A Solid Exploration of Condensed Mathematics

Wannes Malfait

Contents

1 Introduction

Most of the contents of this paper will be based on the lecture notes by Peter Scholze, who developed the theory of condensed mathematics together with Dustin Clausen ([Sch19]).

2 Sheaves

A central role in this paper will be played by sheaves. It therefore makes sense to study these objects more closely, and attempt to gain some intuition before jumping straight into the heart of the matter. The first step, will be to convince our selves, that sheaves are interesting enough to study, and that they form a nice type of category. As such, we will dive into sheaf cohomology and some applications of it in algebraic topology. But first, we must define what a sheaf is.

2.1 On Topological Spaces

Consider the following situation in algebraic geometry. We have some (affine) algebraic variety X, with the Zariski Topology. Associated to X is an ideal I of all the polynomials that vanish on X. Then, the quotient ring $\mathcal{O}_X(X) = k[x_1, \dots, x_n]/I$, is the coordinate ring of X. The elements of $\mathcal{O}_X(X)$ are the regular functions on X. For an open $U \subseteq X$ we can also consider the regular functions on U, which we will denote as $\mathcal{O}_X(U)$. This mapping between open subsets of X and rings, given by $\mathcal{O}_X$, is precisely a sheaf. In this case, $\mathcal{O}_X$ is known as the *structure sheaf*. In general, a sheaf will give some global data that can be defined locally.

Definition 1 (Presheaves)**.** Let X be a topological space. A *presheaf of sets* $\mathcal{F}$ on X consists of two things:

1. For each open $U \subseteq X$ a set $\mathcal{F}(U)$. These are known as the *sections* of $\mathcal{F}$ over U.

2. For each inclusion $U \subseteq V$ a map $\rho_{U,V} \colon \mathcal{F}(V) \to \mathcal{F}(U)$, such that $\rho_{U,U} = id_{\mathcal{F}(U)}$, and for $U \subseteq V \subseteq W$ we have $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$. We call the $\rho_{U,V}$ the *restrictions*, and if $s \in \mathcal{F}(U)$, we will often denote $\rho_{V,U}(s) = s|_V$.

Remark 2. If we write $\mathbf{Open}(X)$ for the category of open sets of X, with morphisms given by the inclusion maps, then a presheaf is precisely a functor $\mathcal{F}\colon \mathbf{Open}(X)^{\mathrm{opp}} \to \mathbf{Set}$.

This only defines what a "presheaf" is. The sheaf condition will ensure that the value of $\mathcal{F}(U)$ can be constructed by defining it locally on elements of a cover $\{U_i\}_{i\in I}$ of U. It is exactly this interaction between local definitions and global objects that makes sheaves so useful.

Definition 3 (Sheaves)**.** Let X be a topological space, and $\mathcal{F}$ a presheaf of sets on X. We say that $\mathcal{F}$ is a sheaf, if it satisfies the following additional properties for any open $U \subseteq X$ and open cover $\{U_i\}_{i\in I}$ of U:

- (uniqueness/locality) If $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.
- (gluing) If $\{s_i \in \mathcal{F}(U_i)\}_{i\in I}$ is a collection of sections such that

  $$s_i|_{U_i \cap U_j} = s_j|_{U_i\cap U_j} \text{ for all } i, j,$$

  then there is a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.

In other words, if we have a bunch of sections that agree on overlaps, then we can uniquely glue them together.

Remark 4. We can reformulate the properties in a more categorical manner. Namely, for any open cover $\{U_i\}_{i\in I}$ of $U \subseteq X$, the following diagram:

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j),$$

has to be an equalizer. This diagram makes sense in any target category that has all limits. In terms of sets, the first map is given by $s \mapsto (s|_{U_i})_{i\in I}$. The pair of maps is given by the two possible restrictions, $(s_i)_{i\in I} \mapsto ((s_i|_{U_i\cap U_j})_{j\in I})_{i\in I}$ and $(s_j)_{j\in I} \mapsto ((s_j|_{U_i\cap U_j})_{i\in I})_{j\in J}$.

