A Condensed Introduction to Condensed Mathematics

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1 Introduction

The goal of this paper is to provide an idea of what condensed mathematics is about, while not assuming too[1] much mathematical background from the reader. The material is based on the lecture notes by Peter Scholze and the master class in condensed mathematics by Peter Scholze and Dustin Clausen ([Sch19, CS20]). Let us now try to motivate why one might consider studying condensed sets. Additionally, we provide a potential train of thought leading to the definition of a condensed set, which hopes to give the reader some intuition where this seemingly strange definition comes from.

2 Why condensed sets?

The modern definition of a topological space only came about in the 1920s. It has proven to be a very interesting and rich field of study, with many applications across mathematics. Even so, it seems that there is something fundamentally wrong with the way we combine algebraic structures with topology. Consider the inclusion map

$$\mathbb{Q} \hookrightarrow \mathbb{R},$$

for example. As a map between abelian groups, this map is injective, but not surjective. However, when we instead consider this as a map between Hausdorff topological abelian groups, where $\mathbb{Q}$ and $\mathbb{R}$ have been equipped with their usual metric topology, then this map also becomes an epimorphism. Indeed, if two continuous maps $f, g \colon \mathbb{R} \to X$ agree on the dense subspace $\mathbb{Q}$, then they must be equal, if X is Hausdorff. One sees that it somehow becomes more difficult to tell apart $\mathbb{Q}$ from $\mathbb{R}$ in the topological setting.

A potential way to fix this would be to consider only a particularly nice class of topological spaces. For example, one could try to restrict to just **CHaus**, the category of compact Hausdorff spaces. There, every continuous map is also closed, and hence a continuous bijection is automatically a homeomorphism. In fact, this category is an abelian category. Indeed, Pontryagin duality gives that discrete abelian groups are dual to compact Hausdorff groups (by considering the group of continuous group characters). So $\mathbf{Ab}^{\mathrm{opp}} \simeq \mathbf{CHaus}$, and **CHaus** is abelian as the opposite category of an abelian category. This means that it makes sense to start talking about long exact sequences and homology in this setting. The problem with this approach is that there are many interesting and non-pathological topological groups that are not compact. For example, the real numbers with the metric topology is not compact. Hence, we will need some way of including a broader class of topological spaces.

At this point it is a good idea to study the literature. The field of algebraic geometry turns out to be particularly fruitful. In [Ste67] it is suggested to look at the category of compactly generated

[1] Throughout this paper it will be assumed that the reader is somewhat familiar with basic notions from category theory, and in particular homological algebra.

spaces. This category satisfies some nice properties relating unions products and quotients, such as $Z^{Y\times X} = (Z^Y)^X$. It is also not too strict: it contains many families of topological spaces, such as locally compact spaces, CW-complexes and first-countable spaces. The product in this category turns out to be the natural product for CW-complexes (see the appendix of [Hat02]). So what does it mean for a topological space to be compactly generated?

Definition 1. A topological space X is said to be *compactly generated* if the topology on X coincides with the final topology generated by all the continuous maps $K \to X$ with K compact Hausdorff. In other words, a map $f\colon X \to Y$ is continuous, if $K \to X \to Y$ is continuous for every continuous map $K \to X$.

So, if we understand the continuous maps $K \to X$, we understand the space X. We now take a small leap in logic, motivated by algebraic geometry, and conclude that X is determined by the functor

$$\underline{X} \overset{\text{def}}{=} \mathrm{Cont}(-, X)\colon\ \mathbf{CHaus}^{\mathrm{opp}} \to \mathbf{Set},$$

which maps any compact Hausdorff space K to the set of continuous functions $K \to X$. This claim is justified in proposition 4. We can of course consider the same type of functor for any topological space X. The functor $\underline{X}$ (with some minor modifications) will be the condensed set associated to X. For this to lead to a good theory, which would replace topological spaces, we want two things:

1. The definition is strong enough. We are able to express the topological properties we want in this new setting.
2. The definition is not too strong. We do not end up with a theory that is just the same as topological spaces. Constructions and invariants that we care about should be valid in this new setting.

The goal of the following sections is to provide evidence that condensed mathematics does indeed behave as wanted.

3 What are condensed sets?

Recall from the previous section that we moved from topological spaces X, to contravariant functors $\underline{X}$. We now want to view these as objects of some category. As morphisms, i.e. maps between condensed sets, we'll take natural transformations. This poses a few problems.

