Fall 2019 - Math 205
Homework 3

Due at the beginning of class on Weds. Sept. 18 (Profs. Zhang and Wu), Thurs. Sept. 19 (Profs. Coll, Weintraub, Recio-Mitter). Write your name and section number on your homework. You must show your work in order to receive full credit.

Neo: What is the Matrix?
Trinity: The answer is out there, Neo, and it's looking for you, and it will find you if you want it to.
Dialogue from "The Matrix"

1. Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Solution: (Graded)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \qquad A_{1,2}(-4), A_{1,3}(-7)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{bmatrix} \qquad M_2(-1/3)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_{2,3}(6)$$

Because the row echelon form has two leading ones, the rank of A is 2.

2. Let A and B be $n \times n$ matrices. Write the following determinants in terms of $\det(A)$ and $\det(B)$.

(a) $\det(5A)$

(b) $\det(AB^2)$

(c) $\det((AB)^{-1})$

Solution:

(a) $\det(5A) = 5^n \det(A)$

(b) $\det(AB^2) = \det(A)\det(B)^2$

(c) $\det((AB)^{-1}) = \frac{1}{\det(A)\det(B)}$

3. Compute the determinant of the following matrices.

(a) $A = \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$

(b) $B = \begin{bmatrix} 3 & -5 \\ 9 & -15 \end{bmatrix}$

(c) $A^{-1}B^2$

Solution: (Graded)

Part (a):

$$\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = 4 - (-3) = 7$$

Part (b):

The determinant of B is 0 because the rows are multiples of each other and therefore the rank is 1.

Part (b):

$$\det(A^{-1}B^2) = \frac{\det(B)^2}{\det(A)} = \frac{0}{7} = 0$$

4. Compute the determinant of the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 7 & 3 & -1 \\ 2 & 1 & 4 \end{bmatrix}.$$

Solution:

By row expansion:

$$\det(A) = 2\begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix} + (-1)\begin{vmatrix} 7 & 3 \\ 2 & 1 \end{vmatrix} = 2 \cdot 13 - 1 = 25.$$

5. Compute the determinant of the matrix

$$A = \begin{bmatrix} -7 & 3 & 1 \\ 2 & 0 & -2 \\ 1 & 0 & 5 \end{bmatrix}.$$

Solution: (Graded)

By column expansion:

$$\det(A) = -3\begin{vmatrix} 2 & -2 \\ 1 & 5 \end{vmatrix} = -3 \cdot 12 = -36.$$

6. (a) Compute the determinant of the matrix

$$A = \begin{bmatrix} 8 & -1 & 4 \\ 6 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix}.$$

(b) Determine the rank of the matrix A.

(c) What is the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$? Justify your answer.

Solution:

Part (a):

By row expansion:

$$\det(A) = 1\begin{vmatrix} -1 & 4 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 8 & 4 \\ 6 & 3 \end{vmatrix} = -7 + 0 = -7.$$

Part (b):

Since the determinant of A is not zero, the rank must be 3, the maximum possible.

Part (c):

Because the coefficient matrix has maximal rank, the solution must be unique, if it exists. Because the system is homogeneous, it admits the trivial solution. Therefore, the solution set is

$$S = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

7. Compute the determinant of the matrix

$$A = \begin{bmatrix} 3 & -3 & 7 & 5 \\ 2 & 0 & 5 & 6 \\ 1 & -1 & 3 & 1 \\ 2 & -2 & 1 & 5 \end{bmatrix}.$$

Hint: Use row operations.

Solution: (Graded)

There are two slightly different solutions using row operations.

Solution 1

$$\det(A) = \begin{vmatrix} 3 & -3 & 7 & 5 \\ 2 & 0 & 5 & 6 \\ 1 & -1 & 3 & 1 \\ 2 & -2 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -2 & 2 \\ 0 & 2 & -1 & 4 \\ 1 & -1 & 3 & 1 \\ 0 & 0 & -5 & 3 \end{vmatrix} \qquad A_{3,2}(-2), A_{3,1}(-3), A_{3,4}(-2)$$

Now compute the determinant of the simplified matrix by expanding along the first column:

$$\det(A) = \begin{vmatrix} 0 & 0 & -2 & 2 \\ 0 & 2 & -1 & 4 \\ 1 & -1 & 3 & 1 \\ 0 & 0 & -5 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -2 & 2 \\ 2 & -1 & 4 \\ 0 & -5 & 3 \end{vmatrix} = -2 \begin{vmatrix} -2 & 2 \\ -5 & 3 \end{vmatrix} = -8$$

Solution 2

$$\det(A) = \begin{vmatrix} 3 & -3 & 7 & 5 \\ 2 & 0 & 5 & 6 \\ 1 & -1 & 3 & 1 \\ 2 & -2 & 1 & 5 \end{vmatrix} = - \begin{vmatrix} 1 & -1 & 3 & 1 \\ 2 & 0 & 5 & 6 \\ 3 & -3 & 7 & 5 \\ 2 & -2 & 1 & 5 \end{vmatrix} \qquad P_{1,3}$$

$$= - \begin{vmatrix} 1 & -1 & 3 & 1 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -5 & 3 \end{vmatrix} \qquad A_{1,2}(-2), A_{1,2}(-3), A_{1,4}(-2)$$

$$= - \begin{vmatrix} 1 & -1 & 3 & 1 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & -2 \end{vmatrix} \qquad A_{3,4}(-5/2)$$

$$= -(1 \cdot 2 \cdot (-2) \cdot (-2)) = -8$$

8. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 \\ 7 & 9 & -5 & 1 \\ 0 & -4 & 3 & 2 \end{bmatrix}.$$

How many solutions does the homogeneous system $A\mathbf{x} = \mathbf{0}$ have?

