A Fluid Introduction to Condensed Mathematics

Wannes Malfait

VRIJE UNIVERSITEIT BRUSSEL

MOTIVATION

Problem: Algebra and topology clash:

$$\mathrm{id}\colon \mathbb{R}_{\text{discrete}} \to \mathbb{R}_{\text{Euclidean}}$$

is epi + mono, but not iso

Solution?

MOTIVATION

Problem: Algebra and topology clash:

$$\mathrm{id}\colon \mathbb{R}_{\text{discrete}} \to \mathbb{R}_{\text{Euclidean}}$$

is epi + mono, but not iso

Solution?
$\rightsquigarrow$ Condensed abelian groups

PRESHEAVES

Idea: Study better behaved objects.

PRESHEAVES

Idea: Study better behaved objects.

Definition

A presheaf on a topological space X is a functor

$$\mathcal{F}\colon \ \mathbf{Open}(X)^{\mathrm{opp}} \to \mathbf{Set}\,/\,\mathbf{Rng}\,/\,\mathbf{Ab}\,/\,\ldots$$

Notation:

- Elements of $\mathcal{F}(U)$ are called sections
- If $i\colon U \hookrightarrow V$ then $\mathcal{F}(i)$ is called the restriction map:

$$s \in \mathcal{F}(V) \mapsto s|_U \in \mathcal{F}(U).$$

Example

We always have the presheaf given by

- $\mathcal{F}(U) \mapsto Y$, and
- $\mathcal{F}(f) \mapsto \mathrm{id}_Y$

Example

We always have the presheaf given by

- $\mathcal{F}(U) \mapsto Y$, and
- $\mathcal{F}(f) \mapsto \mathrm{id}_Y$

Problem:

- Want more structure
- Want global things to be defined locally

$\rightsquigarrow$ sheaf conditions:

SHEAF CONDITIONS

For an open cover $\{U_i\}_{i\in I}$ of U:

- uniqueness/locality: If $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

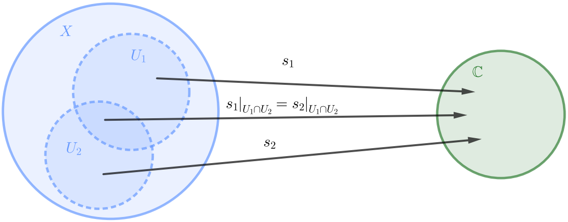
SHEAF CONDITIONS

For an open cover $\{U_i\}_{i \in I}$ of U:

- uniqueness/locality: If $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.
- gluing: If $\{s_i \in \mathcal{F}(U_i)\}_{i \in I}$ is a collection of sections such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \text{ for all } i, j,$$

then there is a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.



EXAMPLES

Typical examples are rings of functions:

- $\mathsf{Cont}(-, \mathbb{C})$
- $C^\infty(-, \mathbb{C})$

EXAMPLES

Typical examples are rings of functions:

- $\mathsf{Cont}(-, \mathbb{C})$
- $C^\infty(-, \mathbb{C})$

The constant functor $U \mapsto Y$ is not a sheaf!
Uniqueness axiom implies $\mathcal{F}(\emptyset) = \{*\}$

LIMITS AND COLIMITS

Proposition

Limits and colimits of presheaves can be computed section-wise, i.e. the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ commutes with all limits and colimits.

Proposition

Limits and colimits of presheaves can be computed section-wise, i.e. the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ commutes with all limits and colimits.

Example

$$(\mathcal{F} \times \mathcal{G})(U) = \mathcal{F}(U) \times \mathcal{G}(U)$$

Can we do the same for sheaves?

SHEAFIFICATION

Can we do the same for sheaves?

- Limits: ok.
- Colimits: not always a sheaf.

There is a solution:

SHEAFIFICATION

Can we do the same for sheaves?

- Limits: ok.
- Colimits: not always a sheaf.

There is a solution:

Theorem

The fully faithful inclusion $\iota\colon\ \mathbf{Sh}(X) \hookrightarrow \mathbf{PSh}(X)$ admits an exact left adjoint: sheafification.

Example

Sheafification of $U \mapsto Y$, is $U \mapsto \mathsf{Cont}(U, Y_{\text{discrete}})$.

ABELIAN SHEAVES

Proposition

The category of abelian sheaves on a topological space X is abelian, and has enough injectives.

Example

If M is an injective object in **Ab**, i.e. a divisible group, then for $x \in X$,

$$U \mapsto \begin{cases} M & x \in U \\ \{*\} & x \notin U \end{cases}$$

is an injective sheaf.

SHEAF COHOMOLOGY

If

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is exact, then globally we only get

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}''(X)$$

SHEAF COHOMOLOGY

If

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is exact, then globally we only get

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}''(X)$$

$\rightsquigarrow$ Right derived functors $H^i(X, -)$ of $\mathcal{F} \mapsto \mathcal{F}(X)$

Definition

The sheaf cohomology of X with coefficients A is $H^i(X, \mathrm{Cont}(-, A))$.

