A Fluid Introduction to Condensed Mathematics

Wannes Malfait

VRIJE
UNIVERSITEIT
BRUSSEL

MOTIVATION

Problem: Algebra and topology clash:

$$\mathrm{id}\colon \mathbb{R}_{\text{discrete}} \to \mathbb{R}_{\text{Euclidean}}$$

is epi + mono, but not iso

Solution?

MOTIVATION

Problem: Algebra and topology clash:

$$\mathrm{id}\colon \mathbb{R}_{\text{discrete}} \to \mathbb{R}_{\text{Euclidean}}$$

is epi + mono, but not iso

Solution?
$\rightsquigarrow$ Condensed abelian groups

PRESHEAVES

Idea: Study better behaved objects.

PRESHEAVES

Idea: Study better behaved objects.

Definition

A presheaf on a topological space X is a functor

$$\mathcal{F}\colon \mathbf{Open}(X)^{\mathrm{opp}} \to \mathbf{Set} \,/\, \mathbf{Rng} \,/\, \mathbf{Ab} \,/\, \ldots$$

Notation:

- Elements of $\mathcal{F}(U)$ are called sections
- If $i\colon U \hookrightarrow V$ then $\mathcal{F}(i)$ is called the restriction map
- $s \in \mathcal{F}(V) \mapsto s|_U \in \mathcal{F}(U)$.

Example

We always have the presheaf given by

- $\mathcal{F}(U) \mapsto Y$, and
- $\mathcal{F}(f) \mapsto \mathrm{id}_Y$

Example

We always have the presheaf given by

- $\mathcal{F}(U) \mapsto Y$, and
- $\mathcal{F}(f) \mapsto \mathrm{id}_Y$

Problem:

- Want more structure
- Want global things to be defined locally

$\rightsquigarrow$ sheaf conditions:

SHEAF CONDITIONS

For an open cover $\{U_i\}_{i \in I}$ of U:

- uniqueness/locality: If $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

SHEAF CONDITIONS

For an open cover $\{U_i\}_{i \in I}$ of U:

- uniqueness/locality: If $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.
- gluing: If $\{s_i \in \mathcal{F}(U_i)\}_{i \in I}$ is a collection of sections such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \text{ for all } i, j,$$

  then there is a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.

EXAMPLES

Typical examples are rings of functions:

- $C(-, \mathbb{C})$
- $C^\infty(-, \mathbb{C})$

EXAMPLES

Typical examples are rings of functions:

- $C(-, \mathbb{C})$
- $C^\infty(-, \mathbb{C})$

The constant functor $U \mapsto Y$ is not a sheaf!
Uniqueness axiom implies $\mathcal{F}(\emptyset) = \{*\}$

Proposition

Limits and colimits of presheaves can be computed section-wise, i.e. the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ *commutes with all limits and colimits.*

LIMITS AND COLIMITS

Proposition

Limits and colimits of presheaves can be computed section-wise, i.e. the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ commutes with all limits and colimits.

Example

$(\mathcal{F} \times \mathcal{G})(U) = \mathcal{F}(U) \times \mathcal{G}(U)$

Can we do the same for sheaves?

SHEAFIFICATION

Can we do the same for sheaves?

- Limits: ok.
- Colimits: not always a sheaf.

There is a solution:

SHEAFIFICATION

Can we do the same for sheaves?

- Limits: ok.
- Colimits: not always a sheaf.

There is a solution:

Theorem

The fully faithful inclusion $\iota \colon \mathbf{Sh}(X) \hookrightarrow \mathbf{PSh}(X)$ admits an exact left adjoint: sheafification.

Example

Sheafification of $U \mapsto Y$, is $U \mapsto C(U, Y_{\text{discrete}})$.

Proposition

The category of abelian sheaves on a topological space X is abelian, and has enough injectives.

Example

If M is an injective object in **Ab**, i.e. a divisible group, then for $x \in X$,

$$U \mapsto \begin{cases} M & x \in U \\ \{*\} & x \notin U \end{cases}$$

is an injective sheaf.