

A Fluid Introduction to Condensed Mathematics

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1 Introduction

Most of the contents of this paper will be based on the lecture notes by Peter Scholze, who developed the theory of condensed mathematics together with Dustin Clausen ([Sch19]).

2 Sheaves

A central role in this paper will be played by sheaves. It therefore makes sense to study these objects more closely, and attempt to gain some intuition before jumping straight into the heart of the matter. The first step, will be to convince our selves, that sheaves are interesting enough to study, and that they form a nice type of category. As such, we will dive into sheaf cohomology and some applications of it in algebraic topology. But first, we must define what a sheaf is.

2.1 On topological spaces

Consider the following situation in algebraic geometry. We have some (affine) algebraic variety X , with the Zariski Topology. Associated to X is an ideal I of all the polynomials that vanish on X . Then, the quotient ring $\mathcal{O}_X(X) = k[x_1, \dots, x_n]/I$, is the coordinate ring of X . The elements of $\mathcal{O}_X(X)$ are the regular functions on X . For an open $U \subseteq X$ we can also consider the regular functions on U , which we will denote as $\mathcal{O}_X(U)$. This mapping between open subsets of X and rings, given by \mathcal{O}_X , is precisely a sheaf. In this case, \mathcal{O}_X is known as the *structure sheaf*. In general, a sheaf will give some global data that can be defined locally.

Definition 2.1 (Presheaves). Let X be a topological space. A *presheaf of sets* \mathcal{F} on X consists of two things:

1. For each open $U \subseteq X$ a set $\mathcal{F}(U)$. These are known as the *sections* of \mathcal{F} over U .

2. For each inclusion $U \subseteq V$ a map $\rho_{U,V}: \mathcal{F}(V) \rightarrow \mathcal{F}(U)$, such that $\rho_{U,U} = \text{id}_{\mathcal{F}(U)}$, and for $U \subseteq V \subseteq W$ we have $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$. We call the $\rho_{U,V}$ the *restrictions*, and if $s \in \mathcal{F}(U)$, we will often denote $\rho_{V,U}(s) = s|_V$.

Remark 2.2. If we write $\mathbf{Open}(X)$ for the category of open sets of X , with morphisms given by the inclusion maps, then a presheaf is precisely a functor $\mathcal{F}: \mathbf{Open}(X)^{\text{opp}} \rightarrow \mathbf{Set}$.

This only defines what a “presheaf” is. The sheaf condition will ensure that the value of $\mathcal{F}(U)$ can be constructed by defining it locally on elements of a cover $\{U_i\}_{i \in I}$ of U . It is exactly this interaction between local definitions and global objects that makes sheaves so useful.

Definition 2.3 (Sheaves). Let X be a topological space, and \mathcal{F} a presheaf of sets on X . We say that \mathcal{F} is a sheaf, if it satisfies the following additional properties for any open $U \subseteq X$ and open cover $\{U_i\}_{i \in I}$ of U :

- (uniqueness/locality) If $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.
- (gluing) If $\{s_i \in \mathcal{F}(U_i)\}_{i \in I}$ is a collection of sections such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \text{ for all } i, j,$$

then there is a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.

In other words, if we have a bunch of sections that agree on overlaps, then we can uniquely glue them together.

Remark 2.4. We can reformulate the properties in a more categorical manner. Namely, for any open cover $\{U_i\}_{i \in I}$ of $U \subseteq X$, the following diagram:

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i, j \in I} \mathcal{F}(U_i \cap U_j),$$

has to be an equalizer. This diagram makes sense in any target category that has all limits. In terms of sets, the first map is given by $s \mapsto (s|_{U_i})_{i \in I}$. The pair of maps is given by the two possible restrictions, $(s_i)_{i \in I} \mapsto ((s_i|_{U_i \cap U_j})_{j \in I})_{i \in I}$ and $(s_j)_{j \in I} \mapsto ((s_j|_{U_i \cap U_j})_{i \in I})_{j \in I}$.

In this way, we can define sheaves on rings or abelian groups, or other categories with all (set-indexed) limits. For example, a presheaf of abelian groups on X , is a functor $\mathcal{F}: \mathbf{Open}(X)^{\text{opp}} \rightarrow \mathbf{Ab}$.

In many cases, defining some structure on sheaves, will come down to defining it on $\mathcal{F}(U)$ in a compatible way. As an example, we can look at morphisms of sheaves.

Definition 2.5 (Morphisms of sheaves). Let \mathcal{F}, \mathcal{G} be presheaves on a topological space X . A *morphism of presheaves* $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ assigns to each open $U \subset X$ a morphism $\varphi_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ compatible with the restriction maps, i.e. for $U \subseteq V \subseteq X$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \\ \downarrow \rho_{U,V}^{\mathcal{F}} & & \downarrow \rho_{U,V}^{\mathcal{G}} \\ \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \end{array}$$

In other words, φ is a natural transformation between \mathcal{F} and \mathcal{G} . Using a bit of abusive notation, the compatibility can be read as $\varphi(s|_U) = \varphi(s)|_U$. A *morphism of sheaves* is a morphism of the underlying presheaves.

For a topological space X , we now have (pre)sheaves and morphisms between them. These form a category. We will write $\mathbf{PSh}(X)$ for the category of presheaves of sets on X , and $\mathbf{Sh}(X)$ for the category of sheaves of sets on X . In the same way, we write $\mathbf{Ab}(X)$, $\mathbf{Ring}(X)$, $\mathbf{Vect}(X)$, \dots for the category of sheaves of abelian groups, rings, vector spaces, \dots on X .

Remark 2.6. We can "recover" the underlying space, by taking $X = \{*\}$, the one-point space. We have $\mathbf{Sh}(*) = \mathbf{Set}$, $\mathbf{Ab}(*) = \mathbf{Ab}$, \dots

2.1.1 Examples

To solidify our understanding of sheaves, it will be beneficial to look at some examples.

Example 2.7. The simplest examples of sheaves are those where $\mathcal{F}(U)$ is not just a set, but a set of functions, and the restrictions correspond to actual restrictions.

Example 2.8. Let Y be another topological space, then we can define $\mathcal{F}(U) \stackrel{\text{def}}{=} \text{Cont}(U, Y)$ where $\text{Cont}(U, Y) = \{f: U \rightarrow Y \mid f \text{ continuous}\}$. The restrictions are defined as actual restrictions, and we get a presheaf. To see that it is a sheaf, we need to verify the gluing condition. Let $\{U_i\}$ be an open covering of $U \subseteq X$, and $f_i: U_i \rightarrow Y$ is a continuous map for each $i \in I$, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. We can now define a map $f: U \rightarrow Y$ via $f(u) = f_i(u)$ for any $i \in I$ such that $u \in U_i$. By assumption, this map is well-defined. To see that it is continuous, for any $V \subseteq Y$ open, we can write

$$f^{-1}(V) = U \cap f^{-1}(V) = \bigcup_{i \in I} (U_i \cap f^{-1}(V)) = \bigcup_{i \in I} f_i^{-1}(V),$$

which is open since all the f_i are continuous.

So we see that \mathcal{F} is a sheaf. In the case that Y has the discrete topology, we call \mathcal{F} the *constant sheaf with value Y* . The sections of \mathcal{F} over U are the *locally constant* functions, i.e. at each point $x \in U$ we can find an open neighborhood V of x , such that f is constant on V .

Example 2.9. Let $f: X \rightarrow Y$ be a continuous map, then we get a sheaf on X by the rule

$$\Gamma(Y/X)(U) = \{s: U \rightarrow Y \mid f \circ s = 1_U\}.$$

The gluing construction is the same as in the previous example. To see that this yields another section, note that for any $u \in U$, $s(u) = s_i(u)$ for some $i \in I$, and hence $f(s(u)) = f(s_i(u)) = u$. We call $\Gamma(Y/X)$ the *sheaf of sections of f* .

Example 2.10. As we remarked above, the structure sheaf \mathcal{O}_X is also a sheaf, where again, the restrictions are actual restrictions of functions. To show that the gluing of regular functions is again regular, requires some machinery from algebraic geometry, which falls outside the scope of this paper.

As it turns out, it is possible to generalize the definition of sheaves to categories which also have the notion of a covering. We will need this notion when discussing condensed sets.

2.2 On a site

The idea of a site was first introduced by Alexander Grothendieck, and has proven to be very useful in algebraic geometry. For this subsection we largely follow [Sta23, Tag 00UZ].

