A Fluid Introduction to Condensed Mathematics

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Contents

1 Introduction

Most of the contents of this paper will be based on the lecture notes by Peter Scholze, who developed the theory of condensed mathematics together with Dustin Clausen ([Sch19]).

2 Sheaves

A central role in this paper will be played by sheaves. It therefore makes sense to study these objects more closely, and attempt to gain some intuition before jumping straight into the heart of the matter. The first step, will be to convince our selves, that sheaves are interesting enough to study, and that they form a nice type of category. As such, we will dive into sheaf cohomology and some applications of it in algebraic topology. But first, we must define what a sheaf is.

2.1 On Topological Spaces

Consider the following situation in algebraic geometry. We have some (affine) algebraic variety X, with the Zariski Topology. Associated to X is an ideal I of all the polynomials that vanish on X. Then, the quotient ring $\mathcal{O}_X(X) = k[x_1, \ldots, x_n]/I$, is the coordinate ring of X. The elements of $\mathcal{O}_X(X)$ are the regular functions on X. For an open $U \subseteq X$ we can also consider the regular functions on U, which we will denote as $\mathcal{O}_X(U)$. This mapping between open subsets of X and rings, given by $\mathcal{O}_X$, is precisely a sheaf. In this case, $\mathcal{O}_X$ is known as the *structure sheaf*. In general, a sheaf will give some global data that can be defined locally.

Definition 2.1 (Presheaves)**.** Let X be a topological space. A *presheaf of sets* $\mathcal{F}$ on X consists of two things:

1. For each open $U \subseteq X$ a set $\mathcal{F}(U)$. These are known as the *sections* of $\mathcal{F}$ over U.
2. For each inclusion $U \subseteq V$ a map $\rho_{U,V} \colon \mathcal{F}(V) \to \mathcal{F}(U)$, such that $\rho_{U,U} = id_{\mathcal{F}(U)}$, and for $U \subseteq V \subseteq W$ we have $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$. We call the $\rho_{U,V}$ the *restrictions*, and if $s \in \mathcal{F}(U)$, we will often denote $\rho_{V,U}(s) = s|_V$.

Remark 2.2. If we write $\mathbf{Open}(X)$ for the category of open sets of X, with morphisms given by the inclusion maps, then a presheaf is precisely a functor $\mathcal{F}\colon \mathbf{Open}(X)^{\mathrm{opp}} \to \mathbf{Set}$.

This only defines what a "presheaf" is. The sheaf condition will ensure that the value of $\mathcal{F}(U)$ can be constructed by defining it locally on elements of a cover $\{U_i\}_{i\in I}$ of U. It is exactly this interaction between local definitions and global objects that makes sheaves so useful.

Definition 2.3 (Sheaves)**.** Let X be a topological space, and $\mathcal{F}$ a presheaf of sets on X. We say that $\mathcal{F}$ is a sheaf, if it satisfies the following additional properties for any open $U \subseteq X$ and open cover $\{U_i\}_{i\in I}$ of U:

- (uniqueness/locality) If $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.
- (gluing) If $\{s_i \in \mathcal{F}(U_i)\}_{i\in I}$ is a collection of sections such that
$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \text{ for all } i, j,$$
then there is a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.

In other words, if we have a bunch of sections that agree on overlaps, then we can uniquely glue them together.

Remark 2.4. We can reformulate the properties in a more categorical manner. Namely, for any open cover $\{U_i\}_{i\in I}$ of $U \subseteq X$, the following diagram:

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j),$$

has to be an equalizer. This diagram makes sense in any target category that has all limits. In terms of sets, the first map is given by $s \mapsto (s|_{U_i})_{i\in I}$. The pair of maps is given by the two possible restrictions, $(s_i)_{i\in I} \mapsto ((s_i|_{U_i\cap U_j})_{j\in I})_{i\in I}$ and $(s_j)_{j\in I} \mapsto ((s_j|_{U_i\cap U_j})_{i\in I})_{j\in J}$.

In this way, we can define sheaves on rings or abelian groups, or other categories with all (set-indexed) limits. For example, a presheaf of abelian groups on X, is a functor $\mathcal{F}\colon \mathbf{Open}(X)^{\mathrm{opp}} \to \mathbf{Ab}$.

In many cases, defining some structure on sheaves, will come down to defining it on $\mathcal{F}(U)$ in a compatible way. As an example, we can look at morphisms of sheaves.

Definition 2.5 (Morphisms of sheaves)**.** Let $\mathcal{F}, \mathcal{G}$ be presheaves on a topological space X. A *morphism of presheaves* $\varphi\colon \mathcal{F} \to \mathcal{G}$ assigns to each open $U \subset X$ a morphism $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with the restriction maps, i.e. for $U \subseteq V \subseteq X$ the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "\rho^{\mathcal{F}}_{U,V}"] & \mathcal{G}(V) \arrow[d, "\rho^{\mathcal{G}}_{U,V}"] \\ \mathcal{F}(U) \arrow[r, "\varphi_U"] & \mathcal{G}(U) \end{tikzcd}.$$

In other words, φ is a natural transformation between $\mathcal{F}$ and $\mathcal{G}$. Using a bit of abusive notation, the compatibility can be read as $\varphi(s|_U) = \varphi(s)|_U$. A *morphism of sheaves* is a morphism of the underlying presheaves.

For a topological space X, we now have (pre)sheaves and morphisms between them. These form a category. We will write $\mathbf{PSh}(X)$ for the category of presheaves of sets on X, and $\mathbf{Sh}(X)$ for the category of sheaves of sets on X. In the same way, we write $\mathbf{Ab}(X), \mathbf{Ring}(X), \mathbf{Vect}(X), \ldots$ for the category of sheaves of abelian groups, rings, vector spaces, ... on X.