In this way, we can define sheaves on rings or abelian groups, or other categories with all (set-indexed) limits. For example, a presheaf of abelian groups on X, is a functor $\mathcal{F}\colon \mathbf{Open}(X)^{\mathrm{opp}} \to \mathbf{Ab}$.

In many cases, defining some structure on sheaves, will come down to defining it on $\mathcal{F}(U)$ in a compatible way. As an example, we can look at morphisms of sheaves.

Definition 5 (Morphisms of sheaves)**.** Let $\mathcal{F}, \mathcal{G}$ be presheaves on a topological space X. A *morphism of presheaves* $\varphi\colon \mathcal{F} \to \mathcal{G}$ assigns to each open $U \subset X$ a morphism $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with the restriction maps, i.e. for $U \subseteq V \subseteq X$ the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "\rho^{\mathcal{F}}_{U,V}"] & \mathcal{G}(V) \arrow[d, "\rho^{\mathcal{G}}_{U,V}"] \\
\mathcal{F}(U) \arrow[r, "\varphi_U"] & \mathcal{G}(U)
\end{tikzcd}.$$

In other words, φ is a natural transformation between $\mathcal{F}$ and $\mathcal{G}$. Using a bit of abusive notation, the compatibility can be read as $\varphi(s|_U) = \varphi(s)|_U$. A *morphism of sheaves* is a morphism of the underlying presheaves.

For a topological space X, we now have (pre)sheaves and morphisms between them. These form a category. We will write $\mathbf{PSh}(X)$ for the category of presheaves of sets on X, and $\mathbf{Sh}(X)$ for the category of sheaves of sets on X. In the same way, we write $\mathbf{Ab}(X), \mathbf{Ring}(X), \mathbf{Vect}(X), \ldots$ for the category of sheaves of abelian groups, rings, vector spaces, $\ldots$ on X.

2.1.1 Examples

To solidify our understanding of sheaves, it will be beneficial to look at some examples.

Example 6. The simplest examples of sheaves are those were $\mathcal{F}(U)$ is not just a set, but a set of functions, and the restrictions correspond to actual restrictions.

Example 7. Let Y be another topological space, then we can define $\mathcal{F}(U) \stackrel{\text{def}}{=} \mathrm{Cont}(U, Y)$ where $\mathrm{Cont}(U, Y) = \{f\colon U \to Y \mid f \text{ continuous }\}$. The restrictions are defined as actual restrictions, and we get a presheaf. To see that it is a sheaf, we need to verify the gluing condition. Let $\{U_i\}$ be an open covering of $U \subseteq X$, and $f_i\colon U_i \to Y$ is a continuous map for each $i \in I$, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. We can now define a map $f\colon U \to Y$ via $f(u) = f_i(u)$ for any $i \in I$ such that $u \in U_i$. By assumption, this map is well-defined. To see that it is continuous, for any $V \subseteq Y$ open, we can write

$$f^{-1}(V) = U \cap f^{-1}(V) = \bigcup_{i \in I} (U_i \cap f^{-1}(V)) = \bigcup_{i \in I} f_i^{-1}(V),$$

which is open since all the f_i are continuous.

So we see that $\mathcal{F}$ is a sheaf. In the case that Y has the discrete topology, we call $\mathcal{F}$ the *constant sheaf with value* Y. The sections of $\mathcal{F}$ over U are the *locally* constant functions, i.e. at each point $x \in U$ we can find an open neighborhood V of x, such that f is constant on V.

Example 8. Let $f\colon X \to Y$ be a continuous map, then we get a sheaf on X by the rule

$$\Gamma(Y/X)(U) = \{s\colon U \to Y \mid f \circ s = 1_U\}.$$

The gluing construction is the same as in the previous example. To see that this yields another section, note that for any $u \in U$, $s(u) = s_i(u)$ for some $i \in I$, and hence $f(s(u)) = f(s_i(u)) = u$. We call $\Gamma(Y/X)$ the *sheaf of sections of* f.

Example 9. As we remarked above, the structure sheaf $\mathcal{O}_X$ is also a sheaf, where again, the restrictions are actual restrictions of functions. To show that the gluing of regular functions is again regular, requires some machinery from algebraic geometry, which falls outside the scope of this paper.

As it turns out, it is possible to generalize the definition of sheaves to categories which also have the notion of a covering. We will need this notion when discussing condensed sets.

