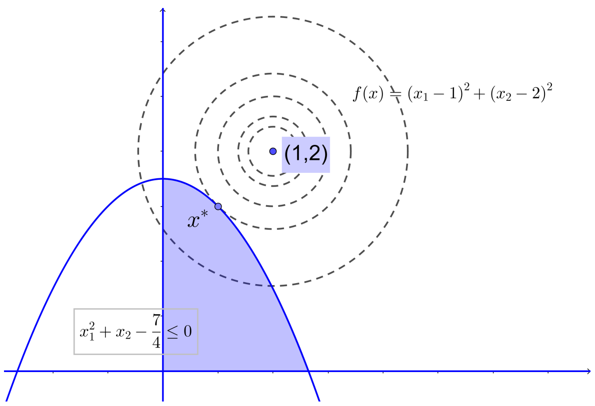
实用优化算法习题

0. 考虑问题

$$\begin{aligned} \min\ & (x_1-1)^2+(x_2-2)^2 \\ \text{s.t. } & x_1^2+x_2-\frac{7}{4}\le 0, \\ & x_1,x_2\ge 0. \end{aligned}$$

画出问题的可行域和等高线图, 据此求出问题的解.

【解:】见第一章课件例2.6.



1. 用斐波那契法求函数$f(x)=x^2-x+2$ 在区间$[-1,3]$ 上的极小点, 要求精度为0.1, 至少迭代3 次.

【解:】先确定迭代次数：

$$\frac{3-(-1)}{0.1}=40.$$

写出斐波那契数列

$$F_0=1,F_1=1,F_2=2,F_3=3,F_4=5,F_5=8,F_6=13,F_7=21,F_8=34,F_9=55.$$

故需迭代$n=9-1=8$ 次.

第一次迭代$a=-1,\ b=3,$

$$\begin{aligned} x_1=a+\tfrac{F_7}{F_9}(b-a)=-1+\tfrac{21}{55}\times 4=0.5273,\ \ f_1=f(x_1)=1.7507; \\ x_2=a+\tfrac{F_8}{F_9}(b-a)=-1+\tfrac{34}{55}\times 4=1.4727,\ \ f_2=f(x_2)=2.6962. \end{aligned}$$

因为$f_1<f_2$, 故而$b=x_2=1.4727,\ x_2=x_1=0.5273,\ f_2=f_1=1.7507.$

第二次迭代$a=-1,\ b=1.4727$.

$$x_1=a+\frac{F_6}{F_8}(b-a)=-1+\frac{13}{34}(1.4727-(-1))=-0.0545,\ \ f_1=f(x_1)=2.0575;$$
$$x_2=0.5273,\ \ f_2=1.7507.$$

因为$f_1>f_2$, 故而$a=x_1=-0.0545,\ x_1=x_2=0.5273,\ f_1=f_2=1.7507$.

第三次迭代$a=-0.0545,\ b=1.4727$.

$$x_1=0.5273,\ \ f_1=1.7507;$$
$$x_2=a+\frac{F_6}{F_7}(b-a)=0.8909,\ \ f_2=f(x_2)=1.9028.$$

因$f_1<f_2$, 故而$b=x_2=0.8909,\ x_2=x_1=0.5273,\ f_2=f_1=1.7507$.

2. 用黄金分割法求
$$f(x)=-2x^3+21x^2-60x+50$$
在区间$[-1,4]$ 内的最小值, 迭代三次.

【解:】第一次迭代:

$$a=-1,b=4;x_1=a+0.382(b-a)=0.91,f_1=f(x_1)=11.2830;$$
$$x_2=2.09,f_2=f(x_2)=-1.9286.$$

因为$f_1>f_2$, 故而$a=x_1=0.91,\ x_1=x_2=2.09,\ f_1=f_2=-1.9286$.

第二次迭代:

$$a=0.91,b=4;x_1=2.09,f_1=-1.9286;$$
$$x_2=a+0.618(b-a)=2.8196,f_2=f(x_2)=2.9448.$$

因$f_1<f_2$, 故而$b=x_2=2.8196,\ x_2=x_1=2.09,\ f_2=f_1=-1.9268$.

第三次迭代:

$$a=0.91,b=2.8196;x_2=2.09,f_2=-1.9286;$$
$$x_1=a+0.382(b-a)=1.6395,f_1=f(x_1)=-0.7364.$$

因$f_1>f_2$, 故而$a=x_1=1.6395,\ x_1=x_2=2.09,\ f_1=f_2=-1.9628$.

3. 考虑函数
$$f(\boldsymbol{x})=\frac{1}{3}x_1^3-16x_1+\frac{1}{3}x_2^3-x_2$$
写出$\min\ \ f(\boldsymbol{x})$ 的一阶必要条件并利用该条件求$f(\boldsymbol{x})$ 的极小点.

