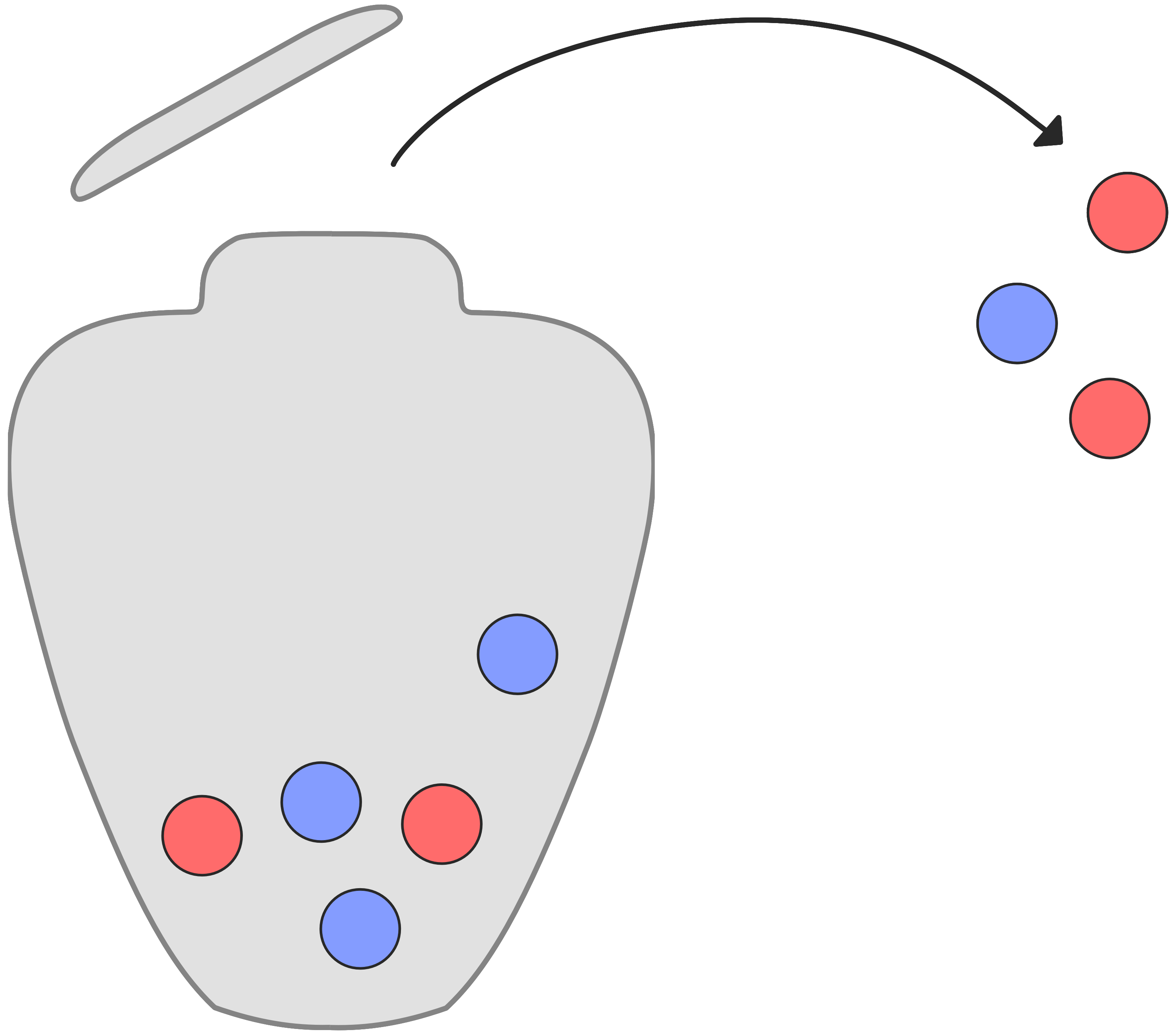

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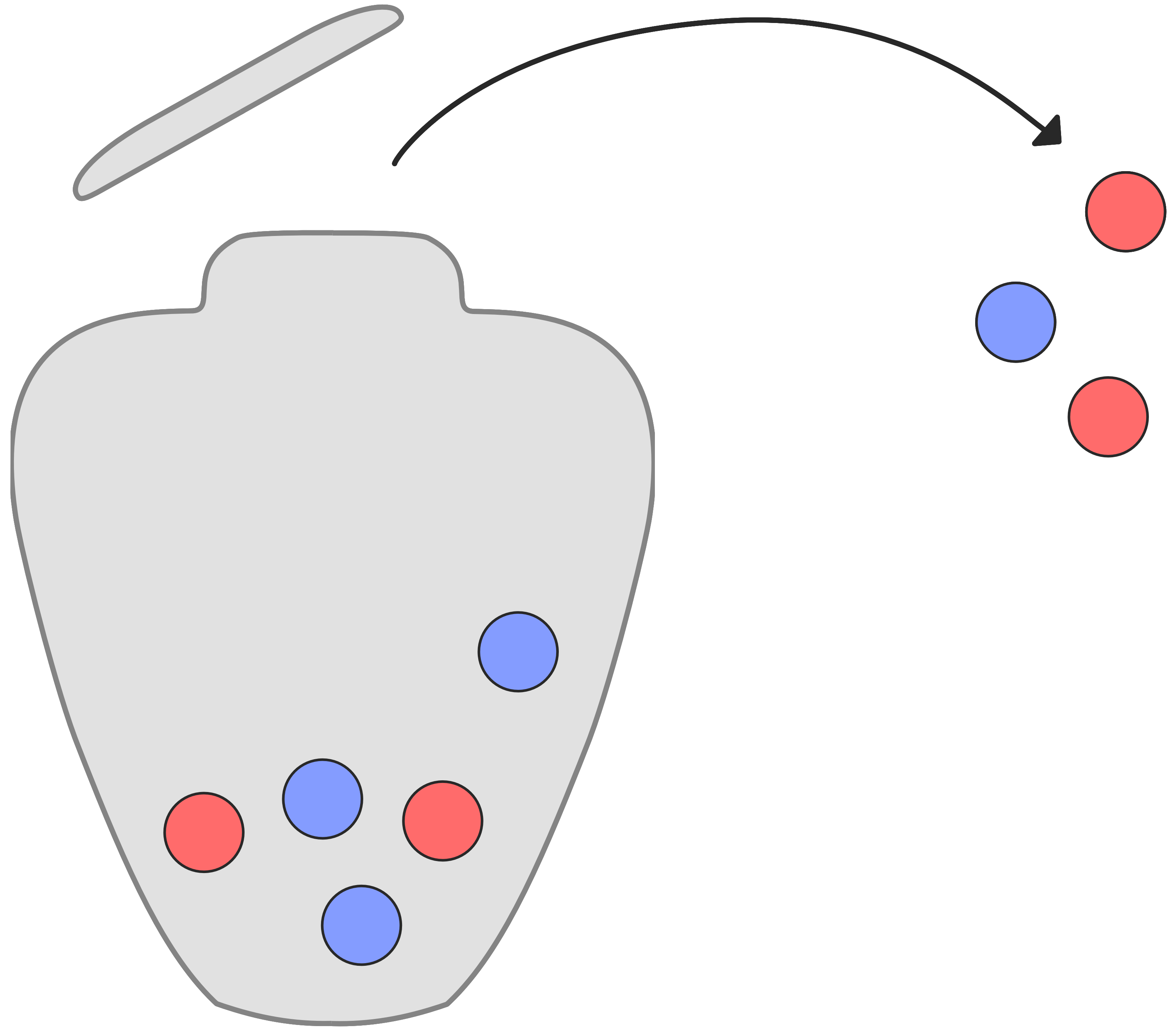