2.2.1 Coverings and sites

To define a site, we collect all the essential properties that coverings of topological spaces have:

Definition 2.11 (Coverings and Sites). A *site* is a small category \mathcal{C} together with a set $\text{Cov}(\mathcal{C})$ of *coverings* of \mathcal{C} , where the following axioms hold:¹

1. If $V \rightarrow U$ is an isomorphism, then $\{V \rightarrow U\} \in \text{Cov}(\mathcal{C})$.
2. If $\{U_i \rightarrow U\}_{i \in I}$ is a covering, and for each $i \in I$, $\{V_{ij} \rightarrow U_i\}_{j \in J_i}$ is a covering as well, then so is the composition $\{V_{ij} \rightarrow U\}_{i \in I, j \in J_i}$.
3. If $\{U_i \rightarrow U\}_{i \in I}$ is a covering and $V \rightarrow U$ is a morphism of \mathcal{C} , then the pullback $U_i \times_U V$ exists for all i and $\{U_i \times_U V \rightarrow V\}_{i \in I}$ is a covering.

To make sense of this definition, let us look at the canonical example.

Example 2.12. Let X be a topological space, then $\mathbf{Open}(X)$ is a site with coverings given by the open covers. Let us verify that the axioms hold:

1. The only isomorphisms in $\mathbf{Open}(X)$ are the identity maps. Since $\{U\}$ is an open cover of U for any $U \in \mathbf{Open}(X)$, the first axiom is satisfied.
2. If $U = \bigcup_{i \in I} U_i$ is an open cover of U , and for each $i \in I$, $U_i = \bigcup_{j \in J_i} U_{ij}$ is an open cover of U_i , then $U = \bigcup_{i \in I} \bigcup_{j \in J_i} U_{ij}$ is an open cover of U .
3. If $U = \bigcup_{i \in I} U_i$ is an open cover, and $V \subseteq U$, then $V = \bigcup_{i \in I} V \cap U_i$ is an open cover of V .

Here we used that $U \times_W V = U \cap V$ for $U, V \subseteq W$, which follows from $S \subseteq V, S \subseteq U \implies S \subseteq V \cap U$.

The following example is still quite similar to the previous example. The underlying site of a condensed set will be similar to this site.

Example 2.13. Let G be a group, then we can consider the category of all G -sets, i.e. sets with a corresponding action by the group G . Morphisms are given by maps $f: X \rightarrow Y$ satisfying $g \cdot f(x) = f(g \cdot x)$, i.e. G -equivariant maps. To make this into a site², we define the covers to be the families of jointly surjective maps. In other words $\{f_i: X_i \rightarrow X\}_{i \in I}$ is a cover if $\bigcup_{i \in I} f_i(X_i) = X$. Let us verify the axioms:

¹We force $\text{Cov}(\mathcal{C})$ to be a set since we will take limits over all coverings. It is possible to let $\text{Cov}(\mathcal{C})$ be a class, and then show that it can be replaced with a set of coverings that gives rise to the same category of sheaves. See [Sta23, Tag 00VI].

²Technically, this is not a site, since the category of all G -sets is a proper class. The example is simply meant to illustrate what covers could look like. There is a way around this problem though. See [Sta23, Example 00VK].

1. If $f: X \rightarrow Y$ is an isomorphism, then in particular it is surjective, so $\{f: X \rightarrow Y\}$ is a cover of Y
2. If $\{f_i: X_i \rightarrow Y\}_{i \in I}$ is jointly surjective, and for each $i \in I$ the family $\{f_{ij}: X_{ij} \rightarrow X_i\}_{j \in J_i}$ is jointly surjective, then so is $\{f_i \circ f_{ij}: X_{ij} \rightarrow Y\}_{i \in I, j \in J_i}$.
3. Let $\{f_i: X_i \rightarrow Y\}_{i \in I}$ be a cover, and $f: T \rightarrow Y$ a G -equivariant map. We claim that the pullback $X_i \times_Y T$ exists and is given by $S = \{(x, t) \in X_i \times T \mid f_i(x) = f(t)\}$. This is a G -set, as if $f_i(x) = f(t)$, then

$$f_i(g \cdot x) = g \cdot f_i(x) = g \cdot f(t) = f(g \cdot t).$$

Since S is the pullback in **Set** and any G -equivariant map is in particular a map of sets, the claim follows. To see that the projection maps $\{X_i \times_Y T \rightarrow T\}$ are jointly surjective, take $t \in T$. There exists $i \in I$ such that $f(t) \in f_i(X_i)$, so $(x_i, t) \in X_i \times_Y T$ for some $x_i \in X_i$.

In this new context, we have to reword the definitions of presheaves and sheaves in a more abstract manner.

Definition 2.14 (Presheaves and Sheaves on a Site). Let \mathcal{C} be a small category. A *presheaf on \mathcal{C}* is a contravariant functor $\mathcal{F}: \mathcal{C} \rightarrow \mathbf{Set}$. If \mathcal{C} is a site, then a *sheaf on \mathcal{C}* is a presheaf \mathcal{F} , such that for any covering $\{U_i \rightarrow U\}_{i \in I}$, the following diagram is an equalizer:

$$\mathcal{F}(U) \xrightarrow{e} \prod_{i \in I} \mathcal{F}(U_i) \xrightleftharpoons[p_1^*]{p_0^*} \prod_{i, j \in I} \mathcal{F}(U_i \times_U U_j) .$$

The map e is given by $s \mapsto (s|_{U_i})_{i \in I}$. For the second maps, take $i, j \in I$. Then we have projections $p_0^{(i, j)}: U_i \times_U U_j \rightarrow U_i$ and $p_1^{(i, j)}: U_i \times_U U_j \rightarrow U_j$. These result in maps $p_0^{(i, j), *}: \mathcal{F}(U_i) \rightarrow \mathcal{F}(U_i \times_U U_j)$ and $p_1^{(i, j), *}: \mathcal{F}(U_j) \rightarrow \mathcal{F}(U_i \times_U U_j)$. The maps p_0^* and p_1^* are then given at component i, j by mapping $(s_k)_{k \in I}$ to $p_0^{(i, j), *}(s_i)$ and $p_1^{(i, j), *}(s_j)$ respectively.

2.2.2 Sheafification

In a lot of constructions, the natural thing we do, will end up being a presheaf, but not a sheaf in general. So, we will need some way to turn presheaves into sheaves.

Proposition 2.15. *The fully faithful inclusion*

$$\iota: \mathbf{Sh}(\mathcal{C}) \hookrightarrow \mathbf{PSh}(\mathcal{C}),$$

admits a left adjoint $\mathcal{F} \rightarrow \mathcal{F}^\sharp$, “sheafification”.

We will give an explicit construction of \mathcal{F}^\sharp . As it turns out, “sheafification” is actually a two-step process. The first step is making the presheaf separated, and then turning the separated presheaf into a sheaf.

Definition 2.16. We say that a presheaf \mathcal{F} is *separated* if $\mathcal{F}(U) \rightarrow \prod_{i \in I} \mathcal{F}(U_i)$ is injective for any cover $\{U_i \rightarrow U\}_{i \in I} \in \mathbf{Cov}(\mathcal{C})$.

Remark 2.17. Since any equalizer is a monomorphism, it follows from the definition of sheaves on a site, that a sheaf is separated.

Example 2.18. The presheaf \mathcal{F} on a topological space X , which maps every open to the same set $Y \neq \{*\}$ is not separated, as

$$Y = \mathcal{F}(\emptyset) \rightarrow \prod_{i \in \emptyset} = \{*\},$$

is not injective if $Y \neq \{*\}$.

On the other hand, the presheaf \mathcal{G} on a topological space X which maps every open U to the constant functions $U \rightarrow Y$ for some space Y , is separated but not a sheaf. After all, if $U = \cup_i U_i$ is an open cover of U , and $f, g \in \mathcal{G}(U)$ are such that $f|_{U_i} = g|_{U_i}$ then $f = g$, since any $x \in U$ is contained in some U_i , and hence $f(x) = f|_{U_i}(x) = g|_{U_i}(x) = g(x)$. Since there is only a single function $\emptyset \rightarrow Y$ there are no problems with empty coverings. \mathcal{G} fails to be a sheaf in general, since if $U = U_1 \cup U_2$ with $U_1 \cap U_2 = \emptyset$ and $U_1 \neq \emptyset \neq U_2$, then we can't "glue" the functions $f: U_1 \rightarrow Y, f(x) = y_1$ and $g: U_2 \rightarrow Y, g(x) = y_2$ together to a constant function on U if $y_1 \neq y_2$.