【解:】一阶必要条件

$$\nabla f(\boldsymbol{x})=\begin{pmatrix}x_1^2-16\\x_2^2-1\end{pmatrix}=0.$$

求解上述方程组得
$$x_1=\pm4,\ \ x_2=\pm1.$$

由于Hesse 矩阵
$$\nabla^2 f(\boldsymbol{x})=\begin{pmatrix}2x_1 & \\ & 2x_2\end{pmatrix}$$

当$x_1=4,\ x_2=1$ 时正定, 故而$\boldsymbol{x}=(4,1)^T$ 为问题的极小点.

4. 考虑下列函数

(1) $f(\boldsymbol{x}) = 2x_1^2 + x_2^2 - 2x_1x_2 + 2x_1^3 + x_1^4$

(2) $f(\boldsymbol{x}) = 2x_1^3 - 3x_1^2 - 6x_1x_2\left(x_1 - x_2 - 1\right).$

的所有驻点.哪些是极小点，是否是整体极小点.

【解：】(1) 函数都梯度和Hesse 矩阵分别为

$$\nabla f(\boldsymbol{x}) = \begin{pmatrix} 4x_1 - 2x_2 + 6x_1^2 + 4x_1^3 \\ 2x_2 - 2x_1 \end{pmatrix},\ G(\boldsymbol{x}) = \begin{pmatrix} 4 + 12x_1 + 12x_1^2 & -2 \\ -2 & 2 \end{pmatrix}.$$

令$\nabla f(\boldsymbol{x}) = 0$, 得

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \quad 或 \begin{cases} x_1 = -1 \\ x_2 = -1 \end{cases} \quad 或 \begin{cases} x_1 = -\frac{1}{2} \\ x_2 = -\frac{1}{2} \end{cases} \quad .$$

在这三个点处, Hesse 矩阵分别为

$$\begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix},\ \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix},\ \begin{pmatrix} 1 & -2 \\ -2 & 2 \end{pmatrix}.$$

其中前两个矩阵是正定矩阵, 第三个矩阵不是正定的,也不是半正定的. 故而$(0,0)^T$, $(-1,-1)^T$ 为局部极小点.

又因为$f(0,0) = 0$, $f(-1,-1) = 0$, 故而这两个点均为整体极小点.

(2) 函数的梯度和Hesse矩阵分别为

$$\begin{aligned} \nabla f(\boldsymbol{x}) &= \begin{pmatrix} 6x_1^2 - 12x_1x_2 - 6x_1 + 6x_2^2 + 6x_2 \\ 12x_1x_2 - 6x_1^2 + 6x_1 \end{pmatrix}, \\ G(\boldsymbol{x}) &= \begin{pmatrix} 12x_1 - 12x_2 - 6 & 12x_2 - 12x_1 + 6 \\ 12x_2 - 12x_1 + 6 & 12x_1 \end{pmatrix} \end{aligned} .$$

令$\nabla f(\boldsymbol{x}) = 0$, 得

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \text{ 或 } \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

在这四个点处的Hesse矩阵分别是

$$G = \begin{pmatrix} -6 & 6 \\ 6 & 0 \end{pmatrix}, \begin{pmatrix} 6 & -6 \\ -6 & 12 \end{pmatrix}, \begin{pmatrix} 6 & -6 \\ -6 & 0 \end{pmatrix},\ 和 \begin{pmatrix} -6 & 6 \\ 6 & -12 \end{pmatrix}.$$

这四个矩阵分别是鞍点矩阵, 正定矩阵, 鞍点矩阵和负定矩阵. 所以上述四个点分别为鞍点、极小点、鞍点、极大点. 其中, $(1,0)$ 也是整体极小点.

5. 对正定二次函数$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T G\boldsymbol{x} + c^T\boldsymbol{x}$, 在点$\boldsymbol{x}_k$ 处, 求出沿方向d_k 做精确搜索的步长α_k.

【解：】该二次函数的梯度为

$$\nabla f(\boldsymbol{x}) = G\boldsymbol{x} + c.$$

定义$\varphi(\alpha)=f(\boldsymbol{x}_k+\alpha d_k)$, 则$\alpha_k$ 满足

$$\varphi'(\alpha)=0,$$

也就是

$$\nabla f(\boldsymbol{x}_k+\alpha_k d_k)^T d_k=0.$$

又

$$\begin{aligned}&\nabla f(\boldsymbol{x}_k+\alpha_k d_k)^T d_k=(G(\boldsymbol{x}_k+\alpha_k d_k)+c)^T d_k\\=&(G\boldsymbol{x}_k+c+\alpha_k G d_k)^T d_k\\=&(\nabla f_k+\alpha_k G d_k)^T d_k\\=&\nabla f_k^T d_k+\alpha_k d_k^T G d_k=0.\end{aligned}$$

从而有

$$\alpha_k=-\frac{\nabla f_k^T d_k}{d_k^T G d_k}.$$

6. 用最速下降法求解

$$\min\ f(\boldsymbol{x})=\frac{1}{2}x_1^2+\frac{9}{2}x_2^2,$$

设初始点为$(9,1)^T$.