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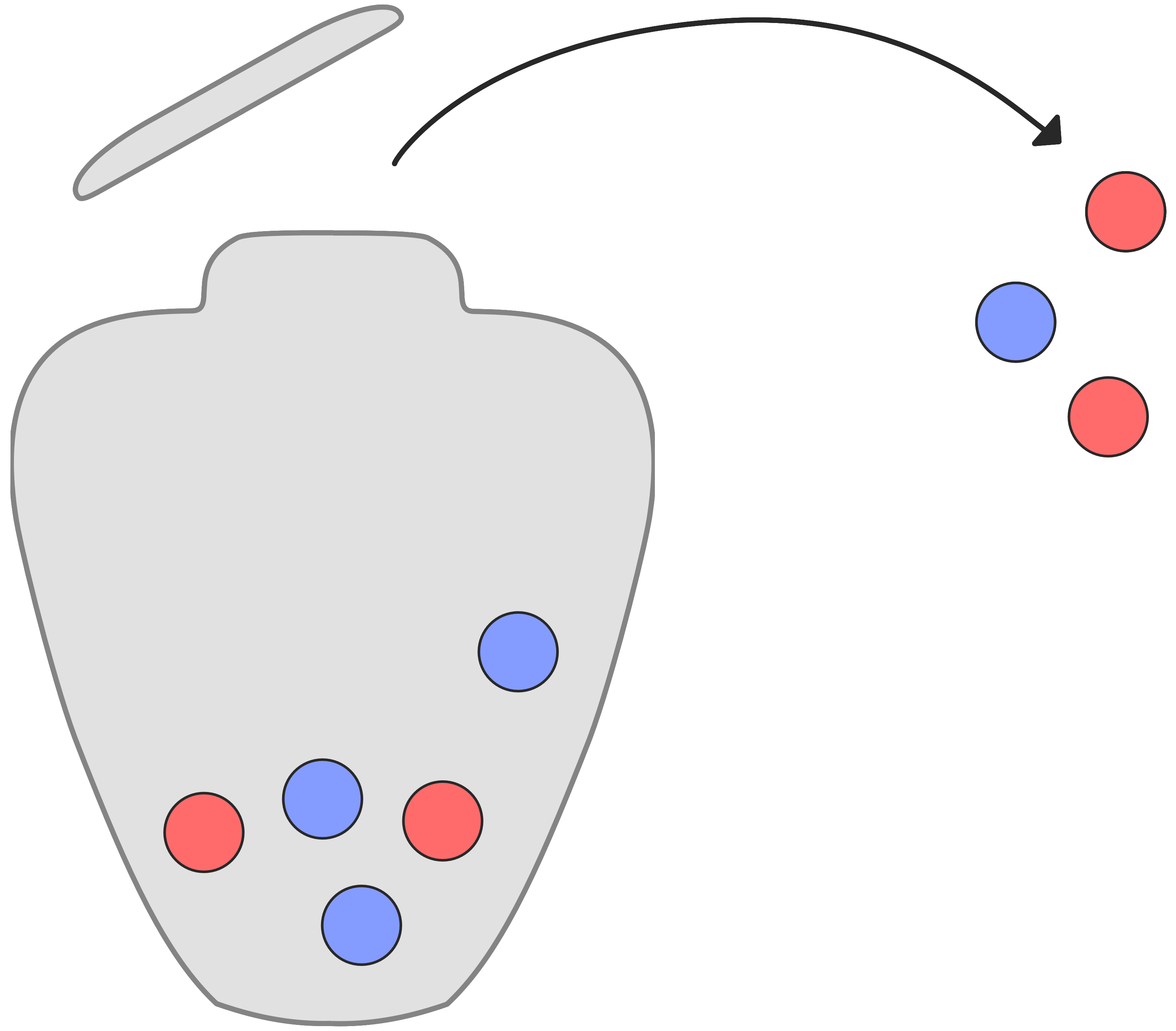

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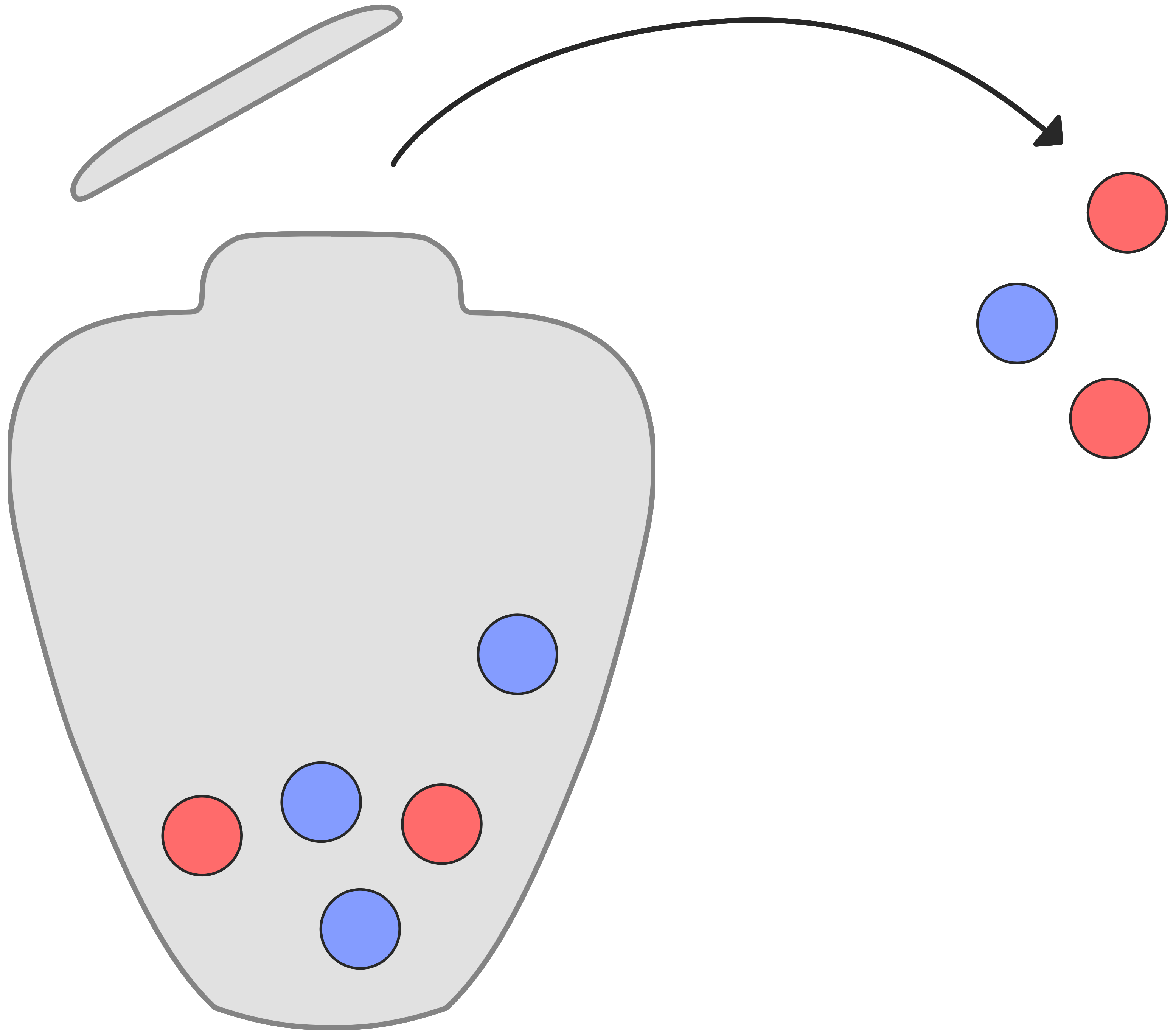

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$$\vdots$$

$$X_t \sim \text{Unif}\left((x_1, \ldots, x_N) \backslash X_1^{t-1}\right)$$

Want to estimate $\mu := \frac{1}{N}\sum_{i=1}^{N} x_i$

Goal: construct a game so that $\left(K_t(\mu)\right)_{t=0}^{N}$ is a martingale under WoR sampling.

$$X_t \sim \mathrm{Unif}\left((x_1, \ldots, x_N) \backslash X_1^{t-1}\right) \implies \mathbb{E}(X_t \mid X_1^{t-1}) = \underbrace{\frac{N\mu - \sum_{i=1}^{t-1} X_i}{N - t + 1}}_{=: \, \mu_t^{\mathrm{WoR}}}$$

Consider a “candidate mean” $m \in [0,1]$

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Gambler chooses bet $\lambda_t \in (-1/(1-m_t^{\text{WoR}}),\ 1/m_t^{\text{WoR}})$

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Observe X_t

$K_t \leftarrow K_{t-1} \ + \ K_{t-1} \cdot \lambda_t \cdot (X_t - m_t^{\mathrm{WoR}})$

EndFor

$$m_t^{\mathrm{WoR}} = \frac{Nm - \sum_{i=1}^{t-1} X_i}{N - t + 1}$$

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$K_0 \leftarrow \$1$

For $t = 1,2,3,\ldots$:

Gambler chooses bet $\lambda_t \in (-1/(1-m),\ 1/m)$

Observe X_t

$K_t \leftarrow K_{t-1} + K_{t-1} \cdot \lambda_t \cdot (X_t - m)$

EndFor

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Observe X_t

$K_t \leftarrow K_{t-1} \ + \ K_{t-1} \cdot \lambda_t \cdot (X_t - m_t^{\text{WoR}})$