We first have a few lemmas regarding limits of presheaves and sheaves.

Lemma 2.19. *Let \mathcal{C} be a site, then limits and colimits exist in $\mathbf{PSh}(\mathcal{C})$. Additionally, for any $U \in \mathcal{C}$, the functor $\mathbf{PSh}(\mathcal{C}) \rightarrow \mathbf{Set}: \mathcal{F} \mapsto \mathcal{F}(U)$ commutes with all limits and colimits.*

Proof. We only prove the case of limits. The colimit case is analogous.

Let $\mathcal{F}: \mathcal{I} \rightarrow \mathbf{PSh}(\mathcal{C})$ be a diagram. We get a cone $(\mathcal{F}_{\lim}, p^i)$ by moving to the images in \mathbf{Set} . In other words we take $\mathcal{F}_{\lim}(U) = \lim_{i \in \mathcal{I}} \mathcal{F}_i(U)$, and $p_U^i: \lim_{i \in \mathcal{I}} \mathcal{F}_i(U) \rightarrow \mathcal{F}_i(U)$. If $U \rightarrow V$ is a morphism in \mathcal{C} , we get a unique map $\mathcal{F}_{\lim}(V) \rightarrow \mathcal{F}_{\lim}(U)$, given by the fact that $\mathcal{F}_{\lim}(U)$ is a limit, and $\mathcal{F}_{\lim}(V)$ is a cone on all the $\mathcal{F}_i(U)$ by composing the maps $\mathcal{F}_{\lim}(V) \rightarrow \mathcal{F}_i(V)$ with the maps $\mathcal{F}_i(V) \rightarrow \mathcal{F}_i(U)$. So we see that \mathcal{F}_{\lim} is indeed a presheaf.

$$\begin{array}{ccccc}
 & & \mathcal{F}_{\lim}(V) & & \\
 & \swarrow p_V^i & \downarrow & \searrow p_V^{i'} & \\
 & & \mathcal{F}_{\lim}(U) & & \\
 & \swarrow p_U^i & & \searrow p_U^{i'} & \\
 \mathcal{F}_i(V) \rightarrow \mathcal{F}_i(U) & \xrightarrow{\quad} & \mathcal{F}_{i'}(U) \leftarrow \mathcal{F}_{i'}(V)
 \end{array}$$

To see that the maps p^i are morphisms of presheaves, we need to verify that the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{F}_{\lim}(V) & \xrightarrow{p_V^i} & \mathcal{F}_i(V) \\
 \downarrow & & \downarrow \\
 \mathcal{F}_{\lim}(U) & \xrightarrow{p_U^i} & \mathcal{F}_i(U)
 \end{array},$$

but that already follows from the previous diagram.

Let us now verify that \mathcal{F}_{\lim} is actually a limit. If (\mathcal{G}, g^i) is another cone, then for each $U \in \mathcal{C}$, we get a unique map $\mathcal{G}(U) \rightarrow \mathcal{F}_{\lim}(U)$, such that the corresponding cone diagrams

commute. It suffices to show that these maps combine to form a map of presheaves. Using the universal property of limits, this comes down to showing

$$\begin{array}{ccc} \mathcal{G}(V) & \longrightarrow & \mathcal{F}_{\lim}(V) \\ \downarrow & & \downarrow \\ \mathcal{G}(U) & \longrightarrow & \mathcal{F}_{\lim}(U) \end{array} \iff \begin{array}{ccc} \mathcal{G}(V) & \xrightarrow{g_V^i} & \mathcal{F}_i(V) \\ \downarrow & & \downarrow \\ \mathcal{G}(U) & \xrightarrow{g_U^i} & \mathcal{F}_i(U) \end{array} \quad \forall i \in \mathcal{I}.$$

The commutativity of these last diagrams, is just saying that the maps g^i are presheaf morphisms, which is true by (\mathcal{G}, g^i) being a cone. \square

The big difference between sheaves and presheaves, is that we can glue things together defined on a cover. The trick will be to force the presheaf to behave as we want. Let us try and make this more precise.

So far we have just talked about coverings as objects. We can also consider maps between coverings. If $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$ and $\mathcal{V} = \{V_j \rightarrow V\}_{j \in J}$ are two coverings, a morphism of coverings between \mathcal{U} and \mathcal{V} is a morphism $U \rightarrow V$ and a map $\alpha: I \rightarrow J$ together with morphisms $U_i \rightarrow V_{\alpha(i)}$ such that

$$\begin{array}{ccc} U_i & \longrightarrow & V_{\alpha(i)} \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

commutes. If $U = V$ and $U \rightarrow V$ is the identity, we call \mathcal{U} a *refinement* of \mathcal{V} . For a $U \in \mathcal{C}$ the coverings together with refinements gives a category $\text{Cov}(U)$. This allows the following reformulation:

Lemma 2.20. *Let \mathcal{F} be a presheaf on a site \mathcal{C} . For $U \in \mathcal{C}$ and a cover $\mathcal{U} \in \text{Cov}(U)$, define $\mathcal{F}(\mathcal{U})$ as the following equalizer:*

$$\mathcal{F}(\mathcal{U}) = \{(s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(U_i) \mid s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}\}.$$

If $\mathcal{U} \rightarrow \mathcal{V}$ is a morphism of coverings, there is an induced map $\mathcal{F}(\mathcal{V}) \rightarrow \mathcal{F}(\mathcal{U})$. This construction is functorial. Furthermore, since $\{1_U\}$ is a cover by the axioms of a site, this gives a map $\mathcal{F}(U) \cong \mathcal{F}(\{1_U\}) \rightarrow \mathcal{F}(\mathcal{U})$. The presheaf \mathcal{F} is a sheaf if and only if the map $\mathcal{F}(U) \rightarrow \mathcal{F}(\mathcal{U})$ is bijective for every cover \mathcal{U} .

Proof. Let $\mathcal{U} \rightarrow \mathcal{V}$ be a map of coverings. We have the following commutative diagram:

$$\begin{array}{ccccc} & & U_i & \longrightarrow & V_{\alpha(i)} \\ & \nearrow & \downarrow & & \nwarrow \\ U_i \times_U U_j & & U & \longrightarrow & V \\ & \searrow & \uparrow & & \nearrow \\ & & U_j & \longrightarrow & V_{\alpha(j)} \end{array} \quad , \quad V_{\alpha(i)} \times_V V_{\alpha(j)}$$

which gives a unique map $U_i \times_U U_j \rightarrow V_{\alpha(i)} \times_V V_{\alpha(j)}$ making the diagram commute. Applying \mathcal{F} gives the diagram:

$$\begin{array}{ccc}
 \mathcal{F}(U_i) & \longleftarrow & \mathcal{F}(V_{\alpha(i)}) \\
 \downarrow & & \downarrow \\
 \mathcal{F}(U_i \times_U U_j) & \longleftarrow & \mathcal{F}(V_{\alpha(i)} \times_V V_{\alpha(j)}) \\
 \uparrow & & \uparrow \\
 \mathcal{F}(U_j) & \longleftarrow & \mathcal{F}(V_{\alpha(j)})
 \end{array} \quad (1)$$

We can now define $\mathcal{F}(\mathcal{V}) \rightarrow \mathcal{F}(\mathcal{U})$ by

$$(s_j)_{j \in J} \mapsto ((s_{\alpha(i)})|_{U_i})_{i \in I}.$$

That this map is well-defined follows from eq. (1). Indeed, if $(s_j)_{j \in J} \in \mathcal{F}(\mathcal{V})$, then

$$(s_{\alpha(j)}|_{V_{\alpha(i)} \times_U V_{\alpha(j)}})|_{U_i \times U_j} = (s_{\alpha(i)}|_{V_{\alpha(i)} \times_U V_{\alpha(j)}})|_{U_i \times U_j},$$

and hence

$$(s_{\alpha(i)}|_{U_i})|_{U_i \times_U U_j} = (s_{\alpha(j)}|_{U_j})|_{U_i \times_U U_j}.$$

