Homework 9[1] (Autumn 2019)

Due before class on November 4, Monday (Professor Wu, Zhang)

Due before class on November 5, Tuesday

(Professor Coll, Professor Recio-Mitter and Professor Weintraub)

There are 3 problems in the first part of this homework assignment.

Consider the following second order nonhomogeneous linear differential equations

$$\begin{aligned}
&(1) \qquad y'' + 100y = 36\cos(8x) + 72\sin(8x),\\
&(2) \qquad y'' - 4y' + 3y = (2 - 12x)e^x + (60 + 40x)e^{3x},\\
&(3) \qquad y'' - 2\alpha y' + \alpha^2 y = 60xe^{\alpha x} + 60x^2e^{\alpha x},
\end{aligned}$$

where $\alpha \neq 0$ is a real nonozero constant. Use the method of undetermined coefficients to find a particular solution for each equation. Then solve each equation for real general solution.

Solutions of (1), (2), (3): Recall that for the nonhomogeneous second order linear differential equations, the general solution is equal to the sum of the complementary solution and a particular solution. First of all, let us find the complementary solutions for all of the three differential equations. The auxiliary equations and their solutions of these differential equations are given by, respectively

$$\begin{aligned}
&(1) \qquad \lambda^2 + 100 = 0, \qquad \lambda = \pm 10\mathrm{i},\\
&(2) \qquad \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0, \qquad \lambda_1 = 1, \lambda_2 = 3,\\
&(3) \qquad \lambda^2 - 2\alpha\lambda + \alpha^2 = (\lambda - \alpha)^2 = 0, \qquad \lambda = \alpha.
\end{aligned}$$

Therefore, the real complementary solutions are given by

$$\begin{aligned}
&(1) \qquad y_c(x) = C_1\cos(10x) + C_2\sin(10x),\\
&(2) \qquad y_c(x) = C_1e^x + C_2e^{3x},\\
&(3) \qquad y_c(x) = C_1e^{\alpha x} + C_2xe^{\alpha x}.
\end{aligned}$$

Next let us find a particular solution for each differential equation. Let us use the method of undetermined coefficients to find a particular solution of equation (1). Suppose that

$$y_p(x) = A\cos(8x) + B\sin(8x),$$

is a particular solution, where A and B are real constants, to be determined later. Then

$$y_p'(x) = -8A\sin(8x) + 8B\cos(8x),$$

and

$$y_p''(x) = -64A\cos(8x) - 64B\sin(8x).$$

[1]For Section 8.3: the Method of Undetermined Coefficients. Section: 1.5: Some Simple Population Models. Section 8.5: Oscillations of a Mechanical System. Section 8.6: RLC Circuits.

Let us plug $y_p(x)$ and $y_p''(x)$ back into the nonohomogeneous differential equation

$$y'' + 100y = 36\cos(8x) + 72\sin(8x).$$

We have

$$\begin{aligned} & y_p''(x) + 100y_p(x) \\ = \; & -64A\cos(8x) - 64B\sin(8x) + 100A\cos(8x) + 100B\sin(8x) \\ = \; & 36A\cos(8x) + 36B\sin(8x) = 36\cos(8x) + 72\sin(8x). \end{aligned}$$

Let us compare the coefficients of $\cos(8x)$ and $\sin(8x)$ to get

$$A = 1, \qquad B = 2.$$

Now the particular solution is

$$y_p(x) = \cos(8x) + 2\sin(8x).$$

As before, we will use the method of undetermined coefficients to find a particular solution of equation (2). Suppose that

$$y_p(x) = (Ax + Bx^2)e^x + (Cx + Dx^2)e^{3x},$$

is a particular solution of the differential equation, where A and B are real constants to be determined. Let us recall that

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x),$$

and

$$[f(x)g(x)]'' = f''(x)g(x) + f(x)g''(x) + 2f'(x)g'(x),$$

for all differentiable functions f and g. Then

$$\begin{aligned} y_p'(x) &= (Ax + Bx^2)e^x + (A + 2Bx)e^x \\ &+ 3(Cx + Dx^2)e^{3x} + (C + 2Dx)e^{3x}, \end{aligned}$$

and

$$\begin{aligned} y_p''(x) &= (Ax + Bx^2)e^x + 2Be^x + 2(A + 2Bx)e^x \\ &+ 9(Cx + Dx^2)e^{3x} + 2De^{3x} + 6(C + 2Dx)e^{3x}. \end{aligned}$$