EndFor

$$m_t^{\text{WoR}} = \frac{Nm - \sum_{i=1}^{t-1} X_i}{N - t + 1}$$

Then,

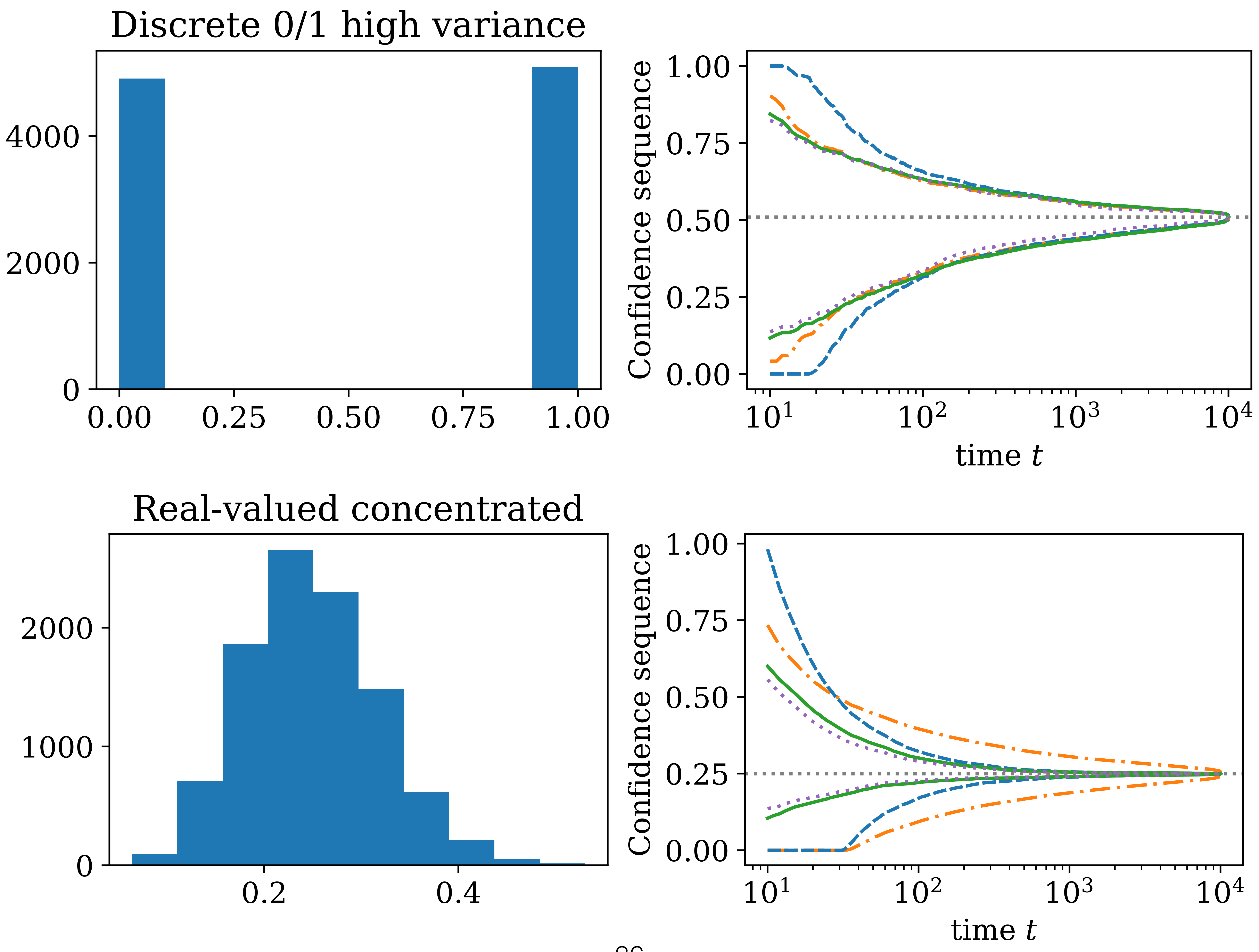
$$K_t^{\mathrm{WoR}}(\mu) := \prod_{i=1}^{t} (1 + \lambda_i \cdot (X_i - \mu_t^{\mathrm{WoR}}))$$

forms a nonnegative martingale, and

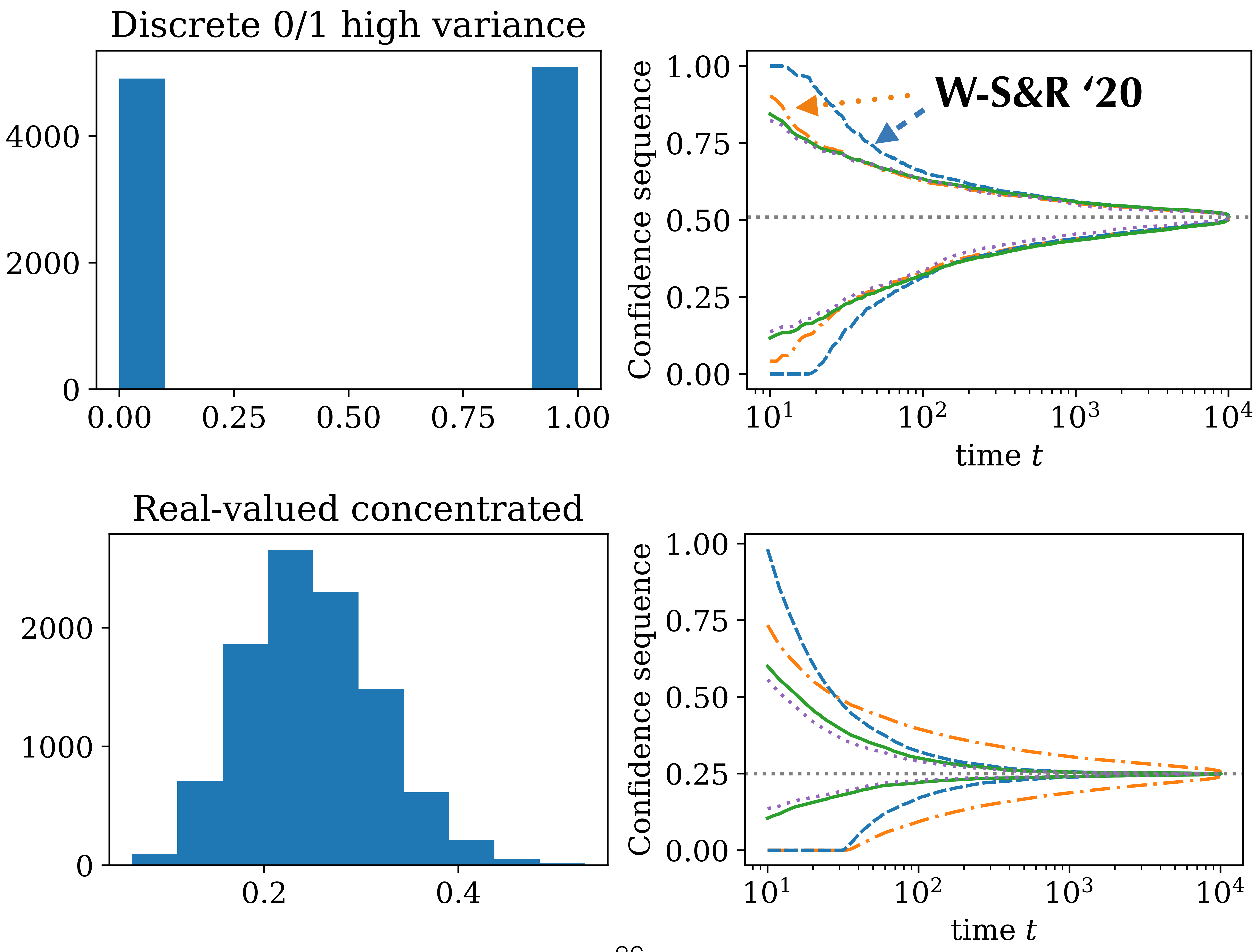
$$C_t^{\mathrm{WoR}} := \left\{ m \in [0,1] : K_t^{\mathrm{WoR}}(m) < \frac{1}{\alpha} \right\}$$

forms a $(1 - \alpha)$-confidence sequence.

