1. 7^{2046} 十进制

$7^1 \equiv 7 \pmod{10}$

$7^2 \equiv 9 \pmod{10}$

$7^3 \equiv 3 \pmod{10}$

$7^4 \equiv 1 \pmod{10}$

$7^{2046} \equiv (7^4)^{511} \cdot 7^2 \equiv 7^2 \equiv 9 \pmod{10}$

$\therefore$ 个位数是9

2. $1^5 + 2^5 + \cdots + 99^5$ 被4除的余数

$\because (2n)^5 = 32n^5 \quad 4 \mid 32n^5$

$\therefore$ 偶数项无余数

$1^5 \equiv 1 \pmod 4$

$3^5 \equiv (4-1)^5 \equiv -1 \pmod 4$

$5^5 \equiv (4+1)^5 \equiv 1 \pmod 4$

$7^5 \equiv (4\times 2-1)^5 \equiv -1 \pmod 4$

$9^5 \equiv (4\times 2+1)^5 \equiv 1 \pmod 4$

$\vdots$

$99^5 \equiv (4\times 25-1)^5 \equiv -1 \pmod 4$

原式 $\equiv 1+0+(-1)+1+0+(-1)$

$+\cdots+1+0+(-1) \pmod 4$

$\equiv 0 \pmod 4$ 即无余数

3. 555^{555} 除以7的余数.

$555^{1} \equiv 2 \pmod 7$

$555^{2} \equiv 4 \pmod 7$

$555^{3} \equiv 1 (1 \bmod 7)$

$555^{555} \equiv (555^{3})^{185} \equiv 1 \pmod 7$

4. $r_i \equiv 1 \pmod 3$

$r_i = 3t+1 \quad t=0,1,\ldots,n$.

所求完全剩余系为

$\{1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31\}$

5. (1) $24 = 2^{3} \times 3^{1}$

$\varphi(24) = 24 \times (1-\frac{1}{2}) \times (1-\frac{1}{3}) = 8$

(2) $360 = 2^{3} \times 3^{2} \times 5^{1}$

$\varphi(360) = 360 \times (1-\frac{1}{2}) \times (1-\frac{1}{3}) \times (1-\frac{1}{5})$

$= 96$

6. (1) $a \equiv 9^{794} \pmod{73}$

$\equiv (9^{72})^{11} \cdot 9^{2} \pmod{73}$

由费马小定理 $9^{72} \equiv 1 \pmod{73}$

$\therefore$ 原式 $\equiv (73+8) \pmod{73}$

$\equiv 8 \pmod{73}$

$\therefore a = 8$

6. (2) $x^{86} \equiv 6 \pmod{29}$

由费马小定理得 $x^{28} \equiv 1 \pmod{29}$

$x^{86} \equiv (x^{28})^{3} \cdot x^{2} \equiv 6 \pmod{29}$

$x^{2} \equiv 6 \pmod{29}$

$x^{2} \equiv 64 \pmod{29}$

$x \equiv 8$ 或 $x \equiv 21 \pmod{29}$

$\therefore x = 8$ 或 $x = 21$

7. (1) $27x \equiv 12 \pmod{15}$

$9x \equiv 4 \pmod{5}$

$4x \equiv 4 \pmod{5}$

$\because (4, 5) = 1$

$\therefore x \equiv 1 \pmod{5}$

$\therefore x = 1, 6, 11$

7. (2) $24x \equiv 6 \pmod{81}$

$8x \equiv 2 \pmod{27}$

$\because (2, 27) = 1$

$\therefore 4x \equiv 1 \pmod{27}$

$x \equiv 7 \pmod{27}$

$\therefore x = 7, 34, 61$

8. $y \equiv 7x+3 \pmod{26}$

$7x \equiv y-3 \pmod{26}$

$\varphi(26) = 26\times(1-\frac{1}{2})\times(1-\frac{1}{13}) = 12$

$\therefore\ 7^{12} \equiv 1 \pmod{26}$

$\therefore\ 7^{12}\cdot x \equiv 7^{11}(y-3) \pmod{26}$

$x \equiv 7\cdot(52-3)^5\ (y-3) \pmod{26}$

$\equiv 7\cdot(-3)\cdot(78+3)\ (y-3) \pmod{26}$

$= 15\ (y-3) \pmod{26}$

9.(1) $\begin{cases} x \equiv 9 \pmod{12} \\ x \equiv 6 \pmod{25} \end{cases}$

$M_1 = 25 \quad M_2 = 12$

$M_1' M_1 \equiv 1 \pmod{12}$

$M_1' \equiv 1 \pmod{12}$

$M_2' M_2 \equiv 1 \pmod{25}$

$M_2' \equiv 23 \pmod{25}$

$$x \equiv 25 \times 1 \times 9 + 23 \times 12 \times 6 \pmod{300}$$

$$\equiv 1881 \pmod{300}$$

$$\equiv 81 \pmod{300}$$

9(2). $\begin{cases} x \equiv 2 \pmod{9} \\ 3x \equiv 4 \pmod{5} \\ 4x \equiv 3 \pmod{7} \end{cases}$