【见教材、课件例题】

7. 试用最速下降法求解

$$\min\ f(\boldsymbol{x})=2x_1^2+x_2^2,$$

设初始点取为$\boldsymbol{x}_0=(4,4)^T$, 迭代2次, 并验证相邻两次迭代的搜索方向是正交的.

【解:】记

$$g(\boldsymbol{x})=\nabla f(\boldsymbol{x})=\begin{pmatrix}4x_1\\2x_2\end{pmatrix}.$$

在初始点处$f(\boldsymbol{x}_0)=48$, $g(\boldsymbol{x}_0)=(16,8)^T$, 搜索方向为$d_0=-g(\boldsymbol{x}_0)=(-16,-8)$. 考虑

$$\min_{a\geq 0} f(\boldsymbol{x}_0+\alpha d_0)=16(3-20\alpha+36\alpha_2).$$

其极小点为$\alpha_0=5/18$. （或者$\alpha_0=-\frac{g_0^T d_0}{d_0^T G d_0}=\frac{5}{18}$）

则

$$\boldsymbol{x}_1=\boldsymbol{x}_0+\alpha_0 d_0=\begin{pmatrix}-4/9\\16/9\end{pmatrix}.$$

此时

$$g_1=g(\boldsymbol{x}_1)=\begin{pmatrix}-16/9\\32/9\end{pmatrix}, d_1=-g_1=-\begin{pmatrix}-16/9\\32/9\end{pmatrix}.$$

考虑

$$\min_{a\geq 0} f(\boldsymbol{x}_1+\alpha d_1)=\frac{32}{81}(9-40\alpha+48\alpha_2).$$

其极小点为$\alpha_1 = 5/12$.（或者$\alpha_1 = -\frac{g_1^T d_1}{d_1^T G d_1} = \frac{5}{12}$）则

$$\boldsymbol{x}_2 = \boldsymbol{x}_1 + \alpha_1 d_1 = \begin{pmatrix} 8/27 \\ 8/27 \end{pmatrix}.$$

此时

$$g_2 = g(\boldsymbol{x}_2) = \begin{pmatrix} 32/27 \\ 16/27 \end{pmatrix}, d_2 = -g_2 = \begin{pmatrix} -32/27 \\ -16/27 \end{pmatrix}.$$

考虑

$$\min_{a \geq 0} f(\boldsymbol{x}_2 + \alpha d_2) = \left(\frac{8}{27}\right)(3 - 20\alpha + 36\alpha_2).$$

其极小点为$\alpha_2 = 5/18$.（或者$\alpha_2 = -\frac{g_2^T d_2}{d_2^T G d_2} = \frac{5}{18}$）则

$$\boldsymbol{x}_3 = \boldsymbol{x}_2 + \alpha_2 d_2 = \begin{pmatrix} -8/243 \\ 32/243 \end{pmatrix}.$$

8. 用Newton 法求解

$$\min \ f(\boldsymbol{x}) = \frac{1}{2}x_1^2 + \frac{9}{2}x_2^2.$$

【解：】Newton 法用于求解严格凸二次函数时可任选初始点，这里选$\boldsymbol{x}_0 = (1,1)^T$. 函数都梯度和Hesse矩阵分别是

$$\nabla f(\boldsymbol{x}) = \begin{pmatrix} x_1 \\ 9x_2 \end{pmatrix}, \ G = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}.$$

令

$$d_0 = -G^{-1}\nabla f(\boldsymbol{x}_0) = -\begin{pmatrix} 1 & \\ & -\frac{1}{9} \end{pmatrix}\begin{pmatrix} 1 \\ 9 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

则

$$\boldsymbol{x}_1 = \boldsymbol{x}_0 + d_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

算法终止.

9. 用FR 共轭梯度法求解

$$\min \ f(\boldsymbol{x}) = \frac{3}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1x_2 - 2x_1,$$

取初始点$\boldsymbol{x}_0 = (0,0)^T$.

【解:】记

$$G = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}, \ b = \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

则$f(\boldsymbol{x})$, $\nabla f(\boldsymbol{x})$ 可表示成

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T G\boldsymbol{x} + b^T\boldsymbol{x}, \ \nabla f(\boldsymbol{x}) = G\boldsymbol{x} + b.$$

第一次迭代. $\boldsymbol{x}_0 = (0,0)^T$.

$$d_0 = -\nabla f(\boldsymbol{x}_0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix},\ \alpha_0 = -\frac{\nabla f(\boldsymbol{x}_0)^T d_0}{d_0^T G d_0} = \frac{1}{3}.$$

故

$$\boldsymbol{x}_1 = \boldsymbol{x}_0 + \alpha_0 d_0 = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}.$$

因

$$\nabla f(\boldsymbol{x}_1) = \begin{pmatrix} 0 \\ -\frac{2}{3} \end{pmatrix} \neq 0,$$

故算法不终止.

