A Fluid Introduction to Condensed Mathematics

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Contents

1 Introduction

Most of the contents of this paper will be based on the lecture notes by Peter Scholze, who developed the theory of condensed mathematics together with Dustin Clausen ([Sch19]).

2 Sheaves

A central role in this paper will be played by sheaves. It therefore makes sense to study these objects more closely, and attempt to gain some intuition before jumping straight into the heart of the matter. The first step will be to convince ourselves that sheaves are interesting enough to study, and that they form a nice type of category. As such, we will look at sheaf cohomology, and compare it to singular cohomology. But first, we must define what a sheaf is.

2.1 On topological spaces

Consider the following situation in algebraic geometry. We have some (affine) algebraic variety X, with the Zariski Topology. Associated to X is an ideal I of all the polynomials that vanish on X. Then, the quotient ring $\mathcal{O}_X(X) = k[x_1, \ldots, x_n]/I$, is the coordinate ring of X. The elements of $\mathcal{O}_X(X)$ are the regular functions on X. For an open $U \subseteq X$ we can also consider the regular functions on U, which we will denote as $\mathcal{O}_X(U)$. This mapping between open subsets of X and rings, given by $\mathcal{O}_X$, is precisely a sheaf. In this case, $\mathcal{O}_X$ is known as the *structure sheaf*. In general, a sheaf will give some global data that can be defined locally.

Definition 2.1 (Presheaves)**.** Let X be a topological space. A *presheaf of sets* $\mathcal{F}$ on X consists of two things:

1. For each open $U \subseteq X$ a set $\mathcal{F}(U)$. These are known as the *sections* of $\mathcal{F}$ over U.

2. For each inclusion $U \subseteq V$ a map $\rho_{U,V} : \mathcal{F}(V) \to \mathcal{F}(U)$, such that $\rho_{U,U} = id_{\mathcal{F}(U)}$, and for $U \subseteq V \subseteq W$ we have $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$. We call the $\rho_{U,V}$ the *restrictions*, and if $s \in \mathcal{F}(U)$, we will often denote $\rho_{V,U}(s) = s|_V$.

Remark 2.2. If we write $\mathbf{Open}(X)$ for the category of open sets of X, with morphisms given by the inclusion maps, then a presheaf is precisely a functor $\mathcal{F} \colon \mathbf{Open}(X)^{\mathrm{opp}} \to \mathbf{Set}$.

This only defines what a "presheaf" is. The sheaf condition will ensure that the value of $\mathcal{F}(U)$ can be constructed by defining it locally on elements of a cover $\{U_i\}_{i \in I}$ of U. It is exactly this interaction between local definitions and global objects that makes sheaves so useful.

Definition 2.3 (Sheaves)**.** Let X be a topological space, and $\mathcal{F}$ a presheaf of sets on X. We say that $\mathcal{F}$ is a sheaf, if it satisfies the following additional properties for any open $U \subseteq X$ and open cover $\{U_i\}_{i \in I}$ of U:

- (uniqueness/locality) If $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.
- (gluing) If $\{s_i \in \mathcal{F}(U_i)\}_{i \in I}$ is a collection of sections such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \text{ for all } i, j,$$

  then there is a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.

In other words, if we have a bunch of sections that agree on overlaps, then we can uniquely glue them together.

Remark 2.4. We can reformulate the properties in a more categorical manner. Namely, for any open cover $\{U_i\}_{i \in I}$ of $U \subseteq X$, the following diagram:

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j),$$

has to be an equalizer. This diagram makes sense in any target category that has all limits. In terms of sets, the first map is given by $s \mapsto (s|_{U_i})_{i \in I}$. The pair of maps is given by the two possible restrictions, $(s_i)_{i \in I} \mapsto ((s_i|_{U_i \cap U_j})_{j \in I})_{i \in I}$ and $(s_j)_{j \in I} \mapsto ((s_j|_{U_i \cap U_j})_{i \in I})_{j \in J}$.

In this way, we can define sheaves on rings or abelian groups, or other categories with all (set-indexed) limits. For example, a presheaf of abelian groups on X, is a functor $\mathcal{F} \colon \mathbf{Open}(X)^{\mathrm{opp}} \to \mathbf{Ab}$.

In many cases, defining some structure on sheaves, will come down to defining it on $\mathcal{F}(U)$ in a compatible way. As an example, we can look at morphisms of sheaves.

Definition 2.5 (Morphisms of sheaves)**.** Let $\mathcal{F}, \mathcal{G}$ be presheaves on a topological space X. A *morphism of presheaves* $\varphi \colon \mathcal{F} \to \mathcal{G}$ assigns to each open $U \subset X$ a morphism $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with the restriction maps, i.e. for $U \subseteq V \subseteq X$ the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "\rho^{\mathcal{F}}_{U,V}"] & \mathcal{G}(V) \arrow[d, "\rho^{\mathcal{G}}_{U,V}"] \\ \mathcal{F}(U) \arrow[r, "\varphi_U"] & \mathcal{G}(U) \end{tikzcd}.$$

In other words, φ is a natural transformation between $\mathcal{F}$ and $\mathcal{G}$. Using a bit of abusive notation, the compatibility can be read as $\varphi(s|_U) = \varphi(s)|_U$. A *morphism of sheaves* is a morphism of the underlying presheaves.

For a topological space X, we now have (pre)sheaves and morphisms between them. These form a category. We will write $\mathbf{PSh}(X)$ for the category of presheaves of sets on X, and $\mathbf{Sh}(X)$ for the category of sheaves of sets on X. In the same way, we write $\mathbf{Ab}(X), \mathbf{Ring}(X), \mathbf{Vect}(X), \ldots$ for the category of sheaves of abelian groups, rings, vector spaces, ... on X.

Remark 2.6. We can "recover" the underlying space, by taking $X = \{*\}$, the one-point space. We have $\mathbf{Sh}(*) = \mathbf{Set}, \mathbf{Ab}(*) = \mathbf{Ab}, \ldots$

2.1.1 Examples

To solidify our understanding of sheaves, it will be beneficial to look at some examples.

Example 2.7. The simplest examples of sheaves are those were $\mathcal{F}(U)$ is not just a set, but a set of functions, and the restrictions correspond to actual restrictions.

Example 2.8. Let Y be another topological space, then we can define $\mathcal{F}(U) \stackrel{\text{def}}{=} \mathrm{Cont}(U, Y)$ where $\mathrm{Cont}(U, Y) = \{f \colon U \to Y \mid f \text{ continuous }\}$. The restrictions are defined as actual restrictions, and we get a presheaf. To see that it is a sheaf, we need to verify the gluing condition. Let $\{U_i\}$ be an open covering of $U \subseteq X$, and $f_i \colon U_i \to Y$ is a continuous map for each $i \in I$, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. We can now define a map $f \colon U \to Y$ via $f(u) = f_i(u)$ for any $i \in I$ such that $u \in U_i$. By assumption, this map is well-defined. To see that it is continuous, for any $V \subseteq Y$ open, we can write

$$f^{-1}(V) = U \cap f^{-1}(V) = \bigcup_{i \in I} (U_i \cap f^{-1}(V)) = \bigcup_{i \in I} f_i^{-1}(V),$$

which is open since all the f_i are continuous.

So we see that $\mathcal{F}$ is a sheaf. In the case that Y has the discrete topology, we call $\mathcal{F}$ the *constant sheaf with value* Y. The sections of $\mathcal{F}$ over U are the *locally* constant functions, i.e. at each point $x \in U$ we can find an open neighborhood V of x, such that f is constant on V.

Example 2.9. Let $f \colon X \to Y$ be a continuous map, then we get a sheaf on X by the rule

$$\Gamma(Y/X)(U) = \{s \colon U \to Y \mid f \circ s = 1_U\}.$$

The gluing construction is the same as in the previous example. To see that this yields another section, note that for any $u \in U$, $s(u) = s_i(u)$ for some $i \in I$, and hence $f(s(u)) = f(s_i(u)) = u$. We call $\Gamma(Y/X)$ the *sheaf of sections of* f.

Example 2.10. As we remarked above, the structure sheaf $\mathcal{O}_X$ is also a sheaf, where again, the restrictions are actual restrictions of functions. To show that the gluing of regular functions is again regular, requires some machinery from algebraic geometry, which falls outside the scope of this paper.

As it turns out, it is possible to generalize the definition of sheaves to categories which also have the notion of a covering. We will need this notion when discussing condensed sets.

2.2 On a site

The idea of a site was first introduced by Alexander Grothendieck, and has proven to be very useful in algebraic geometry. For this subsection we largely follow [Sta23, Tag 00UZ].

2.2.1 Coverings and sites

To define a site, we collect all the essential properties that coverings of topological spaces have:

Definition 2.11 (Coverings and Sites)**.** A *site* is a small category $\mathcal{C}$ together with a set $\mathrm{Cov}(\mathcal{C})$ of *coverings* of $\mathcal{C}$, where the following axioms hold: [1]

1. If $V \to U$ is an isomorphism, then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
2. If $\{U_i \to U\}_{i \in I}$ is a covering, and for each $i \in I$, $\{V_{ij} \to U_i\}_{j \in J_i}$ is a covering as well, then so is the composition $\{V_{ij} \to U\}_{i \in I, j \in J_i}$.
3. If $\{U_i \to U\}_{i \in I}$ is a covering and $V \to U$ is a morphism of $\mathcal{C}$, then the pullback $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i \in I}$ is a covering.

To make sense of this definition, let us look at the canonical example.

Example 2.12. Let X be a topological space, then $\mathbf{Open}(X)$ is a site with coverings given by the open covers. Let us verify that the axioms hold:

1. The only isomorphisms in $\mathbf{Open}(X)$ are the identity maps. Since $\{U\}$ is an open cover of U for any $U \in \mathbf{Open}(X)$, the first axiom is satisfied.
2. If $U = \bigcup_{i \in I} U_i$ is an open cover of U, and for each $i \in I$, $U_i = \bigcup_{j \in J_i} U_{ij}$ is an open cover of U_i, then $U = \bigcup_{i \in I} \bigcup_{j \in J_i} U_{ij}$ is an open cover of U.
3. If $U = \bigcup_{i \in I} U_i$ is an open cover, and $V \subseteq U$, then $V = \bigcup_{i \in I} V \cap U_i$ is an open cover of V.

Here we used that $U \times_W V = U \cap V$ for $U, V \subseteq W$, which follows from $S \subseteq V, S \subseteq U \implies S \subseteq V \cap U$.

The following example is still quite similar to the previous example. The underlying site of a condensed set will be similar to this site.