It remains to show that this construction is functorial, since the final claim is just the definition of a sheaf of sites. Let, to this purpose, $\mathcal{U} \rightarrow \mathcal{V} \rightarrow \mathcal{W}$ be maps of coverings, with associated maps $\alpha: I \rightarrow J$ and $\beta: J \rightarrow K$. The map $\mathcal{F}(\mathcal{W}) \rightarrow \mathcal{F}(\mathcal{U})$ maps $(s_k)_{k \in K}$ to $(s_{\beta(\alpha(i))}|_{U_i})_{i \in I}$. The other map is given by first sending $(s_k)_{k \in K}$ to $(s_{\beta(j)}|_{V_j})_{j \in J}$, which is then sent to $((s_{\beta(\alpha(i))}|_{V_{\alpha(i)}})|_{U_i})_{i \in I}$. The result now follows by functoriality of \mathcal{F} . \square

We now have the following construction:

Lemma 2.21. *Given a presheaf of \mathcal{F} on a site \mathcal{C} , we can construct a new presheaf \mathcal{F}^+ by setting*

$$\mathcal{F}^+(U) = \operatorname{colim}_{\mathcal{U} \in \operatorname{Cov}(U)} \mathcal{F}(\mathcal{U}),$$

Proof. Let us first show how \mathcal{F}^+ acts on morphisms. Consider a morphism $U \rightarrow V$. Then this gives a map $\operatorname{Cov}(V) \rightarrow \operatorname{Cov}(U)$ by mapping a cover $\{V_i \rightarrow V\}_{i \in I}$ of V to the cover $\{V_i \times_V U \rightarrow U\}_{i \in I}$ of U , which exists by the third axiom of coverings. From the previous lemma there are induced maps $\mathcal{F}(\mathcal{U}) \rightarrow \mathcal{F}(\mathcal{V})$. Since we have maps $\mathcal{F}(\mathcal{V}) \rightarrow \mathcal{F}^+(V)$, we obtain maps $\mathcal{F}(\mathcal{U}) \rightarrow \mathcal{F}^+(V)$. By the universal property of the colimit, this gives a unique map $\mathcal{F}^+(V) \rightarrow \mathcal{F}^+(U)$, making the cone diagrams commute.

$$\begin{array}{ccccc}
 & & \mathcal{F}^+(U) & & \\
 & \nearrow & \uparrow & \nwarrow & \\
 & & \mathcal{F}^+(V) & & \\
 \mathcal{F}(\mathcal{U}) & \rightarrow & \mathcal{F}(\mathcal{V}) & \rightarrow & \mathcal{F}(\mathcal{V}') \leftarrow \mathcal{F}(\mathcal{U}')
 \end{array}$$

It remains to show that this defines a functor $\mathcal{C}^{\text{opp}} \rightarrow \mathbf{Set}$. If we have morphisms $U \rightarrow V \rightarrow W$, then

$$W_i \times_W U \cong (W_i \times_W V) \times_V U,$$

since both are pullbacks of $W_i \rightarrow W \leftarrow U$. So $\text{Cov}(-)$ is a functor, and since $\mathcal{U} \mapsto \mathcal{F}(\mathcal{U})$ is a functor by the previous lemma, the result now follows. \square

It is possible to be more explicit about how \mathcal{F}^+ looks like. For two coverings $\mathcal{U}, \mathcal{U}'$ of $U \in \mathcal{C}$, we have a common refinement $\{U_i \times_U U'_j \rightarrow U\}_{i \in I, j \in J}$ which exists by the second and third axioms. Furthermore, one can show that if $f, g: \mathcal{U} \rightarrow \mathcal{V}$ are refinements, then $\mathcal{F}(f) = \mathcal{F}(g)$ ([Sta23, Tag 00W7]). This gives that $\text{Cov}(U)^{\text{opp}} \rightarrow \mathbf{Set}$ is a filtered diagram. So

$$\mathcal{F}^+(U) = \left(\coprod_{\mathcal{U} \in \text{Cov}(U)} \mathcal{F}(\mathcal{U}) \right) / \sim,$$

Where $s \sim s'$ if and only if there are covers $\mathcal{U}, \mathcal{U}'$ with $s \in \mathcal{F}(\mathcal{U}), s' \in \mathcal{F}(\mathcal{U}')$ and a common refinement \mathcal{V} such that

$$s_{\alpha(i)}|_{V_i} = s'_{\beta(i)}|_{V_i}, \forall i \in I.$$

We now come to the main theorem, from which proposition 2.15 will follow.

Theorem 2.22. *Let \mathcal{F} be a presheaf on a site. Then*

1. *The presheaf \mathcal{F}^+ is separated.*
2. *If \mathcal{F} is separated, then \mathcal{F}^+ is a sheaf.*
3. *If \mathcal{F} is a sheaf, then $\mathcal{F} \rightarrow \mathcal{F}^+$ is an isomorphism.*

Proof.

1. We need to show that $s \mapsto (s|_{U_i})_{i \in I}$ is injective for any cover $\{U_i \rightarrow U\}_{i \in I} \in \text{Cov}(\mathcal{C})$. Take $s, s' \in \mathcal{F}^+(U)$ such that $(s|_{U_i})_{i \in I} = (s'|_{U_i})_{i \in I}$ for some cover $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$. By the description above of \mathcal{F}^+ , we know that we can find covers \mathcal{V} and \mathcal{V}' of U , such that $s \in \mathcal{F}(\mathcal{V}) / \sim$ and $s' \in \mathcal{F}(\mathcal{V}') / \sim$. Let \mathcal{W} be a common refinement of the three covers $\mathcal{U}, \mathcal{V}, \mathcal{V}'$. Since it is a refinement of \mathcal{U} , we have that

$$s'|_{W_j} = (s|_{U_{\alpha(j)}})|_{W_j} = (s|_{U_{\alpha(j)}})|_{W_j} = s'|_{W_j},$$

so that $s = s'$.

2. We need to verify the sheaf condition. Let $\{U_i \rightarrow U\}_{i \in I}$ be a cover of U , and for each $i \in I$, $s_i \in \mathcal{F}^+(U_i)$ such that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for every $i, j \in I$. Since \mathcal{F} is separated, the map $s \mapsto (s|_{U_i})_{i \in I}$ is injective. It is hence enough to show that there is some $s \in \mathcal{F}^+(U)$ with $s|_{U_i} = s_i$ for all $i \in I$. For each $i \in I$ we can find a cover $\mathcal{U}_i = \{U_{ij} \rightarrow U_i\}$ such that $s_i \in \mathcal{F}(\mathcal{U}_i) / \sim$, and hence $s_{ij} \in \mathcal{F}(U_{ij})$ such that $s_i|_{U_{ij}} = s_{ij} / \sim$. In the same way as lemma 2.20 we get that

$$s_{ij}|_{U_{ij} \times_U U_{i'j'}} = s_{i'j'}|_{U_{ij} \times_U U_{i'j'}}.$$

So, $(s_{ij})_{i,j \in I} \in \mathcal{F}(\{U_{ij} \rightarrow U\}_{i,j \in I})$, and we can take $s = (s_{ij})_{i,j \in I} / \sim$. We just need to verify that $s|_{U_i} = s_i$. This follows from $(s|_{U_i})|_{U_{ij}} = s|_{U_{ij}} = s_{ij} = s_i|_{U_{ij}}$, as \mathcal{F}^+ is also separated.

3. This follows from lemma 2.20. □

With this we are now ready to prove proposition 2.15. We define $\mathcal{F}^\# = \mathcal{F}^{++}$.

