Practice exam 1 answer key

1. $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & 1 & 2 & 3 \\ 2 & 3 & 7 & 6 & 19 \end{array}\right] \xrightarrow[\text{to equation 3}]{\text{Add } -2\cdot\text{equation 1}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 3 & 3 & 6 & 9 \end{array}\right]$

$\xrightarrow[\text{to equation 3}]{\text{Add } -3\cdot\text{equation 2}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Note this matrix is in RREF, and the system is consistent. It gives the equations

$$\begin{aligned} x_1 \quad + 2x_3 \quad &= 5 \\ x_2 + x_3 + 2x_4 &= 3 \end{aligned}$$

with solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5-2x_3 \\ 3-x_3-2x_4 \\ x_3 \\ x_4 \end{bmatrix}$, x_3, x_4 arbitrary.

2. We perform row reduction:

$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & -4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Interchange rows 1 and 2}}$

$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & -4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Add } -2\cdot\text{row 1 to row 3}}$

$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 2 & -4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Add } -1\cdot\text{row 2 to row 4}}$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 2 & -3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Multiply row 3 by } -1}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Add } -2\cdot\text{row 3 To row 4}}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -4 & 2 & 1 \end{array}\right] \xrightarrow{\text{Multiply row 4 by } -1}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 4 & -2 & -1 \end{array}\right] \xrightarrow{\text{Add } 1\cdot\text{row 4 to row 3}}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 6 & -3 & -1 \\ 0 & 0 & 0 & 1 & 1 & 4 & -2 & -1 \end{array}\right] \xrightarrow{\text{Add } 1\cdot\text{row 4 to row 2}}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 4 & -2 & -1 \\ 0 & 0 & 1 & 0 & 1 & 6 & -3 & -1 \\ 0 & 0 & 0 & 1 & 1 & 4 & -2 & -1 \end{array}\right] \xrightarrow{\text{Add } -1\cdot\text{row 4 to row 1}}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & -1 & -3 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 & 4 & -2 & -1 \\ 0 & 0 & 1 & 0 & 1 & 6 & -3 & -1 \\ 0 & 0 & 0 & 1 & 1 & 4 & -2 & -1 \end{array}\right] \xrightarrow{\text{Add } -1\cdot\text{row 3 to row 1}}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & -9 & 5 & 2 \\ 0 & 1 & 0 & 0 & 2 & 4 & -2 & -1 \\ 0 & 0 & 1 & 0 & 1 & 6 & -3 & -1 \\ 0 & 0 & 0 & 1 & 1 & 4 & -2 & -1 \end{array}\right],$$

so $$\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 3 \\ 0 & 1 & 2 & -4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & -9 & 5 & 2 \\ 2 & 4 & -2 & -1 \\ 1 & 6 & -3 & -1 \\ 1 & 4 & -2 & -1 \end{bmatrix}$$

3. $$\begin{vmatrix} 0 & 2 & 3 & 5 \\ 2 & 1 & 1 & 1 \\ 0 & -1 & 2 & 1 \\ 4 & 2 & 2 & 5 \end{vmatrix} \overset{\text{Add } -2\cdot\text{row } 2 \text{ to row } 4}{=} \begin{vmatrix} 0 & 2 & 3 & 5 \\ 2 & 1 & 1 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{vmatrix}$$

Expand by minors of col 1

$$= (-2)\begin{vmatrix} 2 & 3 & 5 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{vmatrix}$$

Expand by minors of row 3

$$= (-2)(3)\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}$$

$$= (-2)(3)\left(2\cdot 2 - 3\cdot -1\right) = (-2)(3)(7) = -42$$

4. $W = \{ p(x) \in P_3 \mid p(0) = 0 \}$

(0) The zero polynomial $z(x) = 0$ is in W as $z(0) = 0$

(1) If $f(x) \in W$, then $f(0) = 0$

If $g(x) \in W$, then $g(0) = 0$

Then $(f+g)(0) = f(0) + g(0) = 0$, so $(f+g)(x) \in W$

(2) If $f(x) \in W$, then $f(0) = 0$.

If c is a scalar, then

$(cf)(0) = c \cdot f(0) = c \cdot 0 = 0$, so $(cf)(x) \in W$.

Hence W is a subspace of P_3.

5. Let $S = \{v_1, \ldots, v_k\}$ be a set of vectors in a vector space V.

(a) S is linearly independent if the <u>only</u> linear combination of the vectors in S that is equal to 0 is the trivial one, i.e., if the equation

$$c_1 v_1 + \cdots + c_k v_k = 0$$

<u>only</u> has the solution $c_1 = \cdots = c_k = 0$.

(b) S spans V if <u>every</u> vector $v \in V$ can be expressed as a linear combination of the vectors in S, i.e., if the equation

$$c_1 v_1 + \cdots + c_k v_k = v$$

has a solution for <u>every</u> $v \in V$

(c) S is a basis of V if

(1) S is linearly independent

and (2) S spans V.

6. We want to see whether

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \\ 4 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} + c_4 \begin{bmatrix} 4 \\ 8 \\ 11 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

has a solution for <u>every</u> a, b, c, d.

We row-reduce and obtain

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & a \\ 2 & 5 & 2 & 8 & b \\ 2 & 4 & 1 & 11 & c \\ 1 & 3 & 2 & 5 & d \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & a \\ 0 & 1 & 2 & 0 & -2a+b \\ 0 & 0 & 1 & 3 & -2a+c \\ 0 & 0 & 0 & 1 & a-b+d \end{array}\right]$$

We can stop here, as this matrix is in row-echelon form with <u>no</u> zero rows, so this system has a solution for <u>every</u> right hand side. Thus we conclude S spans $\mathbb{R}^4$.

7. We want to see whether

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>only</u> has the trivial solution $c_1 = c_2 = c_3 = c_4 = 0$

We row-reduce and obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 2 & 1 & 4 & 1 & 0 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & -5 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in row-reduced echelon form and there <u>is</u> a free variable. Hence the equation has a nontrivial solution and S is <u>not</u> linearly independent.

We find as solutions

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = c_4 \begin{bmatrix} 3 \\ 5 \\ -3 \\ 1 \end{bmatrix}$$

with c_4 arbitrary.

Choosing $c_4 = 1$, for convenience, we obtain the linear dependence relation

$$3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + (-3) \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$