Remark 2.6. We can "recover" the underlying space, by taking $X = \{*\}$, the one-point space. We have $\mathbf{Sh}(*) = \mathbf{Set}, \mathbf{Ab}(*) = \mathbf{Ab}, \ldots$

2.1.1 Examples

To solidify our understanding of sheaves, it will be beneficial to look at some examples.

Example 2.7. The simplest examples of sheaves are those were $\mathcal{F}(U)$ is not just a set, but a set of functions, and the restrictions correspond to actual restrictions.

Example 2.8. Let Y be another topological space, then we can define $\mathcal{F}(U) \overset{\text{def}}{=} \mathrm{Cont}(U, Y)$ where $\mathrm{Cont}(U, Y) = \{f : U \to Y \mid f \text{ continuous }\}$. The restrictions are defined as actual restrictions, and we get a presheaf. To see that it is a sheaf, we need to verify the gluing condition. Let $\{U_i\}$ be an open covering of $U \subseteq X$, and $f_i : U_i \to Y$ is a continuous map for each $i \in I$, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. We can now define a map $f : U \to Y$ via $f(u) = f_i(u)$ for any $i \in I$ such that $u \in U_i$. By assumption, this map is well-defined. To see that it is continuous, for any $V \subseteq Y$ open, we can write

$$f^{-1}(V) = U \cap f^{-1}(V) = \bigcup_{i \in I} (U_i \cap f^{-1}(V)) = \bigcup_{i \in I} f_i^{-1}(V),$$

which is open since all the f_i are continuous.

So we see that $\mathcal{F}$ is a sheaf. In the case that Y has the discrete topology, we call $\mathcal{F}$ the *constant sheaf with value* Y. The sections of $\mathcal{F}$ over U are the *locally* constant functions, i.e. at each point $x \in U$ we can find an open neighborhood V of x, such that f is constant on V.

Example 2.9. Let $f : X \to Y$ be a continuous map, then we get a sheaf on X by the rule

$$\Gamma(Y/X)(U) = \{s : U \to Y \mid f \circ s = 1_U\}.$$

The gluing construction is the same as in the previous example. To see that this yields another section, note that for any $u \in U$, $s(u) = s_i(u)$ for some $i \in I$, and hence $f(s(u)) = f(s_i(u)) = u$. We call $\Gamma(Y/X)$ the *sheaf of sections of* f.

Example 2.10. As we remarked above, the structure sheaf $\mathcal{O}_X$ is also a sheaf, where again, the restrictions are actual restrictions of functions. To show that the gluing of regular functions is again regular, requires some machinery from algebraic geometry, which falls outside the scope of this paper.

As it turns out, it is possible to generalize the definition of sheaves to categories which also have the notion of a covering. We will need this notion when discussing condensed sets.

2.2 On a Site

The idea of a site was first introduced by Alexander Grothendieck, and has proven to be very useful in algebraic geometry. For this subsection we largely follow [Sta23, Part 1, Chapter 7].

2.2.1 Coverings and Sites

To define a site, we collect all the essential properties that coverings of topological spaces have:

Definition 2.11 (Coverings and Sites)**.** A *site* is a small category $\mathcal{C}$ together with a set $\mathrm{Cov}(\mathcal{C})$ of *coverings* of $\mathcal{C}$, where the following axioms hold: [1]

1. If $V \to U$ is an isomorphism, then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
2. If $\{U_i \to U\}_{i\in I}$ is a covering, and for each $i \in I$, $\{V_{ij} \to U_i\}_{j\in J_i}$ is a covering as well, then so is the composition $\{V_{ij} \to U\}_{i\in I, j\in J_i}$.
3. If $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a morphism of $\mathcal{C}$, then the pullback $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I}$ is a covering.

To make sense of this definition, let us look at the canonical example.

Example 2.12. Let X be a topological space, then $\mathbf{Open}(X)$ is a site with coverings given by the open covers. Let us verify that the axioms hold:

1. The only isomorphisms in $\mathbf{Open}(X)$ are the identity maps. Since $\{U\}$ is an open cover of U for any $U \in \mathbf{Open}(X)$, the first axiom is satisfied.
2. If $U = \bigcup_{i\in I} U_i$ is an open cover of U, and for each $i \in I$, $U_i = \bigcup_{j\in J_i} U_{ij}$ is an open cover of U_i, then $U = \bigcup_{i\in I} \bigcup_{j\in J_i} U_{ij}$ is an open cover of U.
3. If $U = \bigcup_{i\in I} U_i$ is an open cover, and $V \subseteq U$, then $V = \bigcup_{i\in I} V \cap U_i$ is an open cover of V.

Here we used that $U \times_W V = U \cap V$ for $U, V \subseteq W$, which follows from $S \subseteq V, S \subseteq U \implies S \subseteq V \cap U$.

The following example is still quite similar to the previous example. The underlying site of a condensed set will be similar to this site.

Example 2.13. Let G be a group, then we can consider the category of all G-sets, i.e. sets with a corresponding action by the group G. Morphisms are given by maps $f\colon X \to Y$ satisfying $g \cdot f(x) = f(g \cdot x)$, i.e. G-equivariant maps. To make this into a site, we define the covers to be the families of jointly surjective maps. In other words $\{f_i\colon X_i \to X\}_{i\in I}$ is a cover if $\bigcup_{i\in I} f_i(X_i) = X$. Let us verify the axioms:

1. If $f\colon X \to Y$ is an isomorphism, then in particular it is surjective, so $\{f\colon X \to Y\}$ is a cover of Y
2. If $\{f_i\colon X_i \to Y\}_{i\in I}$ is jointly surjective, and for each $i \in I$ the family $\{f_{ij}\colon X_{ij} \to X_j\}_{j\in J}$ is jointly surjective, then so is $\{f_i \circ f_{ij}\colon X_{ij} \to Y\}_{i\in I, j\in J}$.