Proof of proposition 2.15. We first note that for any map of presheaves $\alpha: \mathcal{F} \rightarrow \mathcal{G}$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ \\ \downarrow \alpha & & \downarrow \alpha^+ \\ \mathcal{G} & \longrightarrow & \mathcal{G}^+ \end{array}$$

where $\alpha_U^+(s/\sim) = \alpha_U(s)/\sim$. That this commutes, is by construction of α^+ . Using this, we can now show that

$$\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathcal{F}^\#, \mathcal{G}) = \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathcal{F}, \iota(\mathcal{G})).$$

The bottom row of the diagram

$$\begin{array}{ccccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ & \longrightarrow & \mathcal{F}^{++} = \mathcal{F}^\# \\ \downarrow & & \downarrow & & \downarrow \\ \iota(\mathcal{G}) & \longrightarrow & \iota(\mathcal{G})^+ & \longrightarrow & \iota(\mathcal{G})^{++} = \mathcal{G}^\# \end{array},$$

consists of isomorphisms by the previous theorem. So, any map $\mathcal{F} \rightarrow \iota(\mathcal{G})$ gives rise to a map $\mathcal{F}^\# \rightarrow \mathcal{G}$. Conversely, since every $s \in \mathcal{F}^\#(U)$ comes from sections in $\mathcal{F}(U)$, we can lift any map $\mathcal{F}^\# \rightarrow \mathcal{G}$ to a map $\mathcal{F} \rightarrow \mathcal{G}$. □

As an immediate consequence, we get that $(-)^{\#}$ commutes with all colimits. So $\mathbf{Sh}(\mathcal{C})$ has all colimits, since $\mathbf{PSh}(\mathcal{C})$ has all colimits by lemma 2.19. We can say more:

Proposition 2.23. *The functor $(-)^{\#}: \mathbf{PSh}(\mathcal{C}) \rightarrow \mathbf{Sh}(\mathcal{C})$ is exact.*

Proof. Since it is a left adjoint, it is right exact. On the other hand, colimits over filtered diagrams commute with finite limits. So, $(-)^{\#}$ is left exact as functor between presheaves. We claim that if $\mathcal{I} \rightarrow \mathbf{Sh}(\mathcal{C})$ is a diagram, then the limit $\mathcal{F} = \lim_i \mathcal{F}_i$ as presheaves is a sheaf. For this we show that $\mathcal{F}(\mathcal{U}) \cong \mathcal{F}(U)$ for any cover $\mathcal{U} = \{U_j \rightarrow U\}$. Take $(s_j)_{j \in J} \in \mathcal{F}(\mathcal{U})$, then by definition of the limit, we can project these to elements $(s_{ij})_{j \in J} \in \mathcal{F}_i(\mathcal{U})$. Since each \mathcal{F}_i is a sheaf, we have unique elements $s_i \in \mathcal{F}_i(U)$ such that $s_i|_{U_j} = s_{ij}$.

We now want an element $s \in \mathcal{F}(U)$ with projections equal to the s_i . Choosing an element of $\mathcal{F}(U)$, is the same as giving a map $\{*\} \rightarrow \mathcal{F}(U)$, which by the universal property of the limit is the same as giving a cone $(\{*\}, \lambda_i)$. Let $\lambda_i(*) = s_i$, then we just need to verify that this defines a cone. We need that for $f: i \rightarrow i'$ in \mathcal{I} , $\mathcal{F}(f)(s_i) = s_{i'}$. This follows, as $s_i|_{U_j}$ is mapped to $s_{i'}|_{U_j}$ for all $j \in J$, and hence s_i is mapped to $s_{i'}$ because $\mathcal{F}_{i'}$ is a sheaf. So we have a unique $s \in \mathcal{F}(U)$, and by the universal property of the limit

$$s|_{U_j} = s_j \iff s_i|_{U_j} = s_{ij} \quad \forall i \in \mathcal{I},$$

which holds by construction. Hence, the claim holds, and $(-)^{\#}$ is also right exact as a functor into sheaves. □

To get a better understanding of sheafification, let us work out the process of sheafification for some presheaves.

Example 2.24. We consider the presheaf \mathcal{F} of example 2.18. To calculate \mathcal{F}^+ we use that

$$\mathcal{F}^+(U) = \left(\coprod_{\mathcal{U} \in \text{Cov}(U)} \mathcal{F}(\mathcal{U}) \right) / \sim.$$

If U is non-empty, then

$$\mathcal{F}(\mathcal{U}) = \{(y_i)_{i \in I} \in \prod_{i \in I} Y \mid y_i = y_j \ \forall i, j \in I\} = \{(y)_{i \in I} \mid y \in Y\}$$

for any cover \mathcal{U} of U . Take $(y)_{i \in I} \in \mathcal{F}(\mathcal{U})$, and $(y')_{j \in J} \in \mathcal{F}(\mathcal{U}')$. For any common refinement \mathcal{V} of \mathcal{U} and \mathcal{U}' , we have $(y_{\alpha(i)})|_{V_i} = y$ and $(y'_{\beta(j)})|_{V_i} = y'$ so that $(y)_{i \in I} \sim (y')_{j \in J} \iff y = y'$. So, we find again that $\mathcal{F}^+(U) = Y$ as long as $U \neq \emptyset$.

Now, when $U = \emptyset$ there are two covers: $\{\emptyset \rightarrow \emptyset\}$, and the empty covering, $\{\}_{i \in \emptyset}$. We have $\mathcal{F}(\{\emptyset \rightarrow \emptyset\}) = \mathcal{F}(\emptyset) = Y$, while $\mathcal{F}(\{\}_{i \in \emptyset}) = \{*\}$. Now every $y \in Y$ is equivalent to $*$, as $\{\}_{i \in \emptyset}$ is a common refinement of the two covers, and the equivalence condition becomes an empty statement in this case. As such we find $\mathcal{F}^+(\emptyset) = \{*\}$. Note that \mathcal{F}^+ is isomorphic to \mathcal{G} from example 2.18. After all, a constant function $U \rightarrow Y$ is the same as choosing an element in Y .

We now have a separated presheaf \mathcal{F}^+ . What does the sheaf $\mathcal{F}^\#$ look like? Let us first consider the case that U is empty. Then for both covers $\{\emptyset \rightarrow \emptyset\}$ and $\{\}_{i \in \emptyset}$ we get $\mathcal{F}^+(\mathcal{U}) = \{*\}$. So $\mathcal{F}^\#(\emptyset) = \{*\}$.

More interesting is what happens when $U \neq \emptyset$. Let $\{U_i \rightarrow U\}_{i \in I}$ be a cover such that none of the U_i are empty. Then

$$\begin{aligned} \mathcal{F}^+(\mathcal{U}) &= \{(y_i)_{i \in I} \in \prod_{i \in I} Y \mid y_i|_{U_i \cap U_j} = y_j|_{U_i \cap U_j} \ \forall i, j \in I\} \\ &= \{(y_i)_{i \in I} \in \prod_{i \in I} Y \mid y_i = y_j, U_i \cap U_j \neq \emptyset\}, \end{aligned}$$

as $\mathcal{F}^+(\emptyset) = \{*\}$ and hence $y|_{U_i \cap U_j} = *$ if $U_i \cap U_j = \emptyset$ for any $y \in Y$. We can view elements of $\mathcal{F}^+(\mathcal{U})$ as functions $s: U \rightarrow Y$, that are constant on each of the opens U_i . Adding empty sets to the cover does not really change anything: the elements of $\mathcal{F}(\mathcal{U})$ will be $*$ at the indices for which $U_i = \emptyset$. Under \sim such functions will all be the same. Take, $f \in \mathcal{F}^\#(U)$. Then there is some cover $\mathcal{U} = \{U_i\}_{i \in I}$ such that f arises from $\mathcal{F}^+(\mathcal{U})$. Consequently, for each $x \in U$, there is some U_i containing x , such that f is constant on U_i . In other words, f is a locally constant function on U . If we equip Y with the discrete topology then this is equivalent to saying that $f: U \rightarrow Y$ is continuous.

The sheaf $\mathcal{F}^\#$ is (perhaps a little confusingly) called the *constant sheaf with value Y* and sometimes denoted as \underline{Y} .

3 Sheaf cohomology

In the previous section we have explored some fundamental properties of sheaves of sets. We will now be looking at sheaves of abelian groups, which form a very interesting and rich

abelian category. The previous section only proved results for sheaves of sets, but these results still hold for sheaves of abelian groups. This can be shown by either verifying that all the proofs still hold in the abelian case, or through a more abstract approach like in [Sta23, Section 00YR]

Proposition 3.1. *Let \mathcal{C} be a site. The category $\mathbf{Ab}(\mathcal{C})$ of abelian sheaves on \mathcal{C} is abelian.*

Proof. Let \mathcal{F} and \mathcal{G} be two abelian sheaves on \mathcal{C} . For natural transformations $\alpha, \beta \in \mathrm{Hom}_{\mathbf{Ab}(\mathcal{C})}(\mathcal{F}, \mathcal{G})$ we define $\alpha + \beta$ via

$$(\alpha + \beta)_U \stackrel{\mathrm{def}}{=} \alpha_U + \beta_U,$$

where the $+$ on the right-hand side is the $+$ in \mathbf{Ab} . Let us verify that $\alpha + \beta$ is a natural transformation. If $f: U \rightarrow V$ is a morphism in \mathcal{C} we have:

$$\begin{aligned} (\alpha + \beta)_U \circ \mathcal{F}(f) &= (\alpha_U + \beta_U) \circ \mathcal{F}(f) \\ &= \alpha_U \circ \mathcal{F}(f) + \beta_U \circ \mathcal{F}(f) \\ &= \mathcal{G}(f) \circ \alpha_V + \mathcal{G}(f) \circ \beta_V \\ &= \mathcal{G}(f) \circ (\alpha_V + \beta_V) \\ &= \mathcal{G}(f) \circ (\alpha + \beta)_V, \end{aligned}$$

so that $\alpha + \beta$ is indeed an abelian category. With this operation, the hom-sets become abelian groups. Using lemma 2.19 and proposition 2.23 we find that $\mathbf{Ab}(\mathcal{C})$ has finite limits and colimits. Explicitly, limits in sheaves are the limits in presheaves and colimits in sheaves are the sheafifications of the colimits in presheaves. As such, we find:

