Math 205 Exam 1 Answer Key

1. Solve the linear system
$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 5 & 6 & 1 \\ 3 & 5 & -1 & 9 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$
(As a check on your arithmetic, this system is consistent and the matrix of this system has rank 2.)

Solution:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 0 \\ 2 & 5 & 6 & 1 & 1 \\ 3 & 5 & -1 & 9 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 0 \\ 0 & 1 & 4 & -3 & 1 \\ 0 & -1 & -4 & 3 & -1 \end{array}\right] \qquad A_{1,2}(-2), A_{1,3}(-3)$$
$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 0 \\ 0 & 1 & 4 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad A_{2,3}(1)$$
$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -7 & 8 & -2 \\ 0 & 1 & 4 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad A_{2,1}(-2)$$

Because the (reduced) row echelon form of the coefficient matrix has two leading ones, the rank is 2.

The reduced row echelon form of the augmented matrix corresponds to the linear system
$$\begin{array}{rrrrr} x_1 & & -\,7x_3 & +\,8x_4 & = & -2 \\ & x_2 & +\,4x_3 & -\,3x_4 & = & 1 \end{array}$$

The variables x_3 and x_4 are free variables and the solution set is
$$S = \left\{ \begin{bmatrix} -2 + 7x_3 - 8x_4 \\ 1 - 4x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} \middle| \, x_3, x_4 \in \mathbb{R} \right\}.$$

2. Let

$$A = \begin{pmatrix} 1 & 1 & 3 & 1 \\ 3 & 3 & 11 & 3 \\ 4 & 5 & 13 & 4 \\ 2 & 2 & 6 & 3 \end{pmatrix}.$$

Find the inverse matrix of A and then solve the system

$$A\mathbf{x} = \begin{pmatrix} 2 \\ 4 \\ 1 \\ 5 \end{pmatrix}.$$

Solution: Let us perform elementary row operations to the matrix (A, I), so that we may get (I, A^{-1}). We have

$$\begin{aligned}
&\begin{pmatrix} 1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 3 & 3 & 11 & 3 & 0 & 1 & 0 & 0 \\ 4 & 5 & 13 & 4 & 0 & 0 & 1 & 0 \\ 2 & 2 & 6 & 3 & 0 & 0 & 0 & 1 \end{pmatrix} \\
\to &\begin{pmatrix} 1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{pmatrix} \\
\to &\begin{pmatrix} 1 & 0 & 0 & 0 & 10 & -1 & -1 & -1 \\ 0 & 0 & 2 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{pmatrix} \\
\to &\begin{pmatrix} 1 & 0 & 0 & 0 & 10 & -1 & -1 & -1 \\ 0 & 0 & 2 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -5/2 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{pmatrix} \\
\to &\begin{pmatrix} 1 & 0 & 0 & 0 & -10 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -5/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1 & 0 & -3/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Therefore, we obtain the inverse matrix

$$A^{-1} = \begin{pmatrix} 10 & -1 & -1 & -1 \\ -5/2 & -1/2 & 1 & 0 \\ -3/2 & 1/2 & 0 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}.$$

The solution of the system $A\mathbf{x} = \mathbf{b}$ is given by

$$
\begin{aligned}
\mathbf{x} &= A^{-1}\mathbf{b} \\
&= \begin{pmatrix} 10 & -1 & -1 & -1 \\ -5/2 & -1/2 & 1 & 0 \\ -3/2 & 1/2 & 0 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \\ -1 \\ 1 \end{pmatrix}.
\end{aligned}
$$

3. $\begin{vmatrix} 0 & 5 & 6 & 7 \\ 2 & 4 & 6 & 5 \\ 4 & 8 & 12 & 13 \\ 0 & 10 & 14 & 13 \end{vmatrix} \xrightarrow{\text{Add } -2\cdot\text{row 2 to row 3}} \begin{vmatrix} 0 & 5 & 6 & 7 \\ 2 & 4 & 6 & 5 \\ 0 & 0 & 0 & 3 \\ 0 & 10 & 14 & 13 \end{vmatrix}$