Example 2.13. Let G be a group, then we can consider the category of all G-sets, i.e. sets with a corresponding action by the group G. Morphisms are given by maps $f\colon X \to Y$ satisfying $g \cdot f(x) = f(g \cdot x)$, i.e. G-equivariant maps. To make this into a site[2], we define the covers to be the families of jointly surjective maps. In other words $\{f_i\colon X_i \to X\}_{i \in I}$ is a cover if $\bigcup_{i \in I} f_i(X_i) = X$. Let us verify the axioms:

[1] We force $\mathrm{Cov}(\mathcal{C})$ to be a set since we will take limits over all coverings. It is possible to let $\mathrm{Cov}(\mathcal{C})$ be a class, and then show that it can be replaced with a set of coverings that gives rise to the same category of sheaves. See [Sta23, Tag 00VI].

[2] Technically, this is not a site, since the category of all G-sets is a proper class. The example is simply meant to illustrate what covers could look like. There is a way around this problem though. See [Sta23, Example 00VK].

1. If $f\colon X \to Y$ is an isomorphism, then in particular it is surjective, so $\{f\colon X \to Y\}$ is a cover of Y
2. If $\{f_i\colon X_i \to Y\}_{i\in I}$ is jointly surjective, and for each $i \in I$ the family $\{f_{ij}\colon X_{ij} \to X_i\}_{j\in J_i}$ is jointly surjective, then so is $\{f_i \circ f_{ij}\colon X_{ij} \to Y\}_{i\in I, j\in J_i}$.
3. Let $\{f_i\colon X_i \to Y\}_{i\in I}$ be a cover, and $f\colon T \to Y$ a G-equivariant map. We claim that the pullback $X_i \times_Y T$ exists and is given by $S = \{(x,t) \in X_i \times T \mid f_i(x) = f(t)\}$. This is a G-set, as if $f_i(x) = f(t)$, then
$$f_i(g\cdot x) = g\cdot f_i(x) = g\cdot f(t) = f(g\cdot t).$$
Since S is the pullback in **Set** and any G-equivariant map is in particular a map of sets, the claim follows. To see that the projection maps $\{X_i \times_Y T \to T\}$ are jointly surjective, take $t \in T$. There exists $i \in I$ such that $f(t) \in f_i(X_i)$, so $(x_i, t) \in X_i \times_Y T$ for some $x_i \in X_i$.

In this new context, we have to reword the definitions of presheaves and sheaves in a more abstract manner.

Definition 2.14 (Presheaves and Sheaves on a Site)**.** Let $\mathcal{C}$ be a small category. A *presheaf on* $\mathcal{C}$ is a contravariant functor $\mathcal{F}\colon \mathcal{C} \to \mathbf{Set}$. If $\mathcal{C}$ is a site, then a *sheaf on* $\mathcal{C}$ is a presheaf $\mathcal{F}$, such that for any covering $\{U_i \to U\}_{i\in I}$, the following diagram is an equalizer:

$$\mathcal{F}(U) \xrightarrow{e} \prod_{i\in I} \mathcal{F}(U_i) \underset{p_1^*}{\overset{p_0^*}{\rightrightarrows}} \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)\ .$$

The map e is given by $s \mapsto (s|_{U_i})_{i\in I}$. For the second maps, take $i, j \in I$. Then we have projections $p_0^{(i,j)}\colon U_i \times_U U_j \to U_i$ and $p_1^{(i,j)}\colon U_i \times_U U_j \to U_j$. These result in maps $p_0^{(i,j),*}\colon \mathcal{F}(U_i) \to \mathcal{F}(U_i \times_U U_j)$ and $p_1^{(i,j),*}\colon \mathcal{F}(U_j) \to \mathcal{F}(U_i \times_U U_j)$. The maps p_0^* and p_1^* are then given at component i, j by mapping $(s_k)_{k\in I}$ to $p_0^{(i,j),*}(s_i)$ and $p_1^{(i,j),*}(s_j)$ respectively.

2.2.2 Sheafification

In a lot of constructions, the natural thing we do, will end up being a presheaf, but not a sheaf in general. So, we will need some way to turn presheaves into sheaves.

Proposition 2.15. *The fully faithful inclusion*
$$\iota\colon \mathbf{Sh}(\mathcal{C}) \hookrightarrow \mathbf{PSh}(\mathcal{C}),$$
admits a left adjoint $\mathcal{F} \to \mathcal{F}^\sharp$, "sheafification".

We will give an explicit construction of $\mathcal{F}^\sharp$. As it turns out, "sheafification" is actually a two-step process. The first step is making the presheaf separated, and then turning the separated presheaf into a sheaf.

Definition 2.16. We say that a presheaf $\mathcal{F}$ is *separated* if $\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i)$ is injective for any cover $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

Remark 2.17. Since any equalizer is a monomorphism, it follows from the definition of sheaves on a site, that a sheaf is separated.

Example 2.18. The presheaf $\mathcal{F}$ on a topological space X, which maps every open to the same set $Y \neq \{*\}$ is not separated, as

$$Y = \mathcal{F}(\emptyset) \to \prod_{i \in \emptyset} = \{*\},$$

is not injective if $Y \neq \{*\}$.

On the other hand, the presheaf $\mathcal{G}$ on a topological space X which maps every open U to the constant functions $U \to Y$ for some space Y, is separated but not a sheaf. After all, if $U = \cup_i U_i$ is an open cover of U, and $f, g \in \mathcal{G}(U)$ are such that $f|_{U_i} = g|_{U_i}$ then $f = g$, since any $x \in U$ is contained in some U_i, and hence $f(x) = f|_{U_i}(x) = g|_{U_i}(x) = g(x)$. Since there is only a single function $\emptyset \to Y$ there are no problems with empty coverings. $\mathcal{G}$ fails to be a sheaf in general, since if $U = U_1 \cup U_2$ with $U_1 \cap U_2 = \emptyset$ and $U_1 \neq \emptyset \neq U_2$, then we can't "glue" the functions $f\colon U_1 \to Y, f(x) = y_1$ and $g\colon U_2 \to Y, f(x) = y_2$ together to a constant function on U if $y_1 \neq y_2$.

We first have a few lemmas regarding limits of presheaves and sheaves.

Lemma 2.19. *Let $\mathcal{C}$ be a site, then limits and colimits exist in $\mathbf{PSh}(\mathcal{C})$. Additionally, for any $U \in \mathcal{C}$, the functor $\mathbf{PSh}(\mathcal{C}) \to \mathbf{Set}\colon \mathcal{F} \mapsto \mathcal{F}(U)$ commutes with all limits and colimits.*

Proof. We only prove the case of limits. The colimit case is analogous.

Let $\mathcal{F}\colon \mathcal{I} \to \mathbf{PSh}(\mathcal{C})$ be a diagram. We get a cone $(\mathcal{F}_{\lim}, p^i)$ by moving to the images in $\mathbf{Set}$. In other words we take $\mathcal{F}_{\lim}(U) = \lim_{i \in \mathcal{I}} \mathcal{F}_i(U)$, and $p^i_U\colon \lim_{i \in \mathcal{I}} \mathcal{F}_i(U) \to \mathcal{F}_i(U)$. If $U \to V$ is a morphism in $\mathcal{C}$, we get a unique map $\mathcal{F}_{\lim}(V) \to \mathcal{F}_{\lim}(U)$, given by the fact that $\mathcal{F}_{\lim}(U)$ is a limit, and $\mathcal{F}_{\lim}(V)$ is a cone on all the $\mathcal{F}_i(U)$ by composing the maps $\mathcal{F}_{\lim}(V) \to \mathcal{F}_i(V)$ with the maps $\mathcal{F}_i(V) \to \mathcal{F}_i(U)$. So we see that $\mathcal{F}_{\lim}$ is indeed a presheaf.

$$\begin{tikzcd}
 & & \mathcal{F}_{\lim}(V) \arrow[dll, bend right, "p^i_V"'] \arrow[d, dashed] \arrow[drr, bend left, "p^{i'}_V"] & & \\
 & & \mathcal{F}_{\lim}(U) \arrow[dl, "p^i_U"'] \arrow[dr, "p^{i'}_U"] & & \\
\mathcal{F}_i(V) \arrow[r] & \mathcal{F}_i(U) \arrow[rr] & & \mathcal{F}_{i'}(U) & \mathcal{F}_{i'}(V) \arrow[l]
\end{tikzcd}$$

To see that the maps p^i are morphisms of presheaves, we need to verify that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}_{\lim}(V) \arrow[r, "p^i_V"] \arrow[d] & \mathcal{F}_i(V) \arrow[d] \\
\mathcal{F}_{\lim}(U) \arrow[r, "p^i_U"] & \mathcal{F}_i(U)
\end{tikzcd},$$

but that already follows from the previous diagram.

Let us now verify that $\mathcal{F}_{\lim}$ is actually a limit. If $(\mathcal{G}, g^i)$ is another cone, then for each $U \in \mathcal{C}$, we get a unique map $\mathcal{G}(U) \to \mathcal{F}_{\lim}(U)$, such that the corresponding cone diagrams commute. It suffices to show that these maps combine to form a map of presheaves. Using the universal property of limits, this comes down to showing

$$\begin{tikzcd} \mathcal{G}(V) \arrow[r] \arrow[d] & \mathcal{F}_{\lim}(V) \arrow[d] \\ \mathcal{G}(U) \arrow[r] & \mathcal{F}_{\lim}(U) \end{tikzcd} \iff \begin{tikzcd} \mathcal{G}(V) \arrow[r, "g^i_V"] \arrow[d] & \mathcal{F}_i(V) \arrow[d] \\ \mathcal{G}(U) \arrow[r, "g^i_U"] & \mathcal{F}_i(U) \end{tikzcd} \quad \forall i \in \mathcal{I}.$$

The commutativity of these last diagrams, is just saying that the maps g^i are presheaf morphisms, which is true by $(\mathcal{G}, g^i)$ being a cone. $\square$

The big difference between sheaves and presheaves, is that we can glue things together defined on a cover. The trick will be to force the presheaf to behave as we want. Let us try and make this more precise.