- The zero object is given by $\underline{0}$, the constant sheaf with value 0.
- The biproduct of \mathcal{F} and \mathcal{G} is given by $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$
- The kernel of a map $\alpha: \mathcal{F} \rightarrow \mathcal{G}$ is given by $\mathrm{Ker}(\alpha)(U) = \mathrm{Ker}(\alpha_U)$.
- The cokernel of $\alpha: \mathcal{F} \rightarrow \mathcal{G}$ is given by the sheafification of the presheaf defined by $\mathcal{F}(\alpha)(U) = \mathrm{Coker}(\alpha_U)$.

For $\mathbf{Ab}(\mathcal{C})$ to be an abelian category, we still need to show that for $\alpha: \mathcal{F} \rightarrow \mathcal{G}$ a morphism of sheaves, $\mathrm{Im}(\alpha) = \mathrm{Coim}(\alpha)$. In the category of abelian presheaves we have

$$\mathrm{Coim}(\iota(\alpha))(U) = \mathrm{Coim}(\iota(\alpha)_U) = \mathrm{Im}(\iota(\alpha)_U) = \mathrm{Im}(\iota(\alpha))(U),$$

where ι is the inclusion functor of sheaves into presheaves. Since $(-)^{\#}$ is exact, we find

$$\begin{aligned}
\mathrm{Coim}(\alpha) &= \mathrm{Coker}(\mathrm{Ker}(\alpha)) \\
&= \mathrm{Coker}^{\#}(\iota(\mathrm{Ker}(\alpha))) \\
&= \mathrm{Coker}^{\#}(\mathrm{Ker}(\iota(\alpha))) \\
&= \mathrm{Coim}^{\#}(\iota(\alpha)) \\
&= \mathrm{Im}^{\#}(\iota(\alpha)) \\
&= \mathrm{Ker}^{\#}(\mathrm{Coker}(\iota(\alpha))) \\
&= \mathrm{Ker}(\mathrm{Coker}^{\#}(\iota(\alpha))) \\
&= \mathrm{Ker}(\mathrm{Coker}(\alpha)) \\
&= \mathrm{Im}(\alpha).
\end{aligned}$$

□

TODO: example where Coker really needs the sheafification.

We will now restrict ourselves to the special case where $\mathcal{C} = \mathbf{Open}(X)$ for some topological space X . We follow the general ideas presented in [CS20, Session 1], while filling out the details.

Proposition 3.2. *Let $f: X \rightarrow Y$ be a continuous map. Then there is a pair of adjoint functors: the pullback functor, $f^*: \mathbf{Ab}(Y) \rightarrow \mathbf{Ab}(X)$, with right adjoint, the pushforward functor $f_*: \mathbf{Ab}(X) \rightarrow \mathbf{Ab}(Y)$. Furthermore, f^* is exact.*

Proof. We start off by writing down what the pullback and pushforward functors look like.

Let \mathcal{F} be a sheaf on X , and $V \in \mathbf{Open}(Y)$. Then, $(f_*\mathcal{F})(V) \stackrel{\mathrm{def}}{=} \mathcal{F}(f^{-1}(V))$. If \mathcal{G} is a sheaf on Y , then we get a presheaf $V \mapsto \mathrm{colim}_{U \supseteq f(V)} \mathcal{G}(U)$, and $f^*\mathcal{G}$ is defined as the sheafification of this presheaf. There are now a lot of things to verify:

- $f_*\mathcal{F}$ is a sheaf on Y . Since f is continuous, $f^{-1}(V) \in \mathbf{Open}(X)$ for all $V \in \mathbf{Open}(Y)$. Furthermore, if $V \subseteq V' \subseteq V''$ then $f^{-1}(V) \subseteq f^{-1}(V') \subseteq f^{-1}(V'')$. So, f^{-1} is a functor $\mathbf{Open}(Y) \rightarrow \mathbf{Open}(X)$, and $f_*\mathcal{F}$ is a presheaf as composition of functors. If $\{V_i\}_{i \in I}$ is an open cover of V , then $\{f^{-1}(V_i)\}_{i \in I}$ is an open cover of $f^{-1}(V)$. So $(f_*\mathcal{F})(V) \rightarrow (f_*\mathcal{F})(\{V_i\}_{i \in I})$ is a bijection since $\mathcal{F}(f^{-1}(V)) \rightarrow \mathcal{F}(\{f^{-1}(V_i)\}_{i \in I})$ is a bijection by \mathcal{F} being a sheaf. Consequently, $f_*\mathcal{F}$ is a sheaf by lemma 2.20.
- $f^*\mathcal{G}$ is a sheaf on X . Since we take the sheafification, we just need to verify that $G(U) = \mathrm{colim}_{V \supseteq f(U)} \mathcal{G}(V)$ defines a presheaf. First note that if $U \subseteq U'$, then $\{V \supseteq f(U')\} \subseteq \{V \supseteq f(U)\}$. So, $G(U)$ is also a co-cone for $\mathcal{G}(V)_{V \supseteq f(U')}$, and there is hence a unique map $G(U') \rightarrow G(U)$, by the colimit property. The uniqueness of this map gives the functoriality of G . After all, the map $G(U) \rightarrow G(U)$ must be the identity, by uniqueness, and the two maps $G(U'') \rightarrow G(U') \rightarrow G(U)$, for $U \subseteq U' \subseteq U''$ must also be equal by uniqueness.
- f_* is a functor. For $\alpha: \mathcal{F} \rightarrow \mathcal{G}$, we define $f_*\alpha$ via $(f_*\alpha)_V = \alpha_{f^{-1}(V)}$. Then,

$$(f_*\beta \circ f_*\alpha)_V = \beta_{f^{-1}(V)} \circ \alpha_{f^{-1}(V)} = (\beta \circ \alpha)_{f^{-1}(V)} = (f_*(\beta \circ \alpha))_V,$$

so we just need to check that $f_*\alpha$ is a natural transformation. If $V \subseteq V'$, then $f^{-1}(V) \subseteq f^{-1}(V')$, so naturality of $f_*\alpha$ in V, V' follows from naturality of α in $f^{-1}(V), f^{-1}(V')$.

- f^* is a functor. Let $\alpha: \mathcal{G} \rightarrow \mathcal{F}$ be a natural transformation, and G and F be the presheaves from the construction of $f^*\mathcal{G}$ and $f^*\mathcal{F}$. To define $f^*\alpha$ we need to give a map $G(U) \rightarrow F(U)$ for each $U \in \mathbf{Open}(X)$. For each $V \supseteq f(U)$ we have a map $\mathcal{F}(V) \rightarrow F(U)$ which we can compose with α_V to get a map $\mathcal{G}(V) \rightarrow F(U)$. Naturality of α implies that this gives a well-defined co-cone, and hence a unique morphism $G(U) \rightarrow F(U)$ by the universal property of colimits. Uniqueness of this map gives the functoriality of f^* .
- $\mathrm{Hom}_{\mathbf{Ab}(X)}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathbf{Ab}(Y)}(\mathcal{G}, f_*\mathcal{F})$. Let $\alpha: \mathcal{G} \rightarrow f_*\mathcal{F}$ be a morphism in $\mathbf{Ab}(Y)$. So, for each $V \in \mathbf{Open}(Y)$ we have a map

$$\alpha_V: \mathcal{G}(V) \rightarrow \mathcal{F}(f^{-1}(V)).$$

Now, if $V \supseteq f(U)$, then $f^{-1}(V) \supseteq f^{-1}(f(U)) \supseteq U$. So we get a map

$$\beta_U^V \stackrel{\text{def}}{=} \alpha_V|_U: \mathcal{G}(V) \rightarrow \mathcal{F}(f^{-1}(V)) \rightarrow \mathcal{F}(U),$$

