Math 205 - Second Practice Exam - 2019

1. Let

$$A = \begin{pmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ 3 & 3 & 4 & 4 & 0 & 0 \\ 4 & 4 & 5 & 5 & 6 & 6 \end{pmatrix}.$$

(1) Find a basis and the dimension of the null space of A.
(2) Find a basis and the dimension of the row space of A.
Solutions: Let us perform elementary row operations to the matrix A. We have

$$\begin{aligned} &\begin{pmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ 3 & 3 & 4 & 4 & 0 & 0 \\ 4 & 4 & 5 & 5 & 6 & 6 \end{pmatrix} \\ \to &\begin{pmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 & 0 \\ 0 & 0 & 5 & 5 & 6 & 6 \end{pmatrix} \\ \to &\begin{pmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 6 \end{pmatrix} \\ \to &\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \end{aligned}$$

The system of equations corresponding to the last matrix is

$$\begin{aligned} x_1 + x_2 &= 0, \\ x_3 + x_4 &= 0, \\ x_5 + x_6 &= 0. \end{aligned}$$

Let us rewrite this system in a better way. We have

$$\begin{aligned} x_2 &= -x_1, \\ x_4 &= -x_3, \\ x_6 &= -x_5. \end{aligned}$$

Therefore, the solutions of the system of equations are given by

$$\begin{aligned}
&\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \\
= & \begin{pmatrix} x_1 \\ -x_1 \\ x_3 \\ -x_3 \\ x_5 \\ -x_5 \end{pmatrix} \\
= & \ x_1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix},
\end{aligned}$$

where x_1, x_3, x_5 are free variables. Therefore, we find a basis for the null space

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

The dimension of the null space: $\dim NS(A) = 3$.

Moreover, we find a basis of the row space

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \right\}.$$

The dimension of the row space: $\dim RS(A) = 3$.

2. Let $\mathbf{V}$ be a three-dimensional vector space with the basis

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.$$

Let $T : \mathbf{V} \to \mathbf{V}$ be a linear transformation, such that

$$\begin{aligned} T(\mathbf{v}_1) &= \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3, \\ T(\mathbf{v}_2) &= 11\mathbf{v}_1 + 12\mathbf{v}_2 + 13\mathbf{v}_3, \\ T(\mathbf{v}_3) &= 21\mathbf{v}_1 + 22\mathbf{v}_2 + 23\mathbf{v}_3. \end{aligned}$$

(1) Find $T(6\mathbf{v}_1 + 7\mathbf{v}_2 - 8\mathbf{v}_3)$.
(2) Find the matrix of linear transformation relative to the basis $\mathcal{B}$, that is, $[T]_{\mathcal{B}}$.

Solutions: (1) By using the properties of linear transformations and the given information, we have

$$\begin{aligned} & T(6\mathbf{v}_1 + 7\mathbf{v}_2 - 8\mathbf{v}_3) = 6T(\mathbf{v}_1) + 7T(\mathbf{v}_2) - 8T(\mathbf{v}_3) \\ = \ & 6(\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3) + 7(11\mathbf{v}_1 + 12\mathbf{v}_2 + 13\mathbf{v}_3) - 8(21\mathbf{v}_1 + 22\mathbf{v}_2 + 23\mathbf{v}_3) \\ = \ & -85\mathbf{v}_1 - 80\mathbf{v}_2 - 75\mathbf{v}_3. \end{aligned}$$

(2) From the given information, we have

$$(T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)) = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \begin{pmatrix} 1 & 11 & 21 \\ 2 & 12 & 22 \\ 3 & 13 & 23 \end{pmatrix}.$$

Therefore, we have

$$[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 11 & 21 \\ 2 & 12 & 22 \\ 3 & 13 & 23 \end{pmatrix}.$$

3. Define the following subspace of $\mathbb{R}^3$:

$$\mathbf{W} = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}.$$

We know that

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$$

is a basis of $\mathbf{W}$.

(1) Is $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \in \mathbf{W}$? If so, then find the coordinate vector $[\mathbf{v}_1]_{\mathcal{B}}$.

(2) Is $\mathbf{v}_2 = \begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix} \in \mathbf{W}$? If so, then find the coordinate vector $[\mathbf{v}_2]_{\mathcal{B}}$.

Solutions: (1) Let

$$\begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

for two real constants α and β. However, there exists no solution to this system. Therefore

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \notin \mathbf{W}.$$

(2) Let

$$\begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix},$$

for two real constants α and β. Solving the system, we find the unique solution

$$\alpha = -3, \qquad \beta = 3.$$

Therefore, we have

$$\mathbf{v}_2 = \begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix} \in \mathbf{W}, \qquad [\mathbf{v}_2]_{\mathcal{B}} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}.$$

4. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation, given by

$$T\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} 3 & 4 & -1 & 5 \\ 1 & 2 & -1 & 3 \\ -2 & -2 & 2 & -7 \\ -4 & -3 & -2 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix}.$$

Let

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\},$$

be a basis of $\mathbb{R}^4$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix},$$

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

(1) Let the vector coordinate $[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 2 \\ 3 \\ 3 \end{pmatrix}$. Find the vector $\mathbf{v}$.