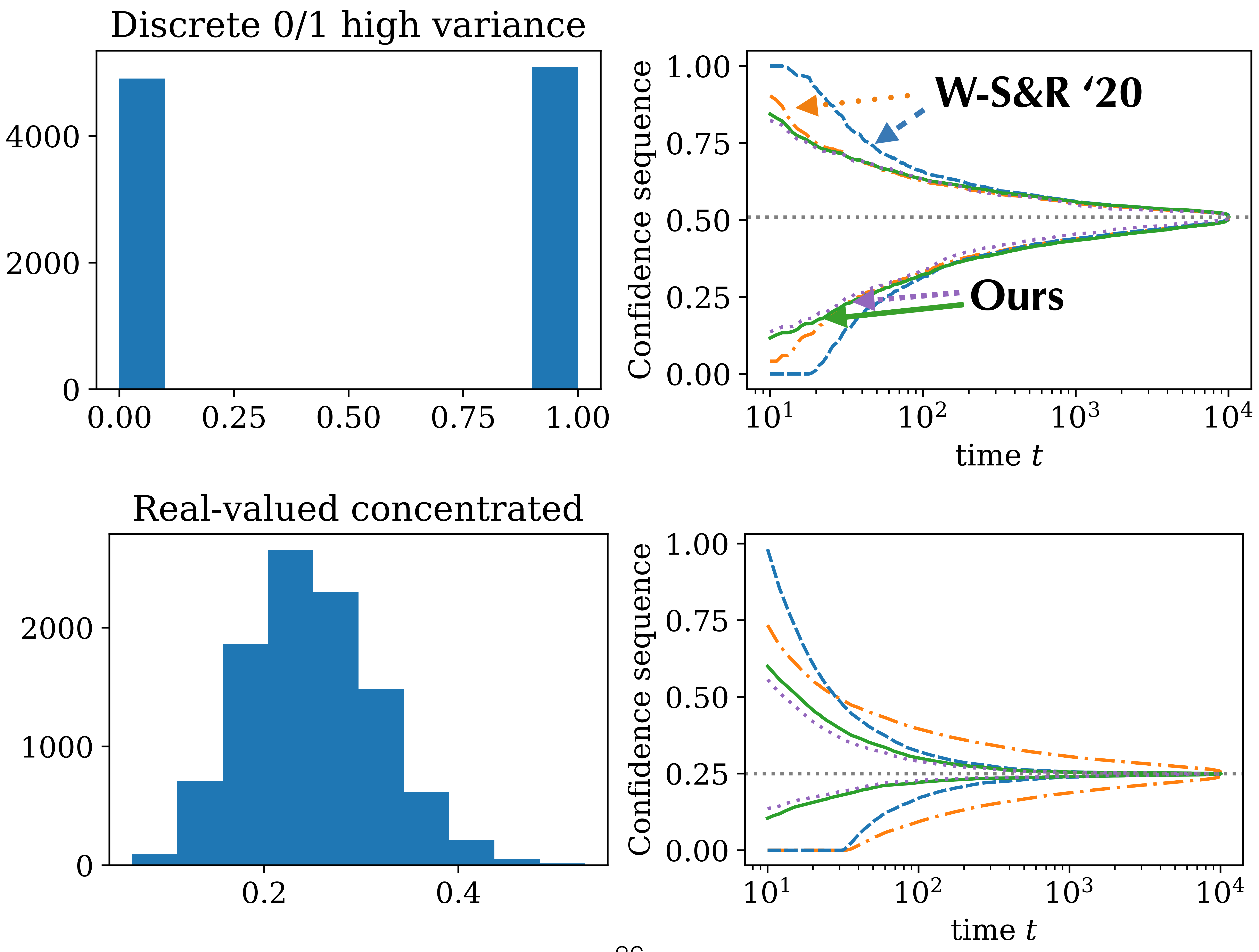
Confidence sequences for sampling WoR



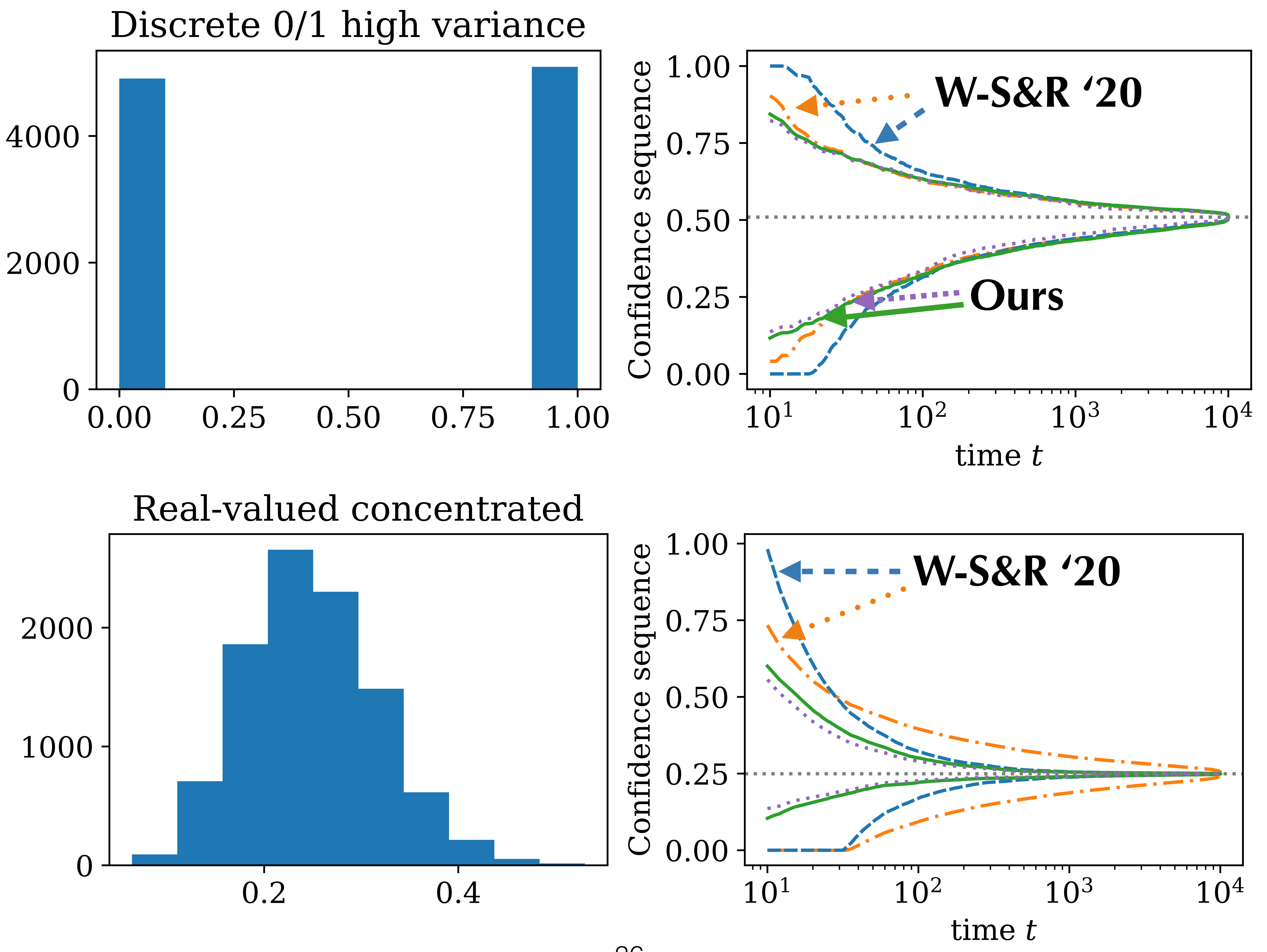
Confidence sequences for sampling WoR



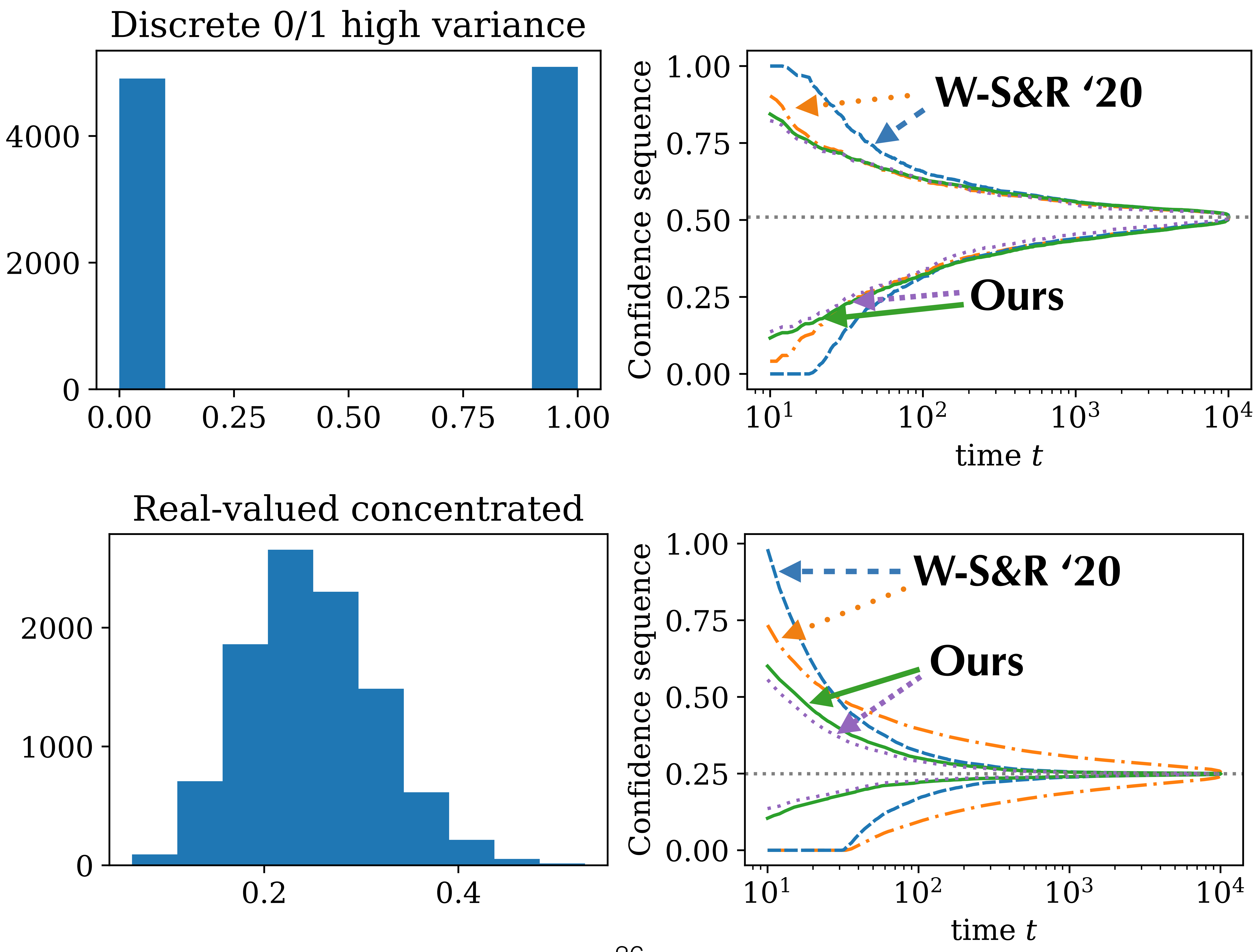
Confidence sequences for sampling WoR



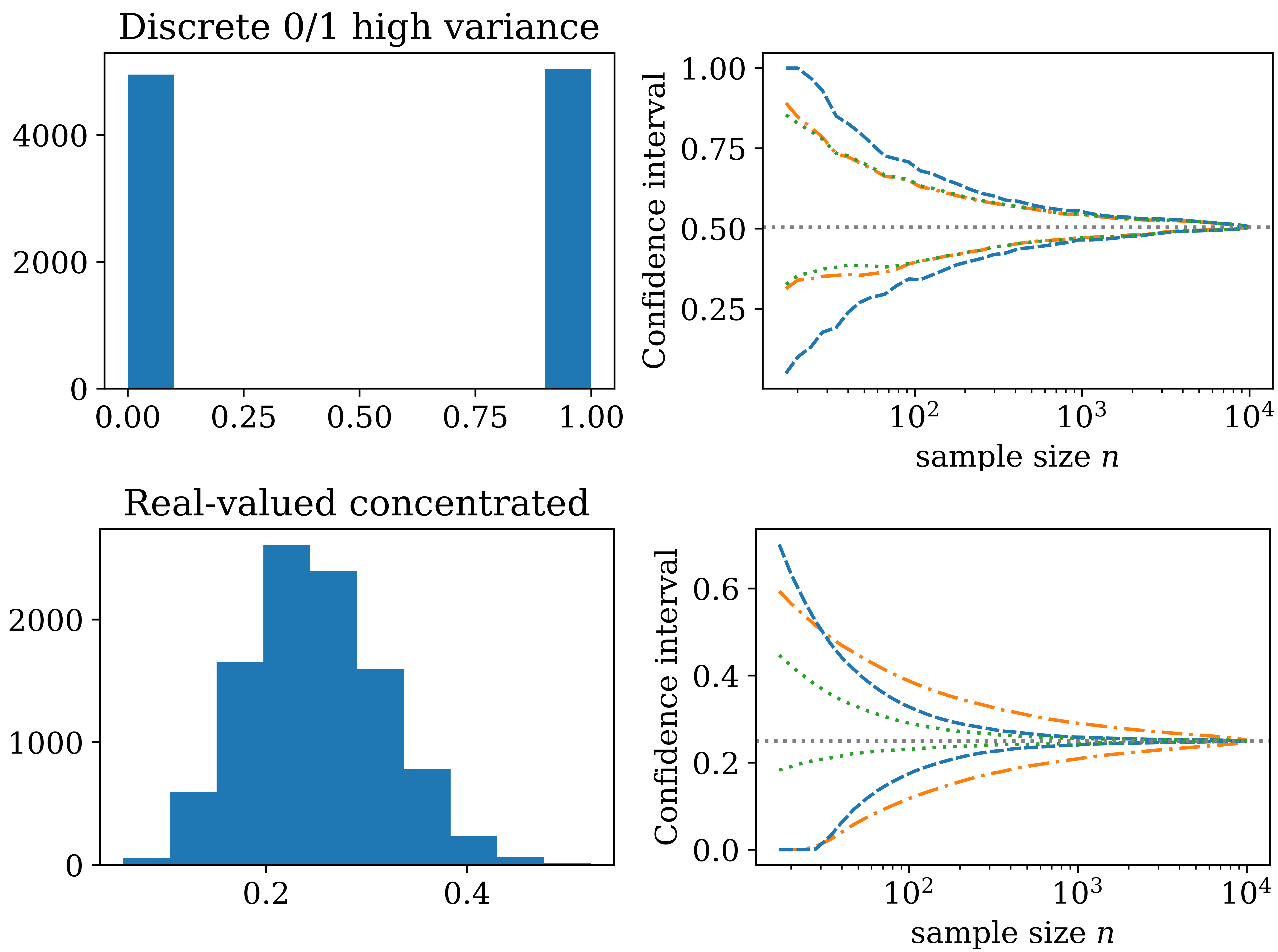
Confidence sequences for sampling WoR



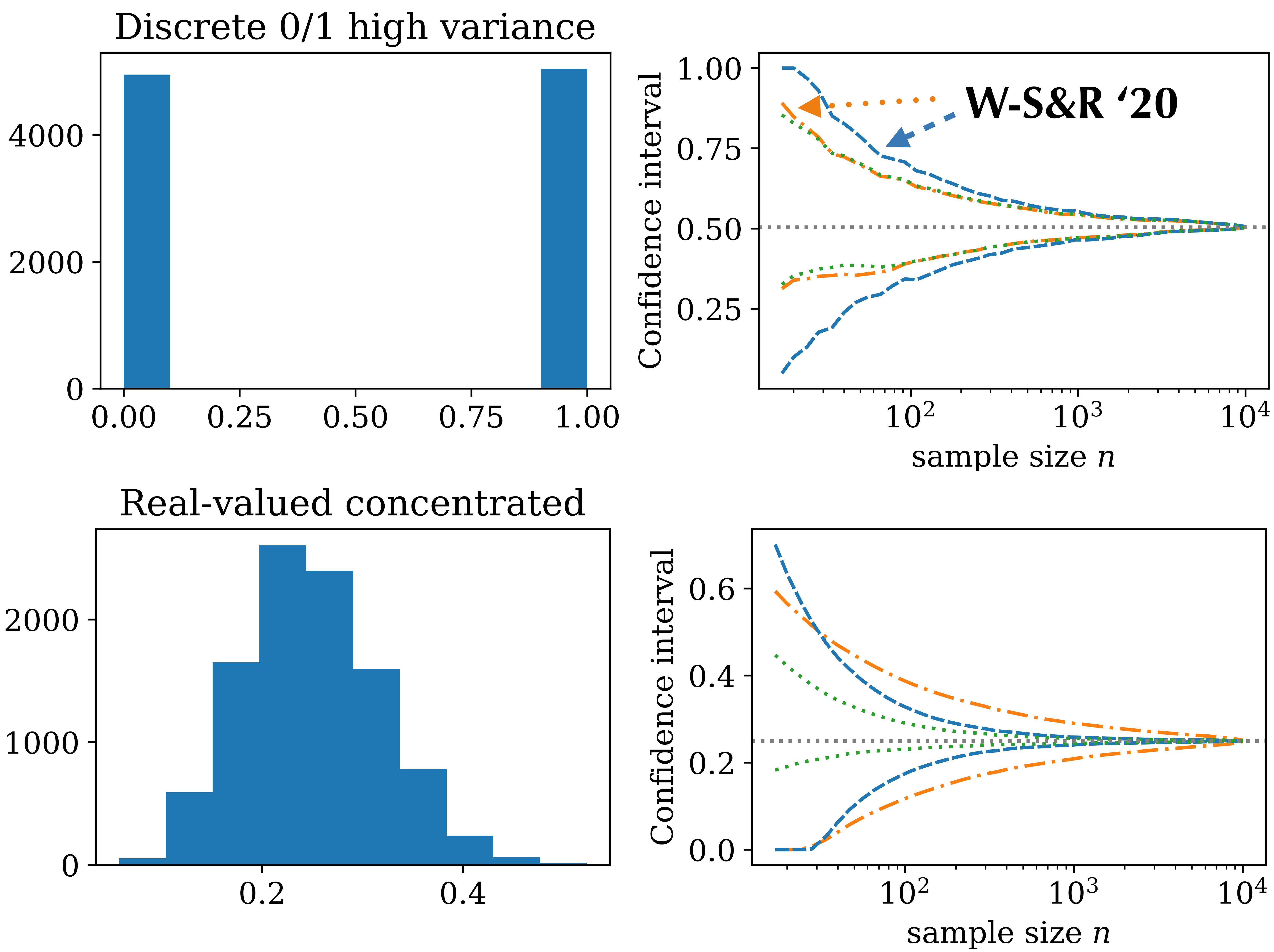
Confidence sequences for sampling WoR



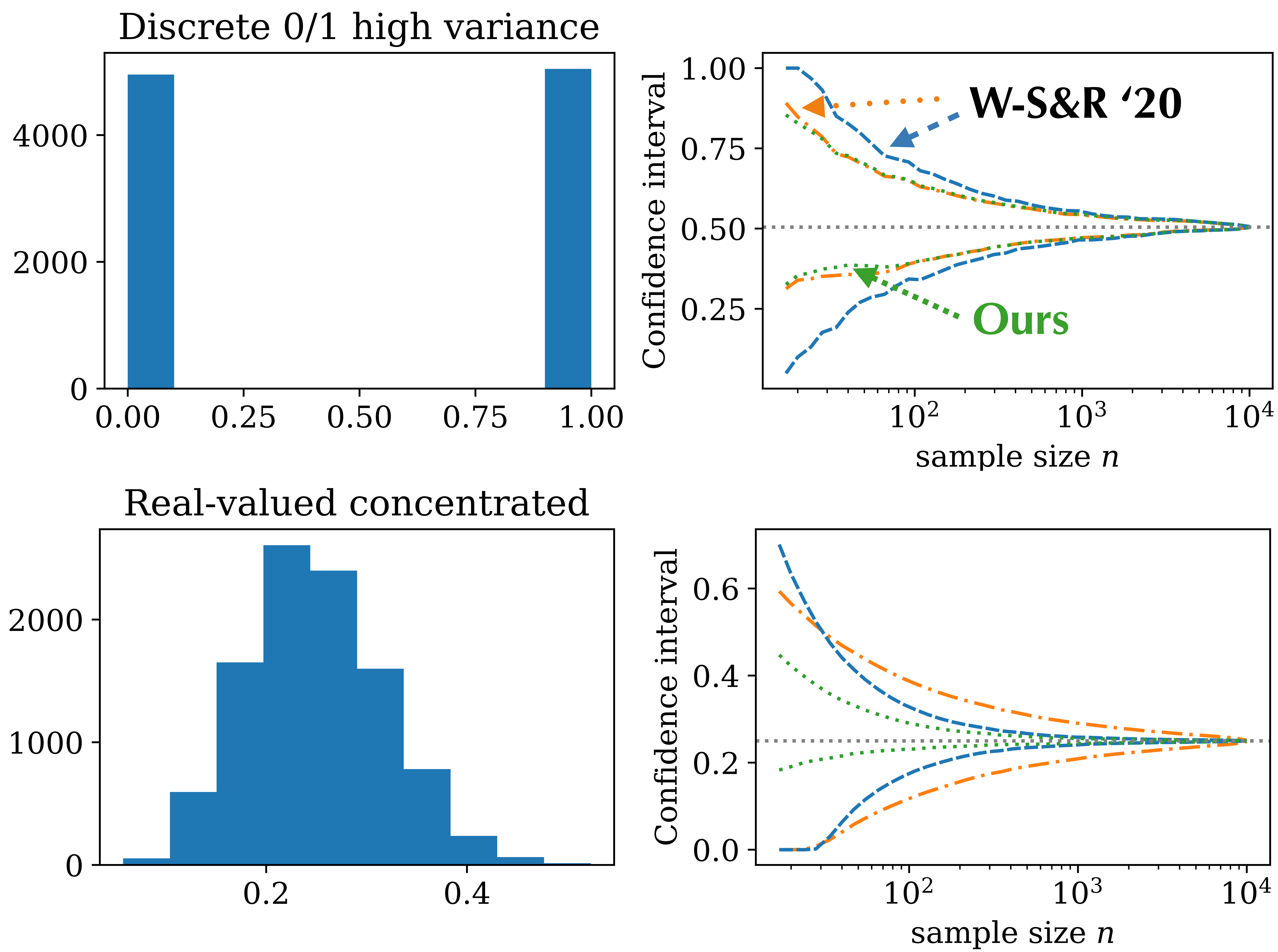