for each $V \supseteq f(U)$. In order for $(\mathcal{F}(U), \beta_U^V)$ to form a co-cone we need that

$$\begin{array}{ccccc} & & \mathcal{F}(U) & & \\ & \nearrow |_U & & \nwarrow |_U & \\ \mathcal{F}(f^{-1}(V)) & \xrightarrow{|_{f^{-1}(V')}} & \mathcal{F}(f^{-1}(V')) & & \\ \alpha_V \uparrow & & & & \uparrow \alpha_{V'} \\ \mathcal{G}(V) & \xrightarrow{|_{V'}} & \mathcal{G}(V') & & \end{array}$$

commutes for $V \supseteq V' \supseteq f(U)$. The top triangle commutes by functoriality of \mathcal{F} , while the bottom square commutes by naturality of α . The universal property of the colimit gives a unique map

$$\tilde{\beta}_U: \mathrm{colim}_{V \supseteq f(U)} \mathcal{G}(V) \rightarrow \mathcal{F}(U),$$

making the co-cone diagrams commute. We now need to verify that this gives a map of presheaves $\tilde{\beta}: G \rightarrow \mathcal{F}$, where G is the presheaf in the construction of $f^*\mathcal{G}$. Since

$$\begin{array}{ccc} G(U) \xrightarrow{\tilde{\beta}_U} \mathcal{F}(U) & & \mathcal{G}(V) \xrightarrow{\alpha_V} \mathcal{F}(f^{-1}(V)) \rightarrow \mathcal{F}(U) \\ \downarrow & \iff & \downarrow \\ G(U') \xrightarrow{\tilde{\beta}_{U'}} \mathcal{F}(U') & & \mathcal{G}(V) \xrightarrow{\alpha_{V'}} \mathcal{F}(f^{-1}(V)) \rightarrow \mathcal{F}(U) \end{array}, \forall V \supseteq f(U), V' \supseteq f(U'),$$

this is indeed the case. After all, the right-hand side commutes by naturality of α and functoriality of \mathcal{F} . Finally, applying sheafification gives the map $\beta: f^*\mathcal{G} \rightarrow \mathcal{F}$ we needed.

Conversely, given $\beta: f^*\mathcal{G} \rightarrow \mathcal{F}$, we have maps

$$\beta_U^V: \mathcal{G}(V) \rightarrow \mathcal{F}(U)$$

for every $U \in \mathbf{Open}(X)$ and $V \supseteq f(U)$. Since $V \supseteq f(f^{-1}(V))$, we can define

$$\alpha_V \stackrel{\text{def}}{=} \beta_{f^{-1}(V)}^V : \mathcal{G}(V) \rightarrow \mathcal{F}(f^{-1}(V)).$$

By naturality of β ,

$$\begin{array}{ccc} f^*\mathcal{G}(f^{-1}(V)) & \longrightarrow & f^*\mathcal{G}(f^{-1}(V')) \\ \beta_{f^{-1}(V)} \downarrow & & \downarrow \beta_{f^{-1}(V')} \\ \mathcal{F}(f^{-1}(V)) & \longrightarrow & \mathcal{F}(f^{-1}(V')) \end{array}$$

commutes, and hence in particular we get

$$\begin{array}{ccc} \mathcal{G}(V) & \longrightarrow & \mathcal{G}(V') \\ \beta_{f^{-1}(V)}^V \downarrow & & \downarrow \beta_{f^{-1}(V')}^{V'} \\ \mathcal{F}(f^{-1}(V)) & \longrightarrow & \mathcal{F}(f^{-1}(V')) \end{array}$$

which shows the naturality of α .

We now need to check that the two constructions are inverses of each other. Starting from $\alpha : \mathcal{G} \rightarrow f_*\mathcal{F}$ we obtain $\beta : f^*\mathcal{G} \rightarrow \mathcal{F}$. Let $\gamma : \mathcal{G} \rightarrow f_*\mathcal{F}$ be the map we obtain from β . We want to show that $\gamma = \alpha$. Take $V \in \mathbf{Open}(X)$, then

$$\gamma_V = \beta_{f^{-1}(V)}^V = \alpha_V|_{f^{-1}(V)} = \alpha_V,$$

as desired. Conversely, given $\beta : f^*\mathcal{G} \rightarrow \mathcal{F}$, we obtain $\alpha : \mathcal{G} \rightarrow f_*\mathcal{F}$ from β , and $\eta : f^*\mathcal{G} \rightarrow \mathcal{F}$ from α . We have,

$$\eta_U^V = \alpha_V|_U = \beta_{f^{-1}(V)}^V|_U,$$

for $V \supseteq f(U)$. Since $U \subseteq f^{-1}(V)$, naturality of β gives $\beta_{f^{-1}(V)}^V|_U = \beta_U^V$, so $\eta_U^V = \beta_U^V$. TODO: naturality in \mathcal{F} and \mathcal{G} .

- f^* is exact. Since it is a left adjoint, it is right exact. We still need to show that it is left exact, i.e. f^* preserves monomorphisms. Let $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ be a monomorphism. Since $\text{Ker}(\alpha)(V) = \text{Ker}(\alpha_V)$ we have that α is mono if and only if all the α_V are mono. Since colimits over filtered diagrams commute with finite limits, it follows that $F(U) \rightarrow G(U)$ is a mono if and only if all the α_V for $V \supseteq f(U)$ are mono. Consequently, $f^*\alpha$ is a mono if α is a mono.

□

This result allows us to prove the crucial property needed to be able to do cohomology in $\mathbf{Ab}(X)$.

Lemma 3.3. $\mathbf{Ab}(X)$ has enough injectives.

Proof. Take $x \in X$. Then $i_x : \{x\} \hookrightarrow X$ is a continuous map and hence induces the pushforward functor $i_{x,*} : \mathbf{Ab}(\{x\}) \rightarrow \mathbf{Ab}(X)$. Note that $\mathbf{Ab}(\{x\})$ is equivalent to \mathbf{Ab} as category, since a sheaf on one point is completely determined by the value at that point. Now if M is an injective object in \mathbf{Ab} , we claim that $i_{x,*}M$ is injective in $\mathbf{Ab}(X)$. Let $\alpha : \mathcal{F} \rightarrow \mathcal{G}$

be a monomorphism and $\beta: \mathcal{F} \rightarrow i_{x,*}M$ any morphism of sheaves. We need to find a map $\gamma: \mathcal{G} \rightarrow i_{x,*}M$ such that $\beta = \gamma \circ \alpha$. Since i_x^* is a left adjoint to $i_{x,*}$ we get a map $b: i_x^*(\mathcal{F}) \rightarrow M$. Since i_x^* is exact, the map $a: i_x^*\mathcal{F} \rightarrow i_x^*\mathcal{G}$ is still a monomorphism. So, since M is injective, there is a map $g: i_x^*\mathcal{G} \rightarrow M$ such that $b = g \circ a$. This gives a map $\gamma: \mathcal{G} \rightarrow i_{x,*}M$, and $\gamma \circ \alpha$ is the same as the induced map by $g \circ a = b$ since an adjunction is natural in both arguments. Thus, $\gamma \circ \alpha = \beta$ as desired.

What remains to show is that for every sheaf \mathcal{F} , there is a monomorphism $\mathcal{F} \hookrightarrow i_{x,*}M$, for some $x \in X$ and injective $M \in \mathbf{Ab}$. Since \mathbf{Ab} has enough injectives, there is a monomorphism $a_x: i_x^*\mathcal{F} \rightarrow M_x$, for every sheaf \mathcal{F} . The adjunction gives a map $\alpha_x: \mathcal{F} \rightarrow i_{x,*}M_x$. The problem is that this map need not be a monomorphism, since $(i_{x,*}M)(U) = M(i_x^{-1}(U)) = \{*\}$ if $x \notin U$. Since the product of injectives is again injective, we can solve this problem by considering the map

$$\alpha = (\alpha_x)_{x \in X}: \mathcal{F} \rightarrow \prod_{x \in X} i_{x,*}M_x,$$

instead. Take $V \in \mathbf{Open}(X)$. If $V = \emptyset$, then $\mathcal{F}(V) = \{*\}$ and α_V is a monomorphism. Otherwise, take $x \in V$. We claim that $(\alpha_x)_V: \mathcal{F}(V) \rightarrow M_x$ is a monomorphism. By definition, $(\alpha_x)_V$ is given by the composition of $\mathcal{F}(V) \rightarrow i_x^*\mathcal{F}(V)$ and $i_x^*\mathcal{F}(V) \rightarrow M_x$. Since filtered diagrams commute with finite limits, we have that $(\alpha_x)_V$ is indeed a monomorphism. So, if $x, y \in V$ with $\alpha_V(x) = \alpha_V(y)$, then $x = y$, which shows that α is a monomorphism. TODO: Make this proof clearer. \square

In the previous proof we considered the inclusion map. If we instead consider the projection map $f: X \rightarrow \{*\}$, then the pullback f^* becomes the map $\mathbf{Ab} \rightarrow \mathbf{Ab}(X)$, which sends an abelian group M to the constant sheaf with value M . The pushforward $f_*: \mathbf{Ab}(X) \rightarrow \mathbf{Ab}$ maps a sheaf, \mathcal{F} , to its global sections, $\mathcal{F}(X)$.

