Solution 4

Problem 1

Let $S = \left\{ \mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \begin{pmatrix} 2k \\ -3k \end{pmatrix}, k \in \mathbb{R} \right\}$. Show that S is a subspace of $\mathbb{R}^2$.

Solution:

1. When $k = 0$, $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$. Hence, zero vector belongs to S.
2. Assume $\mathbf{x}_1 = \begin{pmatrix} 2k_1 \\ -3k_1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 2k_2 \\ -3k_2 \end{pmatrix}$ both belong to S. Then $\mathbf{x}_1 + \mathbf{x}_2 = \begin{pmatrix} 2(k_1 + k_2) \\ -3(k_1 + k_2) \end{pmatrix}$ also belongs to S.
3. Assume $\mathbf{x} = \begin{pmatrix} 2k \\ -3k \end{pmatrix}$ belongs to S and $a \in \mathbb{R}$. Then $a\mathbf{x} = \begin{pmatrix} 2(ak) \\ -3(ak) \end{pmatrix}$ also belongs to S.

Since S is nonempty, and closed under addition and scalar multiplication, we know S is a subspace of $\mathbb{R}^2$.

Problem 2

Let $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x^2 = y^2 \right\}$. Determine whether S is a subspace of $\mathbb{R}^2$ and explain your answer.

Solution: S is not a subspace of $\mathbb{R}^2$. Consider $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ both belonging to S. However, $\mathbf{v}_1 + \mathbf{v}_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ does not belong to S.

Problem 3

Find the null space of $A = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 2 & 3 \end{bmatrix}$.

Solution: Solving $A\mathbf{x} = 0$, we have the solution space is

$$\mathbf{x} = k \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, k \in \mathbb{R}.$$

Problem 4

Show that $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ span $\mathbb{R}^2$, and express the vector $\mathbf{v} = \begin{pmatrix} 3 \\ 18 \end{pmatrix}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Solution: For any $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, we intend to find c_1, c_2 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{v}$, which is

$$c_1 \begin{pmatrix} 1 \\ -5 \end{pmatrix} + c_2 \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Hence,

$$\begin{bmatrix} 1 & 6 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Since $\begin{vmatrix} 1 & 6 \\ -5 & 3 \end{vmatrix} = 33 \neq 0$, the above equation always has unique solution. Hence, any vector in $\mathbb{R}^2$ can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. In particular, $\begin{pmatrix} 3 \\ 18 \end{pmatrix} = -3\mathbf{v}_1 + \mathbf{v}_2$.

Problem 5

Prove that e^x does not belong to $P_1(\mathbb{R})$.

Solution: Assume that e^x belong to $P_1(\mathbb{R})$. Then there exist $c_1, c_2 \in \mathbb{R}$ such that $\mathrm{e}^x = c_1 + c_2 x$ for any $x \in \mathbb{R}$. We take $x = 0, 1, 2$, then $c_1 = 1$, $c_1 + c_2 = \mathrm{e}$ and $c_1 + 2c_2 = \mathrm{e}^2$. This set of equations does not have solution, so such c_1, c_2 do not exist.

Problem 6

Determine whether the following set of vectors are linearly independent or linearly dependent in $\mathbb{R}^3$: $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Solution: Since $\begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} \neq 0$, the set of vectors is linearly independent.

Problem 7

Show that the vectors $p_1(x) = a+bx$ and $p_2(x) = c+dx$ are linearly independent in $P_1(\mathbb{R})$ if and only if the constants satisfy $ad - bc \neq 0$.

Solution: p_1 and p_2 are linearly independent if and only if $k_1, k_2 \in \mathbb{R}$ satisfying $k_1p_1 + k_2p_2 = 0$ leads to $k_1 = k_2 = 0$. We may rewrite

$$k_1p_1 + k_2p_2 = (ak_1 + ck_2) + (bk_1 + dk_2)x = 0.$$

Since 1 and x are linearly independent, the above equality is true yields that $ak_1 + ck_2 = 0$ and $bk_1 + dk_2 = 0$. Then this set of equation has unique zero solution if and only if the determinant $ad - bc \neq 0$.

Problem 8

Determine whether $S = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

Solution: We first verify whether this set of vectors is linearly independent. Assume there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This forms a linear system, which has infinitely many solutions. We pick one of them $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$. This implies $\mathbf{v}_3 = -\mathbf{v}_1 + 2\mathbf{v}_2$. Hence, we know the vectors are linearly dependent, so S is not a basis of $\mathbb{R}^3$. (Certainly, it is also acceptable to directly verify the determinant is zero.)

Problem 9

Find the dimension of the null space of $A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$.

Solution: Solving $A\mathbf{x} = 0$, we have the solution space is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, the dimension of the null space is 0.

Problem 10

Find a vector $\mathbf{v} \in \mathbb{R}^3$ such that $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v} \right\}$ constitutes a basis of $\mathbb{R}^3$.

Solution: Assume $\mathbf{v} = (x, y, z)$. Then we only need $\begin{vmatrix} 1 & 0 & x \\ 1 & 1 & y \\ 0 & 1 & z \end{vmatrix} \neq 0$. This is equivalent to $x - y + z \neq 0$. Hence, actually we have infinitely many choices for $\mathbf{v}$. For example, we may take $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.