Math 205 - Second Practice Exam - 2019

1. Let

$$A = \begin{pmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ 3 & 3 & 4 & 4 & 0 & 0 \\ 4 & 4 & 5 & 5 & 6 & 6 \end{pmatrix}.$$

(1) Find a basis and the dimension of the null space of A.
(2) Find a basis and the dimension of the row space of A.

2. Let $\mathbf{V}$ be a three-dimensional vector space with the basis

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.$$

Let $T : \mathbf{V} \to \mathbf{V}$ be a linear transformation, such that

$$\begin{aligned} T(\mathbf{v}_1) &= \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3, \\ T(\mathbf{v}_2) &= 11\mathbf{v}_1 + 12\mathbf{v}_2 + 13\mathbf{v}_3, \\ T(\mathbf{v}_3) &= 21\mathbf{v}_1 + 22\mathbf{v}_2 + 23\mathbf{v}_3. \end{aligned}$$

(1) Find $T(6\mathbf{v}_1 + 7\mathbf{v}_2 - 8\mathbf{v}_3)$.
(2) Find the matrix of linear transformation relative to the basis $\mathcal{B}$, that is, $[T]_{\mathcal{B}}$.

3. Define the following subspace of $\mathbb{R}^3$:

$$\mathbf{W} = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}.$$

We know that

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$$

is a basis of $\mathbf{W}$.

(1) Is $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \in \mathbf{W}$? If so, then find the coordinate vector $[\mathbf{v}_1]_{\mathcal{B}}$.

(2) Is $\mathbf{v}_2 = \begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix} \in \mathbf{W}$? If so, then find the coordinate vector $[\mathbf{v}_2]_{\mathcal{B}}$.

4. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation, given by

$$T \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} 3 & 4 & -1 & 5 \\ 1 & 2 & -1 & 3 \\ -2 & -2 & 2 & -7 \\ -4 & -3 & -2 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix}.$$

Let

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\},$$

be a basis of $\mathbb{R}^4$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix},$$
$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

(1) Let the vector coordinate $[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 2 \\ 3 \\ 3 \end{pmatrix}$. Find the vector $\mathbf{v}$.

(2) Let the vector $\mathbf{v} = \begin{pmatrix} 18 \\ -4 \\ 0 \\ -2 \end{pmatrix}$. Find the vector coordinate $[\mathbf{v}]_{\mathcal{B}}$.

(3) Find $[T]_{\mathcal{B}}$.

5. Let

$$A = \begin{pmatrix} 1 & 4 & 1 \\ 3 & 2 & 1 \\ 3 & 4 & -1 \end{pmatrix}.$$

(1) Find all eigenvalues and all corresponding eigenvectors of A.
(2) Find a diagonal matrix D and an invertible matrix T, such that $T^{-1}AT = D$.

6. Solve the following nonhomogeneous linear differential equations

$$\begin{array}{ll}
(1) & y'' - 24y' + 169y = -13\cos(13x) + 13\sin(13x), \\
(2) & y'' - 15y' + 56y = -e^{7x} + e^{8x}, \\
(3) & y'' - 16y' + 64y = 12xe^{8x} + 24x^2e^{8x}, \\
(4) & y'' + 100y = 718 + 800x + 900x^2, \\
(5) & y''' - y'' + y' - y = 24x - 12x^2 + 4x^3 - x^4, \\
(6) & y''' - 3y'' + 3y' - y = 24xe^{x}, \\
(7) & (D-7)(D-8)(D^2+25)y = e^{7x} + e^{8x} + \cos(5x) + \sin(5x), \\
(8) & (D^2+25)^2 y = \cos(5x) + \sin(5x) + x\cos(5x) + x\sin(5x).
\end{array}$$

Moreover, find the particular solution of equation (2) if the initial conditions $y(0) = 7$ and $y'(0) = 55$ are given.