第二次迭代.

$$\beta_0 = \frac{\|\nabla f(\boldsymbol{x}_1)\|^2}{\|\nabla f(\boldsymbol{x}_0)\|^2} = \frac{1}{9},\ d_1 = -\nabla f(\boldsymbol{x}_1) + \beta_0 d_0 = \begin{pmatrix} \frac{2}{9} \\ \frac{2}{3} \end{pmatrix}.$$

计算步长和新迭代点

$$\alpha_1 = -\frac{\nabla f(\boldsymbol{x}_1)^T d_1}{d_1^T G d_1} = \frac{3}{2},\ \boldsymbol{x}_2 = \boldsymbol{x}_1 + \alpha_1 d_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

因$\nabla f(\boldsymbol{x}_1) = (0,0)^T$, 故而算法终止.

函数$f(\boldsymbol{x})$ 的极小点为$\boldsymbol{x}^* = (1,1)^T$.

10. 考虑函数

$$f(\boldsymbol{x}) = 1/2x_1^2 + 1/2x_2^2$$

设初始点为$\boldsymbol{x}_0 = (1,1)^T$，取$d_0 = (-1,0)^T$,

(a) 沿方向d_0 进行精确一维搜索得α_0. 令$\boldsymbol{x}_1 = \boldsymbol{x}_0 + \alpha_0 d_0$, 用FR 公式求$d_1$.

(b) 设G 为目标函数的Hesse 矩阵，证明d_0 与d_1 不是关于G 共轭的. 试说明原因。

【解：】在初始点处

$$g_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

沿d_0 方向进行精确搜索得最优步长为

$$\alpha_0 = -\frac{(g_0)^T d_0}{(d_0)^T G d_0} = 1.$$

则$\boldsymbol{x}_1 = \boldsymbol{x}_0 + \alpha_0 d_0 = (0,1)^T$. 计算

$$g_1 = \nabla f(\boldsymbol{x}_1) = (0,1)^T,\ \beta_0 = \frac{\|g_1\|^2}{\|g_0\|^2} = \frac{1}{2}.$$

则

$$d_1 = -g_1 + \beta_0 d_0 = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}.$$

由于

$$(d_0)^T G d_1 = (d_0) d_1 = \frac{1}{2} \neq 0$$

所以d_0 与d_1 不是关于

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

共轭的.

11. 用DFP 算法求解

$$\min\ f(\boldsymbol{x}) = x_1^2 + 2x_2^2 - 2x_1x_2 - 4x_1,$$

取$\boldsymbol{x}_0 = (1,1)^T$, $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

【解:】记$c = (-4,0)^T$. 则目标函数可记为$f = \frac{1}{2}\boldsymbol{x}^T G\boldsymbol{x} + c^T\boldsymbol{x}$.

1) $g_0 = \nabla f(\boldsymbol{x}_0) = G\boldsymbol{x}_0 + c = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$. 搜索方向$d_0 = -H_0 g_0 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$. 用精确搜索，步长为

$$\alpha_0 = -\frac{g_0^T d_0}{d_0^T G d_0} = \frac{1}{4}.$$

2) $\boldsymbol{x}_1 = \boldsymbol{x}_0 + \alpha_0 d_0 = \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}$, $g_1 = G\boldsymbol{x}_1 + c = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

$$s_0 = \alpha_0 d_0, y_0 = g_1 - g_0 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, H_1 = H_0 - \frac{H_0 y_0 y_0^T H_0}{y_0^T H_0 y_0} + \frac{s_0 s_0^T}{y_0^T s_0} = \begin{pmatrix} \frac{21}{25} & \frac{19}{50} \\ \frac{19}{50} & \frac{41}{100} \end{pmatrix}.$$

$$d_1 = -H_1 g_1 = \begin{pmatrix} \frac{8}{5} \\ \frac{6}{5} \end{pmatrix}, \text{用精确搜索}\alpha_1 = \frac{5}{4}.$$

3) $\boldsymbol{x}_2 = \boldsymbol{x}_1 + \alpha_1 d_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $g_2 = G\boldsymbol{x}_2 + c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

由于$d_0^T G d_1 = 0$, 故而d_0, d_1 关于G 共轭.

12. 用DFP 算法求解

$$\min f(\boldsymbol{x}) = \frac{1}{2}\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 7.$$

初始点为$x_0 = (-1,0)^T$, 初始矩阵为单位矩阵.