Confidence intervals for sampling WoR



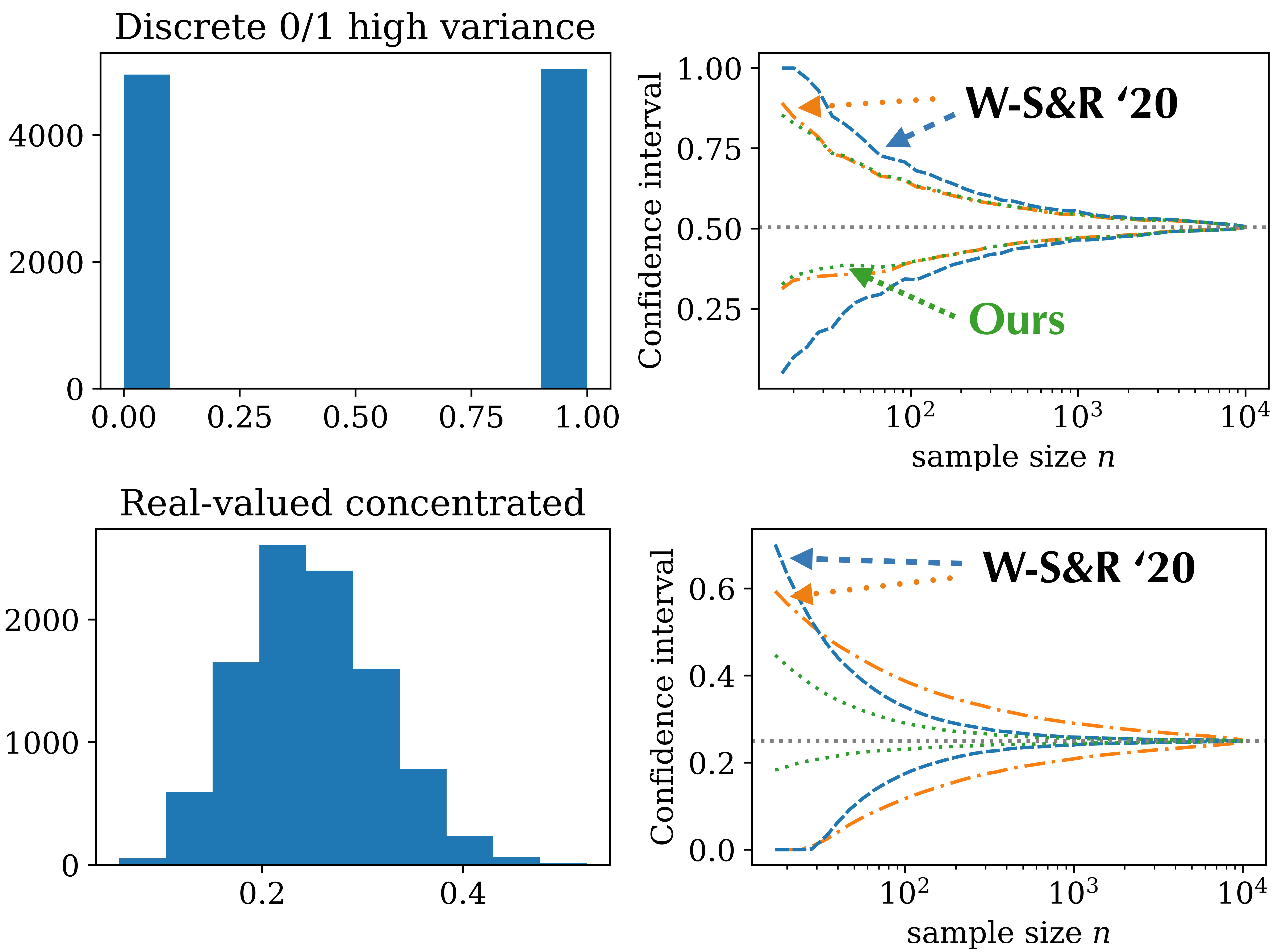
Confidence intervals for sampling WoR



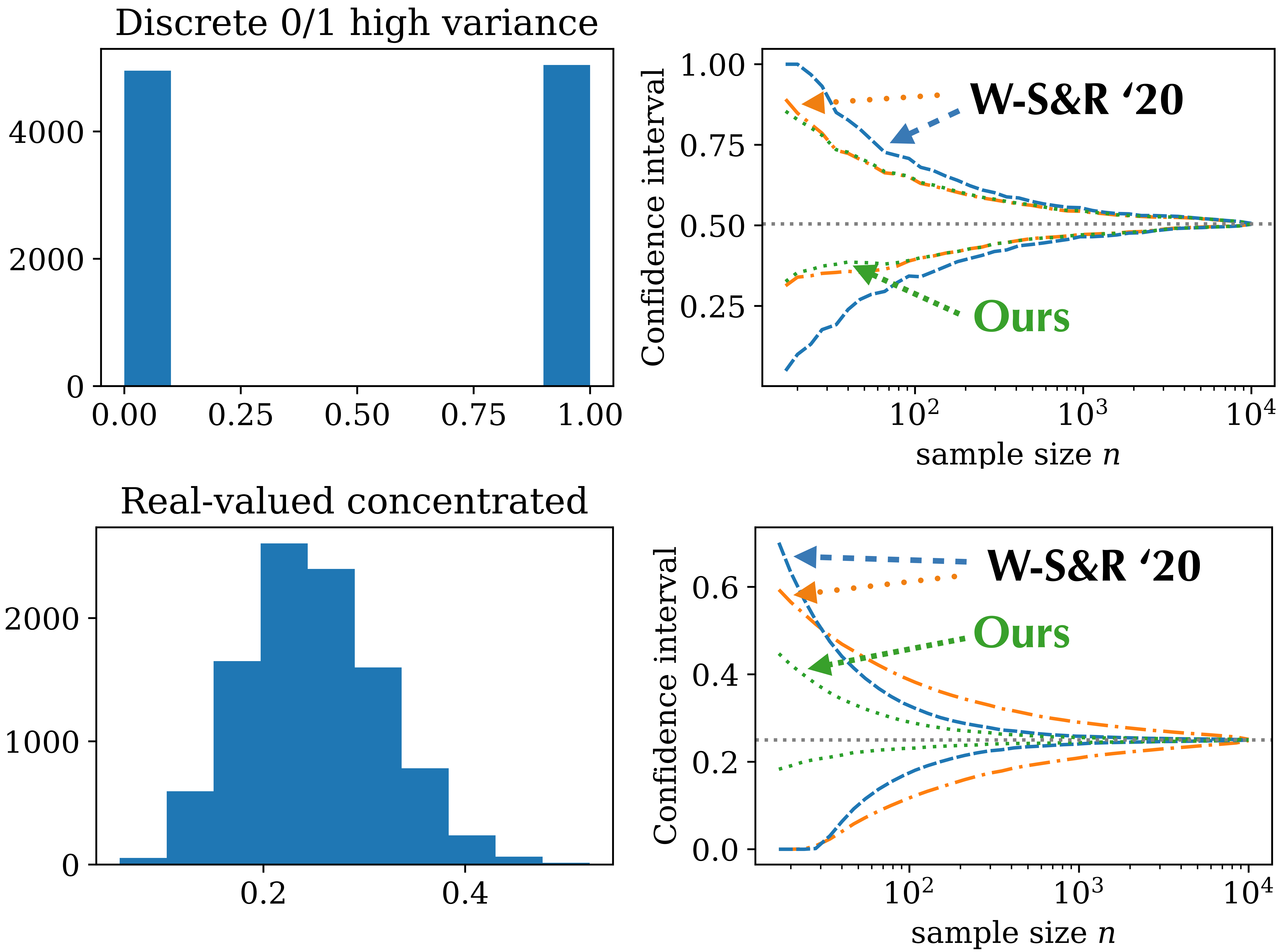
Confidence intervals for sampling WoR



Confidence intervals for sampling WoR



Confidence intervals for sampling WoR



Closed-form empirical Bernstein confidence sequences & confidence intervals

$$C_t := \left\{ m \in [0,1] : \prod_{i=1}^{t} (1 + \lambda_i(m) \cdot (X_i - m) \right\}$$