(2) Let the vector $\mathbf{v} = \begin{pmatrix} 18 \\ -4 \\ 0 \\ -2 \end{pmatrix}$. Find the vector coordinate $[\mathbf{v}]_{\mathcal{B}}$.

(3) Find $[T]_{\mathcal{B}}$.

Solutions: (1) This is easy, because

$$\begin{aligned} & \mathbf{v} = 2\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3 + 3\mathbf{v}_4 \\ = \;\; & 2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \\ = \;\; & \begin{pmatrix} 10 \\ -2 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

(2) Let

$$\begin{pmatrix} 18 \\ -4 \\ 0 \\ -2 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + \gamma\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + \delta\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix},$$

for real constants α, β, γ, δ. Solving the system, we obtain the unique solution

$$\alpha = 3, \qquad \beta = 4, \qquad \gamma = 5, \qquad \delta = 6.$$

Therefore

$$[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}.$$

(3) By using the properties of matrices for linear transformations, we have

$$\begin{aligned}
[T]_{\mathcal{B}} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}^{-1} \\
&\cdot \begin{pmatrix} 3 & 4 & -1 & 5 \\ 1 & 2 & -1 & 3 \\ -2 & -2 & 2 & -7 \\ -4 & -3 & -2 & -8 \end{pmatrix} \\
&\cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.
\end{aligned}$$

5. Let

$$A = \begin{pmatrix} 1 & 4 & 1 \\ 3 & 2 & 1 \\ 3 & 4 & -1 \end{pmatrix}.$$

(1) Find all eigenvalues and all corresponding eigenvectors of A.
(2) Find a diagonal matrix D and an invertible matrix T, such that $T^{-1}AT = D$.
Solution: Performing elementary row operations to the matrix $A - \lambda I$, we have

$$\begin{aligned} A - \lambda I = & \begin{pmatrix} 1-\lambda & 4 & 1 \\ 3 & 2-\lambda & 1 \\ 3 & 4 & -1-\lambda \end{pmatrix} \\ \to & \begin{pmatrix} 1-\lambda & 4 & 1 \\ \lambda+2 & -\lambda-2 & 0 \\ \lambda+2 & 0 & -\lambda-2 \end{pmatrix}. \end{aligned}$$

Now let us compute the determinant $A - \lambda I$:

$$\begin{aligned} \det(A - \lambda I) &= \begin{pmatrix} 1-\lambda & 4 & 1 \\ \lambda+2 & -\lambda-2 & 0 \\ \lambda+2 & 0 & -\lambda-2 \end{pmatrix} \\ &= (\lambda+2)^2 \det \begin{pmatrix} 1-\lambda & 4 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\ &= (\lambda+2)^2 \det \begin{pmatrix} 6-\lambda & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\ &= -(\lambda+2)^2(\lambda-6). \end{aligned}$$

There are two eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 6$. The eigenvectors are given by

$$\begin{aligned} & p \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + q \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}, \qquad \text{for } \lambda = -2, \\ & r \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \text{for } \lambda = 6, \end{aligned}$$

where p, q, r are real constants, such that $(p, q) \neq (0, 0)$ and $r \neq 0$. The eigenspaces are

$$\begin{aligned} E_{\lambda=-2} &= \operatorname{span} \left\{ \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} \right\}, \\ E_{\lambda=6} &= \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

Let

$$T = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 3 & 4 & 1 \end{pmatrix}.$$

Then

$$T^{-1}AT = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

6. Solve the following nonhomogeneous linear differential equations

$$
\begin{array}{ll}
(1) & y'' - 24y' + 169y = -13\cos(13x) + 13\sin(13x), \\
(2) & y'' - 15y' + 56y = -e^{7x} + e^{8x}, \\
(3) & y'' - 16y' + 64y = 12xe^{8x} + 24x^2e^{8x}, \\
(4) & y'' + 100y = 718 + 800x + 900x^2, \\
(5) & y''' - y'' + y' - y = 24x - 12x^2 + 4x^3 - x^4, \\
(6) & y''' - 3y'' + 3y' - y = 24xe^x, \\
(7) & (D-7)(D-8)(D^2+25)y = e^{7x} + e^{8x} + \cos(5x) + \sin(5x), \\
(8) & (D^2+25)^2y = \cos(5x) + \sin(5x) + x\cos(5x) + x\sin(5x).
\end{array}
$$