COMPUTING HOMOLOGY

We have

$$0 \longrightarrow \mathrm{Cont}(\mathbb{Z}, -) \longrightarrow \mathrm{Cont}(\mathbb{R}, -) \longrightarrow \mathrm{Cont}(\mathbb{R}/\mathbb{Z}, -) \longrightarrow 0$$

since every map $U \subseteq S^1 \to S^1$ can be lifted locally to a map $U \to \mathbb{R}$.

But, there is no global lift of $\mathrm{id}\colon S^1 \to S^1$ to $S^1 \to \mathbb{R}$. So $H^1(S^1, \mathbb{Z}) \neq 0$.

COMPUTING HOMOLOGY

We have

$$0 \longrightarrow \mathrm{Cont}(\mathbb{Z}, -) \longrightarrow \mathrm{Cont}(\mathbb{R}, -) \longrightarrow \mathrm{Cont}(\mathbb{R}/\mathbb{Z}, -) \longrightarrow 0$$

since every map $U \subseteq S^1 \to S^1$ can be lifted locally to a map $U \to \mathbb{R}$.

But, there is no global lift of $\mathrm{id}\colon S^1 \to S^1$ to $S^1 \to \mathbb{R}$. So $H^1(S^1, \mathbb{Z}) \neq 0$.

Proposition

If S is a CW-complex:

$$H^i_{sheaf}(S, \mathbb{Z}) = H^i_{singular}(S, \mathbb{Z}).$$

SITES

- Need a slight abstraction of sheaves on a space.
- Want sheaves on categories, not just spaces.
- Crucial ingredient: coverings.

SITES

- Need a slight abstraction of sheaves on a space.
- Want sheaves on categories, not just spaces.
- Crucial ingredient: coverings.

$\rightsquigarrow$ Definition of a site $\approx$ "Category + notion of coverings".

CONDENSED SETS

Definition

A condensed set* is a sheaf on the site of profinite sets with coverings given by jointly surjective maps.

CONDENSED SETS

Definition

A condensed set* is a sheaf on the site of profinite sets with coverings given by jointly surjective maps.

For a topological space X, we have an associated condensed set $\underline{X} = \mathrm{Cont}(-, X)$.

PROFINITE SETS

Definition

A profinite set is a compact, Hausdorff, totally disconnected space.

Example

A finite discrete space

PROFINITE SETS

Definition

A profinite set is a compact, Hausdorff, totally disconnected space.

Example

A finite discrete space

Proposition

A limit of profinite sets is profinite. In particular, products of profinite sets are again profinite.

THE MOTIVATING PROBLEM

Claim: for condensed sets the map:

$$\underline{\mathrm{id}} : \underline{\mathbb{R}_{\text{discrete}}} \to \underline{\mathbb{R}_{\text{Euclidean}}}$$

is no longer an epimorphism.

THE MOTIVATING PROBLEM

Claim: for condensed sets the map:

$$\underline{\mathrm{id}} : \underline{\mathbb{R}_{\text{discrete}}} \to \underline{\mathbb{R}_{\text{Euclidean}}}$$

is no longer an epimorphism.

Let $S \subset \mathbb{R}_{\text{Euclidean}}$ be the cantor set. $S \cong \prod_{n \in \mathbb{N}} \{0, 1\}$, so S is profinite. Hence:

$$\mathrm{Cont}(S, \mathbb{R}_{\text{discrete}}) \subsetneq \mathrm{Cont}(S, \mathbb{R}_{\text{Euclidean}})$$

CONDENSED COHOMOLOGY

For a condensed set T, let $\mathbb{Z}[T]$ be the sheafification of $S \mapsto \mathbb{Z}[T(S)]$.

Definition

The cohomology of S with coefficients A is

$$H^i(S, A) = \mathrm{Ext}^i(\mathbb{Z}[\underline{S}], \underline{A})$$

CONDENSED COHOMOLOGY

For a condensed set T, let $\mathbb{Z}[T]$ be the sheafification of $S \mapsto \mathbb{Z}[T(S)]$.

Definition

The cohomology of S with coefficients A is

$$H^i(S, A) = \mathrm{Ext}^i(\mathbb{Z}[\underline{S}], \underline{A})$$

Theorem ([Sch19], Theorem 3.2)

Let S be a compact Hausdorff space. Then

$$H^i_{cond}(S, A) \cong H^i_{sheaf}(S, A)$$

REFERENCES

Rodrigo Marlasca Aparicio.
Condensed mathematics: The internal hom of condensed sets and condensed abelian groups and a prismatic construction of the real numbers, 2021.

Dustin Clausen and Peter Scholze.
Masterclass in condensed mathematics.
https://www.math.ku.dk/english/calendar/events/condensed-mathematics/, November 2020.

Peter Scholze.
Lectures on condensed mathematics.
https://www.math.uni-bonn.de/people/scholze/Condensed.pdf, 2019.

The Stacks Project Authors.
The stacks project.
https://stacks.math.columbia.edu/, 2023.