Expand by minors of column 1

$$= -2 \begin{vmatrix} 5 & 6 & 7 \\ 0 & 0 & 3 \\ 10 & 14 & 13 \end{vmatrix}$$

Expand by minors of row 2

$$= (-2)(-3) \begin{vmatrix} 5 & 6 \\ 10 & 14 \end{vmatrix}$$

$$= (-2)(-3)\left(5\cdot 14 - 10\cdot 6\right) = 60$$

4. (a) S is linearly independent if the <u>only</u> linear combination of the vectors in S that is equal to 0 is the trivial one, i.e., if the equation

$$c_1 v_1 + \cdots + c_k v_k = 0$$

<u>only</u> has the solution $c_1 = \cdots = c_k = 0$.

(b) S spans V if <u>every</u> vector $v \in V$ can be expressed as a linear combination of the vectors in S, i.e., if the equation

$$c_1 v_1 + \cdots + c_k v_k = v$$

has a solution for <u>every</u> $v \in V$

(c) S is a basis of V if

(1) S is linearly independent

and (2) S spans V.

(5) (16 points) Let

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 9 \\ 7 \end{bmatrix} \right\}.$$

Determine whether S spans $\mathbb{R}^4$. If not, find a vector v in $\mathbb{R}^4$ that is not in the span of S.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 2 & b \\ 2 & 5 & 9 & c \\ 3 & 5 & 7 & d \end{array}\right) \xrightarrow{\text{Gaussian Elimination}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 2 & b \\ 0 & 0 & 1 & c-2a-b \\ 0 & 0 & 0 & d-3a+b \end{array}\right) = S'$$

Steps.

$-2R_1 + R_3 \to R_3$
$-3R_1 + R_4 \to R_4$
$R_2 + R_4 \to R_4$
$-R_2 + R_3 \to R_3$

Note: the notation $\alpha R_i + R_j \to R_j$ is the same as the book's $A_{ij}(\alpha)$.

$\therefore$ S is a set of three vectors in $\mathbb{R}^4$ so cannot span $\mathbb{R}^4$.

To find a vector NOT in the span of S we need only have an inconsistent system represented by S'. Therefore any vector for which $d-3a+b \neq 0$ will do. Let $a=b=c=0$, $d=1$.
So $v = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is not in the span of S.

6. Let

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} \right\}.$$

Determine whether S is linearly independent. If not, find a linear dependence relation.
Solution: To start, note that S consists of 4 vectors from $\mathbb{R}^3$. Thus, since $\mathbb{R}^3$ has dimension 3, the 4 vectors of S cannot be linearly independent.

Now, to determine a dependence relation we must form a matrix M whose columns are the vectors of S. Any nonzero element of the null space of M gives rise to a dependence relation between elements of S. So,

$$M = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 4 & 7 & 6 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

and applying the sequence of row operations

$$-2R_1 + R_2 \to R_2$$

$$-R_1 + R_3 \to R_3$$

$$\text{swap } R_2 \text{ and } R_3$$

$$-R_3 + R_2 \to R_2$$

$$-2R_2 + R_1 \to R_1$$

$$-3R_3 + R_1 \to R_1$$

results in the reduced matrix

$$\begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Thus, elements of the null space of M must be of the form

$$s \begin{bmatrix} 6 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

for $s \in \mathbb{R}$. Taking $s = 1$ gives the element

$$\begin{bmatrix} 6 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

of the null space of M as well as the dependence relation:

$$6 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} + (-2) \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Note, any nonzero choice of s would lead to a dependence relation in the above way.

Problem 7.

We use elementary row operations to reduce the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 10 & 12 \\ 4 & 6 & 12 & 15 \end{pmatrix} \xrightarrow[A_{13}(-3),\, A_{14}(-4)]{A_{12}(-2)} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 7 & 6 \\ 0 & 2 & 8 & 7 \end{pmatrix} \xrightarrow[A_{24}(-2)]{A_{23}(-2)} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{A_{34}(-2)} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since there is no zero rows and we have four vectors, the vectors spans $\mathbb{R}^4$. Also, since there are no free variables, the vectors are linearly independent. Hence, they are a basis of $\mathbb{R}^4$.

Remark: It is also available to compute determinant. However, proper explanation is necessary.