Moreover, find the particular solution of equation (2) if the initial conditions $y(0) = 7$ and $y'(0) = 55$ are given.
Solutions: The auxiliary equations of the corresponding homogeneous differential equations and their solutions are given by, respectively

$$
\begin{array}{ll}
(1) & \lambda^2 - 24\lambda + 169 = (\lambda-12)^2 + 5^2 = 0, \qquad \lambda_1 = 12 + 5\mathrm{i}, \lambda_2 = 12 - 5\mathrm{i}, \\
(2) & \lambda^2 - 15\lambda + 56 = (\lambda-7)(\lambda-8) = 0, \qquad \lambda_1 = 7, \lambda_2 = 8, \\
(3) & \lambda^2 - 16\lambda + 64 = (\lambda-8)^2 = 0, \qquad \lambda_1 = \lambda_2 = 8, \\
(4) & \lambda^2 + 100 = 0, \qquad \lambda_1 = 10\mathrm{i}, \lambda_2 = -10\mathrm{i}, \\
(5) & \lambda^3 - \lambda^2 + \lambda - 1 = (\lambda-1)(\lambda^2+1), \lambda_1 = \mathrm{i}, \lambda_2 = -\mathrm{i}, \lambda_3 = 1, \\
(6) & \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda-1)^3 = 0, \lambda = 1, \\
(7) & (\lambda-7)(\lambda-8)(\lambda^2+25) = 0, \lambda_1 = 7, \lambda_2 = 8, \lambda_3 = 5\mathrm{i}, \lambda_4 = -5\mathrm{i}, \\
(8) & (\lambda^2+25)^2 = 0, \lambda_1 = 5\mathrm{i}, \lambda_2 = -5\mathrm{i}.
\end{array}
$$

The solutions of the homogeneous differential equations are given by

$$
\begin{array}{ll}
(1) & y_c(x) = C_1\exp(12x)\cos(5x) + C_2\exp(12x)\sin(5x), \\
(2) & y_c(x) = C_1e^{7x} + C_2e^{8x}, \\
(3) & y_c(x) = C_1e^{8x} + C_2xe^{8x}, \\
(4) & y_c(x) = C_1\cos(10x) + C_2\sin(10x), \\
(5) & y_c(x) = C_1\cos(x) + C_2\sin(x) + C_3e^x, \\
(6) & y_c(x) = C_1e^x + C_2xe^x + C_3x^2e^x, \\
(7) & y_c(x) = C_1e^{7x} + C_2e^{8x} + C_3\cos(5x) + C_4\sin(5x), \\
(8) & y_c(x) = C_1\cos(5x) + C_2\sin(5x) + C_3x\cos(5x) + C_4x\sin(5x),
\end{array}
$$

where C_1, C_2, C_3 and C_4 are real constants. The particular solutions of these differential equations are given by

$$
\begin{aligned}
&(1) \qquad y_p(x) = \frac{1}{24}\cos(13x) + \frac{1}{24}\sin(13x),\\
&(2) \qquad y_p(x) = xe^{7x} + xe^{8x},\\
&(3) \qquad y_p(x) = 2x^3e^{8x} + 2x^4e^{8x},\\
&(4) \qquad y_p(x) = 7 + 8x + 9x^2,\\
&(5) \qquad y_p(x) = x^4,\\
&(6) \qquad y_p(x) = x^4e^x,\\
&(7) \qquad y_p(x) = x^2e^{7x} + x^2e^{8x} + x^2\cos(5x) + x^2\sin(5x),\\
&(8) \qquad y_p(x) = x^2\cos(5x) + x^2\cos(5x).
\end{aligned}
$$

Therefore, the general solutions of these differential equations are given by, respectively

$$
\begin{aligned}
&(1) \qquad y(x) = C_1\exp(12x)\cos(5x) + C_2\exp(12x)\sin(5x)\\
&\qquad\quad +\frac{1}{24}\cos(13x) + \frac{1}{24}\sin(13x),\\
&(2) \qquad y(x) = C_1e^{7x} + C_2e^{8x} + xe^{7x} + xe^{8x},\\
&(3) \qquad y(x) = C_1e^{8x} + C_2xe^{8x} + 2x^3e^{8x} + 2x^4e^{8x},\\
&(4) \qquad y(x) = C_1\cos(10x) + C_2\sin(10x) + 7 + 8x + 9x^2,\\
&(5) \qquad y(x) = C_1\cos(x) + C_2\sin(x) + C_3e^x + x^4,\\
&(6) \qquad y(x) = C_1e^x + C_2x\exp + C_3x^2e^x + x^4e^x,\\
&(7) \qquad y(x) = C_1e^{7x} + C_2e^{8x} + C_3\cos(5x) + C_4\sin(5x) + x^2e^{7x} + x^2e^{8x} + x^2\cos(5x) + x^2\sin(5x),\\
&(8) \qquad y(x) = C_1\cos(5x) + C_2\sin(5x) + C_3x\cos(5x) + C_4x\sin(5x) + x^2\cos(5x) + x^2\sin(5x),
\end{aligned}
$$

where C_1, C_2, C_3 and C_4 are real constants.

To find the constants C_1 and C_2 of the initial value problem for equation (2), let us compute the derivative of the general solution. We have

$$y'(x) = 7C_1e^{7x} + 8C_2e^{8x} + 7xe^{7x} + 8xe^{8x} + e^{7x} + e^{8x}.$$

Let $x = 0$ in both $y(x)$ and $y'(x)$ and use the initial conditions, we get

$$C_1 + C_2 = 7, \qquad 7C_1 + 8C_2 + 2 = 55.$$

Solving the system, we find the unique solution

$$C_1 = 3, \qquad C_2 = 4.$$

Therefore, the solution of the initial value problem is

$$y(x) = 3e^{7x} + 4e^{8x} + xe^{7x} + xe^{8x}.$$