While C_t is easy to compute, it is not closed-form.

However,

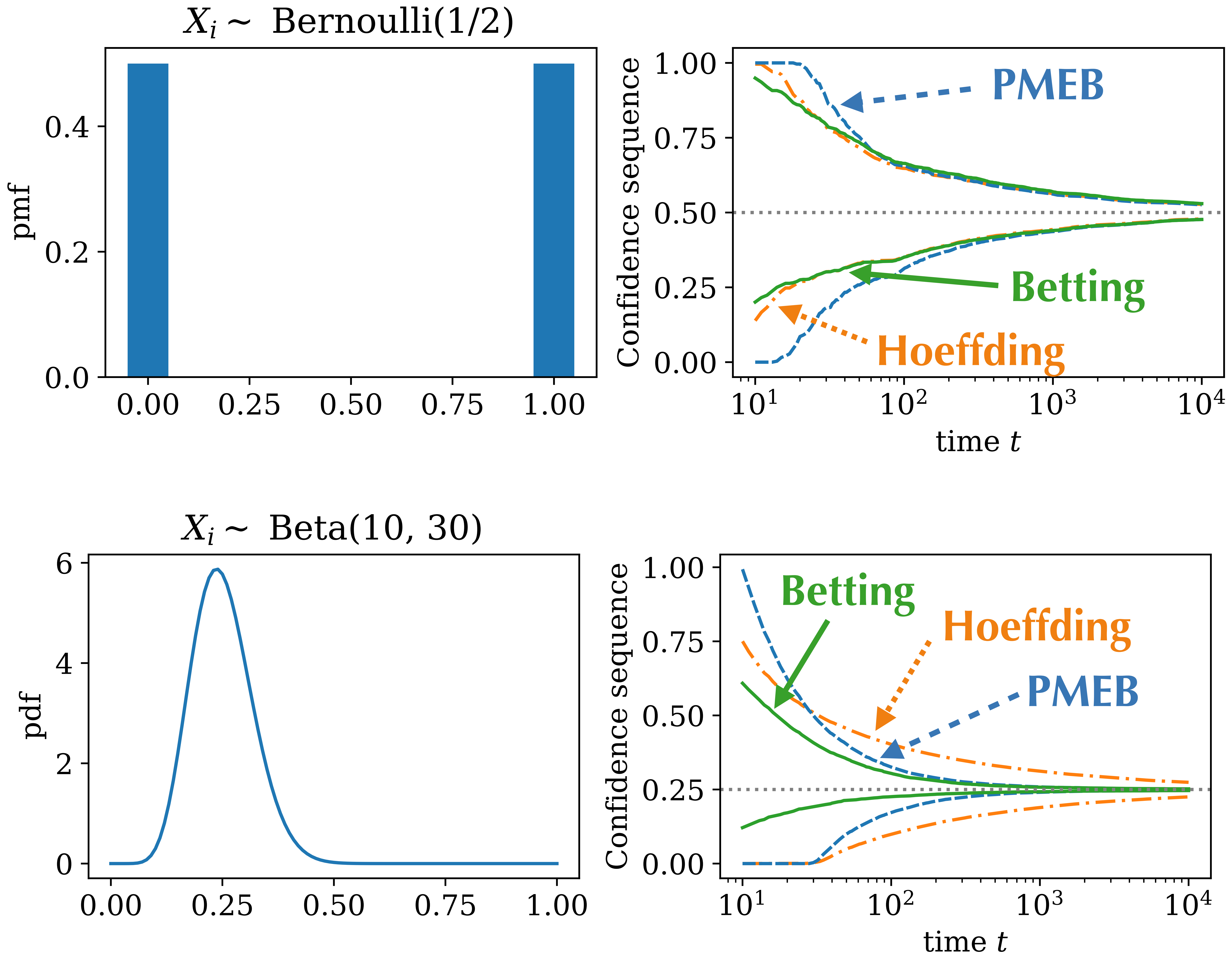
$$C_t^{\text{PMEB}} := \left\{ m \in [0,1] : \prod_{i=1}^{t} \exp\left\{ \lambda_i(X_i - m) - 4(X_i - \widehat{\mu}_{i-1})^2 \psi_E(\lambda_i) \right\} \right\}$$ **is!**

$$C_t^{\mathrm{PMEB}} := \left(\frac{\sum_{i=1}^t \lambda_i X_i}{\sum_{i=1}^t \lambda_i} \pm \frac{\log(2/\alpha) + 4\sum_{i=1}^t (X_i - \widehat{\mu}_{i-1})^2 \psi_E(\lambda_i)}{\sum_{i=1}^t \lambda_i} \right)$$

where $\quad \psi_E(\lambda) := -(\log(1-\lambda) - \lambda)/4,$

$$\lambda_t := \sqrt{\frac{2\log(2/\alpha)}{\widehat{\sigma}_{t-1} t \log(1+t)}} \wedge \frac{1}{2},$$

$$\widehat{\sigma}_t^2 := \frac{1/4 + \sum_{i=1}^t (X_i - \widehat{\mu}_i^2)^2}{t+1}, \quad \text{and} \quad \widehat{\mu}_t := \frac{1/2 + \sum_{i=1}^t X_i}{t+1}.$$