【解：】记

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},\ b = \begin{pmatrix} -1 \\ 1 \end{pmatrix},\ c = 7.$$

第一次迭代.

$$\boldsymbol{x}_0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix},\ \nabla f(\boldsymbol{x}_0) = \begin{pmatrix} -2 \\ 1 \end{pmatrix},\ d_0 = -\nabla f(\boldsymbol{x}_0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

$$\alpha_0 = -\frac{\nabla f_0^T d_0}{d_0^T G d_0} = \frac{5}{6},\ \boldsymbol{x}_1 = \boldsymbol{x}_0 + \alpha_0 d_0 = \begin{pmatrix} \frac{2}{3} \\ -\frac{5}{6} \end{pmatrix}.$$

因为

$$\nabla f(\boldsymbol{x}_1) = \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} \neq 0,$$

故算法不终止.

第二次迭代.

$$s_0 = \alpha_0 d_0 = \begin{pmatrix} \frac{5}{3} \\ -\frac{5}{6} \end{pmatrix},\ y_0 = \begin{pmatrix} \frac{5}{3} \\ -\frac{5}{3} \end{pmatrix}.$$

DFP 修正：

$$H_1 = H_0 - \frac{H_0 y_0 y_0^T H_0}{y_0^T H_0 y_0} + \frac{s_0 s_0^T}{y_0^T s_0} = \begin{pmatrix} \frac{7}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} \end{pmatrix}.$$

计算搜索方向和步长

$$d_1 = -H_1 \nabla f_1 = \left(\frac{1}{2}, \frac{1}{2}\right),\ \alpha_1 = -\frac{\nabla f_1^T d_1}{d_1^T G d_1} = \frac{2}{3}.$$

故

$$\boldsymbol{x}_2 = \boldsymbol{x}_1 + \alpha_1 d_1 = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}.$$

因

$$\nabla f(\boldsymbol{x}_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

故而算法终止.

13. 用DFP 算法求解

$$\min\ x_1^2 - x_1 x_2 + x_2^2 + 2x_1 - 4x_2$$

初始点取为$\boldsymbol{x}_0 = (2, 2)^T$, 初始矩阵取为单位矩阵，并验证算法所产生的两个方向关于$G = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ 共轭的.

【解:】记$c = (2, -4)^T$. 则目标函数可记为$f = \frac{1}{2}\boldsymbol{x}^T G \boldsymbol{x} + c^T \boldsymbol{x}$.

1) $g_0 = \nabla f(\boldsymbol{x}_0) = G\boldsymbol{x}_0 + c = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$. 搜索方向$d_0 = -H_0 g_0 = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$. 用精确搜索，步长为

$$\alpha_0 = -\frac{g_0^T d_0}{d_0^T G d_0} = \frac{5}{14}.$$

2) $\boldsymbol{x}_1 = \boldsymbol{x}_0 + \alpha_0 d_0 = \begin{pmatrix} \frac{4}{7} \\ \frac{19}{7} \end{pmatrix}$, $g_1 = G\boldsymbol{x}_1 + c = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix}$.

$$s_0 = \alpha_0 d_0, y_0 = g_1 - g_0 = \begin{pmatrix} -\frac{25}{7} \\ \frac{20}{7} \end{pmatrix}, H_1 = H_0 - \frac{H_0 y_0 y_0^T H_0}{y_0^T H_0 y_0} + \frac{s_0 s_0^T}{y_0^T s_0} = \begin{pmatrix} \frac{194}{287} & \frac{99}{287} \\ \frac{99}{287} & \frac{391}{574} \end{pmatrix}.$$

$$d_1 = -H_1 g_1 = \begin{pmatrix} -\frac{24}{41} \\ -\frac{30}{41} \end{pmatrix},\ \text{用精确搜索}\alpha_1 = \frac{41}{42}.$$

3) $\boldsymbol{x}_2 = \boldsymbol{x}_1 + \alpha_1 d_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $g_2 = G\boldsymbol{x}_2 + c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

由于$d_0^T G d_1 = 0$, 故而d_0, d_1 关于G 共轭.

14. 考虑问题

$$\begin{aligned} \min\ & f(\boldsymbol{x}) = x_1^2 + x_2^2 \\ \text{s.t. } & c(\boldsymbol{x}) = x_1^2 + 2x_2^2 - 1 = 0. \end{aligned}$$

的最优性条件和极值点.（相当于用Lagrange 乘子法求极小点）

【解：】见课件第四章例题2.1.

15. 考虑优化问题

$$\begin{aligned} \min\ & f(\boldsymbol{x}) = -(x_1 + x_2) \\ \text{s.t. } & c_1(\boldsymbol{x}) = 1 - x_1^2 - 4x_2^2 \geq 0, \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{aligned}$$

初始点为$x_0 = \left[\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$. 试分别判断方向向量$d_1 = [1,0]^T$, $d_2 = [1,-0.5]^T$, $d_3 = [0,-1]^T$ 是否是初始点处的下降方向？是否是可行方向？是否是可行下降方向？

【解：】见课件第四章例题2.4.