2.2 On a Site

The idea of a site was first introduced by Alexander Grothendieck, and has proven to be very useful in algebraic geometry. For this subsection we largely follow [Sta23, Part 1, Chapter 7].

2.2.1 Coverings and Sites

To define a site, we collect all the essential properties that coverings of topological spaces have:

Definition 10 (Coverings and Sites)**.** A *site* is a category $\mathcal{C}$ together with a set $\mathrm{Cov}(\mathcal{C})$ of *coverings* of $\mathcal{C}$, where the following axioms hold: [1]

1. If $V \to U$ is an isomorphism, then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
2. If $\{U_i \to U\}_{i\in I}$ is a covering, and for each $i \in I$, $\{V_{ij} \to U_i\}_{j\in J_i}$ is a covering as well, then so is the composition $\{V_{ij} \to U\}_{i\in I, j\in J_i}$.
3. If $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a morphism of $\mathcal{C}$, then the pullback $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I}$ is a covering.

To make sense of this definition, let us look at the canonical example.

Example 11. Let X be a topological space, then $\mathbf{Open}(X)$ is a site with coverings given by the open covers. Let us verify that the axioms hold:

1. The only isomorphisms in $\mathbf{Open}(X)$ are the identity maps. Since $\{U\}$ is an open cover of U for any $U \in \mathbf{Open}(X)$, the first axiom is satisfied.
2. If $U = \bigcup_{i\in I} U_i$ is an open cover of U, and for each $i \in I$, $U_i = \bigcup_{j\in J_i} U_{ij}$ is an open cover of U_i, then $U = \bigcup_{i\in I} \bigcup_{j\in J_i} U_{ij}$ is an open cover of U.
3. If $U = \bigcup_{i\in I} U_i$ is an open cover, and $V \subseteq U$, then $V = \bigcup_{i\in I} V \cap U_i$ is an open cover of V.

Here we used that $U \times_W V = U \cap V$ for $U, V \subseteq W$, which follows from $S \subseteq V, S \subseteq U \implies S \subseteq V \cap U$.

The following example is very close to the underlying site of condensed sets.

Example 12. TODO: example of G-sets.

2.2.2 Sheafification

In a lot of constructions, the natural thing we do, will end up being a presheaf, but not a sheaf in general. So, we will need some way to turn presheaves into sheaves.

Proposition 13. *The fully faithful inclusion*

$$\iota\colon \mathbf{Sh}(X) \hookrightarrow \mathbf{PSh}(X),$$

admits a left adjoint $\mathcal{F} \to \mathcal{F}^\sharp$, "sheafification".

We will give an explicit construction of $\mathcal{F}^\sharp$. As it turns out, "sheafification" is actually a two-step process. The first step is making the presheaf separated, and then turning the separated presheaf into a sheaf.

[1] We force $\mathrm{Cov}(\mathcal{C})$ to be a set since we will take limits over all coverings. It is possible to let $\mathrm{Cov}(\mathcal{C})$ be a class, and then show that it can be replaced with a set of coverings that gives rise to the same category of sheaves. See [Sta23, Remark 7.6.3].

Definition 14. We say that a presheaf $\mathcal{F}$ is *separated* if $\mathcal{F}(U) \hookrightarrow \prod\limits_i \mathcal{F}(U_i)$ for any cover $U = \bigcup_i U_i$.

We first have a few lemmas regarding limits of presheaves and sheaves.

Lemma 15. *Let X be a topological space, then limits and colimits exist in $\mathbf{PSh}(X)$. Additionally, for any $U \subseteq X$, the functor $\mathbf{PSh}(X) \to \mathbf{Set}\colon \mathcal{F} \mapsto \mathcal{F}(U)$ commutes with all limits and colimits.*