So far we have just talked about coverings as objects. We can also consider maps between coverings. If $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_j \to V\}_{j \in J}$ are two coverings, a morphism of coverings between $\mathcal{U}$ and $\mathcal{V}$ is a morphism $U \to V$ and a map $\alpha \colon I \to J$ together with morphisms $U_i \to V_{\alpha(i)}$ such that

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_{\alpha(i)} \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

commutes. If $U = V$ and $U \to V$ is the identity, we call $\mathcal{U}$ a *refinement* of $\mathcal{V}$. For a $U \in \mathcal{C}$ the coverings together with refinements gives a category $\mathrm{Cov}(U)$. This allows the following reformulation:

Lemma 2.20. *Let $\mathcal{F}$ be a presheaf on a site $\mathcal{C}$. For $U \in \mathcal{C}$ and a cover $\mathcal{U} \in \mathrm{Cov}(U)$, define $\mathcal{F}(\mathcal{U})$ as the following equalizer:*

$$\mathcal{F}(\mathcal{U}) = \{(s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(U_i) \mid s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}\}.$$

If $\mathcal{U} \to \mathcal{V}$ is a morphism of coverings, there is an induced map $\mathcal{F}(\mathcal{V}) \to \mathcal{F}(\mathcal{U})$. This construction is functorial. Furthermore, since $\{1_U\}$ is a cover by the axioms of a site, this gives a map $\mathcal{F}(U) \cong \mathcal{F}(\{1_U\}) \to \mathcal{F}(\mathcal{U})$. The presheaf $\mathcal{F}$ is a sheaf if and only if the map $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$ is bijective for every cover $\mathcal{U}$.

Proof. Let $\mathcal{U} \to \mathcal{V}$ be a map of coverings. We have the following commutative diagram:

$$\begin{tikzcd} & U_i \arrow[r] \arrow[d] & V_{\alpha(i)} \arrow[d] & \\ U_i \times_U U_j \arrow[ur] \arrow[dr] & U \arrow[r] & V & V_{\alpha(i)} \times_V V_{\alpha(j)} \arrow[ul] \arrow[dl] \\ & U_j \arrow[r] \arrow[u] & V_{\alpha(j)} \arrow[u] & \end{tikzcd},$$

which gives a unique map $U_i \times_U U_j \to V_{\alpha(i)} \times_V V_{\alpha(j)}$ making the diagram commute. Applying $\mathcal{F}$ gives the diagram:

$$\begin{tikzcd}
\mathcal{F}(U_i) \arrow[d] & \mathcal{F}(V_{\alpha(i)}) \arrow[l] \arrow[d] \\
\mathcal{F}(U_i \times_U U_j) & \mathcal{F}(V_{\alpha(i)} \times_V V_{\alpha(j)}) \arrow[l] \\
\mathcal{F}(U_j) \arrow[u] & \mathcal{F}(V_{\alpha(j)}) \arrow[l] \arrow[u]
\end{tikzcd} . \tag{1}$$

We can now define $\mathcal{F}(\mathcal{V}) \to \mathcal{F}(\mathcal{U})$ by

$$(s_j)_{j \in J} \mapsto ((s_{\alpha(i)})|_{U_i})_{i \in I}.$$

That this map is well-defined follows from eq. (1). Indeed, if $(s_j)_{j \in J} \in \mathcal{F}(\mathcal{V})$, then

$$(s_{\alpha(j)}|_{V_{\alpha(i)} \times_U V_{\alpha(j)}})|_{U_i \times U_j} = (s_{\alpha(i)}|_{V_{\alpha(i)} \times_U V_{\alpha(j)}})|_{U_i \times U_j},$$

and hence

$$(s_{\alpha(i)}|_{U_i})|_{U_i \times_U U_j} = (s_{\alpha(j)}|_{U_j})|_{U_i \times_U U_j}.$$

It remains to show that this construction is functorial, since the final claim is just the definition of a sheaf of sites. Let, to this purpose, $\mathcal{U} \to \mathcal{V} \to \mathcal{W}$ be maps of coverings, with associated maps $\alpha \colon I \to J$ and $\beta \colon J \to K$. The map $\mathcal{F}(\mathcal{W}) \to \mathcal{F}(\mathcal{U})$ maps $(s_k)_{k \in K}$ to $(s_{\beta(\alpha(i))}|_{U_i})_{i \in I}$. The other map is given by first sending $(s_k)_{k \in K}$ to $(s_{\beta(j)}|_{V_j})_{j \in J}$, which is then sent to $((s_{\beta(\alpha(i))}|_{V_{\alpha(i)}})|_{U_i})_{i \in I}$. The result now follows by functoriality of $\mathcal{F}$. $\square$

We now have the following construction:

Lemma 2.21. *Given a presheaf of $\mathcal{F}$ on a site $\mathcal{C}$, we can construct a new presheaf $\mathcal{F}^+$ by setting*

$$\mathcal{F}^+(U) = \operatorname{colim}_{\mathcal{U} \in \mathrm{Cov}(U)} \mathcal{F}(\mathcal{U}),$$

Proof. Let us first show how $\mathcal{F}^+$ acts on morphisms. Consider a morphism $U \to V$. Then this gives a map $\mathrm{Cov}(V) \to \mathrm{Cov}(U)$ by mapping a cover $\{V_i \to V\}_{i \in I}$ of V to the cover $\{V_i \times_V U \to U\}_{i \in I}$ of U, which exists by the third axiom of coverings. From the previous lemma there are induced maps $\mathcal{F}(\mathcal{U}) \to \mathcal{F}(\mathcal{V})$. Since we have maps $\mathcal{F}(\mathcal{V}) \to \mathcal{F}^+(V)$, we obtain maps $\mathcal{F}(\mathcal{U}) \to \mathcal{F}^+(V)$. By the universal property of the colimit, this gives a unique map $\mathcal{F}^+(V) \to \mathcal{F}^+(U)$, making the cone diagrams commute.

$$\begin{tikzcd}
 & & \mathcal{F}^+(U) & & \\
 & & \mathcal{F}^+(V) \arrow[u, dashed] & & \\
\mathcal{F}(\mathcal{U}) \arrow[r] \arrow[urru, bend left] & \mathcal{F}(\mathcal{V}) \arrow[rr] \arrow[ur] & & \mathcal{F}(\mathcal{V}') \arrow[ul] & \mathcal{F}(\mathcal{U}') \arrow[l] \arrow[ullu, bend right]
\end{tikzcd}$$

It remains to show that this defines a functor $\mathcal{C}^{\text{opp}} \to \mathbf{Set}$. If we have morphisms $U \to V \to W$, then

$$W_i \times_W U \cong (W_i \times_W V) \times_V U,$$

since both are pullbacks of $W_i \longrightarrow W \longleftarrow U$. So $\text{Cov}(-)$ is a functor, and since $\mathcal{U} \mapsto \mathcal{F}(\mathcal{U})$ is a functor by the previous lemma, the result now follows. $\square$

It is possible to be more explicit about how $\mathcal{F}^+$ looks like. For two coverings $\mathcal{U},\mathcal{U}'$ of $U \in \mathcal{C}$, we have a common refinement $\{U_i \times_U U'_j \to U\}_{i\in I, j\in J}$ which exists by the second and third axioms. Furthermore, one can show that if $f, g \colon \mathcal{U} \to \mathcal{V}$ are refinements, then $\mathcal{F}(f) = \mathcal{F}(g)$ ([Sta23, Tag 00W7]). This gives that $\text{Cov}(U)^{\text{opp}} \to \mathbf{Set}$ is a filtered diagram. So

$$\mathcal{F}^+(U) = \Big(\coprod_{\mathcal{U}\in\text{Cov}(U)} \mathcal{F}(\mathcal{U})\Big)/\sim,$$

Where $s \sim s'$ if and only if there are covers $\mathcal{U},\mathcal{U}'$ with $s \in \mathcal{F}(\mathcal{U}), s' \in \mathcal{F}(\mathcal{U}')$ and a common refinement $\mathcal{V}$ such that

$$s_{\alpha(i)}|_{V_i} = s'_{\beta(i)}|_{V_i}, \forall i \in I.$$

We now come to the main theorem, from which proposition 2.15 will follow.

Theorem 2.22. *Let $\mathcal{F}$ be a presheaf on a site. Then*

1. *The presheaf $\mathcal{F}^+$ is separated.*
2. *If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf.*
3. *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism.*

Proof.

1. We need to show that $s \mapsto (s|_{U_i})_{i\in I}$ is injective for any cover $\{U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})$. Take $s, s' \in \mathcal{F}^+(U)$ such that $(s|_{U_i})_{i\in I} = (s'|_{U_i})_{i\in I}$ for some cover $\mathcal{U} = \{U_i \to U\}_{i\in I}$. By the description above of $\mathcal{F}^+$, we know that we can find covers $\mathcal{V}$ and $\mathcal{V}'$ of U, such that $s \in \mathcal{F}(\mathcal{V})/\sim$ and $s' \in \mathcal{F}(\mathcal{V}')/\sim$. Let $\mathcal{W}$ be a common refinement of the three covers $\mathcal{U}, \mathcal{V}, \mathcal{V}'$. Since it is a refinement of $\mathcal{U}$, we have that

   $$s'|_{W_j} = (s|_{U_{\alpha(j)}})|_{W_j} = (s|_{U_{\alpha(j)}})|_{W_j} = s'|_{W_j},$$

   so that $s = s'$.

2. We need to verify the sheaf condition. Let $\{U_i \to U\}_{i\in I}$ be a cover of U, and for each $i \in I$, $s_i \in \mathcal{F}^+(U_i)$ such that $s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$ for every $i, j \in I$. Since $\mathcal{F}$ is separated, the map $s \to (s|U_i)_{i\in I}$ is injective. It is hence enough to show that there is some $s \in \mathcal{F}^+(U)$ with $s|_{U_i} = s_i$ for all $i \in I$. For each $i \in I$ we can find a cover $\mathcal{U}_i = \{U_{ij} \to U_i\}$ such that $s_i \in \mathcal{F}(\mathcal{U}_i)/\sim$, and hence $s_{ij} \in \mathcal{F}(U_{ij})$ such that $s_i|_{U_{ij}} = s_{ij}/\sim$. In the same way as lemma 2.20 we get that

   $$s_{ij}|_{U_{ij}\times_U U_{i'j'}} = s_{i'j'}|_{U_{ij}\times_U U_{i'j'}}.$$

   So, $(s_{ij})_{i,j\in I} \in \mathcal{F}(\{U_{ij} \to U\}_{i,j\in I})$, and we can take $s = (s_{ij})_{i,j\in I}/\sim$. We just need to verify that $s|_{U_i} = s_i$. This follows from $(s|_{U_i})|_{U_{ij}} = s|_{U_{ij}} = s_i|_{U_{ij}}$, as $\mathcal{F}^+$ is also separated.

3. This follows from lemma 2.20.

□

With this we are now ready to prove proposition 2.15. We define $\mathcal{F}^\sharp = \mathcal{F}^{++}$.