[1] We force $\mathrm{Cov}(\mathcal{C})$ to be a set since we will take limits over all coverings. It is possible to let $\mathrm{Cov}(\mathcal{C})$ be a class, and then show that it can be replaced with a set of coverings that gives rise to the same category of sheaves. See [Sta23, Remark 7.6.3].

3. Let $\{f_i \colon X_i \to Y\}_{i \in I}$ be a cover, and $f \colon T \to Y$ a G-equivariant map. We claim that the pullback $X_i \times_Y T$ exists and is given by $S = \{(x,t) \in X_i \times T \mid f_i(x) = f(t)\}$. This is a G-set, as if $f_i(x) = f(t)$, then

$$f_i(g \cdot x) = g \cdot f_i(x) = g \cdot f(t) = f(g \cdot t).$$

Since S is the pullback in **Set** and any G-equivariant map is in particular a map of sets, the claim follows. To see that the projection maps $\{X_i \times_Y T \to T\}$ are jointly surjective, take $t \in T$. There exists $i \in I$ such that $f(t) \in f_i(X_i)$, so $(x_i, t) \in X_i \times_Y T$ for some $x_i \in X_i$.

In this new context, we have to reword the definitions of presheaves and sheaves in a more abstract manner.

Definition 2.14 (Presheaves and Sheaves on a Site)**.** Let $\mathcal{C}$ be a small category. A *presheaf on* $\mathcal{C}$ is a contravariant functor $\mathcal{F} \colon \mathcal{C} \to \mathbf{Set}$. If $\mathcal{C}$ is a site, then a *sheaf on* $\mathcal{C}$ is a presheaf $\mathcal{F}$, such that for any covering $\{U_i \to U\}_{i \in I}$, the following diagram is an equalizer:

$$\mathcal{F}(U) \xrightarrow{\ e\ } \prod_{i \in I} \mathcal{F}(U_i) \overset{p_0^*}{\underset{p_1^*}{\rightrightarrows}} \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j)\ .$$

The map e is given by $s \mapsto (s|_{U_i})_{i \in I}$. For the second maps, take $i, j \in I$. Then we have projections $p_0^{(i,j)} \colon U_i \times_U U_j \to U_i$ and $p_1^{(i,j)} \colon U_i \times_U U_j \to U_j$. These result in maps $p_0^{(i,j),*} \colon \mathcal{F}(U_i) \to \mathcal{F}(U_i \times_U U_j)$ and $p_1^{(i,j),*} \colon \mathcal{F}(U_j) \to \mathcal{F}(U_i \times_U U_j)$. The maps p_0^* and p_1^* are then given at component i, j by mapping $(s_k)_{k \in I}$ to $p_0^{(i,j),*}(s_i)$ and $p_1^{(i,j),*}(s_j)$ respectively.

2.2.2 Sheafification

In a lot of constructions, the natural thing we do, will end up being a presheaf, but not a sheaf in general. So, we will need some way to turn presheaves into sheaves.

Proposition 2.15. *The fully faithful inclusion*

$$\iota \colon \mathbf{Sh}(\mathcal{C}) \hookrightarrow \mathbf{PSh}(\mathcal{C}),$$

admits a left adjoint $\mathcal{F} \to \mathcal{F}^\sharp$, "sheafification".

We will give an explicit construction of $\mathcal{F}^\sharp$. As it turns out, "sheafification" is actually a two-step process. The first step is making the presheaf separated, and then turning the separated presheaf into a sheaf.

Definition 2.16. We say that a presheaf $\mathcal{F}$ is *separated* if $\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$ is injective for any cover $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$.

Remark 2.17. Since any equalizer is a monomorphism, it follows from the definition of sheaves on a site, that a sheaf is separated.

Example 2.18. The presheaf $\mathcal{F}$ on a topological space X, which maps every open to the same set $Y \neq \{*\}$ is not separated, as

$$Y = \mathcal{F}(\emptyset) \to \prod_{i \in \emptyset} = \{*\},$$

is not injective if $Y \neq \{*\}$.

On the other hand, the presheaf $\mathcal{G}$ on a topological space X which maps every open U to the constant functions $U \to Y$ for some space Y, is separated but not a sheaf. After all, if $U = \cup_i U_i$ is an open cover of U, and $f, g \in \mathcal{G}(U)$ are such that $f|_{U_i} = g|_{U_i}$ then $f = g$, since any $x \in U$ is contained in some U_i, and hence $f(x) = f|_{U_i}(x) = g|_{U_i}(x) = g(x)$. Since there is only a single function $\emptyset \to Y$ there are no problems with empty coverings. $\mathcal{G}$ fails to be a sheaf in general, since if $U = U_1 \cup U_2$ with $U_1 \cap U_2 = \emptyset$ and $U_1 \neq \emptyset \neq U_2$, then we can't "glue" the functions $f\colon U_1 \to Y, f(x) = y_1$ and $g\colon U_2 \to Y, f(x) = y_2$ together to a constant function on U if $y_1 \neq y_2$.

We first have a few lemmas regarding limits of presheaves and sheaves.