Firstly, we need to be careful to not end up with set-theoretical issues. Indeed, the category **CHaus** has a proper class of objects, and hence the natural transformations between two functors $\underline{X}$ and $\underline{Y}$ will fail to be a set in general. To solve this, we fix some uncountable strong limit cardinal κ. This is a cardinal $|\mathbb{N}| < \kappa$ such that $2^\lambda < \kappa$ for any cardinal $\lambda < \kappa$. This means that if X and Y are sets with cardinality smaller than κ, then so will their product, union, and respective powersets. We will then restrict to just those compact Hausdorff spaces which are κ-small, i.e. with cardinality less than κ. In practice this is not a big restriction. Most sets one encounters "in the wild" have cardinality less than κ. For example, the set of real numbers, or even the set of functions $\mathbb{R} \to \mathbb{R}$ all have cardinality less than κ. Indeed, $|\mathbb{R}| = 2^{|\mathbb{N}|} < \kappa$ and

$$|\mathbb{R}^{\mathbb{R}}| \leq 2^{|\mathbb{R}\times\mathbb{R}|} = 2^{|\mathbb{R}|} < \kappa.$$

With this, we get a set of natural transformations between any two functors.

The second problem is that the category doesn't have all the nice properties that we want yet. For example, if we take two topological spaces X and Y, and consider the associated condensed sets $\underline{X}$ and $\underline{Y}$, then what would be the coproduct $\underline{X} \sqcup \underline{Y}$? We can't really write this as $\underline{Z}$ for some space Z. So, we will need to consider more objects than just the *representable* functors $\underline{X}$. From the example it becomes apparent that we need to somehow add colimits of the representable functors to our category. We can once again reap our solution from the field of algebraic topology: sheaves[2][3]. Sheaves are a very general object, and appear in various contexts. A condensed set is a special type of sheaf. We omit the definition of what a sheaf is here, and instead describe what it means in this context. For the interested readers, [Dag21, Section 1.2] gives a detailed description of condensed sets in terms of sheaves on a site.

Definition 2. A κ-*condensed set* is a functor

$$T\colon \mathbf{CHaus}_\kappa^{\mathrm{opp}} \to \mathbf{Set},$$

such that $T(\emptyset) = \{*\}$, and

1. For any two compact Hausdorff spaces K_1 and K_2 with cardinality smaller than κ, the natural map
$$T(K_1 \sqcup K_2) \to T(K_1) \times T(K_2),$$
is a bijection.

2. For any surjection $K' \twoheadrightarrow K$ of compact Hausdorff spaces with cardinality smaller than κ, let p_1 and p_2 be the two projections $K' \times_K K' \to K'$. Then the map
$$T(K) \to \{x \in T(K') \mid Tp_1(x) = Tp_2(x)\},$$
is also a bijection.

Similarly, one defines condensed abelian groups or rings, by considering functors into **Ab** or **Ring** instead.

Let us now try to unpack this definition. For the first condition, note that there are two inclusion maps $K_1 \to K_1 \sqcup K_2$ and $K_2 \to K_1 \sqcup K_2$, which give maps $T(K_1 \sqcup K_2) \to T(K_1)$ and $T(K_1 \sqcup K_2) \to T(K_2)$. By the universal property of the product we obtain a map $T(K_1 \sqcup K_2) \to T(K_1) \times T(K_2)$.

In the second property, let $f\colon K' \twoheadrightarrow K$ be the surjection. We then have the fiber product $K' \times_K K' = \{(x, y) \in K' \times K' \mid f(x) = f(y)\}$, together with the projection maps p_1, p_2 onto the two components. Since $f \circ p_1 = f \circ p_2$, we have $Tp_1 \circ Tf = Tp_2 \circ Tf$ which shows that the map in the definition is well-defined.

Let us verify that the functors $\underline{X}$ restrict to κ-condensed sets. Since there is a unique function $\emptyset \to X$, we see that $T(\emptyset) = \{*\}$ as this function is also continuous. For the other two properties:

1. By the universal property of the coproduct, giving a continuous map $K_1 \sqcup K_2 \to X$ is the same as giving a pair of continuous maps $K_1 \to X$ and $K_2 \to X$. So $\mathrm{Cont}(K_1 \sqcup K_2, X) \to \mathrm{Cont}(K_1, X) \times \mathrm{Cont}(K_2, X)$ is a bijection.