Proof of proposition 2.15. We first note that for any map of presheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "\alpha"] & \mathcal{F}^+ \arrow[d, "\alpha^+"] \\ \mathcal{G} \arrow[r] & \mathcal{G}^+ \end{tikzcd},$$

where $\alpha^+_U(s/\sim) = \alpha_U(s)/\sim$. That this commutes, is by construction of α^+. Using this, we can now show that

$$\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathcal{F}^\sharp, \mathcal{G}) = \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathcal{F}, \iota(\mathcal{G})).$$

The bottom row of the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^+ \arrow[r] \arrow[d] & \mathcal{F}^{++} = \mathcal{F}^\sharp \arrow[d] \\ \iota(\mathcal{G}) \arrow[r] & \iota(\mathcal{G})^+ \arrow[r] & \iota(\mathcal{G})^{++} = \mathcal{G}^\sharp \end{tikzcd},$$

consists of isomorphisms by the previous theorem. So, any map $\mathcal{F} \to \iota(\mathcal{G})$ gives rise to a map $\mathcal{F}^\sharp \to \mathcal{G}$. Conversely, since every $s \in \mathcal{F}^\sharp(U)$ comes from sections in $\mathcal{F}(U)$, we can lift any map $\mathcal{F}^\sharp \to \mathcal{G}$ to a map $\mathcal{F} \to \mathcal{G}$. □

As an immediate consequence, we get that $(-)^\sharp$ commutes with all colimits. So $\mathbf{Sh}(\mathcal{C})$ has all colimits, since $\mathbf{PSh}(\mathcal{C})$ has all colimits by lemma 2.19. We can say more:

Proposition 2.23. *The functor $(-)^\sharp\colon \mathbf{PSh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C})$ is exact.*

Proof. Since it is a left adjoint, it is right exact. On the other hand, colimits over filtered diagrams commute with finite limits. So, $(-)^\sharp$ is left exact as functor between presheaves. We claim that if $\mathcal{I} \to \mathbf{Sh}(\mathcal{C})$ is a diagram, then the limit $\mathcal{F} = \lim_i \mathcal{F}_i$ as presheaves is a sheaf. For this we show that $\mathcal{F}(\mathcal{U}) \cong \mathcal{F}(U)$ for any cover $\mathcal{U} = \{U_j \to U\}$. Take $(s_j)_{j\in J} \in \mathcal{F}(\mathcal{U})$, then by definition of the limit, we can project these to elements $(s_{ij})_{j\in J} \in \mathcal{F}_i(\mathcal{U})$. Since each $\mathcal{F}_i$ is a sheaf, we have unique elements $s_i \in \mathcal{F}_i(U)$ such that $s_i|_{U_j} = s_{ij}$.

We now want an element $s \in \mathcal{F}(U)$ with projections equal to the s_i. Choosing an element of $\mathcal{F}(U)$, is the same as giving a map $\{*\} \to \mathcal{F}(U)$, which by the universal property of the limit is the same as giving a cone $(\{*\}, \lambda_i)$. Let $\lambda_i(*) = s_i$, then we just need to verify that this defines a cone. We need that for $f\colon i \to i'$ in $\mathcal{I}$, $\mathcal{F}(f)(s_i) = s_{i'}$. This follows, as $s_i|_{U_j}$ is mapped to $s_{i'}|_{U_j}$ for all $j \in J$, and hence s_i is mapped to $s_{i'}$ because $\mathcal{F}_{i'}$ is a sheaf. So we have a unique $s \in \mathcal{F}(U)$, and by the universal property of the limit

$$s|_{U_j} = s_j \iff s_i|_{U_j} = s_{ij}\ \forall i \in I,$$

which holds by construction. Hence, the claim holds, and $(-)^\sharp$ is also right exact as a functor into sheaves. □

To get a better understanding of sheafification, let us work out the process of sheafification for some presheaves.

Example 2.24. We consider the presheaf $\mathcal{F}$ of example 2.18. To calculate $\mathcal{F}^+$ we use that

$$\mathcal{F}^+(U) = \Big(\coprod_{\mathcal{U} \in \mathrm{Cov}(U)} \mathcal{F}(\mathcal{U}) \Big) / \sim .$$

If U is non-empty, then

$$\mathcal{F}(\mathcal{U}) = \{(y_i)_{i\in I} \in \prod_{i \in I} Y \mid y_i = y_j \ \forall i, j \in I\} = \{(y)_{i \in I} \mid y \in Y\}$$

for any cover $\mathcal{U}$ of U. Take $(y)_{i\in I} \in \mathcal{F}(\mathcal{U})$, and $(y')_{j\in J} \in \mathcal{F}(\mathcal{U}')$. For any common refinement $\mathcal{V}$ of $\mathcal{U}$ and $\mathcal{U}'$, we have $(y_{\alpha(i)})|_{V_i} = y$ and $(y'_{\beta(i)})|_{V_i} = y'$ so that $(y)_{i\in I} \sim (y')_{j \in J} \iff y = y'$. So, we find again that $\mathcal{F}^+(U) = Y$ as long as $U \neq \emptyset$.

Now, when $U = \emptyset$ there are two covers: $\{\emptyset \to \emptyset\}$, and the empty covering, $\{\}_{i\in\emptyset}$. We have $\mathcal{F}(\{\emptyset \to \emptyset\}) = \mathcal{F}(\emptyset) = Y$, while $\mathcal{F}(\{\}_{i\in\emptyset}) = \{*\}$. Now every $y \in Y$ is equivalent to $*$, as $\{\}_{i\in\emptyset}$ is a common refinement of the two covers, and the equivalence condition becomes an empty statement in this case. As such we find $\mathcal{F}^+(\emptyset) = \{*\}$. Note that $\mathcal{F}^+$ is isomorphic to $\mathcal{G}$ from example 2.18. After all, a constant function $U \to Y$ is the same as choosing an element in Y.

We now have a separated presheaf $\mathcal{F}^+$. What does the sheaf $\mathcal{F}^\sharp$ look like? Let us first consider the case that U is empty. Then for both covers $\{\emptyset \to \emptyset\}$ and $\{\}_{i\in\emptyset}$ we get $\mathcal{F}^+(\mathcal{U}) = \{*\}$. So $\mathcal{F}^\sharp(\emptyset) = \{*\}$.

More interesting is what happens when $U \neq \emptyset$. Let $\{U_i \to U\}_{i\in I}$ be a cover such that none of the U_i are empty. Then

$$\begin{aligned} \mathcal{F}^+(\mathcal{U}) &= \{(y_i)_{i\in I} \in \prod_{i\in I} Y \mid y_i|_{U_i \cap U_j} = y_j|_{U_i\cap U_j} \ \forall i, j \in I\} \\ &= \{(y_i)_{i\in I} \in \prod_{i\in I} Y \mid y_i = y_j,\, U_i \cap U_j \neq \emptyset\}, \end{aligned}$$

as $\mathcal{F}^+(\emptyset) = \{*\}$ and hence $y|_{U_i\cap U_j} = *$ if $U_i \cap U_j = \emptyset$ for any $y \in Y$. We can view elements of $\mathcal{F}^+(\mathcal{U})$ as functions $s\colon U \to Y$, that are constant on each of the opens U_i. Adding empty sets to the cover does not really change anything: the elements of $\mathcal{F}(\mathcal{U})$ will be $*$ at the indices for which $U_i = \emptyset$. Under $\sim$ such functions will all be the same. Take, $f \in \mathcal{F}^\sharp(U)$. Then there is some cover $\mathcal{U} = \{U_i\}_{i\in I}$ such that f arises from $\mathcal{F}^+(\mathcal{U})$. Consequently, for each $x \in U$, there is some U_i containing x, such that f is constant on U_i. In other words, f is a locally constant function on U. If we equip Y with the discrete topology then this is equivalent to saying that $f\colon U \to Y$ is continuous.

The sheaf $\mathcal{F}^\sharp$ is (perhaps a little confusingly) called the *constant sheaf with value* Y and sometimes denoted as $\underline{Y}$.

3 Sheaf cohomology

In the previous section we have explored some fundamental properties of sheaves of sets. We will now be looking at sheaves of abelian groups, which form a very interesting and rich abelian category. The previous section only proved results for sheaves of sets, but these results still hold for sheaves of abelian groups. This can be shown by either verifying that all the proofs still hold in the abelian case, or through a more abstract approach like in [Sta23, Section 00YR]

Proposition 3.1. *Let $\mathcal{C}$ be a site. The category $\mathbf{Ab}(\mathcal{C})$ of abelian sheaves on $\mathcal{C}$ is abelian.*

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be two abelian sheaves on $\mathcal{C}$. For natural transformations $\alpha, \beta \in \operatorname{Hom}_{\mathbf{Ab}(\mathcal{C})}(\mathcal{F}, \mathcal{G})$ we define $\alpha + \beta$ via

$$(\alpha + \beta)_U \overset{\text{def}}{=} \alpha_U + \beta_U,$$

where the $+$ on the right-hand side is the $+$ in $\mathbf{Ab}$. Let us verify that $\alpha + \beta$ is a natural transformation. If $f \colon U \to V$ is a morphism in $\mathcal{C}$ we have:

$$\begin{aligned}(\alpha + \beta)_U \circ \mathcal{F}(f) &= (\alpha_U + \beta_U) \circ \mathcal{F}(f) \\ &= \alpha_U \circ \mathcal{F}(f) + \beta_U \circ \mathcal{F}(f) \\ &= \mathcal{G}(f) \circ \alpha_V + \mathcal{G}(f) \circ \beta_V \\ &= \mathcal{G}(f) \circ (\alpha_V + \beta_V) \\ &= \mathcal{G}(f) \circ (\alpha + \beta)_V,\end{aligned}$$

so that $\alpha + \beta$ is indeed an abelian category. With this operation, the hom-sets become abelian groups. Using lemma 2.19 and proposition 2.23 we find that $\mathbf{Ab}(\mathcal{C})$ has finite limits and colimits. Explicitly, limits in sheaves are the limits in presheaves and colimits in sheaves are the sheafifications of the colimits in presheaves. As such, we find:

- The zero object is given by $\underline{0}$, the constant sheaf with value 0.
- The biproduct of $\mathcal{F}$ and $\mathcal{G}$ is given by $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$
- The kernel of a map $\alpha \colon \mathcal{F} \to \mathcal{G}$ is given by $\operatorname{Ker}(\alpha)(U) = \operatorname{Ker}(\alpha_U)$.
- The cokernel of $\alpha \colon \mathcal{F} \to \mathcal{G}$ is given by the sheafification of the presheaf defined by $\mathcal{F}(\alpha)(U) = \operatorname{Coker}(\alpha_U)$.

