Problems to be graded: 2, 3, 5, 7, 9

# Homework Assignment 5 answer key.

1. $A$ reduces to $R = \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

which gives the set of homogeneous equations

$$\begin{aligned} x_1 + 2x_2 \qquad + 4x_5 &= 0 \\ x_3 \qquad + 6x_5 &= 0 \\ x_4 - 5x_5 &= 0 \end{aligned}$$

with solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_5 \\ x_2 \\ -6x_5 \\ 5x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 0 \\ -6 \\ 5 \\ 1 \end{bmatrix}$, $x_2, x_5$ arbitrary

so Nullspace of $A$ has basis $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -6 \\ 5 \\ 1 \end{bmatrix} \right\}$

$R$ has leading entries in columns 1, 3, and 4, so these columns of $A$ are a basis for its column space,

so Column space of $A$ has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

2. B row-reduces to $R = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

which gives the homogeneous system
$$\begin{aligned} x_1 \quad + x_3 \quad + 3x_5 &= 0 \\ x_2 + 2x_3 \quad - x_5 &= 0 \\ x_4 &= 0 \\ x_6 &= 0 \end{aligned}$$

with solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -x_3 - 3x_5 \\ -2x_3 + x_5 \\ x_3 \\ 0 \\ x_5 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $x_3, x_5$ arbitrary

so Nullspace (A) has basis $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

R has leading entries in columns 1, 2, 4, and 6, so these columns of A are a basis for its column space,

so Column space (A) has basis $\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} \right\}$

3. C is $7 \times 9$

(a) rank(C) + nullity (C) = 9, 3 + nullity (C) = 9, nullity (C) = 6

(b) rank (C) + nullity (C) = 9, rank (C) + 5 = 9, rank (C) = 4

4. (a) Rank($8 \times 5$ matrix) $\leq$ min (8, 5) = 5 so rank = 6 is impossible

(b) Nullity ($3 \times 9$ matrix) = 5 gives Rank ($3 \times 9$ matrix) = 4

But Rank($3 \times 9$ matrix) $\leq$ min (3, 9) = 3 so rank = 4 is impossible.

5 (a) $0 = 0v_1 + 0v_2 + \cdots + 0v_n$ so $[0]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

(b) $v_i = 0v_1 + 0v_2 + \cdots + 1v_i + \cdots + 0v_n$ so $[v_i]_{\mathcal{B}} = \begin{array}{r} \\ \\ \text{row } i \to \\ \\ \\ \end{array}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

6 (a) $[v]_{\mathcal{B}} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$ gives $v = 7\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 11\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 47 \\ 123 \end{bmatrix}$

(b) Method 1: $[w]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ gives the linear system $c_1\begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 24 \end{bmatrix}$

with solution $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -32 \\ 23 \end{bmatrix}$

Method 2: $[w]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \mathcal{E}}[w]_{\mathcal{E}} = (P_{\mathcal{E} \leftarrow \mathcal{B}})^{-1}[w]_{\mathcal{E}} = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 24 \end{bmatrix}$

$= \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 24 \end{bmatrix} = \begin{bmatrix} -32 \\ 23 \end{bmatrix}$.

7 (a) $[v]_{\mathcal{B}} = \begin{bmatrix} -6 \\ 17 \end{bmatrix}$ gives $v = (-6)\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 17\begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 101 \\ 129 \end{bmatrix}$

(b) Method 1: $[w]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ gives the linear system $c_1\begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2\begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} -8 \\ 29 \end{bmatrix}$

with solution $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 275 \\ -119 \end{bmatrix}$

Method 2. $[w]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \mathcal{E}}[w]_{\mathcal{E}} = (P_{\mathcal{E} \leftarrow \mathcal{B}})^{-1}[w]_{\mathcal{E}} = \begin{bmatrix} 3 & 7 \\ 4 & 9 \end{bmatrix}^{-1}\begin{bmatrix} -8 \\ 29 \end{bmatrix}$

$= \begin{bmatrix} -9 & 7 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} -8 \\ 29 \end{bmatrix} = \begin{bmatrix} 275 \\ -119 \end{bmatrix}$

8 (a) $[v]_B = \begin{bmatrix} 5 \\ 7 \\ 0 \end{bmatrix}$ gives $v = 5\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 7\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 19 \\ 26 \\ 45 \end{bmatrix}$.

(b) Method 1: $[w]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ gives $c_1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} -9 \\ 14 \\ 3 \end{bmatrix}$

with solution $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -53 \\ 25 \\ 2 \end{bmatrix}$

Method 2: $[w]_B = P_{B\leftarrow\mathcal{E}}[w]_{\mathcal{E}} = (P_{\mathcal{E}\leftarrow B})^{-1}[w]_{\mathcal{E}} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 5 & 8 \end{bmatrix}^{-1}\begin{bmatrix} -9 \\ 14 \\ 3 \end{bmatrix}$

$$= \begin{bmatrix} 4 & -1 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -9 \\ 14 \\ 3 \end{bmatrix} = \begin{bmatrix} -53 \\ 25 \\ 2 \end{bmatrix}$$

9 (a) $[v]_B = \begin{bmatrix} 9 \\ 5 \\ 2 \end{bmatrix}$ gives $v = 9\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + 5\begin{bmatrix} 3 \\ 7 \\ 13 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 6 \\ 13 \end{bmatrix} = \begin{bmatrix} 32 \\ 65 \\ 127 \end{bmatrix}$.

(b) Method 1: $[w]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ gives $c_1\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 7 \\ 13 \end{bmatrix} + c_3\begin{bmatrix} 4 \\ 6 \\ 13 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$

with solution $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -76 \\ 13 \\ 10 \end{bmatrix}$

Method 2: $[w]_B = P_{B\leftarrow\mathcal{E}}[w]_{\mathcal{E}} = (P_{\mathcal{E}\leftarrow B})^{-1}[w]_{\mathcal{E}} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & 6 \\ 4 & 13 & 13 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$

$$= \begin{bmatrix} -13 & -13 & 10 \\ 2 & 3 & -2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -76 \\ 13 \\ 10 \end{bmatrix}$$

10 (a) $[v]_B = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ gives $v = 2(1+2x+3x^2) + 3(x+x^2) + 5(1+2x+4x^2)$

$= 7 + 17x + 29x^2$

(b) Method 1: $[w]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ gives $c_1(1+2x+3x^2) + c_2(x+x^2) + c_3(1+2x+4x^2)$

$= 2 + x^2$

with solution $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}$

Method 2: $[w]_B = P_{B\leftarrow\mathcal{E}}[w]_{\mathcal{E}} = (P_{\mathcal{E}\leftarrow B})^{-1}[w]_{\mathcal{E}} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 1 & -1 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}$$