2. Let $e\colon K' \twoheadrightarrow K$ be a surjection of κ-small compact Hausdorff spaces. This gives a map
$$\underline{e}\colon\ \mathrm{Cont}(K, X) \to \mathrm{Cont}(K', X)\colon g \mapsto g \circ e.$$

[2]It can be shown that a sheaf is a colimit of (sheafifications of) representable sheaves (see [Sta23, Lemma 0GLW]).
[3]The pun with agricultural sheaves and fields *is* intended.

Since e is a surjection, the map $\underline{e}$ is an injection as $g_1 \circ e = g_2 \circ e$ implies $g_1 = g_2$. We now need to show that the image of $\underline{e}$ is precisely

$$S = \{h \in \mathrm{Cont}(K', X) \mid h \circ p_1 = h \circ p_2\}.$$

We already saw that $\mathrm{Im}(\underline{e}) \subseteq S$. On the other hand, if $h \in S$, then $e(x) = e(y) \implies h(x) = h(y)$ for any $x, y \in K'$. This implies that there is a well-defined map of sets $s\colon K \to X$ with $s(k) = h(e^{-1}(\{k\}))$. By construction, $h = s \circ e$. Since e is a surjection of compact Hausdorff spaces, it is closed. So s is continuous, as for any closed $W \subseteq X$, $s^{-1}(W) = e(h^{-1}(W))$ is closed. This shows that $h \in \underline{e}$.

This gives some confidence that the definition of a κ-condensed set is a good one. We have now seen that we can go from a topological space to a condensed set by sending X to $\mathrm{Cont}(-, X)$. Is there also a way to retrieve a topological space from a condensed set? From a condensed set T, we can obtain the "underlying set", by evaluating the functor in the singleton: $T(\{*\})$. Of course, not all condensed sets are of the form $\mathrm{Cont}(-, X)$ for some topological space X, and there are non-trivial condensed sets for which $T(\{*\}) = \{*\}$. The question still remains how we can equip the underlying set with a topology. We arrived at the definition of a condensed set by looking at compactly generated spaces. This also gives us the intuition for how to define a topology on the underlying set.

Let T be a κ-condensed set, and K a κ-small compact Hausdorff space. Note that every element $x \in T(K)$ induces a map (of sets) $f_x\colon K \to T(\{*\})$. Indeed, for every $k \in K$ we have an inclusion map $i_K\colon \{k\} \to K$ which gives a map $Ti_k\colon T(K) \to T(\{k\}) \cong T(\{*\})$. Then $f_x(k)$ is simply defined as $Ti_k(x)$. To make this clearer, we should look at the special case $T = \underline{X}$ for some topological space X. For $h \in \underline{X}(K) = \mathrm{Cont}(K, S)$, we have $f_h = h$. After all, $f_h(k) = \underline{X}i_k(h) = h \circ i_k = h(k)$. So, we can think of the elements of $T(K)$ as functions $K \to T(\{*\})$.

Definition 3. Let T be a κ-condensed set. We equip the underlying set $T(\{*\})$ with the final topology induced by all the maps $f_x\colon K \to T(\{*\})$ for every κ-small compact Hausdorff space K and $x \in K$. This topological space is denoted as $T(\{*\})_{\mathrm{top}}$.

Let us call a topological space κ-compactly generated, if it is compactly generated by just the κ-small compact Hausdorff spaces. From the way we constructed the topology on the underlying set we find that if X is κ-compactly generated, then $\underline{X}(\{*\})_{\mathrm{top}} \cong X$. We can say more:

Proposition 4 ([Sch19, Proposition 1.7])**.** *The functor $X \mapsto \underline{X}$ is faithful, and fully faithful on κ-compactly generated condensed sets. The functor $T \mapsto T(\{*\})_{top}$ is a left adjoint to $X \mapsto \underline{X}$.*

Proof. To see that it is faithful, let $f\colon X \to Y$ be a continuous function between topological spaces. Then we get a morphism

$$\underline{f}\colon \mathrm{Cont}(-, X) \to \mathrm{Cont}(-, Y)\colon h \mapsto f \circ h.$$

Evaluating in $\{*\}$ gives back the original function: $\underline{f}_{\{*\}}\colon X \to Y\colon x \mapsto f(x)$. The fullness for κ-compactly generated spaces follows immediately from the adjunction:

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(\underline{X}(\{*\})_{\mathrm{top}}, Y) \cong \mathrm{Hom}(\underline{X}, \underline{Y}).$$

Hence, it remains to show this adjunction. Let T be a κ-condensed set, and X a topological space. A continuous map $h\colon T(\{*\})_{\mathrm{top}} \to X$ is the same as a set map $h\colon T(\{*\}) \to X$ such that for every κ-small compact Hausdorff space K and $x \in T(K)$, the composition $h \circ f_x\colon K \to T(\{*\}) \to X$ is continuous.