For $\mathbf{Ab}(\mathcal{C})$ to be an abelian category, we still need to show that for $\alpha \colon \mathcal{F} \to \mathcal{G}$ a morphism of sheaves, $\operatorname{Im}(\alpha) = \operatorname{Coim}(\alpha)$. In the category of abelian presheaves we have

$$\operatorname{Coim}(\iota(\alpha))(U) = \operatorname{Coim}(\iota(\alpha)_U) = \operatorname{Im}(\iota(\alpha)_U) = \operatorname{Im}(\iota(\alpha))(U),$$

where ι is the inclusion functor of sheaves into presheaves. Since $(-)^\sharp$ is exact, we find

$$\begin{aligned}
\operatorname{Coim}(\alpha) &= \operatorname{Coker}(\operatorname{Ker}(\alpha)) \\
&= \operatorname{Coker}^\sharp(\iota(\operatorname{Ker}(\alpha))) \\
&= \operatorname{Coker}^\sharp(\operatorname{Ker}(\iota(\alpha))) \\
&= \operatorname{Coim}^\sharp(\iota(\alpha)) \\
&= \operatorname{Im}^\sharp(\iota(\alpha)) \\
&= \operatorname{Ker}^\sharp(\operatorname{Coker}(\iota(\alpha))) \\
&= \operatorname{Ker}(\operatorname{Coker}^\sharp(\iota(\alpha))) \\
&= \operatorname{Ker}(\operatorname{Coker}(\alpha)) \\
&= \operatorname{Im}(\alpha).
\end{aligned}$$

□

Taking the sheafification is really needed when taking colimits. This is illustrated in the following example.

Example 3.2. Consider the two sheaves $\operatorname{Cont}(-,\mathbb{R})$ and $\operatorname{Cont}(-,2\pi\mathbb{Z})$ on the circle $S^1 \subseteq \mathbb{C}$. We claim that the quotient $\operatorname{Cont}(-,\mathbb{R})/\operatorname{Cont}(-,2\pi\mathbb{Z})$ as presheaves is not a sheaf. For this, consider the two open sets $U_1 = \{e^{i\theta} \in S^1 \mid \theta \in (0,2\pi)\}$ and $U_2 = \{e^{i\theta} \in S^1 \mid \theta \in (-\varepsilon,\varepsilon)\}$. We have continuous functions $f_i \colon U_i \to \mathbb{R}$ given by $f_1(e^{i\theta}) = \theta \in (0,2\pi)$, and $f_2(e^{i\theta}) = \theta \in (-\varepsilon,\varepsilon)$. Let $V = U_1 \cap U_2 = \{e^{i\theta} \in S^1 \mid \theta \in (-\varepsilon,0)\cup(0,\varepsilon)\}$. Then

$$(f_1|_V - f_2|_V)(e^{i\theta}) = \begin{cases} 2\pi & \theta \in (-\varepsilon,0) \\ 0 & \theta \in (0,\varepsilon) \end{cases},$$

so $f_1|_V - f_2|_V \in \operatorname{Cont}(V,2\pi\mathbb{Z})$ as it is locally constant. Consequently, the two functions agree on overlaps in the quotient. If $\operatorname{Cont}(-,\mathbb{R})/\operatorname{Cont}(-,2\pi\mathbb{Z})$ were a sheaf, then we would be able to glue the two functions together to a function $\bar{f} \in \operatorname{Cont}(S^1,\mathbb{R})/\operatorname{Cont}(S^1,2\pi\mathbb{Z})$ such that $\bar{f_1} = \bar{f}|_{U_1}$ and $\bar{f_2} = \bar{f}|_{U_2}$. Since S^1 is connected, the elements of $\operatorname{Cont}(S^1,2\pi\mathbb{Z})$ are constant functions. So, $f_1|_V - f|_V$, and $f_2|_V - f|_V$ would have to be constants, and hence also $f_1|_V - f_2|_V$, a contradiction.

We will now restrict ourselves to the special case where $\mathcal{C} = \mathbf{Open}(X)$ for some topological space X. We follow the general ideas presented in [CS20, Session 1], while filling out the details.

Proposition 3.3. *Let $f \colon X \to Y$ be a continuous map. Then there is a pair of adjoint functors: the pullback functor, $f^* \colon Ab(Y) \to Ab(X)$, with right adjoint, the pushforward functor $f_* \colon Ab(X) \to Ab(Y)$. Furthermore, f^* is exact.*

Proof. We start off by writing down what the pullback and pushforward functors look like. Let $\mathcal{F}$ be a sheaf on X, and $V \in \mathbf{Open}(Y)$. Then, $(f_*\mathcal{F})(V) \stackrel{\text{def}}{=} \mathcal{F}(f^{-1}(V))$. If $\mathcal{G}$ is a sheaf on Y, then we get a presheaf $V \mapsto \operatorname{colim}_{U \supseteq f(V)} \mathcal{G}(U)$, and $f^*\mathcal{G}$ is defined as the sheafification of this presheaf. There are now a lot of things to verify:

- $f_*\mathcal{F}$ is a sheaf on Y. Since f is continuous, $f^{-1}(V) \in \mathbf{Open}(X)$ for all $V \in \mathbf{Open}(Y)$. Furthermore, if $V \subseteq V' \subseteq V''$ then $f^{-1}(V) \subseteq f^{-1}(V') \subseteq f^{-1}(V'')$. So, f^{-1} is a functor $\mathbf{Open}(Y) \to \mathbf{Open}(X)$, and $f_*\mathcal{F}$ is a presheaf as composition of functors. If $\{V_i\}_{i\in I}$ is an open cover of V, then $\{f^{-1}(V_i)\}_{i\in I}$ is an open cover of $f^{-1}(V)$. So $(f_*\mathcal{F})(V) \to (f_*\mathcal{F})(\{V_i\}_{i\in I})$ is a bijection since $\mathcal{F}(f^{-1}(V)) \to \mathcal{F}(\{f^{-1}(V_i)\}_{i\in I})$ is a bijection by $\mathcal{F}$ being a sheaf. Consequently, $f_*\mathcal{F}$ is a sheaf by lemma 2.20.

- $f^*\mathcal{G}$ is a sheaf on X. Since we take the sheafification, we just need to verify that $G(U) = \operatorname{colim}_{V \supseteq f(U)} \mathcal{G}(V)$ defines a presheaf. First note that if $U \subseteq U'$, then $\{V \supseteq f(U')\} \subseteq \{V \supseteq f(U)\}$. So, $G(U)$ is also a co-cone for $\mathcal{G}(V)_{V\supseteq f(U')}$, and there is hence a unique map $G(U') \to G(U)$, by the colimit property. The uniqueness of this map gives the functoriality of G. After all, the map $G(U) \to G(U)$ must be the identity, by uniqueness, and the two maps $G(U'') \to G(U') \to G(U)$, for $U \subseteq U' \subseteq U''$ must also be equal by uniqueness.

- f_* is a functor. For $\alpha\colon \mathcal{F} \to \mathcal{G}$, we define $f_*\alpha$ via $(f_*\alpha)_V = \alpha_{f^{-1}(V)}$. Then,

$$(f_*\beta \circ f_*\alpha)_V = \beta_{f^{-1}(V)} \circ \alpha_{f^{-1}(V)} = (\beta \circ \alpha)_{f^{-1}(V)} = (f_*(\beta \circ \alpha))_V,$$

so we just need to check that $f_*\alpha$ is a natural transformation. If $V \subseteq V'$, then $f^{-1}(V) \subseteq f^{-1}(V')$, so naturality of $f_*\alpha$ in V, V' follows from naturality of α in $f^{-1}(V), f^{-1}(V')$.

- f^* is a functor. Let $\alpha\colon \mathcal{G} \to \mathcal{F}$ be a natural transformation, and G and F be the presheaves from the construction of $f^*\mathcal{G}$ and $f^*\mathcal{F}$. To define $f^*\alpha$ we need to give a map $G(U) \to F(U)$ for each $U \in \mathbf{Open}(X)$. For each $V \supseteq f(U)$ we have a map $\mathcal{F}(V) \to F(U)$ which we can compose with α_V to get a map $\mathcal{G}(V) \to F(U)$. Naturality of α implies that this gives a well-defined co-cone, and hence a unique morphism $G(U) \to F(U)$ by the universal property of colimits. Uniqueness of this map gives the functoriality of f^*.

- $\operatorname{Hom}_{\mathbf{Ab}(X)}(f^*\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_{\mathbf{Ab}(Y)}(\mathcal{G}, f_*\mathcal{F})$. Let $\alpha\colon \mathcal{G} \to f_*\mathcal{F}$ be a morphism in $\mathbf{Ab}(Y)$. So, for each $V \in \mathbf{Open}(Y)$ we have a map

$$\alpha_V\colon \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V)).$$

Now, if $V \supseteq f(U)$, then $f^{-1}(V) \supseteq f^{-1}(f(U)) \supseteq U$. So we get a map

$$\beta_U^V \stackrel{\text{def}}{=} \alpha_V|_U\colon \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U),$$

for each $V \supseteq f(U)$. In order for $(\mathcal{F}(U), \beta_U^V)$ to form a co-cone we need that

$$\begin{tikzcd}
 & \mathcal{F}(U) & \\
\mathcal{F}(f^{-1}(V)) \arrow[ur, "|_U"] \arrow[rr, "|_{f^{-1}(V')}"] & & \mathcal{F}(f^{-1}(V')) \arrow[ul, "|_U"'] \\
\mathcal{G}(V) \arrow[u, "\alpha_V"] \arrow[rr, "|_{V'}"] & & \mathcal{G}(V') \arrow[u, "\alpha_{V'}"']
\end{tikzcd},$$

commutes for $V \supseteq V' \supseteq f(U)$. The top triangle commutes by functoriality of $\mathcal{F}$, while the bottom square commutes by naturality of α. The universal property of the colimit gives a unique map

$$\tilde{\beta}_U \colon \operatorname{colim}_{V \supseteq f(U)} \mathcal{G}(V) \to \mathcal{F}(U),$$

making the co-cone diagrams commute. We now need to verify that this gives a map of presheaves $\tilde{\beta} \colon G \to \mathcal{F}$, where G is the presheaf in the construction of $f^*\mathcal{G}$. Since

$$\begin{array}{ccc} G(U) & \xrightarrow{\tilde{\beta}_U} & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ G(U') & \xrightarrow{\tilde{\beta}_{U'}} & \mathcal{F}(U') \end{array} \iff \begin{array}{ccccc} \mathcal{G}(V) & \xrightarrow{\alpha_V} & \mathcal{F}(f^{-1}(V)) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G}(V) & \xrightarrow{\alpha_{V'}} & \mathcal{F}(f^{-1}(V)) & \longrightarrow & \mathcal{F}(U) \end{array}, \forall V \supseteq f(U), V' \supseteq f(U'),$$

this is indeed the case. After all, the right-hand side commutes by naturality of α and functoriality of $\mathcal{F}$. Finally, applying sheafification gives the map $\beta \colon f^*\mathcal{G} \to \mathcal{F}$ we needed.