16. 问题

$$\begin{aligned} \min\ & x_1^2 + x_2^2, \\ s.t.\ & (x_1 - 1)^2 + x_2^2 \leq 1, \\ & x_2^2 - x_1 + 1 \leq 0 \end{aligned} \tag{1}$$

的最优解是$x^* = (1,0)^T$. 求点x^* 处的有效集$\mathcal{I}^*$.

【解：】$\mathcal{I}^* = \{2\}$.

17. 求问题

$$\begin{aligned} \min\ & f(\boldsymbol{x}) = (x_1 - 1)^2 + (x_2 - 1)^2, \\ \text{s.t. } & c_1(\boldsymbol{x}) = 1 - x_1 - x_2 \geq 0, \\ & c_2(\boldsymbol{x}) = x_1 \geq 0, \\ & c_3(\boldsymbol{x}) = x_2 \geq 0. \end{aligned}$$

的KKT 点.

【解：】见课件第四章例题2.6.

18. 求问题

$$\begin{aligned} \min\ & f(\boldsymbol{x}) = (x_1+x_2)^2 + 2x_1 + x_2^2, \\ \text{s.t.}\ & x_1 + 3x_2 \le 4, \\ & 2x_1 + x_2 \le 3, \\ & x_1, x_2 \ge 0 \end{aligned}$$

的KKT 点.

【解:】问题的KKT 条件为

$$\begin{cases} 2x_1 + 2x_2 + 2 + \lambda_1 + 2\lambda_2 - \lambda_3 = 0, \\ 2x_1 + 4x_2 + 3\lambda_1 + \lambda_2 - \lambda_4 = 0, \\ 4 - x_1 - 3x_2 \ge 0,\ \lambda_1 \ge 0,\ \lambda_1(4 - x_1 - 3x_2) = 0, \\ 3 - 2x_1 - x_2 \ge 0,\ \lambda_2 \ge 0,\ \lambda_2(3 - 2x_1 - x_2) = 0, \\ x_1 \ge 0,\ \lambda_3 \ge 0,\ x_1\lambda_3 = 0, \\ x_2 \ge 0,\ \lambda_4 \ge 0,\ x_2\lambda_4 = 0. \end{cases}$$

由于

$$\lambda_3 = 2x_1 + 2x_2 + 2 + \lambda_1 + 2\lambda_2 \ge 2 > 0,$$

故而$x_1 = 0$.

若$\lambda_4 > 0$, 则$x_2 = 0$. 则

$$4 - x_1 - 3x_2 = 4 > 0,\ 3 - 2x_1 - x_2 = 3 > 0,$$

从而$\lambda_1 = 0$, $\lambda_2 = 0$. 那么

$$\lambda_4 = 3\lambda_1 + \lambda_2 = 0,$$

矛盾.

从而$\lambda_4 = 0$. 那么

$$4x_2 + 3\lambda_1 + \lambda_2 = 0.$$

这意味着

$$x_2 = 0,\ \lambda_1 = 0,\ \lambda_2 = 0.$$

所以原问题的解为

$$x_1 = 0,\ x_2 = 0,$$

相应的Lagrange 乘子为

$$\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 2,\ \lambda_4 = 0.$$

19. 已知约束问题

$$\begin{aligned}\min\ & f(\boldsymbol{x}) = -3x_1^2 - x_2^2 - 2x_3^2,\\ \text{s.t. } & c_1(x) = x_1^2 + x_2^2 + x_3^2 - 3 = 0,\\ & c_2(\boldsymbol{x}) = -x_1 + x_2 \geq 0,\\ & c_3(\boldsymbol{x}) = x_1 \geq 0,\\ & c_4(\boldsymbol{x}) = x_2 \geq 0,\\ & c_5(\boldsymbol{x}) = x_3 \geq 0.\end{aligned}$$

试验证最优解$\boldsymbol{x}^* = (1,1,1)^T$ 为KKT 点.

【解：】见课件第四章例题2.10.

20. 用外罚函数法求解约束优化问题

$$\begin{aligned}\min\ & f(\boldsymbol{x}) = x_1^2 + x_2^2,\\ s.t.\ & x_1 + x_2 - 2 = 0.\end{aligned}$$

【解：】见课件第四章例题3.1.

21. 对问题

$$\begin{aligned}\min\ & x_2^2 - 3x_1,\\ \text{s.t. } & x_1 + x_2 = 1\\ & x_1 - x_2 = 0\end{aligned}$$

考虑外罚函数法，求出问题的局部最优解和相应的Lagrange 乘子.

