Homework Assignment 6

Due at the start of class, Weds. Oct. 9 (Profs. Zhang and Wu), Thurs. Oct. 10 (Profs. Coll, Weintraub, Recio-Mitter).

1. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis of a vector space V and let $T : V \longrightarrow V$ be the linear transformation with

$$T(\mathbf{v}_1) = 2\mathbf{v}_1 + 3\mathbf{v}_2,$$
$$T(\mathbf{v}_2) = 5\mathbf{v}_1 + 7\mathbf{v}_2.$$

Find $[T]_{\mathcal{B}}$, the matrix of T in the basis $\mathcal{B}$.

2. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis of a vector space V and let $T : V \longrightarrow V$ be the linear transformation with

$$T(\mathbf{v}_1) = 3\mathbf{v}_1 + 4\mathbf{v}_2 + 5\mathbf{v}_3,$$
$$T(\mathbf{v}_2) = 7\mathbf{v}_1 + 8\mathbf{v}_2 + 9\mathbf{v}_3,$$
$$T(\mathbf{v}_3) = -\mathbf{v}_1 - 2\mathbf{v}_2 - 3\mathbf{v}_3.$$

Find $[T]_{\mathcal{B}}$, the matrix of T in the basis $\mathcal{B}$.

3. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 16 \end{bmatrix} \right\}.$$

$\mathcal{B}$ is a basis of $\mathbb{R}^2$.

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation with

$$T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 9 \\ 31 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 5 \\ 16 \end{bmatrix} \right) = \begin{bmatrix} 38 \\ 123 \end{bmatrix}.$$

Find $[T]_{\mathcal{B}}$, the matrix of T in the basis $\mathcal{B}$ *directly from the definition of* $[T]_{\mathcal{B}}$.

4. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 11 \end{bmatrix} \right\}.$$

$\mathcal{B}$ is a basis of $\mathbb{R}^2$.

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation with

$$T\left(\begin{bmatrix} 1 \\ 6 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ -3 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 2 \\ 11 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 10 \end{bmatrix}.$$

Find $[T]_{\mathcal{B}}$, the matrix of T in the basis $\mathcal{B}$ *directly from the definition of* $[T]_{\mathcal{B}}$.

5. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} \right\}.$$

$\mathcal{B}$ is a basis of $\mathbb{R}^3$.

Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation given by $T(\mathbf{v}) = A\mathbf{v}$ where A is the matrix

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 6 & 9 \\ 0 & 2 & 5 \end{bmatrix}.$$

Find $[T]_{\mathcal{B}}$, the matrix of T in the basis $\mathcal{B}$. *You may leave your answer expressed as a product of matrices and their inverses.*

6. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 3 \end{bmatrix} \right\}.$$

$\mathcal{B}$ is a basis of $\mathbb{R}^3$.

Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation given by $T(\mathbf{v}) = A\mathbf{v}$ where A is the matrix

$$A = \begin{bmatrix} 5 & 1 & 4 \\ 2 & -1 & 3 \\ 6 & 2 & -7 \end{bmatrix}.$$

Find $[T]_{\mathcal{B}}$, the matrix of T in the basis $\mathcal{B}$. *You may leave your answer expressed as a product of matrices and their inverses.*

7. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of the n-dimensional vector space V. Let $T : V \longrightarrow V$ be the linear transformation with

$$\begin{aligned} T(\mathbf{v}_1) &= a_1\mathbf{v}_1, \\ T(\mathbf{v}_2) &= a_2\mathbf{v}_2, \\ &\cdots \\ T(\mathbf{v}_n) &= a_n\mathbf{v}_n, \end{aligned}$$

where $a_1, a_2, \ldots, a_n$ are fixed scalars. Find $[T]_{\mathcal{B}}$, the matrix of T in the basis $\mathcal{B}$.

8. Let P be a fixed invertible n-by-n matrix and let $T : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ be defined by $T(A) = P^{-1}AP$. Show that T is a linear transformation.

In problems 9 and 10, let $D : P_4(\mathbb{R}) \longrightarrow P_4(\mathbb{R})$ be the linear transformation $D(p(x)) = p'(x)$ (the derivative of $p(x)$).

9. Let $\mathcal{E} = \{1, x, x^2, x^3, x^4\}$. Find $[D]_{\mathcal{E}}$, the matrix of D in the basis $\mathcal{E}$.

10. (a) Find a basis for $\mathrm{Ker}(D)$.
(b) Find a basis for $\mathrm{Im}(D) = \mathrm{Rng}(D)$.