Conversely, given $\beta \colon f^*\mathcal{G} \to \mathcal{F}$, we have maps

$$\beta_U^V \colon \mathcal{G}(V) \to \mathcal{F}(U)$$

for every $U \in \mathbf{Open}(X)$ and $V \supseteq f(U)$. Since $V \supseteq f(f^{-1}(V))$, we can define

$$\alpha_V \overset{\text{def}}{=} \beta^V_{f^{-1}(V)} \colon \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V)).$$

By naturality of β,

$$\begin{tikzcd} f^*\mathcal{G}(f^{-1}(V)) \arrow[r] \arrow[d, "\beta_{f^{-1}(V)}"'] & f^*\mathcal{G}(f^{-1}(V')) \arrow[d, "\beta_{f^{-1}(V')}"] \\ \mathcal{F}(f^{-1}(V)) \arrow[r] & \mathcal{F}(f^{-1}(V')) \end{tikzcd}$$

commutes, and hence in particular we get

$$\begin{tikzcd} \mathcal{G}(V) \arrow[r] \arrow[d, "\beta^V_{f^{-1}(V)}"'] & \mathcal{G}(V') \arrow[d, "\beta^{V'}_{f^{-1}(V')}"] \\ \mathcal{F}(f^{-1}(V)) \arrow[r] & \mathcal{F}(f^{-1}(V')) \end{tikzcd}$$

which shows the naturality of α.

We now need to check that the two constructions are inverses of each other. Starting from $\alpha \colon \mathcal{G} \to f_*\mathcal{F}$ we obtain $\beta \colon f^*\mathcal{G} \to \mathcal{F}$. Let $\gamma \colon \mathcal{G} \to f_*\mathcal{F}$ be the map we obtain from β. We want to show that $\gamma = \alpha$. Take $V \in \mathbf{Open}(X)$, then

$$\gamma_V = \beta^V_{f^{-1}(V)} = \alpha_V|_{f^{-1}(V)} = \alpha_V,$$

as desired. Conversely, given $\beta \colon f^*\mathcal{G} \to \mathcal{F}$, we obtain $\alpha \colon \mathcal{G} \to f_*\mathcal{F}$ from β, and $\eta \colon f^*\mathcal{G} \to \mathcal{F}$ from α. We have,

$$\eta_U^V = \alpha_V|_U = \beta^V_{f^{-1}(V)}|_U,$$

for $V \supseteq f(U)$. Since $U \subseteq f^{-1}(V)$, naturality of β gives $\beta^V_{f^{-1}(V)}|_U = \beta_U^V$, so $\eta_U^V = \beta_U^V$.

The final thing to check is that the transformations are natural in $\mathcal{F}$ and $\mathcal{G}$. Let us denote Φ for the map from $\mathrm{Hom}_{\mathbf{Ab}(X)}(f^*\mathcal{G}, \mathcal{F})$ to $\mathrm{Hom}_{\mathbf{Ab}(Y)}(\mathcal{G}, f_*\mathcal{F})$. Let us start with naturality in $\mathcal{F}$. If $\gamma\colon \mathcal{F} \to \mathcal{F}'$ is a natural transformation, we need to show that $\Phi(\gamma\beta) = f_*\gamma\Phi(\beta)$ for any morphism $\beta\colon f^*\mathcal{G} \to \mathcal{F}$. Indeed,

$$\Phi(\gamma\alpha)_V = (\gamma\alpha)^V_{f^{-1}(V)} = \gamma_{f^{-1}(V)}\alpha^V_{f^{-1}(V)} = (f_*\gamma)_V\Phi(\alpha)_V.$$

For $\eta\colon \mathcal{G} \to \mathcal{G}'$, we need to show that $\Phi(\beta f^*\eta) = \Phi(\beta)\eta$. The following calculation shows that this is indeed the case

$$\Phi(\beta f^*\gamma)_V = (\beta f^*\gamma)^V_{f^{-1}(V)} = \beta^V_{f^{-1}(V)}\gamma_V = \Phi(\beta)_V\gamma_V.$$

- f^* is exact. Since it is a left adjoint, it is right exact. We still need to show that it is left exact, i.e. f^* preserves monomorphisms. Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a monomorphism. Since $\mathrm{Ker}(\alpha)(V) = \mathrm{Ker}(\alpha_V)$ we have that α is mono if and only if all the α_V are mono. Since colimits over filtered diagrams commute with finite limits, it follows that $F(U) \to G(U)$ is a mono if and only if all the α_V for $V \supseteq f(U)$ are mono. Consequently, $f^*\alpha$ is a mono if α is a mono.

$\square$

This result allows us to prove the crucial property needed to be able to do cohomology in $\mathbf{Ab}(X)$.

Lemma 3.4. $\mathbf{Ab}(X)$ *has enough injectives.*

Proof. Take $x \in X$. Then $i_x\colon \{x\} \hookrightarrow X$ is a continuous map and hence induces the pushforward functor $i_{x,*}\colon \mathbf{Ab}(\{x\}) \to \mathbf{Ab}(X)$. Note that $\mathbf{Ab}(\{x\})$ is equivalent to $\mathbf{Ab}$ as category, since a sheaf on one point is completely determined by the value at that point. Now if M is an injective object in $\mathbf{Ab}$, we claim that $i_{x,*}M$ is injective in $\mathbf{Ab}(X)$. Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a monomorphism and $\beta\colon \mathcal{F} \to i_{x,*}M$ any morphism of sheaves. We need to find a map $\gamma\colon \mathcal{G} \to i_{x,*}M$ such that $\beta = \gamma \circ \alpha$. Since i_x^* is a left adjoint to $i_{x,*}$ we get a map $b\colon i_x^*(\mathcal{F}) \to M$. Since i_x^* is exact, the map $a\colon i_x^*\mathcal{F} \to i_x^*\mathcal{G}$ is still a monomorphism. So, since M is injective, there is a map $g\colon i_x^*\mathcal{G} \to M$ such that $b = g \circ a$. This gives a map $\gamma\colon \mathcal{G} \to i_{x,*}M$, and $\gamma \circ \alpha$ is the same as the induced map by $g \circ a = b$ since an adjunction is natural in both arguments. Thus, $\gamma \circ \alpha = \beta$ as desired.

What remains to show is that for every sheaf $\mathcal{F}$, there is a monomorphism $\mathcal{F} \hookrightarrow i_{x,*}M$, for some $x \in X$ and injective $M \in \mathbf{Ab}$. Since $\mathbf{Ab}$ has enough injectives, there is a monomorphism $a_x\colon i_x^*\mathcal{F} \to M_x$, for every sheaf $\mathcal{F}$. The adjunction gives a map $\alpha_x\colon \mathcal{F} \to i_{x,*}M_x$. The problem is that this map need not be a monomorphism, since $(i_{x,*}M)(U) = M(i_x^{-1}(U)) = \{*\}$ if $x \notin U$. Since the product of injectives is again injective, we can solve this problem by considering the map

$$\alpha = (\alpha_x)_{x\in X}\colon \mathcal{F} \to \prod_{x\in X} i_{x,*}M,$$

instead. Take $V \in \mathbf{Open}(X)$. If $V = \emptyset$, then $\mathcal{F}(V) = \{*\}$ and α_V is a monomorphism. Otherwise, take $x \in V$. We claim that $(\alpha_x)_V\colon \mathcal{F}(V) \to M_x$ is a monomorphism. By definition, $(\alpha_x)_V$ is given by the composition of $\mathcal{F}(V) \to i_x^*\mathcal{F}(V)$ and $i_x^*\mathcal{F}(V) \to M_x$. Since filtered diagrams commute with finite limits, we have that $(\alpha_x)_V$ is indeed a monomorphism. So, if $x, y \in V$ with $\alpha_V(x) = \alpha_V(y)$, then $x = y$, which shows that α is a monomorphism. $\square$

In the previous proof we considered the inclusion map. If we instead consider the projection map $f\colon X \to \{*\}$, then the pullback f^* becomes the map $\mathbf{Ab} \to \mathbf{Ab}(X)$, which sends an abelian group M to the constant sheaf with value M. The pushforward $f_*\colon \mathbf{Ab}(X) \to Ab$ maps a sheaf, $\mathcal{F}$, to its global sections, $\mathcal{F}(X)$.

With this we can now do cohomology in $\mathbf{Ab}(X)$. Let $f\colon X \to Y$ be a continuous function. Since f_* is a right adjoint, it is left exact, and we can consider the right derived functors $R^i f_*\colon \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$.

Example 3.5. Let us look at the specific example where $f\colon S^1 \to \{*\}$ is the projection map. We have an exact sequence of sheaves on S^1

$$0 \longrightarrow \mathrm{Cont}(-, 2\pi\mathbb{Z}) \xrightarrow{f} \mathrm{Cont}(-, \mathbb{R}) \xrightarrow{g} \mathrm{Cont}(-, S^1) \longrightarrow 0.$$

To check that f is a monomorphism, we can check it locally. Since each of the maps $f_U\colon \mathrm{Cont}(U, 2\pi\mathbb{Z}) \to \mathrm{Cont}(U, \mathbb{R})$ are injective, so is f. The kernel of g can be computed locally as well. For every open U in S^1, we have that $h \in \mathrm{Cont}(U, \mathbb{R})$ is mapped to $\pi \circ h$ where $\pi\colon \mathbb{R} \to S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ is the quotient map. So the kernel of g_U is given by the functions whose image lies in $2\pi\mathbb{Z}$. In other words, $\mathrm{Ker}(g_U) = \mathrm{Cont}(U, 2\pi\mathbb{Z})$. This shows exactness at the middle. What remains to show is that g is an epimorphism. For this, we need to check that for every open U in S^1, and continuous function $s \in \mathrm{Cont}(U, S^1)$, there is a cover $U_{i i\in I}$ of U such that $s|_{U_i}$ is in the image of g_U. Let $U_1 = U \cap S^1 \setminus \{1\}$ and $U_2 = U \cap S^2 \setminus \{2\}$. Then $\{U_1, U_2\}$ is an open cover of U, and each of the elements in the cover is homeomorphic to an open subset of $\mathbb{R}$. So for $s \in \mathrm{Cont}(U, S^1)$, we have that $s|_{U_1}$ and $s|_{U_2}$ are in the image of g_{U_1} and g_{U_2}.