【解:】构造外罚函数

$$P(\boldsymbol{x};\sigma) = x_2^2 - 3x_1 + \sigma(x_1 + x_2 - 1)^2 + \sigma(x_1 - x_2)^2.$$

$$\begin{cases}\frac{\partial P}{\partial x_1} = -3 + 2\sigma(x_1 + x_2 - 1) + 2\sigma(x_1 - x_2) = 0,\\ \frac{\partial P}{\partial x_2} = 2x_2 + 2\sigma(x_1 + x_2 - 1) - 2\sigma(x_1 - x_2) = 0.\end{cases}$$

解之得

$$\begin{cases}x_1(\sigma) = \frac{3+2\sigma}{4\sigma},\\ x_2(\sigma) = \frac{2\sigma}{4\sigma+2}.\end{cases}$$

令$\sigma \to +\infty$, 得

$$\begin{cases}x_1 = \frac{1}{2},\\ x_2 = \frac{1}{2}.\end{cases}$$

所以, 问题的解为$\boldsymbol{x}^* = (1/2, 1/2)^T$.

Lagrange 乘子.(方法一)

$$\begin{aligned}\lambda_1 &= \lim_{\sigma\to+\infty} -2\sigma(x_1(\sigma) + x_2(\sigma) - 1) = -1;\\ \lambda_2 &= \lim_{\sigma\to+\infty} -2\sigma(x_1(\sigma) - x_2(\sigma)) = -2.\end{aligned}$$

(方法二: 求得$\boldsymbol{x}^*$ 后利用最优性条件)

$$\nabla f(\boldsymbol{x}^*) - \lambda_1 \nabla c_1(\boldsymbol{x}^*) - \lambda_2 \nabla c_2(\boldsymbol{x}^*) = 0,$$

也就是

$$\begin{pmatrix} -3 \\ 1 \end{pmatrix} - \lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

解之得$\lambda_1 = -1, \lambda_2 = -2$.

22. 对问题

$$\begin{array}{ll} \min & 2x_1 + 3x_2 \\ \text{s.t.} & 1 - 2x_1^2 - x_2^2 \geqslant 0 \end{array}$$

考虑对数障碍函数, 求出问题的解.

【解:】构造障碍函数

$$B(\boldsymbol{x}; r) = 2x_1 + 3x_2 - r\ln(1 - 2x_1^2 - x_2^2).$$

$$\begin{cases} \frac{\partial B}{\partial x_1} = 2 + \frac{4rx_1}{1-2x_1^2-x_2^2} = 0, \\ \frac{\partial B}{\partial x_2} = 3 + \frac{2rx_2}{1-2x_1^2-x_2^2} = 0. \end{cases}$$

解之得,

$$\begin{cases} x_1 = \frac{r+\sqrt{r^2+11}}{11}, \\ x_2 = \frac{3(r+\sqrt{r^2+11})}{11} \end{cases} \qquad \begin{cases} x_1 = \frac{r-\sqrt{r^2+11}}{11}, \\ x_2 = \frac{3(r-\sqrt{r^2+11})}{11} \end{cases}$$

由于第一组解不在可行域内, 故而取

$$\begin{cases} x_1(r) = \frac{r-\sqrt{r^2+11}}{11}, \\ x_2(r) = \frac{3(r-\sqrt{r^2+11})}{11}. \end{cases}$$

令$r \to 0$, 得解$\boldsymbol{x}^* = \left(-\frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}}\right)$.

附:求Lagrange 乘子的方法

方法一: 利用内点法乘子迭代公式$\lambda_i = \lim_{r\to 0} r/c_i(\boldsymbol{x}(r))$.

$$\lambda = \lim_{r\to 0} \frac{r}{c(\boldsymbol{x}(r))} = \lim_{r\to 0} \frac{r+\sqrt{r^2+11}}{2} = \frac{\sqrt{11}}{2}.$$

方法二: 利用KKT 条件解方程(注:若有多个约束条件, 由于非有效约束的乘子必为0,故实际上出现在方程中的只有有效约束的乘子)

$$\nabla f(\boldsymbol{x}^*) - \lambda \nabla c(x) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \lambda \begin{pmatrix} \frac{4}{\sqrt{11}} \\ \frac{6}{\sqrt{11}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

解得$\lambda = \frac{\sqrt{11}}{2}$.

23. 写出如下不等式约束最优化问题

$$\begin{aligned} &\min\ x_1^2+x_2^2, \\ &s.t.\ x_1+x_2-2\geq 0. \end{aligned}$$

的外罚函数.

【按定义写,略】

24. 写出如下不等式约束最优化问题

$$\begin{aligned} &\min\ -x_1x_2 \\ &s.t.\ c_1=-x_1-x_2^2+1\geq 0, \\ &\qquad c_2=x_1+x_2\geq 0. \end{aligned}$$

的外罚函数.

