A Condensed Introduction to Condensed Mathematics

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1 Introduction

The goal of this paper is to provide an idea of what condensed mathematics is about, while not assuming too[1] much mathematical background from the reader. The material is based on the lecture notes by Peter Scholze and the master class in condensed mathematics by Peter Scholze and Dustin Clausen ([Sch19, CS20]). Let us now try to motivate why one might consider studying condensed sets. Additionally, a potential train of thought leading to the definition of a condensed set is proposed, which hopes to give the reader some intuition where this seemingly strange definition comes from.

2 Why condensed sets?

The modern definition of a topological space only came about in the 1920s. It has proven to be a very interesting and rich field of study, with many applications across mathematics. Even so, it seems that there is something fundamentally wrong with the way we combine algebraic structures with topology. Consider the inclusion map

$$\mathbb{Q} \hookrightarrow \mathbb{R},$$

for example. As a map between abelian groups, this map is injective, but not surjective. However, when we instead consider this as a map between Hausdorff topological abelian groups, where $\mathbb{Q}$ and $\mathbb{R}$ have been equipped with their usual metric topology, then this map also becomes an epimorphism. Indeed, if two continuous maps $f, g \colon \mathbb{R} \to X$ agree on the dense subspace $\mathbb{Q}$, then they must be equal, if X is Hausdorff. One sees that it somehow becomes more difficult to tell apart $\mathbb{Q}$ from $\mathbb{R}$ in the topological setting.

A potential way to fix this would be to consider only a particularly nice class of topological spaces. For example, one could try to restrict to just **CHaus**, the category of compact Hausdorff spaces. There, every continuous map is also closed, and hence a continuous bijection is automatically a homeomorphism. In fact, this category is an abelian category. Indeed, Pontryagin duality gives that

[1] Throughout this paper it will be assumed that the reader is somewhat familiar with basic notions from category theory, and in particular homological algebra.

discrete abelian groups are dual to compact Hausdorff groups (by considering the group of continuous group characters). So $\mathbf{Ab}^{\mathrm{opp}} \simeq \mathbf{CHaus}$, and $\mathbf{CHaus}$ is abelian as the opposite category of an abelian category. This means that it makes sense to start talking about long exact sequences and homology in this setting. The problem with this approach is that there are many interesting and non-pathological topological groups that are not compact. For example, the real numbers with the metric topology is not compact. Hence, we will need some way of including a broader class of topological spaces.

At this point it is a good idea to study the literature. The field of algebraic geometry turns out to be particularly fruitful. In [Ste67] it is suggested to look at the category of compactly generated spaces. This category satisfies some nice properties relating unions products and quotients, such as $Z^{Y\times X} = (Z^Y)^X$. It is also not too strict: it contains many families of topological spaces, such as locally compact spaces, CW-complexes and first-countable spaces. The product in this category turns out to be the natural product for CW-complexes (see the appendix of [Hat02]). What does it mean for a topological space to be compactly generated?

Definition 1. A topological space X is said to be *compactly generated* if the topology on X coincides with the final topology generated by all the continuous maps $K \to X$ with K compact Hausdorff. In other words, a map $f\colon X \to Y$ is continuous, if $K \to X \to Y$ is continuous for every continuous map $K \to X$.

So, if we understand the continuous maps $K \to X$, we understand the space X. We now take a small leap in logic, motivated by algebraic geometry, and conclude that X is determined by the functor

$$\underline{X} \stackrel{\mathrm{def}}{=} \mathrm{Cont}(-, X)\colon \mathbf{CHaus}^{\mathrm{opp}} \to \mathbf{Set},$$

which maps any compact Hausdorff space K to the set of continuous functions $K \to X$. This claim is justified in proposition 2. We can of course consider the same type of functor for any topological space X. The functor $\underline{X}$ (with some minor modifications) will be the condensed set associated to X. For this to lead to a good theory, which would replace topological spaces, we want two things:

1. The definition is strong enough. We are able to express the topological properties we want in this new setting.
2. The definition is not too strong. We do not end up with a theory that is just the same as topological spaces. Constructions and invariants that we care about should be valid in this new setting.

The goal of the following sections is to provide evidence that condensed mathematics does indeed behave as wanted.

3 What are condensed sets?

Proposition 2.

4 Condensed abelian groups

References

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