Lemma 2.19. *Let $\mathcal{C}$ be a site, then limits and colimits exist in $\mathbf{PSh}(\mathcal{C})$. Additionally, for any $U \in \mathcal{C}$, the functor $\mathbf{PSh}(\mathcal{C}) \to \mathbf{Set}\colon \mathcal{F} \mapsto \mathcal{F}(U)$ commutes with all limits and colimits.*

Proof. Let $\mathcal{F}\colon \mathcal{I} \to \mathbf{PSh}(\mathcal{C})$ be a diagram. We get a cone $(\mathcal{F}_{\lim}, p^i)$ by moving to the images in $\mathbf{Set}$. In other words we take $\mathcal{F}_{\lim}(U) = \lim_{i \in \mathcal{I}} \mathcal{F}_i(U)$, and $p_U^i\colon \lim_{i \in \mathcal{I}} \mathcal{F}_i(U) \to \mathcal{F}_i(U)$. If $U \to V$ is a morphism in $\mathcal{C}$, we get a unique map $\mathcal{F}_{\lim}(V) \to \mathcal{F}_{\lim}(U)$, given by the fact that $\mathcal{F}_{\lim}(U)$ is a limit, and $\mathcal{F}_{\lim}(V)$ is a cone on all the $\mathcal{F}_i(U)$ by composing the maps $\mathcal{F}_{\lim}(V) \to \mathcal{F}_i(V)$ with the maps $\mathcal{F}_i(V) \to \mathcal{F}_i(U)$. So we see that $\mathcal{F}_{\lim}$ is indeed a presheaf.

$$\begin{tikzcd}
 & & \mathcal{F}_{\lim}(V) \arrow[d, dashed] \arrow[ddll, bend right, "p_V^i"'] \arrow[ddrr, bend left, "p_V^{i'}"] & & \\
 & & \mathcal{F}_{\lim}(U) \arrow[dl, "p_U^i"'] \arrow[dr, "p_U^{i'}"] & & \\
\mathcal{F}_i(V) \arrow[r] & \mathcal{F}_i(U) \arrow[rr] & & \mathcal{F}_{i'}(U) & \mathcal{F}_{i'}(V) \arrow[l]
\end{tikzcd}$$

To see that the maps p^i are morphisms of presheaves, we need to verify that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}_{\lim}(V) \arrow[r, "p_V^i"] \arrow[d] & \mathcal{F}_i(V) \arrow[d] \\
\mathcal{F}_{\lim}(U) \arrow[r, "p_U^i"] & \mathcal{F}_i(U)
\end{tikzcd},$$

but that already follows from the previous diagram.

Let us now verify that $\mathcal{F}_{\lim}$ is actually a limit. If $(\mathcal{G}, g^i)$ is another cone, then for each $U \in \mathcal{C}$, we get a unique map $\mathcal{G}(U) \to \mathcal{F}_{\lim}(U)$, such that the corresponding cone diagrams commute. It suffices to show that these maps combine to form a map of presheaves. Using the universal property of limits, this comes down to showing

$$\begin{array}{ccc} \mathcal{G}(V) & \longrightarrow & \mathcal{F}_{\lim}(V) \\ \downarrow & & \downarrow \\ \mathcal{G}(U) & \longrightarrow & \mathcal{F}_{\lim}(U) \end{array} \iff \begin{array}{ccc} \mathcal{G}(V) & \xrightarrow{g^i_V} & \mathcal{F}_i(V) \\ \downarrow & & \downarrow \\ \mathcal{G}(U) & \xrightarrow{g^i_U} & \mathcal{F}_i(U) \end{array} \quad \forall i \in \mathcal{I}.$$

The commutativity of these last diagrams, is just saying that the maps g^i are presheaf morphisms, which is true by $(\mathcal{G}, g^i)$ being a cone. □

The big difference between sheaves and presheaves, is that we can glue things together defined on a cover. The trick will be to force the presheaf to behave as we want. Let us try and make this more precise.

So far we have just talked about coverings as objects. We can also consider maps between coverings. If $\mathcal{U} = \{U_i \to U\}_{i\in I}$ and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ are two coverings, a morphism of coverings between $\mathcal{U}$ and $\mathcal{V}$ is a morphism $U \to V$ and a map $\alpha\colon I \to J$ together with morphisms $U_i \to V_{\alpha(i)}$ such that

$$\begin{array}{ccc} U_i & \longrightarrow & V_{\alpha(i)} \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

commutes. If $U = V$ and $U \to V$ is the identity, we call $\mathcal{U}$ a *refinement* of $\mathcal{V}$. For a $U \in \mathcal{C}$ the coverings together with refinements gives a category $\mathrm{Cov}(U)$. This allows the following reformulation:

Lemma 2.20. *Let $\mathcal{F}$ be a presheaf on a site $\mathcal{C}$. For $U \in \mathcal{C}$ and a cover $\mathcal{U} \in \mathrm{Cov}(U)$, define $\mathcal{F}(\mathcal{U})$ as the following equalizer:*

$$\mathcal{F}(\mathcal{U}) = \{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(U_i) \mid s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}\}.$$

If $\mathcal{U} \to \mathcal{V}$ is a morphism of coverings, there is an induced map $\mathcal{F}(\mathcal{V}) \to \mathcal{F}(\mathcal{U})$. This construction is functorial. Furthermore, since $\{1_U\}$ is a cover by the axioms of a site, this gives a map $\mathcal{F}(U) \cong \mathcal{F}(\{1_U\}) \to \mathcal{F}(\mathcal{U})$. The presheaf $\mathcal{F}$ is a sheaf if and only if the map $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$ is bijective for every cover $\mathcal{U}$.