【按定义写,略】

25. 用障碍函数法

$$\begin{aligned} &\min\ f(x_1,x_2)=\frac{1}{3}(x_1+1)^3+x_2 \\ &s.t.\ 1-x_1\leq 0, \\ &\qquad x_2\geq 0. \end{aligned}$$

【解:】见课件第四章例题3.5.

26. 用内罚函数法求如下问题的最优点。

$$\begin{aligned} &\min\quad f(\boldsymbol{x})=x_1^2+x_2^2 \\ &s.t.\quad x_1-x_2+1\leq 0 \end{aligned}$$

【解: 注意不等号的方向】

构造内罚函数(障碍函数)

$$B(\boldsymbol{x},\sigma)=x_1^2+x_2^2-r\ln(-x_1+x_2-1).$$

令

$$\begin{cases} \frac{\partial B}{\partial x_1}=2x_1+\frac{r}{-x_1+x_2-1}=0, \\ \frac{\partial B}{\partial x_2}=2x_2-\frac{r}{-x_1+x_2-1}=0. \end{cases}$$

两式相加, 得$x_1+x_2=0$. 将其回代入第一式并求解有

$$x_1=-\frac{2\pm\sqrt{4+16r}}{8}.$$

那么

$$x_2=-x_1=\frac{2\pm\sqrt{4+16r}}{8}.$$

如果取

$$x_1 = -\frac{2-\sqrt{4+16r}}{8},$$

则

$$x_2 = -x_1 = \frac{2-\sqrt{4+16r}}{8}.$$

那么约束条件

$$x_1 - x_2 + 1 = \frac{\sqrt{4+16r}}{4} + 1 > 0$$

这说明该解不在可行域内部，不符合内点法的要求. 从而舍弃这个解.

因此取

$$x_1 = -\frac{2+\sqrt{4+16r}}{8},$$

则

$$x_2 = -x_1 = \frac{2+\sqrt{4+16r}}{8}.$$

令$r \to 0$, 得解

$$\boldsymbol{x}^* = (-1/2, 1/2).$$

【注：】求Lagrange 乘子？

27. 用增广Lagrange 函数法求解等式约束最优化问题

$$\begin{aligned} &\min\ x_1^2 + x_2^2, \\ &s.t.\ x_1 + x_2 - 2 = 0. \end{aligned}$$

【解：】见课件第四章例题4.2.

28. 用增广Lagrange 函数法求解等式约束最优化问题

$$\begin{aligned} &\min f(\boldsymbol{x}) = 2x_1^2 - x_2^2 + x_1 - x_2 \\ &\text{s.t.} \quad x_1 - x_2 = 0 \end{aligned}$$

【解:】构造增广Lagrange 函数

$$M(\boldsymbol{x}, \lambda; \sigma) = 2x_1^2 - x_2^2 + x_1 - x_2 - \lambda(x_1 - x_2) + \frac{\sigma}{2}(x_1 - x_2)^2.$$

$$\begin{cases} \frac{\partial M}{\partial x_1} = 4x_1 + 1 - \lambda + \sigma(x_1 - x_2) = 0, \\ \frac{\partial M}{\partial x_2} = -2x_2 - 1 + \lambda - \sigma(x_1 - x_2) = 0. \end{cases}$$

解之得

$$\begin{cases} x_1 = \frac{\lambda - 1}{4 - \sigma}, \\ x_2 = \frac{2\lambda - 2}{4 - \sigma}. \end{cases}$$

将解代入约束条件, 得$\lambda = 1$. 从而得问题的解为$\boldsymbol{x}^* = (0, 0)^T$.

29. 写出不等式约束最优化问题

$$\begin{aligned} &\min\ x_1^2 + x_2^2, \\ &s.t.\ x_1 + x_2 - 2 \geq 0. \end{aligned}$$

的增广Lagrange 函数.

【按定义写,略】

30. 用增广Lagrange 函数法求解等式约束最优化问题

$$\begin{aligned} &\min f(\boldsymbol{x}) = 2x_1^2 - x_2^2 + x_1 - x_2 \\ &\text{s.t.} \quad x_1 - x_2 \leq 0 \end{aligned}$$

(写出增广Lagrange 函数，不需求解).

【按定义写,略】

31. 用有效集法求解

$$\begin{aligned} \min\ & q(\boldsymbol{x}) = (x_1 - 1)^2 + (x_2 - 2.5)^2 \\ \text{s.t.}\ & x_1 - 2x_2 + 2 \geq 0 \\ & -x_1 - 2x_2 + 6 \geq 0 \\ & -x_1 + 2x_2 + 2 \geq 0 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

初始点为$\boldsymbol{x}_0 = (2, 0)^T$. (至少能迭代一次)

【见课件例6.2】