Math 205 - Fall 2019
Homework #2
Answer Key

1. $\left[\begin{array}{ccc|c} 1 & 3 & 3 & 4 \\ 1 & 4 & 3 & 2 \\ 1 & 3 & 4 & 6 \end{array}\right] \overset{A_{12}(-1)}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 4 \\ 0 & 1 & 0 & -2 \\ 1 & 3 & 4 & 6 \end{array}\right]$

$\overset{A_{13}(-1)}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \overset{A_{31}(-3)}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$

$\overset{A_{21}(-3)}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$

2. The system <u>is</u> consistent (with unique solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$).

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3. $\left[\begin{array}{cccc|c} 1 & -3 & -4 & 3 & 0 \\ -1 & 3 & 6 & 1 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 8 & 0 \end{array}\right] \overset{A_{12}(1)}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & -4 & 3 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 8 & 0 \end{array}\right]$

$\overset{M_2(\frac{1}{2})}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & -4 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 8 & 0 \end{array}\right] \overset{A_{23}(-2)}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & -4 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 8 & 0 \end{array}\right]$

$\overset{A_{24}(-4)}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & -4 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \end{array} \left[\begin{array}{cccc|c} \boxed{1} & -3 & 0 & 11 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

The left-hand side of the above augmented matrix is in reduced row-echelon form, represents the equivalent system

$$x_1 - 3x_2 \quad + 11x_4 = 0$$
$$x_3 + 2x_4 = 0$$

The variables x_1 and x_3 (corresponding to the leading 1s) are the dependent variables, while x_2 and x_4 are the independent variables. The system solution is given by

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 11x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} \;\middle|\; x_2, x_4 \text{ arbitrary} \right\}$$

4. Note: For 2x2 Matrices there is an easy way to get the inverse*:

$$\text{If } M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ Then } M^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note that M^{-1} exists precisely when $ad-bc \neq 0$.

$$\text{So, } \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}^{-1} = \frac{1}{1(2)-(-3)(-2)}\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \frac{1}{-4}\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{4} \\ -\frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

5. $$A^2 - 3A - 4I_2 = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -9 \\ -6 & 10 \end{bmatrix} - \begin{bmatrix} 3 & -9 \\ -6 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

* Note: This easy way is nothing more than taking an arbitrary 2x2 matrix, augmenting it by the 2x2 identity matrix I_2 and reducing via a row operations. (cf, problem #9).

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

6.

$$\left[\begin{array}{ccc|ccc} 2 & -4 & 6 & 1 & 0 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ 2 & -5 & 10 & 0 & 0 & 1 \end{array}\right] \overset{M_1(\frac{1}{2})}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & \frac{1}{2} & 0 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ 2 & -5 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{A_{12}(1)}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 5 & \frac{1}{2} & 1 & 0 \\ 2 & -5 & 10 & 0 & 0 & 1 \end{array}\right] \overset{A_{13}(-2)}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 5 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 4 & -1 & 0 & 1 \end{array}\right]$$

$$\overset{P_{23}}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 1 \\ 0 & 0 & 5 & \frac{1}{2} & 1 & 0 \end{array}\right] \overset{M_2(-1)}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 & -1 \\ 0 & 0 & 5 & \frac{1}{2} & 1 & 0 \end{array}\right]$$

$$\overset{M_3(\frac{1}{5})}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{10} & \frac{1}{5} & 0 \end{array}\right] \overset{A_{32}(4)}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{7}{5} & \frac{4}{5} & -1 \\ 0 & 0 & 1 & \frac{1}{10} & \frac{1}{5} & 0 \end{array}\right]$$

$$\overset{A_{31}(-3)}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & \frac{1}{5} & -\frac{3}{5} & 0 \\ 0 & 1 & 0 & \frac{7}{5} & \frac{4}{5} & -1 \\ 0 & 0 & 1 & \frac{1}{10} & \frac{1}{5} & 0 \end{array}\right] \overset{A_{21}(2)}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & -2 \\ 0 & 1 & 0 & \frac{7}{5} & \frac{4}{5} & -1 \\ 0 & 0 & 1 & \frac{1}{10} & \frac{1}{5} & 0 \end{array}\right]$$

So,

$$A^{-1} = \begin{bmatrix} 3 & 1 & -2 \\ \frac{7}{5} & \frac{4}{5} & -1 \\ \frac{1}{10} & \frac{1}{5} & 0 \end{bmatrix}$$

7. First note that if $B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ Then $B^5 = \begin{bmatrix} 2^5 & 0 \\ 0 & (-1)^5 \end{bmatrix}$

$$= \begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix}$$

So, by The note on problem #4, $[B^5]^{-1} = \frac{1}{-32}\begin{bmatrix} -1 & 0 \\ 0 & 32 \end{bmatrix} = \begin{bmatrix} \frac{1}{32} & 0 \\ 0 & -1 \end{bmatrix}$

Alternatively, $[B^5]^{-1} = [B^{-1}]^5$ since

$$B^{-1}B^{-1}B^{-1}B^{-1}B^{-1} \cdot B \cdot B \cdot B \cdot B \cdot B = I_2^5 = I_2$$

Note $B^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}$ so $[B^{-1}]^5 = \begin{bmatrix} (\frac{1}{2})^5 & 0 \\ 0 & (-1)^5 \end{bmatrix} = \begin{bmatrix} \frac{1}{32} & 0 \\ 0 & -1 \end{bmatrix}$

This problem illustrates the nice way diagonal matrices multiply

8. $B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\Rightarrow B^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ from above and $C^{-1} = \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}$

$$[BC]^{-1} = C^{-1}B^{-1} \quad \text{since} \quad BCC^{-1}B^{-1} = I_2$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

9. The matrix equation for the indicated system is

$$(\dagger)\quad A\bar{x} = \bar{c}, \text{ where } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -2 \\ -1 & -7 & -1 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \bar{c} = \begin{bmatrix} 4 \\ a^2 \\ a \end{bmatrix}$$

The system is consistent if for any choice of $\bar{c}$ there is an $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ that solves $(\dagger)$. This certainly is true when A^{-1} exists for then $\bar{x} = A^{-1}\bar{c}$. But A^{-1} does exist in this case since A can be reduced to the identity as follows:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -2 \\ -1 & -7 & -1 \end{bmatrix} \overset{A_{12}(-2)}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -8 \\ -1 & -7 & -1 \end{bmatrix}$$

$$\overset{A_{13}(1)}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -8 \\ 0 & -5 & 2 \end{bmatrix} \overset{A_{23}(-1)}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -8 \\ 0 & 0 & 10 \end{bmatrix}$$

$$\overset{M_3(\frac{1}{10})}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -8 \\ 0 & 0 & 1 \end{bmatrix} \underset{A_{31}(-3)}{\overset{A_{32}(8)}{\sim}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{M_2(\frac{-1}{5})}{\sim} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{A_{21}(-2)}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, any choice of a yields a consistent system

10. To find A^{-1}, we augment A with the 3×3 identity matrix, I_3. Then use elementary row operations to transform A into the 3×3 identity - this transforms I_3 into A^{-1}!

$$\left[\begin{array}{c|c} A & I_3 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{\overset{A_{12}(-1)}{}} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{\overset{A_{13}(-1)}{}} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \underset{\sim}{\overset{A_{31}(-3)}{}} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{\overset{A_{21}(-3)}{}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

and we have the (unique) solution to the given system

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$