Proof. Let $\mathcal{U} \to \mathcal{V}$ be a map of coverings. We have the following commutative diagram:

$$\begin{tikzcd} & U_i \arrow[r] \arrow[d] & V_{\alpha(i)} \arrow[d] & \\ U_i \times_U U_j \arrow[ur] \arrow[dr] & U \arrow[r] & V & V_{\alpha(i)} \times_V V_{\alpha(j)} \arrow[ul] \arrow[dl] \\ & U_j \arrow[r] \arrow[u] & V_{\alpha(j)} \arrow[u] & \end{tikzcd},$$

which gives a unique map $U_i \times_U U_j \to V_{\alpha(i)} \times_V V_{\alpha(j)}$ making the diagram commute. Applying $\mathcal{F}$ gives the diagram:

$$
\begin{tikzcd}
\mathcal{F}(U_i) \arrow[d] & \mathcal{F}(V_{\alpha(i)}) \arrow[l] \arrow[d] \\
\mathcal{F}(U_i \times_U U_j) & \mathcal{F}(V_{\alpha(i)} \times_V V_{\alpha(j)}) \arrow[l] \\
\mathcal{F}(U_j) \arrow[u] & \mathcal{F}(V_{\alpha(j)}) \arrow[l] \arrow[u]
\end{tikzcd}
\quad . \tag{1}
$$

We can now define $\mathcal{F}(\mathcal{V}) \to \mathcal{F}(\mathcal{U})$ by

$$(s_j)_{j\in J} \mapsto ((s_{\alpha(i)})|_{U_i})_{i\in I}.$$

That this map is well-defined follows from eq. (1). Indeed, if $(s_j)_{j\in J} \in \mathcal{F}(\mathcal{V})$, then

$$(s_{\alpha(j)}|_{V_{\alpha(i)}\times_U V_{\alpha(j)}})|_{U_i\times U_j} = (s_{\alpha(i)}|_{V_{\alpha(i)}\times_U V_{\alpha(j)}})|_{U_i\times U_j},$$

and hence

$$(s_{\alpha(i)}|_{U_i})|_{U_i\times_U U_j} = (s_{\alpha(j)}|_{U_j})|_{U_i\times_U U_j}.$$

It remains to show that this construction is functorial, since the final claim is just the definition of a sheaf of sites. Let, to this purpose, $\mathcal{U} \to \mathcal{V} \to \mathcal{W}$ be maps of coverings, with associated maps $\alpha\colon I \to J$ and $\beta\colon J \to K$. The map $\mathcal{F}(\mathcal{W}) \to \mathcal{F}(\mathcal{U})$ maps $(s_k)_{k\in K}$ to $(s_{\beta(\alpha(i))}|_{U_i})_{i\in I}$. The other map is given by first sending $(s_k)_{k\in K}$ to $(s_{\beta(j)}|_{V_j})_{j\in J}$, which is then sent to $((s_{\beta(\alpha(i))}|_{V_{\alpha(i)}})|_{U_i})_{i\in I}$. The result now follows by functoriality of $\mathcal{F}$. □

We now have the following construction:

Lemma 2.21. *Given a presheaf of $\mathcal{F}$ on a site $\mathcal{C}$, we can construct a new presheaf $\mathcal{F}^+$ by setting*

$$\mathcal{F}^+(U) = \operatorname{colim}_{\mathcal{U}\in \mathrm{Cov}(U)} \mathcal{F}(\mathcal{U}),$$

Proof. Let us first show how $\mathcal{F}^+$ acts on morphisms. Consider a morphism $U \to V$. Then this gives a map $\mathrm{Cov}(V) \to \mathrm{Cov}(U)$ by mapping a cover $\{V_i \to V\}_{i\in I}$ of V to the cover $\{V_i \times_V U \to U\}_{i\in I}$ of U, which exists by the third axiom of coverings. From the previous lemma there are induced maps $\mathcal{F}(\mathcal{U}) \to \mathcal{F}(\mathcal{V})$. Since we have maps $\mathcal{F}(\mathcal{V}) \to \mathcal{F}^+(V)$, we obtain maps $\mathcal{F}(\mathcal{U}) \to \mathcal{F}^+(V)$. By the universal property of the colimit, this gives a unique map $\mathcal{F}^+(V) \to \mathcal{F}^+(U)$, making the cone diagrams commute.

$$
\begin{tikzcd}
& & \mathcal{F}^+(U) & & \\
& & \mathcal{F}^+(V) \arrow[u, dashed] & & \\
\mathcal{F}(\mathcal{U}) \arrow[r] \arrow[uurr, bend left] & \mathcal{F}(\mathcal{V}) \arrow[rr] \arrow[ur] & & \mathcal{F}(\mathcal{V}') \arrow[ul] & \mathcal{F}(\mathcal{U}') \arrow[l] \arrow[uull, bend right]
\end{tikzcd}
$$

It remains to show that this defines a functor $\mathcal{C}^{\mathrm{opp}} \to \mathbf{Set}$. If we have morphisms $U \to V \to W$, then

$$W_i \times_W U \cong (W_i \times_W V) \times_V U,$$

since both are pullbacks of $W_i \longrightarrow W \longleftarrow U$. So $\mathrm{Cov}(-)$ is a functor, and since $\mathcal{U} \mapsto \mathcal{F}(\mathcal{U})$ is a functor by the previous lemma, the result now follows. $\square$

It is possible to be more explicit about how $\mathcal{F}^+$ looks like. For two coverings $\mathcal{U}, \mathcal{U}'$ of $U \in \mathcal{C}$, we have a common refinement $\{U_i \times_U U'_j \to U\}_{i\in I, j\in J}$ which exists by the second and third axioms. Furthermore, one can show that if $f, g\colon \mathcal{U} \to \mathcal{V}$ are refinements, then $\mathcal{F}(f) = \mathcal{F}(g)$ ([Sta23, Lemma 7.10.6]). This gives that $\mathrm{Cov}(U)^{\mathrm{opp}} \to \mathbf{Set}$ is a filtered diagram. So

$$\mathcal{F}^+(U) = \Big(\coprod_{\mathcal{U} \in \mathrm{Cov}(U)} \mathcal{F}(\mathcal{U}) \Big) / \sim,$$

Where $s \sim s'$ if and only if there are covers $\mathcal{U}, \mathcal{U}'$ with $s \in \mathcal{F}(\mathcal{U}), s' \in \mathcal{F}(\mathcal{U}')$ and a common refinement $\mathcal{V}$ such that

$$s_{\alpha(i)}|_{V_i} = s'_{\beta(i)}|_{V_i}, \ \forall i \in I.$$

We now come to the main theorem, from which proposition 2.15 will follow.