Let us plug $y_p(x)$, $y_p'(x)$ and $y_p''(x)$ back into the nonhomogeneous differential equation

$$y'' - 4y' + 3y = (2 - 12x)e^x + (60 + 40x)e^{3x}.$$

We have [2]

$$
\begin{aligned}
& y_p''(x) - 4y_p'(x) + 3y_p(x) \\
= \;& (Ax + Bx^2)e^x + 2Be^x + 2(A + 2Bx)e^x \\
- \;& 4(Ax + Bx^2)e^x - 4(A + 2Bx)e^x + 3(Ax + Bx^2)e^x \\
+ \;& 9(Cx + Dx^2)e^{3x} + 2De^{3x} + 6(C + 2Dx)e^{3x} \\
- \;& 12(Cx + Dx^2)e^{3x} - 4(C + 2Dx)e^{3x} + 3(Cx + Dx^2)e^{3x} \\
= \;& 2Be^x - 2(A + 2Bx)e^x + 2De^{3x} + 2(C + 2Dx)e^{3x} \\
= \;& (2B - 2A - 4Bx)e^x + (2D + 2C + 4Dx)e^{3x} \\
= \;& (2 - 12x)e^x + (60 + 40x)e^{3x}.
\end{aligned}
$$

Let us compare the coefficients of e^x, xe^x, e^{3x} and xe^{3x}. We have

$$A = 2, \qquad B = 3, \qquad C = 20, \qquad D = 10.$$

The particular solution is

$$y_p(x) = (2x + 3x^2)e^x + (20x + 10x^2)e^{3x}.$$

Again let us use the method of undetermined coefficients to find a particular solution for the differential equation (3). Suppose that

$$y_p(x) = (Ax^3 + Bx^4)e^{\alpha x}$$

is a particular solution, where A and B are real constants to be determined later. Then

$$y_p'(x) = \alpha(Ax^3 + Bx^4)e^{\alpha x} + (3Ax^2 + 4Bx^3)e^{\alpha x},$$

and

$$
\begin{aligned}
y_p''(x) \;=\;& \alpha^2(Ax^3 + Bx^4)e^{\alpha x} + (6Ax + 12Bx^2)e^{\alpha x} \\
+\;& 2\alpha(3Ax^2 + 4Bx^3)e^{\alpha x}.
\end{aligned}
$$

Let us plug $y_p(x)$, $y_p'(x)$ and $y_p''(x)$ back into the original differential equation

$$y'' - 2\alpha y' + \alpha^2 y = 60xe^{\alpha x} + 60x^2e^{\alpha x}.$$

We have [3]

$$
\begin{aligned}
& y_p''(x) - 2\alpha y_p'(x) + \alpha^2 y_p(x) \\
= \;& \alpha^2(Ax^3 + Bx^4)e^{\alpha x} + (6Ax + 12Bx^2)e^{\alpha x} + 2\alpha(3Ax^2 + 4Bx^3)e^{\alpha x} \\
- \;& 2\alpha^2(Ax^3 + Bx^4)e^{\alpha x} - 2\alpha(3Ax^2 + 4Bx^3)e^{\alpha x} + \alpha^2(Ax^3 + Bx^4)e^{\alpha x} \\
= \;& (6Ax + 12Bx^2)e^{\alpha x} \\
= \;& (60x + 60x^2)e^{\alpha x}.
\end{aligned}
$$

[2] The most complicated functions are cancelled out in the computations.

[3] The most complicated functions are cancelled out in the computations.

Let us compare the coefficients of $xe^{\alpha x}$ and $x^2e^{\alpha x}$ to get

$$A = 10, \qquad B = 5.$$

Now the particular solution is

$$y_p(x) = (10x^3 + 5x^4)e^{\alpha x}.$$

Overall, the particular solutions of the three differential equations are given by

$$\begin{aligned} &(1) \qquad y_p(x) = \cos(8x) + 2\sin(8x), \\ &(2) \qquad y_p(x) = (2x + 3x^2)e^x + (20 + 10x^2)e^{3x}, \\ &(3) \qquad y_p(x) = (10x^3 + 5x^4)e^{\alpha x}. \end{aligned}$$

Therefore, the general solutions of the equation are given by

$$\begin{aligned} &(1) \qquad y(x) = C_1\cos(10x) + C_2\sin(10x) + \cos(8x) + 2\sin(8x), \\ &(2) \qquad y(x) = C_1e^x + C_2e^{3x} + (2x + 3x^2)e^x + (20x + 10x^2)e^{3x}, \\ &(3) \qquad y(x) = C_1e^{\alpha x} + C_2xe^{\alpha x} + (10x^3 + 5x^4)e^{\alpha x}, \end{aligned}$$

where C_1 and C_2 are real constants.

