A Fluid Introduction to Condensed Mathematics

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MOTIVATION

Problem: Algebra and topology clash:

$$\mathrm{id}\colon \mathbb{R}_{\text{discrete}} \to \mathbb{R}_{\text{Euclidean}}$$

is epi + mono, but not iso

Solution?

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Solution?
$\rightsquigarrow$ Condensed abelian groups

PRESHEAVES

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Definition

A presheaf on a topological space X is a functor

$$\mathcal{F}\colon \ \mathbf{Open}(X)^{\mathrm{opp}} \to \mathbf{Set} \,/\, \mathbf{Rng} \,/\, \mathbf{Ab} \,/\, \dots$$

Notation:

- Elements of $\mathcal{F}(U)$ are called sections
- If $i\colon U \hookrightarrow V$ then $\mathcal{F}(i)$ is called the restriction map:

$$s \in \mathcal{F}(V) \mapsto s|_U \in \mathcal{F}(U).$$

Example

We always have the presheaf given by

- $\mathcal{F}(U) \mapsto Y$, and
- $\mathcal{F}(f) \mapsto \mathrm{id}_Y$

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Problem:

- Want more structure
- Want global things to be defined locally

$\rightsquigarrow$ sheaf conditions:

SHEAF CONDITIONS

For an open cover $\{U_i\}_{i \in I}$ of U:

- uniqueness/locality: If $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

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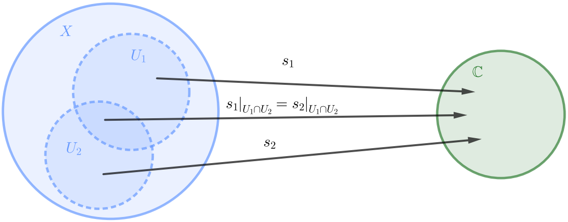
- uniqueness/locality: If $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.
- gluing: If $\{s_i \in \mathcal{F}(U_i)\}_{i \in I}$ is a collection of sections such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \text{ for all } i, j,$$

then there is a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.



EXAMPLES

Typical examples are rings of functions:

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- $C^\infty(-, \mathbb{C})$

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The constant functor $U \mapsto Y$ is not a sheaf!
Sheaf conditions imply $\mathcal{F}(\emptyset) = \{*\}$

Proposition

Limits and colimits of presheaves can be computed section-wise, i.e. the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ commutes with all limits and colimits.

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Example

$(\mathcal{F} \times \mathcal{G})(U) = \mathcal{F}(U) \times \mathcal{G}(U)$

SHEAFIFICATION

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Theorem

The fully faithful inclusion $\iota\colon \mathbf{Sh}(X) \hookrightarrow \mathbf{PSh}(X)$ *admits an exact left adjoint: sheafification.*

Example

Sheafification of $U \mapsto Y$, is $U \mapsto \mathsf{Cont}(U, Y_{\text{discrete}})$.

Proposition

The category of abelian sheaves on a topological space X is abelian, and has enough injectives.

Example

If M is an injective object in **Ab**, i.e. a divisible group, then for $x \in X$,

$$U \mapsto \begin{cases} M & x \in U \\ \{*\} & x \notin U \end{cases}$$

is an injective sheaf.

If

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is exact, then globally we only get

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$\rightsquigarrow$ Right derived functors $H^i(X, -)$ of $\mathcal{F} \mapsto \mathcal{F}(X)$

Definition

The sheaf cohomology of X with coefficients A is $H^i(X, \mathrm{Cont}(-, A))$.

COMPUTING HOMOLOGY

We have

$$0 \longrightarrow \mathsf{Cont}(-,\mathbb{Z}) \longrightarrow \mathsf{Cont}(-,\mathbb{R}) \longrightarrow \mathsf{Cont}(-,\mathbb{R}/\mathbb{Z}) \longrightarrow 0$$

since every map $U \subseteq S^1 \to S^1$ can be lifted locally to a map $U \to \mathbb{R}$.

But, there is no global lift of $\mathsf{id}\colon S^1 \to S^1$ to $S^1 \to \mathbb{R}$. So $H^1(S^1,\mathbb{Z}) \neq 0$.

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Proposition

If X is a CW-complex:

$$H^i_{sheaf}(X,\mathbb{Z}) = H^i_{singular}(X,\mathbb{Z}).$$

SITES

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$\rightsquigarrow$ Definition of a site $\approx$ "Category + notion of coverings".

PROFINITE SETS

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A profinite set is a compact, Hausdorff, totally disconnected space.

Example

A finite discrete space

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A limit of profinite sets is profinite. In particular, products of profinite sets are again profinite.

CONDENSED SETS

Definition

A condensed set is a sheaf on the site of profinite sets with coverings given by finite families of jointly surjective maps.

$$T\colon \{\text{Profinite sets}\}^{\mathrm{opp}} \to \mathbf{Set} \,/\, \mathbf{Ab} \,/\, \mathbf{Rng} \,/\, \ldots$$

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Condensed abelian groups form an abelian category, and limits and colimits can be computed section-wise.

For a topological space X, the associated condensed set $\underline{X}$ is given by $\mathrm{Cont}(-, X)$.

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Claim: As condensed abelian groups, the map:

$$\underline{\mathrm{id}} : \underline{\mathbb{R}_{\text{discrete}}} \to \underline{\mathbb{R}_{\text{Euclidean}}}$$

is no longer an epimorphism.

THE CANTOR SET

Enough to show:

$$\mathsf{Cont}(S, \mathbb{R}_{\text{discrete}}) \subsetneq \mathsf{Cont}(S, \mathbb{R}_{\text{Euclidean}})$$

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Let $S \subset \mathbb{R}_{\text{Euclidean}}$ be the cantor set. $S \cong \prod_{n\in\mathbb{N}}\{0,1\}$, so S is profinite. Hence:

$$\mathrm{Cont}(S, \mathbb{R}_{\text{discrete}}) \subsetneq \mathrm{Cont}(S, \mathbb{R}_{\text{Euclidean}})$$

FREE CONDENSED ABELIAN GROUPS

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For a condensed set T, let $\mathbb{Z}[T]$ be the sheafification of $S \mapsto \mathbb{Z}[T(S)]$.

Proposition

The functor $\mathbb{Z}[-]$ is a left adjoint to the forgetful functor from condensed abelian groups to condensed sets.

CONDENSED COHOMOLOGY

Let X be a topological space, and M a condensed abelian group. By Yoneda's lemma:

$$\mathrm{Hom}_{\mathbf{Cond}(\mathbf{Ab})}(\mathbb{Z}[\underline{X}], M) \cong \mathrm{Hom}_{\mathbf{Cond}(\mathbf{Set})}(\underline{X}, M) \cong M(X)$$

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Theorem ([Sch19], Theorem 3.2)

Let X be a compact Hausdorff space. Then

$$H^i_{cond}(X, A) \cong H^i_{sheaf}(X, A)$$

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