Theorem 2.22. *Let $\mathcal{F}$ be a presheaf on a site. Then*

1. *The presheaf $\mathcal{F}^+$ is separated.*
2. *If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf.*
3. *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism.*

Proof.

1. We need to show that $s \mapsto (s|_{U_i})_{i\in I}$ is injective for any cover $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$. Take $s, s' \in \mathcal{F}^+(U)$ such that $(s|_{U_i})_{i\in I} = (s'|_{U_i})_{i\in I}$ for some cover $\mathcal{U} = \{U_i \to U\}_{i\in I}$. By the description above of $\mathcal{F}^+$, we know that we can find covers $\mathcal{V}$ and $\mathcal{V}'$ of U, such that $s \in \mathcal{F}(\mathcal{V})/\sim$ and $s' \in \mathcal{F}(\mathcal{V}')/\sim$. Let $\mathcal{W}$ be a common refinement of the three covers $\mathcal{U}, \mathcal{V}, \mathcal{V}'$. Since it is a refinement of $\mathcal{U}$, we have that

   $$s'|_{W_j} = (s|_{U_{\alpha(j)}})|_{W_j} = (s|_{U_{\alpha(j)}})|_{W_j} = s'|_{W_j},$$

   so that $s = s'$.

2. We need to verify the sheaf condition. Let $\{U_i \to U\}_{i\in I}$ be a cover of U, and for each $i \in I$, $s_i \in \mathcal{F}^+(U_i)$ such that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for every $i, j \in I$. Since $\mathcal{F}$ is separated, the map $s \to (s|U_i)_{i\in I}$ is injective. It is hence enough to show that there is some $s \in \mathcal{F}^+(U)$ with $s|_{U_i} = s_i$ for all $i \in I$. For each $i \in I$ we can find a cover $\mathcal{U}_i = \{U_{ij} \to U_i\}$ such that $s_i \in \mathcal{F}(\mathcal{U}_i)/\sim$, and hence $s_{ij} \in \mathcal{F}(U_{ij})$ such that $s_i|_{U_{ij}} = s_{ij}/\sim$. In the same way as lemma 2.20 we get that

   $$s_{ij}|_{U_{ij} \times_U U_{i'j'}} = s_{i'j'}|_{U_{ij} \times_U U_{i'j'}}.$$

   So, $(s_{ij})_{i,j\in I} \in \mathcal{F}(\{U_{ij} \to U\}_{i,j\in I})$, and we can take $s = (s_{ij})_{i,j\in I}/\sim$. We just need to verify that $s|_{U_i} = s_i$. This follows from $(s|_{U_i})|_{U_{ij}} = s|_{U_{ij}} = s_i|_{U_{ij}}$, as $\mathcal{F}^+$ is also separated.

3. This follows from lemma 2.20.

$\square$

With this we are now ready to prove proposition 2.15. We define $\mathcal{F}^\sharp = \mathcal{F}^{++}$.

Proof of proposition 2.15. We first note that for any map of presheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "\alpha"] & \mathcal{F}^+ \arrow[d, "\alpha^+"] \\ \mathcal{G} \arrow[r] & \mathcal{G}^+ \end{tikzcd},$$

where $\alpha_U^+(s/\sim) = \alpha_U(s)/\sim$. That this commutes, is by construction of α^+. Using this, we can now show that

$$\mathrm{Hom}_{\mathbf{Sh}(\mathcal{C})}(\mathcal{F}^\sharp, \mathcal{G}) = \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathcal{F}, \iota(\mathcal{G})).$$

The bottom row of the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^+ \arrow[r] \arrow[d] & \mathcal{F}^{++} = \mathcal{F}^\sharp \arrow[d] \\ \iota(\mathcal{G}) \arrow[r] & \iota(\mathcal{G})^+ \arrow[r] & \iota(\mathcal{G})^{++} = \mathcal{G}^\sharp \end{tikzcd},$$

consists of isomorphisms by the previous theorem. So, any map $\mathcal{F} \to \iota(\mathcal{G})$ gives rise to a map $\mathcal{F}^\sharp \to \mathcal{G}$. Conversely, since every $s \in \mathcal{F}^\sharp(U)$ comes from sections in $\mathcal{F}(U)$, we can lift any map $\mathcal{F}^\sharp \to \mathcal{G}$ to a map $\mathcal{F} \to \mathcal{G}$. $\square$

As an immediate consequence, we get that $(-)^\sharp$ commutes with all colimits. So $\mathbf{Sh}(\mathcal{C})$ has all colimits, since $\mathbf{PSh}(\mathcal{C})$ has all colimits by lemma 2.19. We can say more:

Proposition 2.23. *The functor* $(-)^\sharp\colon \mathbf{PSh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C})$ *is exact.*

Proof. Since it is a left adjoint, it is right exact. On the other hand, colimits over filtered diagrams commute with finite limits. So, $(-)^\sharp$ is left exact as functor between presheaves. We claim that if $\mathcal{I} \to \mathbf{Sh}(\mathcal{C})$ is a diagram, then the limit $\mathcal{F} = \lim_i \mathcal{F}_i$ as presheaves is a sheaf. For this we show that $\mathcal{F}(\mathcal{U}) \cong \mathcal{F}(U)$ for any cover $\mathcal{U} = \{U_j \to U\}$. Take $(s_j)_{j\in J} \in \mathcal{F}(\mathcal{U})$, then by definition of the limit, we can project these to elements $(s_{ij})_{j\in J} \in \mathcal{F}_i(\mathcal{U})$. Since each $\mathcal{F}_i$ is a sheaf, we have unique elements $s_i \in \mathcal{F}_i(U)$ such that $s_i|_{U_j} = s_{ij}$.