Solution:

The matrix A is not invertible because the second row is a multiple of the first row: If we perform the row operation of adding the -3 multiple of the first row to the second row we get a matrix row equivalent to A which has a zero row and therefore A is not invertible. One way of seeing this is that the row echelon form will have a zero row and thus the rank is not maximal. Alternatively, we may argue that the determinant is equal to 0.

Because the matrix A is not invertible (equivalently, does not have maximal rank) the system cannot have a unique solution. Because a homogeneous system always has at least one solutions, the system $A\mathbf{x} = \mathbf{0}$ must have infinitely many solutions.

9. (a) Is the set

$$S = \left\{ \begin{bmatrix} 2s + t \\ t - 3s \\ 3t \end{bmatrix} \middle| \; s, t \in \mathbb{R} \right\}$$

a subspace of the vector space $\mathbb{R}^3$?

(b) Is the set

$$S = \left\{ \begin{bmatrix} 2s \\ 3t \\ 1 \end{bmatrix} \middle| \; s, t \in \mathbb{R} \right\}$$

a subspace of the vector space $\mathbb{R}^3$?

Solution: (Graded)

Part (a):

Zero vector check

For $s = t = 0$ we have $\begin{bmatrix} 2s+t \\ t-3s \\ 3t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

This proves that S is not empty.

Closed under addition

We need to show that for any two vectors $\begin{bmatrix} 2s_1+t_1 \\ t_1-3s_1 \\ 3t_1 \end{bmatrix}$ and $\begin{bmatrix} 2s_2+t_2 \\ t_2-3s_2 \\ 3t_2 \end{bmatrix}$ in S, their sum is again in S:

$$\begin{bmatrix} 2s_1+t_1 \\ t_1-3s_1 \\ 3t_1 \end{bmatrix} + \begin{bmatrix} 2s_1+t_1 \\ t_1-3s_1 \\ 3t_1 \end{bmatrix} =$$
$$\begin{bmatrix} 2s_1+t_1+2s_2+t_2 \\ t_1-3s_1+t_2-3s_2 \\ 3t_1+3t_2 \end{bmatrix} = \begin{bmatrix} 2(s_1+s_2)+(t_1+t_2) \\ (t_1+t_2)-3(s_1+s_2) \\ 3(t_1+t_2) \end{bmatrix} = \begin{bmatrix} 2s_3+t_3 \\ t_3-3s_3 \\ 3t_3 \end{bmatrix}.$$

Because $s_3 = s_1 + s_2$ and $t_3 = t_1 + t_2$ are in $\mathbb{R}$, the sum satisfies the condition of the subset S and thus is contained in S.

Closed under scalar multiplication

We need to show that for any scalar $k \in \mathbb{R}$ and any vector $\begin{bmatrix} 2s+t \\ t-3s \\ 3t \end{bmatrix}$ in S, their product is again in S:

$$k \begin{bmatrix} 2s+t \\ t-3s \\ 3t \end{bmatrix} = \begin{bmatrix} k(2s+t) \\ k(t-3s) \\ k3t \end{bmatrix} = \begin{bmatrix} 2(ks)+(kt) \\ (kt)-3(ks) \\ 3(kt) \end{bmatrix} = \begin{bmatrix} 2\tilde{s}+\tilde{t} \\ \tilde{t}-3\tilde{s} \\ 3\tilde{t} \end{bmatrix}.$$

Because $\tilde{s} = ks$ and $\tilde{t} = kt$ are in $\mathbb{R}$, the sum satisfies the condition of the subset S and thus is contained in S.

Part (b):

Zero vector check

The last coordinate of every vector in S has to be 1 by definition. Therefore, the zero vector cannot be in S. This shows that S is not a subspace.

10. The real-valued continuous functions on the real numbers form a vector space $C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \,|\, f \text{ is continuous}\}$ with:

- Addition $(f + g)(x) = f(x) + f(x)$.
- Scalar multiplication $(kf)(x) = kf(x)$.

Is the set

$$S = \{f \in C(\mathbb{R}) \,|\, f(3) = 0\}$$

a subspace of the vector space $C(\mathbb{R})$?

(b) Is the set

$$S = \{f \in C(\mathbb{R}) \,|\, f(3) = 2\}$$

a subspace of the vector space $C(\mathbb{R})$?

Solution:

Part (a):

Zero vector check

The zero function, which is the constant function $z(x) = 0$, clearly satisfies the condition $z(3) = 0$ and thus is contained in S.

This proves that S is not empty.

Closed under addition

We need to show that for any two functions f and g in S, the sum $f + g$ is again in S.

$$(f + g)(3) = f(3) + g(3) = 0 + 0 = 0$$

The sum $f + g$ satisfies the condition of the subset S and thus is contained in S.

Closed under scalar multiplication

We need to show that for any scalar $k \in \mathbb{R}$ and any function f in S, the product kf is again in S.

$$(kf)(3) = kf(3) = k0 = 0$$

The product kf satisfies the condition of the subset S and thus is contained in S.

Part (b):

Zero vector check

Because the zero function $z(x) = 0$ is constantly 0, it satisfies $z(3) = 0 \neq 2$. Therefore, the zero vector cannot be in S. This shows that S is not a subspace.