$m_1 = 35 \quad m_2 = 63 \quad m_3 = 45$

$m_1 m_1' \equiv 1 \pmod{9}$

$m_1' \equiv 8 \pmod{9}$

$m_2 m_2' \equiv 1 \pmod{5}$

$m_2' \equiv 2 \pmod{5}$

$m_3 m_3' \equiv 1 \pmod{7}$

$m_3' \equiv 5 \pmod{7}$

即 $M_1' = 8 \quad M_2' = 2 \quad M_3' = 5$

$$\begin{cases} a \equiv 2 \pmod 9 \\ 3b \equiv 4 \pmod 5 \quad b \equiv 3 \pmod 5 \\ 4c \equiv 3 \pmod 7 \quad c \equiv 6 \pmod 7 \end{cases}$$

$\therefore x \equiv 35 \times 8 \times 2 + 63 \times 2 \times 4 + 45 \times 5 \times 3 \pmod{315}$

$\quad 2288 \pmod{315}$

$\equiv 83 \pmod{315}$

10、 $\begin{pmatrix} 3 & 14 \\ 2 & 19 \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} = \begin{pmatrix} 1 & 14 \\ 11 & 21 \end{pmatrix}$

$$\begin{cases} 3k_{11} + 14k_{21} = 1 \\ 2k_{11} + 19k_{21} = 11 \\ 3k_{12} + 14k_{22} = 14 \\ 2k_{12} + 19k_{22} = 21 \end{cases} \quad 解得 \begin{cases} k_{11} = -\frac{135}{29} \\ k_{12} = -\frac{28}{29} \\ k_{21} = \frac{31}{29} \\ k_{22} = \frac{35}{29} \end{cases}$$

$$\therefore K = \begin{pmatrix} -\frac{135}{29} & -\frac{28}{29} \\ \frac{31}{29} & \frac{35}{29} \end{pmatrix}$$

证明题：

$p \to q$

1. 令 $n = a_0 + 10a_1 + 100a_2 + \cdots + 10^n a_n$ p充分.

① 若 $3 \mid n$

∵ $(3,1) = (3,10) = (3,100) = \cdots = (3,10^n) = 1$

∴ $3 \mid a_0 + a_1 + \cdots a_n$ 证毕

② 若 $3 \mid a_0 + a_1 + \cdots + a_n$

∵ $3 \mid a_0 + a_1 + \cdots + a_n$

∵ $(3,1) = (3,10) = (3,100) = \cdots = (3,10^n) = 1$

∴ $3 \mid a_0 + 10a_1 + \cdots + 10^n a_n$

2. 证：

∵ 模 m 的完全剩余系有 m 个元素

且 $0^2, 1^2 \cdots (m-1)^2$ 共有 m 个

$$m-1 \equiv -1 \pmod m$$

$$(m-1)^2 \equiv 1 \equiv 1^2 \pmod m$$

∴ 仅有 $0^2, \cdots (m-1)^2$ 这 m 个数多无法构成完全剩余系

证毕.

3. $(a,3)=1 \qquad a^7 \equiv a \pmod{63}$

由费马小定理 $a^7 \equiv a \pmod 7$

$\therefore 9a^7 \equiv 9a \pmod{63}$

$\varphi(63) = 63 \times (1-\frac{1}{3}) \times (1-\frac{1}{7}) = 36$

$9^{36} \equiv 1 \pmod{63}$

$\therefore 9^{36} \cdot a^7 \equiv 9^{36} \cdot a \pmod{63}$

$\therefore a^7 \equiv a \pmod{63}$

4. $\varphi(m) = m \cdot (1-\frac{1}{p_1}) \cdot (1-\frac{1}{p_2}) \cdots (1-\frac{1}{p_n})$

由算术基本定理

$p_n = 2k+1 \quad k = 0, 1 \cdots n$

$(1-\frac{1}{p_1}) \cdots (1-\frac{1}{p_n}) = \frac{2k_1}{2k_1+1} \cdot \frac{2k_2}{2k_2+1} \cdots \frac{2k_n}{2k_n+1}$

$= \frac{2^n k_1 \cdots k_n}{(2k_1+1) \cdots (2k_n+1)}$

$\therefore \varphi(m) = 2^n \cdot \frac{m \cdot k_1 \cdots k_n}{(2k_1+1) \cdots (2k_n+1)}$

$\therefore 2 \mid \varphi(m)$ 证毕

5、 p素数 $0<k<p$

证明： $(p-k)!(k-1)! \equiv (-1)^k \pmod p$

∵ p是素数， $0<k<p$

∴ $(k,p)=1$

$$p-k \equiv -k \pmod p$$

$$p-(k+1) \equiv -(k+1) \pmod p$$

$$\vdots$$

$$1 \equiv -(p-1) \pmod p$$

上式相乘，得

$$(p-k)! \equiv (-1)^{p-k} \cdot \frac{(p-1)!}{(k-1)!} \pmod p.$$

∴ $(p-k)!(k-1)! \equiv (-1)^{p-k} \cdot (p-1)! \pmod p.$

由威尔逊定理 $(p-1)! \equiv -1 \pmod p$

∴ $(p-k)!(k-1)! \equiv (-1)^{p+1-k} \pmod p$

∵ $2 \nmid p$ ∴ $2 \mid (p+1)$

∴ $(p-k)!(k-1)! \equiv (-1)^{-k} \equiv (-1)^k \pmod p$

证毕