(4) The number of bacteria in a culture grows at a rate that is proportional to the number present. Initially, there were 100 bacteria in the culture. If the doubling time of the culture is 10 hours, find the number of bacteria that were present after 65 hours.

(4) Solutions: Recall that the solution of the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}y = ky, \qquad y(0) = y_0,$$

is given by

$$y(t) = y_0e^{kt},$$

where k and y_0 are real constants.

The number of bacteria at any time t is governed by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky, \qquad y(0) = 100.$$

where $k \neq 0$ is a real nonzero constant. The solution of the initial value problem is given by

$$y(t) = 100e^{kt}.$$

Letting $t = 10$ and $y(10) = 200$ in the solution, we have

$$200 = y(10) = 100e^{10k}.$$

Solving the equation, we find that

$$k = \frac{\ln 2}{10}.$$

Therefore, the solution of the initial value problem is given by

$$y(t) = 100e^{(\frac{\ln 2}{10})t} = 100 \cdot 2^{t/(10)}.$$

Letting $t = 65$ leads to the final answer

$$y(65) = 100 \cdot 2^{6.5} = 100 \cdot e^{6.5 \ln 2} \approx 100 \cdot 90.496 \approx 9050.$$

(5) The population of a certain city at time t is increasing at a rate that is proportional to the number of residents in the city at that time. On January 1, 2000, the population of the city is 1000 and on January 1, 2010, the population of the city is 10000.
(5-1) What will the population of the city be on January 1, 2020?
(5-2) In what year the population will reach one million?

(5) Solutions: The population at time t obeys the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky, \qquad y(0) = y_0,$$

where k is a real constant and $y_0 = 1000$. The solution is given by

$$y(t) = y_0 e^{kt}.$$

Let $t = 10$ and $y(10) = 10000$ yields

$$10000 = y(10) = 1000e^{10k}.$$

Solving this equation, we find the value

$$k = \frac{1}{10}\ln(10).$$

Therefore, the solution of the initial value problem is

$$y(t) = 1000 \cdot (10)^{t/(10)}.$$

Now letting $t = 20$ leads to the solution to part (1):

$$y(20) = 1000 \cdot (10)^2 = 100000.$$

Let us solve the equation

$$1000000 = 1000 \cdot (10)^{T/(10)}.$$

Therefore, $T = 30$, in the year 2030, the population in the city reach a million.

(6) Consider the spring-mass system whose motion is governed by the initial value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 16y = 0, \qquad y(0) = 24, \qquad y'(0) = -28.$$

Determine the circular frequency of the system and the amplitude.

(6) Solutions: The characteristic equation and its solutions are

$$\lambda^2 + 16 = 0, \qquad \lambda_1 = 4\mathrm{i}, \qquad \lambda_2 = -4\mathrm{i}.$$

The general solution of the differential equation is

$$y(t) = C_1 \cos(4t) + C_2 \sin(4t),$$

where C_1 and C_2 are real constants to be determined later. Let us compute the derivative of the solution

$$y'(t) = -4C_1 \sin(4t) + 4C_2 \cos(4t).$$

Letting $t = 0$ in the solution $y(t)$ and the derivative $y'(t)$, respectively, we have

$$\begin{aligned} C_1 &= 24, \\ 4C_2 &= -28. \end{aligned}$$

Thus $C_1 = 24$ and $C_2 = -7$. The solution is

$$y(t) = 24\cos(4t) - 7\sin(4t) = 25\cos(4t + \alpha),$$

where the angle $0 < \alpha < \frac{\pi}{2}$, such that

$$\cos(\alpha) = \frac{24}{25}, \qquad \sin\alpha = \frac{7}{25}.$$

Therefore, the amplitude is $A = 25$ and the circular frequency is $\omega_0 = 4$.

Problems 7 and 8 are based on the following initial value problems.

Consider the spring-mass system whose motion is governed by the initial value problems

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{c}{m}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{k}{m}y = \frac{1}{m}f(t), \qquad y(0) = y_0, \qquad y'(0) = y_1,$$

where $m > 0$, $c \geq 0$ and $k > 0$ are real constants.

(7) Consider the case where there is no damping: $c = 0$. Let

$$\omega_0 = \sqrt{\frac{k}{m}}, \qquad f(t) = 10m\omega_0[\cos(\omega_0 t) - \sin(\omega_0 t)].$$

Is the motion of the system oscillatory? What happens to the amplitude as $t \to \infty$?