On the other hand, the map $\mathrm{Cont}(S^1, \mathbb{R}) \to \mathrm{Cont}(S^1, S^1)$ is not an epimorphism. After all, $\mathbb{R}$ is contractible, so any function in $\mathrm{Cont}(S^1, \mathbb{R})$ is homotopic to the zero function. Hence, all the functions in the image will also be homotopic to the zero map. Since S^1 has a non-trivial fundamental group, we see that image is not everything.

Consequently, we only get a left exact sequence

$$0 \longrightarrow \mathrm{Cont}(S^1, 2\pi\mathbb{Z}) \xrightarrow{f} \mathrm{Cont}(S^1, \mathbb{R}) \xrightarrow{g} \mathrm{Cont}(S^1, S^1),$$

when we apply f_*. It follows that the derived functor $R^1 f_*$ is non-zero. In fact, one can show that $R^1 f_*(\mathrm{Cont}(-, \mathbb{Z})) = \mathbb{Z}$. More generally, $R^i f_*(\mathrm{Cont}(-, \mathbb{Z})) = H^i(S^1, \mathbb{Z})$, where $H^i(S^1, \mathbb{Z})$ is the singular cohomology.

As in the example, we define the sheaf cohomology of a topological space X with coefficients in an abelian group A as

$$H^i_{\mathrm{sheaf}}(X, A) \overset{\mathrm{def}}{=} R^i f_*(\mathrm{Cont}(-, A)),$$

where A is given the discrete topology and f is the projection map $f\colon X \to \{*\}$.

4 Condensed sets

We now have all the necessary material to be able to say a little about condensed sets. Intuitively, a condensed set is a generalization of a topological space, which contains more information about how it interacts with other spaces. The goal of this section is two-fold:

- We want to define what a condensed set is, and compute some examples.
- We want to define cohomology in the condensed setting and compare it to other types of cohomology.

4.1 What is a condensed set?

As it turns out, the definition of a condensed set is not so straightforward. As is the case in representation theory, it is often useful to study an object by considering representations of the object as simpler objects. In this case we will study topological spaces by considering maps from profinite sets into topological spaces.

Definition 4.1 (Profinite set)**.** A profinite set is a compact, Hausdorff, totally disconnected space. Explicitly, it is a topological space X, such that

- Every open cover of X has a finite subcover.
- For every two different points, $x \neq y \in X$, there exist neighborhoods U and V of x and y such that $U \cap V = \emptyset$.
- The connected components of X are the singletons.

For example, a finite space with the discrete topology is a profinite set. We would now like to turn the category of profinite sets into a site. There is a small problem though. Namely, the category of profinite sets isn't small. To solve this, we take an uncountable strong limit cardinal κ and only consider the profinite sets with cardinality less than κ. Recall that a strong limit cardinal is a cardinal κ such that $2^\lambda < \kappa$ for all cardinals $\lambda < \kappa$.

What are the coverings on this category? A covering is given by a finite family of jointly surjective maps, thus a set of maps $\{f_i\colon X_i \to X\}_{i=0}^n$ such that $X = \bigcup_{i=0}^n f_i(X_i) = X$. This collection of coverings does indeed satisfy the needed axioms:

1. If $f\colon Y \to X$ is an isomorphism, then f is surjective, so $\{f\}$ is a cover.
2. If $\{f_i\colon X_i \to Y\}_{i=0}^n$ is jointly surjective, and for each $i \in \{0, \dots, n\}$, the set $\{f_{ij}\colon X_{ij} \to X_i\}_{j=0}^{n_i}$ is jointly surjective, then so is the composition $\{f_i \circ f_{ij}\colon X_{ij} \to Y\}_{i,j=0}^{n,n_i}$.
3. Finally, let $\{f_i\colon X_i \to Y\}_{i=0}^n$ be a cover, and $f\colon S \to Y$ a morphism. Then, assuming that the pullback $X_i \times_Y S$ is a profinite set, we have that the projection maps $\{X_i \times_Y S \to S\}_{i=0}^n$ are jointly surjective. After all, for $s \in S$, the image $f(s)$ is equal to $f_i(x_i)$ for some $i \in \{0, \dots, n\}$ since the f_i are jointly surjective. So, s is the image of $(x_i, s) \in X_i \times_Y S$.

The only thing left to show is that the pullback of two profinite sets is again a profinite set. In fact, something stronger is true:

Proposition 4.2. *A limit of profinite spaces is again a profinite space.*

Proof. Let $X = \lim_{i \in \mathcal{I}} X_i$ be the limit as topological spaces. We first show that X is still Hausdorff and totally disconnected. Take $x \neq y \in X$. Then for at least one of the projections $f_i\colon X \to X_i$ we must have $f_i(x) \neq f_i(y)$. Since X_i is Hausdorff, we can find open neighborhoods U of $f_i(x)$ and V of $f_i(y)$ such that $U \cap V = \emptyset$. But then $f_i^{-1}(U) \cap f_i^{-1}(V) = \emptyset$, and $x \in f_i^{-1}(U), y \in f_i^{-1}(V)$. Since X_i is totally disconnected, $f_i(x)$ and $f_i(y)$ are in separate components. Since the continuous image of a connected set is connected, we must have that x and y are in separate connected components as well. Finally, one can also show that limits of compact Hausdorff spaces are again compact (see [Sta23, Lemma 08ZV]). Essentially the argument is as follows:

- Limits are made out of products and equalizers.
- Products of compact spaces are compact.
- By above, limits of Hausdorff spaces are Hausdorff.
- Equalizers of functions into Hausdorff spaces are closed.
- So the limit is compact as a closed subspace of a compact space.

□

With this we arrive at the following definition:

Definition 4.3 (κ-condensed sets)**.** The category of κ-condensed sets is the category of sheaves on the site of κ-small profinite sets, with coverings given by jointly surjective maps.

Analogously, we can define condensed abelian groups, rings, vector spaces. . . . As we have seen, the category of abelian sheaves on a site is abelian. This is not true for the category of topological abelian groups. After all, in any abelian group, a morphism that is both an epimorphism and a monomorphism is automatically an isomorphism. The map

$$\mathrm{id}\colon (\mathbb{R}, \mathcal{T}_{\text{discrete}}) \to (\mathbb{R}, \mathcal{T}_{\text{Euclidean}}),$$

is bijective as sets, and hence an epimorphism and a monomorphism. However, since the two topologies don't coincide, the map is not an isomorphism.

How is this issue resolved in the condensed setting? With a topological space X, we can associate a condensed set $\underline{X} = \mathrm{Cont}(-, X)$. The "underlying set" can then be found as $\underline{X}(\{*\})$. The point is that the map

$$\underline{\mathrm{id}}\colon \underline{\mathbb{R}_{\text{discrete}}} \to \underline{\mathbb{R}_{\text{Euclidian}}}$$

is no longer an epimorphism. As we will see later on, the cokernel of this map can be computed "section-wise", i.e. $\mathrm{Coker}(f)(U) = \mathrm{Coker}(f_U)$. So, to show that the cokernel is not trivial, we need to show that there is some profinite set S with $\mathrm{Cont}(S, \mathbb{R}_{\text{discrete}}) \subsetneq \mathrm{Cont}(S, \mathbb{R}_{\text{Euclidean}})$. Consider the Cantor set S, which is homeomorphic to $\prod_{n\in\mathbb{N}}\{0,1\}$ where $\{0,1\}$ has the discrete topology. By proposition 4.2, the Cantor set is a profinite set. It can alternatively be described as the closed subspace of the real numbers (with the Euclidean topology), given by

$$S = [0,1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right).$$

With this description, we get a continuous map from S into the real numbers with the Euclidean topology, namely the inclusion map. On the other hand, this map is not continuous when $\mathbb{R}$ is equipped with the discrete topology. If it were, then every singleton in S would have to be open, as the inverse image of an open set. Since S doesn't have the discrete topology, this is not the case. Consequently, the map f_S is not an epimorphism.

Recall from lemma 2.19 that limits and colimits of presheaves can be taken section-wise. For sheaves, the colimit is not always a sheaf, and we may need to take the sheafification of the colimit as presheaves to obtain a sheaf. For the category of κ-condensed sets, this is no longer necessary. To prove this, we'll need an alternative description of condensed sets.

Definition 4.4. An *extremally disconnected* set is a projective object in the category of compact Hausdorff spaces[3].

One way to construct these extremally disconnected spaces, is through Stone–Čech compactification.

Theorem 4.5 (Stone–Čech compactification)**.** *Let X be a topological space. There is a unique (up to isomorphism) compact Hausdorff space βX together with a continuous map $i_X\colon X \to \beta X$, such that for any continuous map $f\colon X \to K$, with K a compact and Hausdorff space, there is a unique extension of f to a continuous map $\beta f\colon \beta X \to K$:*

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "f"'] & \beta X \arrow[d, dashed, "\beta f"] \\ & K \end{tikzcd}$$

In terms of cardinality, we have $|\beta X| \leq 2^{\left(2^{|X|}\right)}$.

See, for example, [Sta23, Section 0908] for a proof. Now, let X be a compact Hausdorff space, and let X' be the same space equipped with the discrete topology. Then the map $\mathrm{id}\colon X' \to X$ is continuous, and by the universal property of the Stone–Čech compactification, we have a factorization:

$$\begin{tikzcd} X' \arrow[r, "i_{X'}"] \arrow[dr, "\mathrm{id}"'] & \beta X' \arrow[d, dashed, "\beta\,\mathrm{id}"] \\ & X \end{tikzcd} .$$

Since id is bijective, it follows that $i_{X'}$ is injective, and $\beta\,\mathrm{id}$ is surjective.

We claim that $\beta X'$ is extremally disconnected. So, let $e\colon K \twoheadrightarrow S$ be an epimorphism between compact Hausdorff spaces, and $f\colon \beta X' \to S$ be any map. Since any set is projective in **Set**, there is a map (of sets) $g\colon X' \to K$ such that $e \circ g = f \circ i_{X'}$. Since X' has the discrete topology, the map g is continuous. Hence, by the universal property of the Stone–Čech compactification, there is a unique continuous map $\beta g\colon \beta X' \to K$ such that $g = \beta g \circ i_{X'}$.

$$\begin{tikzcd} X' \arrow[r, hook, "i_{X'}"] \arrow[d, "g"] & \beta X \arrow[d, "f"] \arrow[dl, dashed, "\beta g"] \\ K \arrow[r, two heads, "e"'] & S \end{tikzcd}$$

[3]The usual definition is that the closure of every open set must again be open. In this setting the two definitions are equivalent ([Gle58, Theorem 2.5])

So

$$f \circ i_{X'} = e \circ g = e \circ \beta g \circ i_{X'},$$

and hence $f = e \circ \beta g$, since $i_{X'}$ is a monomorphism.