With this we can now do cohomology in $\mathbf{Ab}(X)$. Let $f: X \rightarrow Y$ be a continuous function. Since f_* is a right adjoint, it is left exact, and we can consider the right derived functors $R^i f_*: \mathbf{Ab}(X) \rightarrow \mathbf{Ab}(Y)$.

TODO: example about sheaf cohomology.

4 Condensed sets

We now have all the necessary material to be able to say a little about condensed sets. Intuitively, a condensed set is a generalization of a topological space, which contains more information about how it interacts with other spaces. The goal of this section is two-fold:

- We want to define what a condensed set is, and compute some examples.
- We want to define cohomology in the condensed setting and compare it to other types of cohomology.

4.1 What is a condensed set?

As it turns out, the definition of a condensed set is not so straightforward. As is the case in representation theory, it is often useful to study an object by considering representations of the object as simpler objects. In this case we will study topological spaces by considering maps from profinite sets into topological spaces.

Definition 4.1 (Profinite set). A profinite set is a compact, Hausdorff, totally disconnected space. Explicitly, it is a topological space X , such that

- Every open cover of X has a finite subcover.
- For every two different points, $x \neq y \in X$, there exist neighborhoods U and V of x and y such that $U \cap V = \emptyset$.
- The connected components of X are the singletons.

For example, a finite space with the discrete topology is a profinite set. We would now like to turn the category of profinite sets into a site. There is a small problem though. Namely, the category of profinite sets isn't small. To solve this, we take an uncountable strong limit cardinal κ and only consider the profinite sets with cardinality less than κ . Recall that a strong limit cardinal is a cardinal κ such that $2^\lambda < \kappa$ for all cardinals $\lambda < \kappa$.

What are the coverings on this category? A covering is given by a finite family of jointly surjective maps, thus a set of maps $\{f_i: X_i \rightarrow X\}_{i=0}^n$ such that $X = \bigcup_{i=0}^n f_i(X_i) = X$. This collection of coverings does indeed satisfy the needed axioms:

1. If $f: Y \rightarrow X$ is an isomorphism, then f is surjective, so $\{f\}$ is a cover.
2. If $\{f_i: X_i \rightarrow Y\}_{i=0}^n$ is jointly surjective, and for each $i \in \{0, \dots, n\}$, the set $\{f_{ij}: X_{ij} \rightarrow X_i\}_{j=0}^{n_i}$ is jointly surjective, then so is the composition $\{f_i \circ f_{ij}: X_{ij} \rightarrow Y\}_{i,j=0}^{n,n_i}$.
3. Finally, let $\{f_i: X_i \rightarrow Y\}_{i=0}^n$ be a cover, and $f: S \rightarrow Y$ a morphism. Then, assuming that the pullback $X_i \times_Y S$ is a profinite set, we have that the projection maps $\{X_i \times_Y S \rightarrow S\}_{i=0}^n$ are jointly surjective. After all, for $s \in S$, the image $f(s)$ is equal to $f_i(x_i)$ for some $i \in \{0, \dots, n\}$ since the f_i are jointly surjective. So, s is the image of $(x_i, s) \in X_i \times_Y S$.

The only thing left to show is that the pullback of two profinite sets is again a profinite set. In fact, something stronger is true:

Proposition 4.2. *A limit of profinite spaces is again a profinite space.*

Proof. Let $X = \lim_{i \in \mathcal{I}} X_i$ be the limit as topological spaces. We first show that X is still Hausdorff and totally disconnected. Take $x \neq y \in X$. Then for at least one of the projections $f_i: X \rightarrow X_i$ we must have $f_i(x) \neq f_i(y)$. Since X_i is Hausdorff, we can find open neighborhoods U of $f_i(x)$ and V of $f_i(y)$ such that $U \cap V = \emptyset$. But then $f_i^{-1}(U) \cap f_i^{-1}(V) = \emptyset$, and $x \in f_i^{-1}(U)$, $y \in f_i^{-1}(V)$. Since X_i is totally disconnected, $f_i(x)$ and $f_i(y)$ are in separate components. Since the continuous image of a connected set is connected, we must have that x and y are in separate connected components as well. Finally, one can also show that limits of compact Hausdorff spaces are again compact (see [Sta23, Lemma 08ZV]). Essentially the argument is as follows:

- Limits are made out of products and equalizers.
- Products of compact spaces are compact.
- By above, limits of Hausdorff spaces are Hausdorff.
- Equalizers of functions into Hausdorff spaces are closed.

- So the limit is compact as a closed subspace of a compact space.

□

With this we arrive at the following definition:

Definition 4.3 (κ -condensed sets). The category of κ -condensed sets is the category of sheaves on the site of κ -small profinite sets, with coverings given by jointly surjective maps.

Analogously, we can define condensed abelian groups, rings, vector spaces. . . . As we have seen, the category of abelian sheaves on a site is abelian. This is not true for the category of topological abelian groups. After all, in any abelian group, a morphism that is both an epimorphism and a monomorphism is automatically an isomorphism. The map

$$\text{id}: (\mathbb{R}, \mathcal{T}_{\text{discrete}}) \rightarrow (\mathbb{R}, \mathcal{T}_{\text{Euclidean}}),$$

is bijective as sets, and hence an epimorphism and a monomorphism. However, since the two topologies don't coincide, the map is not an isomorphism.

How is this issue resolved in the condensed setting? With a topological space X , we can associate a condensed set $\underline{X} = \text{Cont}(-, X)$. The “underlying set” can then be found as $\underline{X}(\{*\})$. The point is that the map $f: \mathbb{R}_{\text{discrete}} \rightarrow \mathbb{R}_{\text{Euclidean}}$ is no longer an epimorphism. As we will see later on, the cokernel of this map can be computed “section-wise”, i.e. $\text{Coker}(f)(U) = \text{Coker}(f_U)$. So, to show that the cokernel is not trivial, we need to show that there is some profinite set S with $\text{Cont}(S, \mathbb{R}_{\text{discrete}}) \subsetneq \text{Cont}(S, \mathbb{R}_{\text{Euclidean}})$. Consider the Cantor set S , which is homeomorphic to $\prod_{n \in \mathbb{N}} \{0, 1\}$ where $\{0, 1\}$ has the discrete topology. By proposition 4.2, the Cantor set is a profinite set. It can alternatively be described as the closed subspace of the real numbers (with the Euclidean topology), given by

$$S = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right).$$

With this description, we get a continuous map from S into the real numbers with the Euclidean topology, namely the inclusion map. On the other hand, this map is not continuous when \mathbb{R} is equipped with the discrete topology. If it were, then every singleton in S would have to be open, as the inverse image of an open set. Since S doesn't have the discrete topology, this is not the case. Consequently, the map f_S is not an epimorphism.

Recall from lemma 2.19 that limits and colimits of presheaves can be taken section-wise. For sheaves, the colimit is not always a sheaf, and we may need to take the sheafification of the colimit as presheaves to obtain a sheaf. For the category of κ -condensed sets, this is no longer necessary. To prove this, we'll need an alternative description of condensed sets.

Definition 4.4. An *extremally disconnected* set is a projective object in the category of compact Hausdorff spaces³.

One can show that every compact Hausdorff space is the surjective image of an extremally disconnected set, i.e. the category has enough projectives ([Gle58, Theorem 3.2]). TODO: - Show that category of κ -condensed equivalent to extremally disconnected version. - Show that limits and colimits can be taken section-wise.

³The usual definition is that the closure of every open set must again be open. In this case, the two definitions are equivalent ([Gle58, Theorem 2.5])

4.2 Condensed cohomology

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