We now want an element $s \in \mathcal{F}(U)$ with projections equal to the s_i. Choosing an element of $\mathcal{F}(U)$, is the same as giving a map $\{*\} \to \mathcal{F}(U)$, which by the universal property of the limit is the same as giving a cone $(\{*\}, \lambda_i)$. Let $\lambda_i(*) = s_i$, then we just need to verify that this defines a cone. We need that for $f\colon i \to i'$ in $\mathcal{I}$, $\mathcal{F}(f)(s_i) = s_{i'}$. This follows, as $s_i|_{U_j}$ is mapped to $s_{i'}|_{U_j}$ for all $j \in J$, and hence s_i is mapped to $s_{i'}$ because $\mathcal{F}_{i'}$ is a sheaf. So we have a unique $s \in \mathcal{F}(U)$, and by the universal property of the limit

$$s|_{U_j} = s_j \iff s_i|_{U_j} = s_{ij} \ \forall i \in I,$$

which holds by construction. Hence, the claim holds, and $(-)^\sharp$ is also right exact as a functor into sheaves. $\square$

To get a better understanding of sheafification, let us work out the process of sheafification for some presheaves.

Example 2.24. We consider the presheaf $\mathcal{F}$ of example 2.18. To calculate $\mathcal{F}^+$ we use that

$$\mathcal{F}^+(U) = \Big(\coprod_{\mathcal{U}\in\mathrm{Cov}(U)} \mathcal{F}(\mathcal{U})\Big)/\sim .$$

If U is non-empty, then

$$\mathcal{F}(\mathcal{U}) = \{(y_i)_{i\in I} \in \prod_{i\in I} Y \mid y_i = y_j\ \forall i,j \in I\} = \{(y)_{i\in I} \mid y \in Y\}$$

for any cover $\mathcal{U}$ of U. Take $(y)_{i\in I} \in \mathcal{F}(\mathcal{U})$, and $(y')_{j\in J} \in \mathcal{F}(\mathcal{U}')$. For any common refinement $\mathcal{V}$ of $\mathcal{U}$ and $\mathcal{U}'$, we have $(y_{\alpha(i)})|_{V_i} = y$ and $(y'_{\beta(i)})|_{V_i} = y'$ so that $(y)_{i\in I} \sim (y')_{j\in J} \iff y = y'$. So, we find again that $\mathcal{F}^+(U) = Y$ as long as $U \neq \emptyset$.

Now, when $U = \emptyset$ there are two covers: $\{\emptyset \to \emptyset\}$, and the empty covering, $\{\}_{i\in\emptyset}$. We have $\mathcal{F}(\{\emptyset \to \emptyset\}) = \mathcal{F}(\emptyset) = Y$, while $\mathcal{F}(\{\}_{i\in\emptyset}) = \{*\}$. Now every $y \in Y$ is equivalent to $*$, as $\{\}_{i\in\emptyset}$ is a common refinement of the two covers, and the equivalence condition becomes an empty statement in this case. As such we find $\mathcal{F}^+(\emptyset) = \{*\}$. Note that $\mathcal{F}^+$ is isomorphic to $\mathcal{G}$ from example 2.18. After all, a constant function $U \to Y$ is the same as choosing an element in Y.

We now have a separated presheaf $\mathcal{F}^+$. What does the sheaf $\mathcal{F}^\sharp$ look like? Let us first consider the case that U is empty. Then for both covers $\{\emptyset \to \emptyset\}$ and $\{\}_{i\in\emptyset}$ we get $\mathcal{F}^+(\mathcal{U}) = \{*\}$. So $\mathcal{F}^\sharp(\emptyset) = \{*\}$.

More interesting is what happens when $U \neq \emptyset$. Let $\{U_i \to U\}_{i\in I}$ be a cover such that none of the U_i are empty. Then

$$\begin{aligned}\mathcal{F}^+(\mathcal{U}) &= \{(y_i)_{i\in I} \in \prod_{i\in I} Y \mid y_i|_{U_i\cap U_j} = y_j|_{U_i\cap U_j}\ \forall i,j \in I\} \\ &= \{(y_i)_{i\in I} \in \prod_{i\in I} Y \mid y_i = y_j,\, U_i \cap U_j \neq \emptyset\},\end{aligned}$$

as $\mathcal{F}^+(\emptyset) = \{*\}$ and hence $y|_{U_i\cap U_j} = *$ if $U_i \cap U_j = \emptyset$ for any $y \in Y$. We can view elements of $\mathcal{F}^+(\mathcal{U})$ as functions $s\colon U \to Y$, that are constant on each of the opens U_i. Adding empty sets to the cover does not really change anything: the elements of $\mathcal{F}(\mathcal{U})$ will be $*$ at the indices for which $U_i = \emptyset$. Under $\sim$ such functions will all be the same. Take, $f \in \mathcal{F}^\sharp(U)$. Then there is some cover $\mathcal{U} = \{U_i\}_{i\in I}$ such that f arises from $\mathcal{F}^+(\mathcal{U})$. Consequently, for each $x \in U$, there is some U_i containing x, such that f is constant on U_i. In other words, f is a locally constant function on U. If we equip Y with the discrete topology then this is equivalent to saying that $f\colon U \to Y$ is continuous.