Proof. Let $\mathcal{F}\colon \mathcal{Z} \to \mathbf{PSh}(X)$ be a functor. We get a cone $(\mathcal{F}_{\lim}, p^z)$ by moving to the images in $\mathbf{Set}$. In other words we take $\mathcal{F}_{\lim}(U) = \lim_{z\in\mathcal{Z}} \mathcal{F}_z(U)$, and $p^z_U\colon \lim_{z\in\mathcal{Z}} \mathcal{F}_z(U) \to \mathcal{F}_z(U)$. For the restrictions, if $U \subseteq V$ we get a unique map $\mathcal{F}_{\lim}(V) \to \mathcal{F}_{\lim}(U)$, given by the fact that $\mathcal{F}_{\lim}(U)$ is a limit, and $\mathcal{F}_{\lim}(V)$ is a cone on all the $\mathcal{F}_z(U)$ by composing the maps $\mathcal{F}_{\lim}(V) \to \mathcal{F}_z(V)$ with the restrictions $\mathcal{F}_z(V) \to \mathcal{F}_z(U)$. So we see that $\mathcal{F}_{\lim}$ is indeed a presheaf.

$$\begin{tikzcd}
 & & \mathcal{F}_{\lim}(V) \arrow[d, dashed] \arrow[ddll, bend right, "p^z_V"'] \arrow[ddrr, bend left, "p^{z'}_V"] & & \\
 & & \mathcal{F}_{\lim}(U) \arrow[dl, "p^z_U"'] \arrow[dr, "p^{z'}_U"] & & \\
\mathcal{F}_z(V) \arrow[r] & \mathcal{F}_z(U) \arrow[rr] & & \mathcal{F}_{z'}(U) & \mathcal{F}_{z'}(V) \arrow[l]
\end{tikzcd}$$

To see that the maps p^z are morphisms of presheaves, we need to verify that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}_{\lim}(V) \arrow[r, "p^z_V"] \arrow[d] & \mathcal{F}_z(V) \arrow[d] \\
\mathcal{F}_{\lim}(U) \arrow[r, "p^z_U"] & \mathcal{F}_z(U)
\end{tikzcd},$$

but that already follows from the previous diagram.

Let us now verify that $\mathcal{F}_{\lim}$ is actually a limit. If $(\mathcal{G}, g^z)$ is another cone, then for each $U \subseteq X$, we get a unique map $\mathcal{G}(U) \to \mathcal{F}_{\lim}(U)$, such that the corresponding cone diagrams commute. It suffices to show that these maps combine to form a map of presheaves. Using the universal property of limits, this comes down to showing

$$\begin{tikzcd}
\mathcal{G}(V) \arrow[r] \arrow[d] & \mathcal{F}_{\lim}(V) \arrow[d] \\
\mathcal{G}(U) \arrow[r] & \mathcal{F}_{\lim}(U)
\end{tikzcd}
\iff
\begin{tikzcd}
\mathcal{G}(V) \arrow[r, "g^z_V"] \arrow[d] & \mathcal{F}_z(V) \arrow[d] \\
\mathcal{G}(U) \arrow[r, "g^z_U"] & \mathcal{F}_z(U)
\end{tikzcd}
\quad \forall z \in \mathcal{Z}.$$

The commutativity of these last diagrams, is just saying that the maps g^z are presheaf morphisms, which is true by $(\mathcal{G}, g^z)$ being a cone. $\square$

The big difference between sheaves and presheaves, is that we can glue things together defined on an open cover. The trick will be to force the presheaf to behave as we want. Let us try and make this more precise.

Let $U \subset X$ be an open set of a topological space X. We obtain a category $\mathrm{Cov}(U)$ with objects the open coverings of U, and morphisms given by refinements. If $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in J}$ are open covers of U, we say that $\mathcal{U}$ is a *refinement* of $\mathcal{V}$ if there is a map $\alpha\colon I \to J$ such that $U_i \subseteq V_{\alpha(j)}$ for all $i \in I$. This allows the following reformulation:

Lemma 16. *Let $\mathcal{F}$ be a presheaf on a topological space X. For an open cover $\mathcal{U}$ of $U \subseteq X$, define $\mathcal{F}(\mathcal{U})$ as the following equalizer:*

$$\tilde{\mathcal{F}}(\mathcal{U}) = \{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(U_i) \mid s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}\}.$$

This defines a functor $\tilde{\mathcal{F}}\colon \mathrm{Cov}(U)^{\mathrm{opp}} \to \mathbf{Set}$, and there is a canonical map $\mathcal{F}(U) \to \tilde{\mathcal{F}}(\mathcal{U})$. The presheaf $\mathcal{F}$ is a sheaf if and only if this map is bijective for every cover $\mathcal{U}$.