$X_i \sim$ Bernoulli(1/2)
pmf
0.0
0.2
0.4
0.00
0.25
0.50
0.75
1.00
Confidence sequence
1.00
0.75
0.50
0.25
0.00
PMEB
Betting
Hoeffding
10^1
10^2
10^3
10^4
time t
$X_i \sim$ Beta(10, 30)
pdf
6
4
2
0
0.00
0.25
0.50
0.75
1.00
Confidence sequence
1.00
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0.00
Betting
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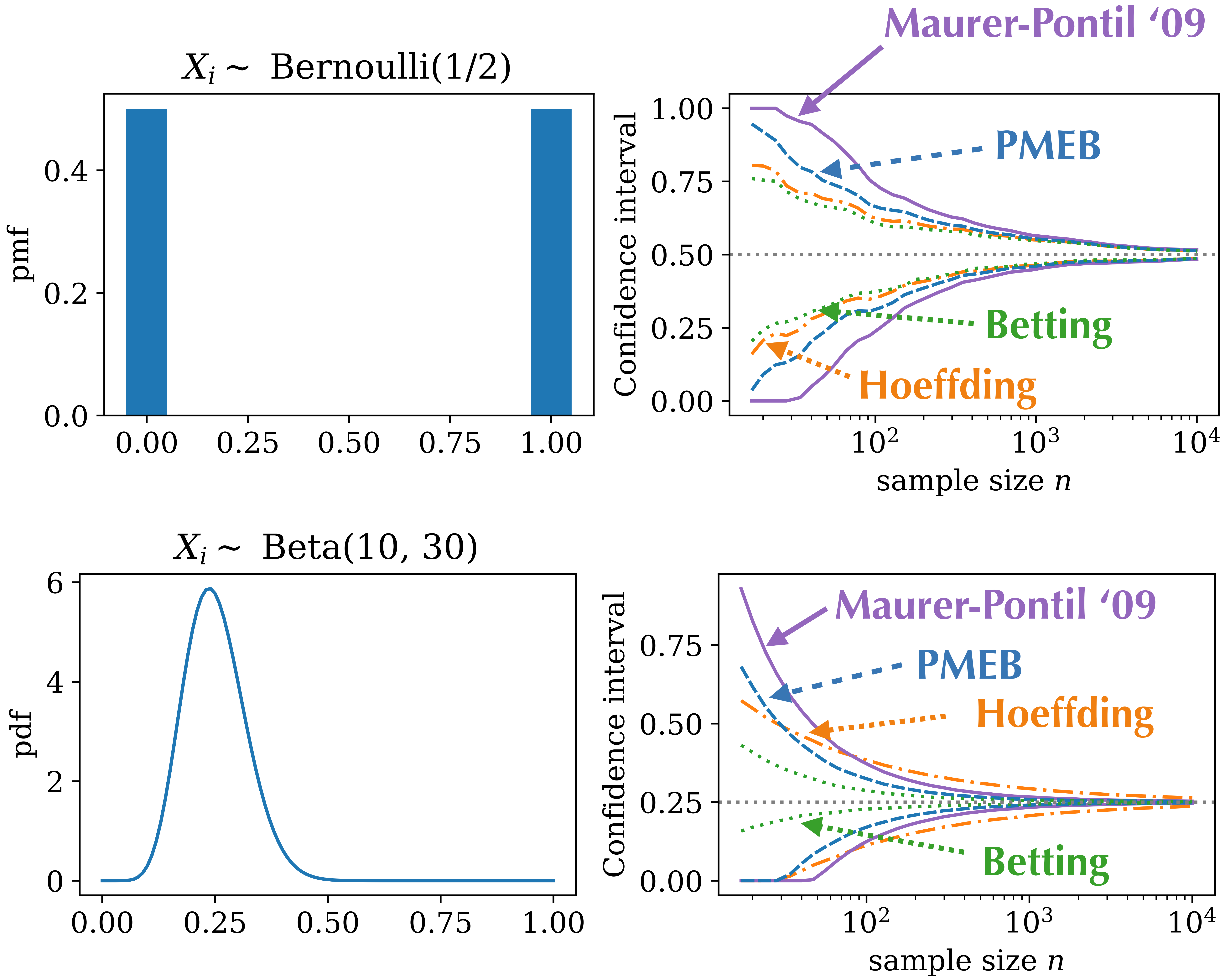
10^1
10^2
10^3
10^4
time t

Similarly for fixed-time confidence intervals:

$$C_n^{\mathrm{PMEB}} := \left(\frac{\sum_{i=1}^{n} \lambda_i X_i}{\sum_{i=1}^{n} \lambda_i} \pm \frac{\log(2/\alpha) + 4 \sum_{i=1}^{n} (X_i - \widehat{\mu}_{i-1})^2 \psi_E(\lambda_i)}{\sum_{i=1}^{n} \lambda_i} \right)$$

but here, $\lambda_i := \sqrt{\dfrac{2 \log(2/\alpha)}{n \widehat{\sigma}_{i-1}}} \wedge \dfrac{1}{2},$

Final bound: $\bigcap_{i \leq n} C_i^{\mathrm{PMEB}}$


$X_i \sim$ Bernoulli(1/2)
pmf
Maurer-Pontil '09
PMEB
Betting
Hoeffding
Confidence interval
sample size n
$X_i \sim$ Beta(10, 30)
pdf
Maurer-Pontil '09
PMEB
Hoeffding
Betting
Confidence interval
sample size n

Choice of $(\lambda_t^+)_{t=1}^n$ and $(\lambda_t^-)_{t=1}^n$ for fixed-time confidence intervals

Why does perform so well?

$$K_n^+(\mu) := \prod_{i=1}^{n} \left(1 + \lambda \cdot (X_i - \mu)\right)$$

$$\gtrsim \prod_{i=1}^{n} \exp\left\{\lambda \cdot (X_i - \mu) - (X_i - \widehat{\mu}_{i-1})^2 \lambda^2 / 2\right\}$$

$$\implies \text{Width}_n := \frac{\log(2/\alpha) + \sum_{i=1}^{n} (X_i - \widehat{\mu}_{i-1})^2/2}{t\lambda}$$

$$\approx \frac{\log(2/\alpha) + n\sigma^2/2}{t\lambda}$$

Why does $\lambda_i^+(m) := \sqrt{\frac{2\log(2/\alpha)}{n\widehat{\sigma}_{i-1}^2}} \wedge \frac{1}{m}$ perform so well?

$$
\begin{aligned}
K_n^+(\mu) &:= \prod_{i=1}^{n} \left(1 + \lambda \cdot (X_i - \mu)\right) \\
&\gtrsim \underbrace{\prod_{i=1}^{n} \exp\left\{\lambda \cdot (X_i - \mu) - (X_i - \widehat{\mu}_{i-1})^2 \lambda^2/2\right\}}
\end{aligned}
$$

$$
\begin{aligned}
\implies \text{Width}_n &:= \frac{\log(2/\alpha) + \sum_{i=1}^n (X_i - \widehat{\mu}_{i-1})^2/2}{t\lambda} \\
&\approx \frac{\log(2/\alpha) + n\sigma^2/2}{t\lambda}
\end{aligned}
$$

$$\underset{\lambda}{\operatorname{argmin}}\ \mathrm{Width}_n = \sqrt{\frac{2\log(2/\alpha)}{n\sigma^2}}$$

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$$\lambda_t^+(m) := \sqrt{\frac{2\log(2/\alpha)}{n\,\widehat{\sigma}_{t-1}^2}} \wedge \frac{1}{m}$$

$$\underset{\lambda}{\operatorname{argmin}}\ \mathrm{Width}_n = \sqrt{\frac{2\log(2/\alpha)}{n\sigma^2}}$$

$$\lambda_t^+(m) := \sqrt{\frac{2\log(2/\alpha)}{n\,\widehat{\sigma}_{t-1}^2}} \wedge \frac{1}{m} \qquad \lambda_t^-(m) := \sqrt{\frac{2\log(2/\alpha)}{n\,\widehat{\sigma}_{t-1}^2}} \wedge \frac{1}{1-m}$$

Brief selective history of betting ideas