Problems to be graded:
2, 4, 6, 7, 10

# Math 205 Homework assignment 6 answer key

1. $[T]_{\mathcal{B}} = \left[ [T(v_1)]_{\mathcal{B}} \middle| [T(v_2)]_{\mathcal{B}} \right] = \left[ [2v_1+3v_2]_{\mathcal{B}} \middle| [5v_1+7v_2]_{\mathcal{B}} \right] = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$

2. $[T]_{\mathcal{B}} = \left[ [T(v_1)]_{\mathcal{B}} \middle| [T(v_2)]_{\mathcal{B}} \middle| [T(v_3)]_{\mathcal{B}} \right]$

$= \left[ [3v_1+4v_2+5v_3]_{\mathcal{B}} \middle| [7v_1+8v_2+9v_3]_{\mathcal{B}} \middle| [-v_1-2v_2-3v_3]_{\mathcal{B}} \right] = \begin{bmatrix} 3 & 7 & -1 \\ 4 & 8 & -2 \\ 5 & 9 & -3 \end{bmatrix}$

3. $[T]_{\mathcal{B}} = \left[ [T(v_1)]_{\mathcal{B}} \middle| [T(v_2)]_{\mathcal{B}} \right] = \left[ \begin{bmatrix} 9 \\ 31 \end{bmatrix}_{\mathcal{B}} \middle| \begin{bmatrix} 38 \\ 123 \end{bmatrix}_{\mathcal{B}} \right]$

$\begin{bmatrix} 9 \\ 31 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Leftrightarrow c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 16 \end{bmatrix} = \begin{bmatrix} 9 \\ 31 \end{bmatrix}$. Solving, we find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 38 \\ 123 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Leftrightarrow c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 16 \end{bmatrix} = \begin{bmatrix} 38 \\ 123 \end{bmatrix}$. Solving, we find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$

Thus $[T]_{\mathcal{B}} = \begin{bmatrix} -11 & -7 \\ 4 & 9 \end{bmatrix}$

4. $[T]_{\mathcal{B}} = \left[ [T(v_1)]_{\mathcal{B}} \middle| [T(v_2)]_{\mathcal{B}} \right] = \left[ \begin{bmatrix} -1 \\ -3 \end{bmatrix}_{\mathcal{B}} \middle| \begin{bmatrix} 1 \\ 10 \end{bmatrix}_{\mathcal{B}} \right]$

$\begin{bmatrix} -1 \\ -3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Leftrightarrow c_1 \begin{bmatrix} 1 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 11 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$. Solving, we find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 10 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Leftrightarrow c_1 \begin{bmatrix} 1 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$. Solving, we find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$

Thus $[T]_{\mathcal{B}} = \begin{bmatrix} 5 & 9 \\ -3 & -4 \end{bmatrix}$

5. $[T]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{E}}\,[T]_{\mathcal{E}}\,P_{\mathcal{E}\leftarrow\mathcal{B}} = \left(P_{\mathcal{E}\leftarrow\mathcal{B}}\right)^{-1}[T]_{\mathcal{E}}\,P_{\mathcal{E}\leftarrow\mathcal{B}}$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 5 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 3 & 7 \\ 1 & 6 & 9 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 5 & 8 \end{bmatrix}.$$

6. By exactly the same logic,

$$[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 3 & 1 \\ 3 & 8 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 1 & 4 \\ 2 & -1 & 3 \\ 6 & 2 & -7 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 \\ 1 & 3 & 1 \\ 3 & 8 & 3 \end{bmatrix}$$

7. $[T]_{\mathcal{B}} = \left[\, [T(v_1)]_{\mathcal{B}} \,\middle|\, [T(v_2)]_{\mathcal{B}} \,\middle|\, \cdots \,\middle|\, [T(v_n)]_{\mathcal{B}} \,\right]$

Now $T(v_i) = a_i v_i = 0v_1 + \cdots + a_i v_i + \cdots + 0v_n$

so $[T(v_i)]_{\mathcal{B}} = [a_i v_i]_{\mathcal{B}} = \begin{array}{r} \\ \\ \text{row } i \to \\ \\ \end{array}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_i \\ 0 \end{bmatrix}$

Thus $[T]_{\mathcal{B}} = \begin{bmatrix} a_1 & 0 & & 0 \\ 0 & a_2 & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & & a_n \end{bmatrix} = \begin{bmatrix} a_1 & & & \\ & a_2 & & 0 \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix},$

the diagonal matrix with diagonal entries $a_1, a_2, \ldots, a_n$.

8. a) We must show $T(A_1+A_2) = T(A_1)+T(A_2)$.
We see $T(A_1+A_2) = P^{-1}(A_1+A_2)P = P^{-1}A_1P + P^{-1}A_2P = T(A_1)+T(A_2)$ as required
(b) We must show $T(cA) = cT(A)$
We see $T(cA) = P^{-1}(cA)P = c(P^{-1}AP) = cT(A)$ as required

9. $[D]_{\mathcal{E}} = \left[ [D(1)]_{\mathcal{E}} \middle| [D(x)]_{\mathcal{E}} \middle| [D(x^2)]_{\mathcal{E}} \middle| [D(x^3)]_{\mathcal{E}} \middle| [D(x^4)]_{\mathcal{E}} \right]$

$= \left[ [0]_{\mathcal{E}} \middle| [1]_{\mathcal{E}} \middle| [2x]_{\mathcal{E}} \middle| [3x^2]_{\mathcal{E}} \middle| [4x^3]_{\mathcal{E}} \right]$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

10. $\ker(D) = \{ p(x) \in P_4(x) \mid D(p(x)) = 0 \}$
$= \{ p(x) \in P_4(x) \mid p'(x) = 0 \}$

We recognize $\ker(D) = \{$ constant functions $\}$ with basis $\{1\}$.

$\operatorname{Im}(D) = \{ q(x) \in P_4(x) \mid q(x) = D(p(x))$ for some $p(x) \in P_4(x) \}$

$= \{ q(x) \in P_4(x) \mid q(x) = p'(x)$ for some $p(x) \in P_4(x) \}$

We recognize $\{ q(x)$ that are derivatives of polynomials of degree $\leq 4 \}$

$= \{$polynomials of degree $\leq 3\}$ with basis $\{1, x, x^2, x^3\}$.

(Of course, other choices of bases are possible.)