Proof. Given a refinement $\mathcal{U}$ of $\mathcal{V}$, we have the map $\tilde{\mathcal{F}}(\mathcal{V}) \to \tilde{\mathcal{F}}(\mathcal{U})$ given by

$$(s_j)_{j\in J} \mapsto ((s_{\alpha(i)})|_{U_i})_{i\in I}.$$

This is well-defined, since

$$\begin{aligned}
(s_{\alpha(i)}|_{U_i})|_{U_i\cap U_j} &= s_{\alpha(i)}|_{U_i\cap U_j} && (U_i \cap U_j \subseteq U_i)\\
&= (s_{\alpha(i)}|_{V_{\alpha(i)}\cap V_{\alpha(j)}})|_{U_i\cap U_j} && (U_i \cap U_j \subseteq V_{\alpha(i)} \cap V_{\alpha(j)})\\
&= (s_{\alpha(j)}|_{V_{\alpha(i)}\cap V_{\alpha(j)}})|_{U_i\cap U_j} && (\text{definition of } \tilde{\mathcal{F}}(\mathcal{V}))\\
&= (s_{\alpha(j)})|_{U_i\cap U_j} && (U_i \cap U_j \subseteq V_{\alpha(i)} \cap V_{\alpha(j)})\\
&= (s_{\alpha(j)}|_{U_j})|_{U_i\cap U_j},
\end{aligned}$$

where we used repeatedly that $(s|_T)|_S = s|_S$ if $S \subseteq T$. Functoriality of $\tilde{\mathcal{F}}$ is now easily verified. Indeed, if $\mathcal{U}$ is a refinement of $\mathcal{V}$ which is a refinement of $\mathcal{W}$, we have maps $\alpha\colon I \to J$ and $\beta\colon J \to K$ such that $U_i \subseteq V_{\alpha(i)}$ and $V_j \subseteq W_{\beta(j)}$ for $i \in I, j \in J$. Hence, $U_i \subseteq V_{\alpha(i)} \subseteq W_{\beta(\alpha(i))}$, so that

$$((s_{\beta(\alpha(i))})|_{V_{\alpha(i)}}|_{U_i})_{i\in I} = ((s_{\beta(\alpha(i))})|_{U_i})_{i\in I},$$

which is what we needed to show.

Since $\{U\}$ is itself a cover of U, and $\mathcal{F}(U) = \tilde{\mathcal{F}}(\{U\})$, we now get a canonical map $\mathcal{F}(U) \to \tilde{\mathcal{F}}(\mathcal{U})$ for any open cover $\mathcal{U}$ of U. The final claim follows from remark 4. □

We now have the following construction:

Lemma 17. *Given a presheaf of $\mathcal{F}$ on a space X, we can construct a new presheaf $\mathcal{F}^+$ by setting*

$$\mathcal{F}^+(U) = \mathrm{colim}_{\mathcal{U}\in\mathrm{Cov}(U)}\, \tilde{\mathcal{F}}(\mathcal{U}),$$

Proof. We omit the details here, since the arguments are very similar to the previous two lemmas.

Let us just show how $\mathcal{F}^+$ acts on morphisms. Consider $U \subseteq V$. Then this gives a map $\mathrm{Cov}(V) \to \mathrm{Cov}(U)$ by mapping a cover $\{V_i\}_{i\in I}$ of V to the cover $\{V_i \cap U\}_{i\in I}$ of U. We get induced maps $\tilde{\mathcal{F}}(\mathcal{U}) \to \tilde{\mathcal{F}}(\mathcal{V})$, and we obtain maps $\tilde{\mathcal{F}}(\mathcal{U}) \to \mathcal{F}^+(V)$. By the universal property of the colimit, this gives a unique map $\mathcal{F}^+(V) \to \mathcal{F}^+(U)$, making the cone diagrams commute. □

Prove theorem 7.10.10 from stacks project...

3 Sheaf Cohomology

Links to Algebraic Topology
Follow [Tu22]? Maybe look at other things as well?

4 Condensed Sets

References

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