So, we have shown that any compact Hausdorff space admits a surjection from and extremally disconnected space, i.e. the category has enough projectives. We now want to describe κ-condensed sets, by restricting to just the category of extremally disconnected spaces. However, there is a small technicality that we have to take care of first. The product of two infinite extremally disconnected spaces is never extremally disconnected. So, we can't take fiber products, and so there is a problem in defining a site on this category. The solution is to consider an alternative description of what a site is.

One defines a *sieve* on an object $X \in \mathcal{C}$ as a subfunctor of its Yoneda embedding $\mathrm{Hom}(-, X)$. So, it is a functor $S\colon \mathcal{C}^{\mathrm{opp}} \to \mathcal{C}$ such that for any $X' \in \mathcal{C}$ we have $S(X') \subseteq \mathrm{Hom}(X', X)$, and for all morphisms $f\colon Y \to Y'$, $S(f)$ is the restriction of $\mathrm{Hom}(f, X)$ to $S(Y')$. A site becomes a category together with a Grothendieck topology. Namely, for each object of $\mathcal{C}$, we choose a collection of sieves on that object, called covering sieves. These must again satisfy axioms similar to those for covers:

1. $\mathrm{Hom}(-, X)$ is a covering sieve for any X in $\mathcal{C}$.

2. If S is a covering sieve on X, T is sieve on X, and for any $f \in S$ the sieve f^*T is a covering sieve, then T is a covering sieve.

3. If S is a covering sieve on X, and $f\colon X \to Y$ is a morphism, then the pullback f^*S is a covering sieve on Y.

Here the pullback of S by $f\colon Y \to X$ is given by $f^*S(Z) = \{g\colon Z \to Y \mid f \circ g \in S(Z)\}$. Importantly, the definition doesn't require the existence of pullbacks. The sheaf condition for a presheaf $\mathcal{F}$, is that for any object X and any covering sieve S on X, the map

$$\mathrm{Hom}(\mathrm{Hom}(-, X), \mathcal{F}) \to \mathrm{Hom}(S, \mathcal{F})$$

induced by the inclusion of S in $\mathrm{Hom}(-, X)$, is a bijection. In other words, every natural transformation from S to $\mathcal{F}$ extends uniquely to a natural transformation from $\mathrm{Hom}(-, X)$ to $\mathcal{F}$.

A collection of covers satisfying the axioms from our original definition of a site, is referred to as a pretopology. Given such a pretopology we can make a Grothendieck topology by taking as covering sieves, precisely those sieves which contain a cover from the pretopology. The crucial part is that these two sites give rise to the same category of sheaves ([Dag21, Corollary 1.1.28]).

So we can define the site of κ-small extremally disconnected spaces, with covers given by finite families of jointly surjective maps. That this generates a Grothendieck topology is shown in [Dag21, Proposition 1.2.12]. Finally, it turns out that restricting to κ-small extremally disconnected spaces gives an equivalent category of sheaves, to the category of κ-condensed sets ([Dag21, Theorem 1.2.16]). Here it is crucial that the cardinality of the Stone–Čech compactification is bounded. Since κ is an uncountable strong limit cardinal, we know that if $|X| < \kappa$ then also $|\beta X| < \kappa$.

Checking that a presheaf is a sheaf, becomes a lot easier now. A sheaf on κ-small extremally disconnected spaces is a functor

$$T\colon \{\text{extremally disconnected spaces}\}^{\text{opp}} \to \mathbf{Set} \,/\, \mathbf{Rng} \,/\, \mathbf{Ab} \,/\, \ldots$$

with $T(\emptyset) = \{*\}$, and such that $T(S_1 \sqcup S_2) \to T(S_1) \times T(S_2)$ is a bijection for any two extremally disconnected sets S_1, S_2 with cardinality smaller than κ ([Dag21, Theorem 1.2.18]). As a result we find:

Corollary 4.6. *The category of κ-condensed sets is abelian. Limits and colimits can be taken section-wise.*

Proof. Note that limits and colimits in **Ab** commute with finite products. Additionally, we already saw that the section-wise limit and colimit of presheaves is again a presheaf (lemma 2.19). Now assume that $(T_i)_{i\in\mathcal{I}}$ are sheaves. Taking their limit or colimit as presheaves will then again be a sheaf, since

$$\begin{aligned}
(\lim_{i\in\mathcal{I}} T_i)(S_1 \sqcup S_2) &= \lim_{i\in\mathcal{I}} T_i(S_1 \sqcup S_2) \\
&= \lim_{i\in\mathcal{I}} (T_i(S_1) \times T_i(S_2)) \\
&= \lim_{i\in\mathcal{I}} T_i(S_1) \times \lim_{i\in\mathcal{I}} T_i(S_2) \\
&= (\lim_{i\in\mathcal{I}} T_i)(S_1) \times (\lim_{i\in\mathcal{I}} T_i)(S_2),
\end{aligned}$$

and similarly for colimits. □

So, problems with colimits, where we need to take the sheafification do not occur in this setting. As a final thing to consider, let us take a look at cohomology in the condensed setting.

4.2 Condensed cohomology

Given all the work that we have done in constructing cohomology for arbitrary categories of sheaves on a topological space, it would not be unreasonable to assume that we would now use this to define cohomology of condensed abelian groups. Although we won't be able to copy over the construction exactly, it will serve as our source of inspiration. There are a few problems that we need to address first.

Ideally, we would want the definition of condensed sets to be independent of an (arbitrary) choice of uncountable limit cardinal. This is possible, and is done in [Sch19, Appendix to Lecture II]. The resulting category is still abelian, and limits and colimits can still be taken section-wise. However, this category is no longer the category of sheaves on a site, and so some properties might no longer be valid. One important one is that the category no longer has enough injectives. In fact, it has no non-zero injectives ([Sch20]). So, we will need to use projective resolutions instead. Let us show that there are enough projectives in the category of κ-condensed abelian groups.

Lemma 4.7. *The forgetful functor U from κ-condensed abelian groups has a left adjoint $\mathbb{Z}[-]$. For extremally disconnected S, $\mathbb{Z}[\underline{S}]$ is projective.*

Proof. Let T be a κ-condensed set, and $\mathbb{Z}[T]$ be the sheafification of the presheaf which sends S to $\mathbb{Z}[T(S)]$, the free abelian group on $T(S)$. To see that this is a left adjoint, note that for each extremally disconnected space S, and condensed abelian group M, we have a natural isomorphism

$$\phi_{T,M,S}\colon \operatorname{Hom}_{\mathbf{Ab}}(\mathbb{Z}[T(S)], M(S)) \cong \operatorname{Hom}_{\mathbf{Set}}(T(S), U(M)(S)),$$

since free abelian groups are a left adjoint to the forgetful functor. This then give the desired isomorphism

$$\phi_{T,M}\colon \operatorname{Hom}_{\mathrm{Cond}(\mathbf{Ab})}(\mathbb{Z}[T], M) \cong \operatorname{Hom}_{\mathrm{Cond}(\mathbf{Set})}(T, U(M)),$$

Since limits and colimits can be taken section-wise, i.e. $M \mapsto M(S)$ commutes with all limits and colimits, it follows that $\mathbb{Z}[\underline{S}]$ is projective. Indeed, in any abelian category, an object P is projective if and only if $\operatorname{Hom}(P, -)$ is exact. So, to check that $\mathbb{Z}[\underline{S}]$ is projective we need that $\operatorname{Hom}_{\mathrm{Cond}(\mathbf{Ab})}(\mathbb{Z}[\underline{S}], -)$ is exact. This is equivalent to saying that the map $M \mapsto M(S)$ is exact, since $\operatorname{Hom}_{\mathrm{Cond}(\mathbf{Ab})}(\mathbb{Z}[\underline{S}], M) = M(S)$ for any condensed abelian group M, by the Yoneda lemma. Since $M \mapsto M(S)$ commutes with limits, it is indeed exact. □

This lemma and the following corollary are still valid in the actual definition of condensed sets, which is independent on a choice of cardinal κ.

Corollary 4.8. *The category of κ-condensed abelian groups has enough projectives.*

Proof. We follow the proof given in [Dag21, Theorem 2.2.6]. Let M be a κ-condensed abelian group. For any extremally disconnected set S, and $x \in M(S)$, we have a morphism $\alpha_x\colon \mathbb{Z}[\underline{S}] \to M$ such that $\alpha_x(\mathrm{id}_S) = x$, by the adjunction and the Yoneda lemma. Then the induced map

$$h\colon \bigoplus_{S} \bigoplus_{x \in M(S)} \mathbb{Z}[\underline{S}] \to M,$$

is surjective. Since all the $\mathbb{Z}[\underline{S}]$ are projective, so is the direct sum, and hence we have found a surjection onto M from a projective object. □

Recall that sheaf cohomology was defined via right-derived functors of $\mathcal{F} \mapsto \mathcal{F}(X)$. We want to define condensed cohomology as derived functors of $M \mapsto M(S)$. Since we don't have injective resolutions, we can't take the right derived functors, the usual way. However, the Yoneda lemma and the above adjunction give us a way around it. We have

$$\operatorname{Hom}_{\mathrm{Cond}(\mathbf{Ab})}(\mathbb{Z}[\underline{S}], M) \cong \operatorname{Hom}_{\mathrm{Cond}\,\mathbf{Set}}(\underline{S}, M) \cong M(S),$$

and hence we can use the Ext functors instead. Concretely, for a topological space X and abelian group A, the condensed cohomology of X with coefficients in A is

$$H^i_{\text{condensed}}(X, A) \overset{\text{def}}{=} \operatorname{Ext}^i(\mathbb{Z}[\underline{X}], \underline{A}).$$

It should come as no surprise that this agrees with sheaf cohomology for nice topological spaces X. We have the following results:

Theorem 4.9 ([Sch19, Theorem 3.2 and 3.3])**.** *If X is a compact Hausdorff space, then*

$$H^i_{\textit{condensed}}(X, \mathbb{Z}) \cong H^i_{\textit{sheaf}}(X, \mathbb{Z}),$$

and

$$H^i_{\textit{condensed}}(X, \mathbb{R}) \cong 0, i > 0, \qquad H^0_{\textit{condensed}}(X, \mathbb{R}) \cong \operatorname{Cont}(X, \mathbb{R}).$$

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