The sheaf $\mathcal{F}^\sharp$ is (perhaps a little confusingly) called the *constant sheaf with value* Y and sometimes denoted as $\underline{Y}$.

3 Sheaf Cohomology

In the previous section we have explored some fundamental properties of sheaves of sets. We will now be looking at sheaves of abelian groups, which form a very interesting and rich abelian category.

Proposition 3.1. *Let $\mathcal{C}$ be a site. The category $\mathbf{Ab}(\mathcal{C})$ of abelian sheaves on $\mathcal{C}$ is abelian.*

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be two abelian sheaves on $\mathcal{C}$. For natural transformations $\alpha, \beta \in \mathrm{Hom}_{\mathbf{Ab}(\mathcal{C})}(\mathcal{F}, \mathcal{G})$ we define $\alpha + \beta$ via

$$(\alpha+\beta)_U \overset{\text{def}}{=} \alpha_U + \beta_U,$$

where the $+$ on the right-hand side is the $+$ in $\mathbf{Ab}$. Let us verify that $\alpha + \beta$ is a natural transformation. If $f\colon U \to V$ is a morphism in $\mathcal{C}$ we have:

$$\begin{aligned}(\alpha+\beta)_U \circ \mathcal{F}(f) &= (\alpha_U + \beta_U) \circ \mathcal{F}(f)\\ &= \alpha_U \circ \mathcal{F}(f) + \beta_U \circ \mathcal{F}(f)\\ &= \mathcal{G}(f) \circ \alpha_V + \mathcal{G}(f) \circ \beta_V\\ &= \mathcal{G}(f) \circ (\alpha_V + \beta_V)\\ &= \mathcal{G}(f) \circ (\alpha+\beta)_V,\end{aligned}$$

so that $\alpha + \beta$ is indeed an abelian category. With this operation, the hom-sets become abelian groups. Using lemma 2.19 and proposition 2.23 we find that $\mathbf{Ab}(\mathcal{C})$ has finite limits and colimits. Explicitly, limits in sheaves are the limits in presheaves and colimits in sheaves are the sheafifications of the colimits in presheaves. As such, we find:

- The zero object is given by $\underline{0}$, the constant sheaf with value 0.
- The biproduct of $\mathcal{F}$ and $\mathcal{G}$ is given by $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$
- The kernel of a map $\alpha\colon \mathcal{F} \to \mathcal{G}$ is given by $\mathrm{Ker}(\alpha)(U) = \mathrm{Ker}(\alpha_U)$.
- The cokernel of $\alpha\colon \mathcal{F} \to \mathcal{G}$ is given by the sheafification of the presheaf defined by $\mathcal{F}(\alpha)(U) = \mathrm{Coker}(\alpha_U)$.

For $\mathbf{Ab}(\mathcal{C})$ to be an abelian category, we still need to show that for $\alpha\colon \mathcal{F} \to \mathcal{G}$ a morphism of sheaves, $\mathrm{Im}(\alpha) = \mathrm{Coim}(\alpha)$. In the category of abelian presheaves we have

$$\mathrm{Coim}(\iota(\alpha))(U) = \mathrm{Coim}(\iota(\alpha)_U) = \mathrm{Im}(\iota(\alpha)_U) = \mathrm{Im}(\iota(\alpha))(U),$$

where ι is the inclusion functor of sheaves into presheaves. Since $(-)^\sharp$ is exact, we find

$$\begin{aligned}\mathrm{Coim}(\alpha) &= \mathrm{Coker}(\mathrm{Ker}(\alpha))\\ &= \mathrm{Coker}^\sharp(\iota(\mathrm{Ker}(\alpha)))\\ &= \mathrm{Coker}^\sharp(\mathrm{Ker}(\iota(\alpha)))\\ &= \mathrm{Coim}^\sharp(\iota(\alpha))\\ &= \mathrm{Im}^\sharp(\iota(\alpha))\\ &= \mathrm{Ker}^\sharp(\mathrm{Coker}(\iota(\alpha)))\\ &= \mathrm{Ker}(\mathrm{Coker}^\sharp(\iota(\alpha)))\\ &= \mathrm{Ker}(\mathrm{Coker}(\alpha))\\ &= \mathrm{Im}(\alpha).\end{aligned}$$

□

We will now restrict ourselves to the special case where $\mathcal{C} = \mathbf{Open}(X)$ for some topological space X. We follow the general ideas presented in [CS20, Session 1], while filling out the details.

Proposition 3.2. *Let $f\colon X \to Y$ be a continuous map. Then there is a pair of adjoint functors: the pullback functor, $f^*\colon Ab(X) \to Ab(Y)$, with right adjoint, the pushforward functor $f_*\colon Ab(Y) \to Ab(X)$.*

Proof. TODO: work this out. □

Lemma 3.3. $\mathbf{Ab}(X)$ *has enough injectives.*

Proof. Take $x \in X$. Then $f\colon X \to \{x\}$ induces the pushforward functor $f_*\colon \mathbf{Ab}(\{x\}) \to \mathbf{Ab}(X)$. We can identify $\mathbf{Ab}(\{x\})$ with $\mathbf{Ab}$. Now if M is an injective object in $\mathbf{Ab}$, then f_*M is injective in $\mathbf{Ab}(X)$. TODO: prove this!. □

With this we can now do cohomology in $\mathbf{Ab}(X)$.

4 Condensed Sets

References

[CS20] Dustin Clausen and Peter Scholze. Masterclass in condensed mathematics. `https://www.math.ku.dk/english/calendar/events/condensed-mathematics/`, November 2020.

[Sch19] Peter Scholze. Lectures on condensed mathematics. `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`, 2019.

[Sta23] The Stacks Project Authors. The stacks project. `https://stacks.math.columbia.edu/`, 2023.