(7) Solutions: The characteristic equation and its solutions are

$$\lambda^2 + \omega_0^2 = 0, \qquad \lambda_1 = \omega_0\mathrm{i}, \qquad \lambda_2 = -\omega_0\mathrm{i}.$$

The complementary solution is

$$y_c(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t),$$

where C_1 and C_2 are real constants.

Suppose that

$$y_p(t) = At \cos(\omega_0 t) + Bt \sin(\omega_0 t)$$

is a particular solution. Plugging it back into the differential equation and then comparing the coefficients of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$,[4] we find that $A = 5$ and $B = 5$. Hence the particular solution is

$$y_p(t) = 5t \cos(\omega_0 t) + 5t \sin(\omega_0 t).$$

Therefore, the general solution is

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + 5t \cos(\omega_0 t) + 5t \sin(\omega_0 t),$$

where C_1 and C_2 are real constants.

As we can easily see, the motion is oscillatory and the amplitude is becoming larger and larger, it is becoming unbounded as $t \to \infty$.

(8) Consider the case there is no damping and the external force is $f(t) = 60\cos(2t) + 60\sin(2t)$. Let $m = 1$ and $k = 16$. Determine the period of the motion for the spring-mass system.

(8) Solutions: The characteristic equation and its solutions are

$$\lambda^2 + 16 = 0, \qquad \lambda_1 = 4\mathrm{i}, \qquad \lambda_2 = -4\mathrm{i}.$$

The complementary solution is

$$y_c(t) = C_1 \cos(4t) + C_2 \sin(4t),$$

where C_1 and C_2 are real constants.

By using the method of undetermined coefficient (the details are simple and they are omitted), a particular solution is

$$y_p(t) = 5\cos(2t) + 5\sin(2t).$$

Therefore, the general solution is

$$y(t) = C_1 \cos(4t) + C_2 \sin(4t) + 5\cos(2t) + 5\sin(2t),$$

where C_1 and C_2 are real constants. The period of the motion is $T = \pi$.

(9) In the RLC circuit

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{LC}q = \frac{1}{L}E(t),$$

[4]The most complicated functions are cancelled out in the computations.

Let

$$R = 4, \qquad L = 1, \qquad C = \frac{1}{4}, \qquad E(t) = 8\cos(2t) - 8\sin(2t).$$

Determine the steady-state current.

(9) Solutions: The characteristic equation and its solution are

$$\lambda^2 + 4\lambda + 4 = 0, \qquad \lambda = -2.$$

The complementary solution is

$$q_c(t) = C_1 e^{-2t} + C_2 t e^{-2t},$$

where C_1 and C_2 are real constants. Suppose that

$$q_p(t) = A\cos(2t) + B\sin(2t)$$

is a particular solution, where A and B are constants to be determined. Plugging the particular solution back into the differential equation and then comparing the coefficients of $\cos(2t)$ and $\sin(2t)$, we find $A = 1$ and $B = 1$. Now the particular solution is $q_p(t) = \cos(2t) + \sin(2t)$. Therefore, the steady-state current (note that $i(t) = q'(t)$) is

$$i(t) = 2\cos(2t) - 2\sin(2t).$$

(10) In the RLC circuit

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{LC}q = \frac{1}{L}E(t),$$

let $R^2 C < 4L$ and $E(t) = 1$. Determine the electric charge.

(10) Solutions: The characteristic equation and its solutions are

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0,$$

and

$$\lambda_1 = \frac{-R + \sqrt{R^2 - 4L/C}}{2L}, \qquad \lambda_2 = \frac{-R - \sqrt{R^2 - 4L/C}}{2L}.$$

These solutions are complex because $CR^2 < 4L$. The complementary solution is

$$\begin{aligned} q_c(t) &= C_1 e^{-Rt/(2L)} \cos\left(\frac{\sqrt{(4L/C) - R^2}}{2L} t\right) \\ &+ C_2 e^{-Rt/(2L)} \sin\left(\frac{\sqrt{(4L/C) - R^2}}{2L} t\right). \end{aligned}$$

By the method of undetermined coefficient, we find a particular solution

$$q_p(t) = C.$$

Therefore, the general solution is

$$\begin{aligned} q(t) &= C_1 e^{-Rt/(2L)} \cos\left(\frac{\sqrt{(4L/C) - R^2}}{2L} t\right) \\ &+ C_2 e^{-Rt/(2L)} \sin\left(\frac{\sqrt{(4L/C) - R^2}}{2L} t\right) \\ &+ C, \end{aligned}$$

where C_1 and C_2 are real constants.