Now, note that any natural transformation $\alpha\colon T \to \underline{X}$ is completely determined by its value on $\{*\}$. Indeed, for any $k \in K$, the diagram

$$\begin{tikzcd} T(K) \arrow[r, "\alpha_K"] \arrow[d, "Ti_k"] & \underline{X}(K) \arrow[d, "-\circ i_k"] \\ T(\{*\}) \arrow[r, "\alpha_{\{*\}}"] & X \end{tikzcd}$$

commutes by naturality of α. So for $x \in T(K)$ we have that

$$(\alpha_K(x))(k) = \alpha_K(x) \circ i_k = \alpha_{\{*\}}(Ti_k(x)) = \alpha_{\{*\}}(f_x(k)).$$

Hence, $\alpha_{\{*\}} \circ f_x = \alpha_k(x) \in \underline{X}(K)$, and we see that $\alpha_{\{*\}}$ is continuous.

Conversely, given a continuous map $h\colon T(\{*\}) \to X$, we can define a natural transformation $\beta\colon T \to \underline{X}$ with $\beta_K\colon T(K) \to \underline{X}(K)\colon x \mapsto h \circ f_x$. To check that this indeed gives a natural transformation, let $g\colon K \to K'$ be a continuous map between κ-small compact Hausdorff spaces. Then

$$\begin{tikzcd} T(K') \arrow[r, "Tg"] \arrow[d, "\beta_{K'}"] & T(K) \arrow[d, "\beta_K"] \\ \underline{X}(K') \arrow[r, "-\circ g"] & \underline{X}(K) \end{tikzcd}$$

commutes, as

$$\begin{aligned} (\beta_{K'}(x) \circ g)(k) = (h \circ f_x)(g(k)) &= h \circ Ti_{g(k)}(x) \\ &= h \circ T(g \circ i_k)(x) \\ &= h \circ Ti_k(Tg(x)) \\ &= h \circ f_{Tg(x)}(k) = \beta_K((Tg(x))), \end{aligned}$$

for all $k \in K$ and $x \in T(K')$. □

We now want to compute limits and colimits in κ-condensed sets. For this we look at extremally disconnected spaces. An *extremally disconnected* space is a compact Hausdorff space such that the closure of every open subset is again open. This is equivalent to being a projective object in **CHaus** ([Gle58, Theorem 2.5]). It turns out that restricting to extremally disconnected spaces still gives rise to the same category of κ-condensed sets. Furthermore, the second condition in the definition of a κ-condensed set turns out to be automatic ([Dag21, Section 1.2]). This gives the following important result.

Theorem 5 ([Sch19, Theorem 2.2])**.** *The category of κ-condensed abelian groups is abelian. Limits and colimits can be computed object-wise on extremally disconnected spaces.*

Let us finish by using this result to show what happens in the condensed setting to the inclusion map $\iota\colon \mathbb{Q} \to \mathbb{R}$. To check whether $\underline{\iota}\colon \underline{\mathbb{Q}} \to \underline{\mathbb{R}}$ is still a monomorphism or an epimorphism we use the previous theorem. This tells us that it is equivalent to checking that $\underline{\iota}_X\colon \mathrm{Cont}(X, \mathbb{Q}) \to \mathrm{Cont}(X, \mathbb{R})$ is a monomorphism or epimorphism for every extremally disconnected space X. In abelian groups, monomorphisms and epimorphism are just injective and surjective group homomorphisms. The maps $\underline{\iota}_X$ are injective, but not always surjective. For example, $\underline{\iota}_{\{*\}}\colon \mathbb{Q} \to \mathbb{R}$ is not surjective. So, we see that moving to condensed abelian groups, the category becomes much better behaved. We have of course only scratched the surface in this paper, but hopefully it has provided enough motivation to the reader to explore more about this fascinating new theory on their own.

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