

A Fluid Introduction to Condensed Mathematics

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MOTIVATION

Problem: Algebra and topology clash:

$$\text{id}: \mathbb{R}_{\text{discrete}} \rightarrow \mathbb{R}_{\text{Euclidean}}$$

is epi + mono, but not iso

Solution?

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\rightsquigarrow Condensed abelian groups

PRESHEAVES

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Definition

A **presheaf** on a topological space X is a functor

$$\mathcal{F}: \mathbf{Open}(X)^{\mathrm{opp}} \rightarrow \mathbf{Set} / \mathbf{Rng} / \mathbf{Ab} / \dots$$

Notation:

- ▶ Elements of $\mathcal{F}(U)$ are called **sections**
- ▶ If $i: U \hookrightarrow V$ then $\mathcal{F}(i)$ is called the **restriction** map:

$$s \in \mathcal{F}(V) \mapsto s|_U \in \mathcal{F}(U).$$

Example

We always have the presheaf given by

- ▶ $\mathcal{F}(U) \mapsto Y$, and
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Problem:

- ▶ Want more structure
- ▶ Want global things to be defined locally

\rightsquigarrow sheaf conditions:

SHEAF CONDITIONS

For an open cover $\{U_i\}_{i \in I}$ of U :

- ▶ **uniqueness/locality**: If $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

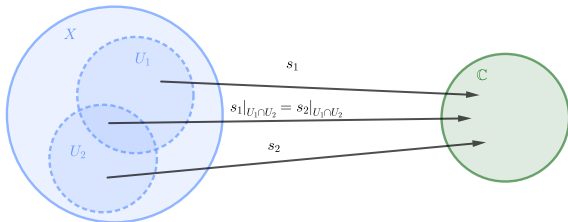
SHEAF CONDITIONS

For an open cover $\{U_i\}_{i \in I}$ of U :

- **uniqueness/locality**: If $\mathbf{s}, \mathbf{t} \in \mathcal{F}(U)$ are sections such that $\mathbf{s}|_{U_i} = \mathbf{t}|_{U_i}$ for all $i \in I$, then $\mathbf{s} = \mathbf{t}$.
- **gluing**: If $\{\mathbf{s}_i \in \mathcal{F}(U_i)\}_{i \in I}$ is a collection of sections such that

$$\mathbf{s}_i|_{U_i \cap U_j} = \mathbf{s}_j|_{U_i \cap U_j} \text{ for all } i, j,$$

then there is a section $\mathbf{s} \in \mathcal{F}(U)$ such that $\mathbf{s}_i = \mathbf{s}|_{U_i}$ for all $i \in I$.



EXAMPLES

Typical examples are rings of functions:

- ▶ $\text{Cont}(-, \mathbb{C})$
- ▶ $C^\infty(-, \mathbb{C})$

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The constant functor $U \mapsto Y$ is **not** a sheaf!

Sheaf conditions imply $\mathcal{F}(\emptyset) = \{*\}$

Proposition

Limits and colimits of presheaves can be computed section-wise, i.e. the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ commutes with all limits and colimits.

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Example

$$(\mathcal{F} \times \mathcal{G})(U) = \mathcal{F}(U) \times \mathcal{G}(U)$$

SHEAFIFICATION

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Theorem

The fully faithful inclusion $\iota: \mathbf{Sh}(X) \hookrightarrow \mathbf{PSh}(X)$ admits an *exact left adjoint: sheafification*.

Example

Sheafification of $U \mapsto Y$, is $U \mapsto \text{Cont}(U, Y_{\text{discrete}})$.

Proposition

The category of abelian sheaves on a topological space X is abelian, and has enough injectives.

Example

If M is an injective object in \mathbf{Ab} , i.e. a divisible group, then for $x \in X$,

$$U \mapsto \begin{cases} M & x \in U \\ \{*\} & x \notin U \end{cases}$$

is an injective sheaf.

If

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is exact, then globally we only get

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\rightsquigarrow Right derived functors $H^i(X, -)$ of $\mathcal{F} \mapsto \mathcal{F}(X)$

Definition

The sheaf cohomology of X with coefficients A is $H^i(X, \text{Cont}(-, A))$.

We have

$$0 \longrightarrow \text{Cont}(-, \mathbb{Z}) \longrightarrow \text{Cont}(-, \mathbb{R}) \longrightarrow \text{Cont}(-, \mathbb{R}/\mathbb{Z}) \longrightarrow 0$$

since every map $U \subseteq S^1 \rightarrow S^1$ can be lifted **locally** to a map $U \rightarrow \mathbb{R}$.

But, there is no global lift of $\text{id}: S^1 \rightarrow S^1$ to $S^1 \rightarrow \mathbb{R}$. So $H^1(S^1, \mathbb{Z}) \neq 0$.

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Proposition

If X is a CW-complex:

$$H_{\text{sheaf}}^i(X, \mathbb{Z}) = H_{\text{singular}}^i(X, \mathbb{Z}).$$

- ▶ Need a slight abstraction of sheaves on a space.
- ▶ Want sheaves on categories, not just spaces.
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↪ Definition of a site \approx "Category + notion of coverings".

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A **profinite set** is a compact, Hausdorff, totally disconnected space.

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A finite discrete space

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A limit of profinite sets is profinite. In particular, products of profinite sets are again profinite.

CONDENSED SETS

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For a topological space X , the associated condensed set \underline{X} is given by $\text{Cont}(-, X)$.

THE MOTIVATING PROBLEM

Recall that

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Claim: As condensed abelian groups, the map:

$$\underline{\text{id}}: \underline{\mathbb{R}_{\text{discrete}}} \rightarrow \underline{\mathbb{R}_{\text{Euclidean}}}$$

is no longer an epimorphism.

THE CANTOR SET

Enough to show:

$$\text{Cont}(\mathbf{S}, \mathbb{R}_{\text{discrete}}) \subsetneq \text{Cont}(\mathbf{S}, \mathbb{R}_{\text{Euclidean}})$$

for some profinite set \mathbf{S} .

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Let $\mathcal{S} \subset \mathbb{R}_{\text{Euclidean}}$ be the cantor set. $\mathcal{S} \cong \prod_{n \in \mathbb{N}} \{0, 1\}$, so \mathcal{S} is profinite. Hence:

$$\text{Cont}(\mathcal{S}, \mathbb{R}_{\text{discrete}}) \subsetneq \text{Cont}(\mathcal{S}, \mathbb{R}_{\text{Euclidean}})$$



CONDENSED COHOMOLOGY

For a condensed set T , let $\mathbb{Z}[T]$ be the sheafification of $S \mapsto \mathbb{Z}[T(S)]$.

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The cohomology of S with coefficients A is

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Theorem ([Sch19], Theorem 3.2)

Let S be a compact Hausdorff space. Then

$$H^i_{\text{cond}}(S, A) \cong H^i_{\text{sheaf}}(S, A)$$

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