Math 205 Exam 2 Answer key

1. Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 3 & 6 & 1 & 10 & 14 \\ 7 & 14 & 2 & 23 & 33 \end{bmatrix}$$

A has reduced row-echelon form

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Find a basis for the nullspace of A.

(b) Find a basis for the column space of A.

Solution:

Part (a)

The reduced row-echelon form corresponds to the following homogeneous linear system

$$\begin{array}{rrrrrl} x_1 & +2x_2 & & +3x_4 & & = 0 \\ & & x_3 & + \; x_4 & & = 0 \; . \\ & & & & x_5 & = 0 \end{array}$$

The variables x_2 and x_4 are free variables and the solution set is

$$S = \left\{ \begin{bmatrix} -2x_2 - 3x_4 \\ x_2 \\ -x_4 \\ x_4 \\ 0 \end{bmatrix} \middle| \, x_2, x_4 \in \mathbb{R} \right\}.$$

This yields a basis for the nullspace:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Part (a)

A basis for the column space is given by those columns of A which correspond to the columns with leading ones in a row echelon form (in this case the first, the third and the fifth):

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 4 \\ 14 \\ 33 \end{bmatrix} \right\}$$

Note that the column space of a reduced row-echelon form of A is in general different from the column space of A, which means that the columns need to be chosen in A!

(2) Let V be a 2-dimensional vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Let $T : V \longrightarrow V$ be a linear transformation with

$$T(\mathbf{v}_1) = 3\mathbf{v}_1 + 2\mathbf{v}_2,$$
$$T(\mathbf{v}_2) = 5\mathbf{v}_1 - 4\mathbf{v}_2.$$

(a) (4 points) Find $T(2\mathbf{v}_1 + \mathbf{v}_2)$.

$$T(2v_1+v_2) = 2T(v_1) + T(v_2)$$
$$= 2(3v_1 + 2v_2) + (5v_1 - 4v_2)$$
$$= 11v_1$$

(b) (4 points) Find $[T]_{\mathcal{B}}$, the matrix of T in the basis $\mathcal{B}$.

$$[T]_{\mathcal{B}} = \Big[[T(v_1)]_{\mathcal{B}} \,\Big|\, [T(v_2)]_{\mathcal{B}} \Big]$$

$$= \Big[[3v_1+2v_2]_{\mathcal{B}} \,\Big|\, [5v_1-4v_2]_{\mathcal{B}} \Big]$$

$$= \begin{bmatrix} 3 & 5 \\ 2 & -4 \end{bmatrix}.$$

(3) Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 14 \end{bmatrix} \right\}.$$

We know that $W = \operatorname{Span}(\mathcal{B})$ is a subspace of $\mathbb{R}^3$, and then $\mathcal{B}$ is a basis of W.

Let

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

(a) (4 points) Is $\mathbf{u}_1$ in W? If so, find the coordinate vector $[\mathbf{u}_1]_{\mathcal{B}}$.

$u_1 \in W \Leftrightarrow u_1 = c_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 7 \\ 14 \end{bmatrix}$, and if this is so, $[u_1]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

This gives the system with augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 7 & 3 \\ 4 & 14 & 4 \end{array}\right]$$

which reduces to $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{array}\right]$ which is inconsistent. Answer: No, $u_1 \notin W$.

(b) (4 points) Is $\mathbf{u}_2$ in W? If so, find the coordinate vector $[\mathbf{u}_2]_{\mathcal{B}}$.

We can see by inspection that $u_2 = v_1$ so $u_2 \in W$ and $[u_2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

If we don't notice that:

$u_2 \in W \Leftrightarrow u_2 = c_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$, and if this is so, $[u_2]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

This gives the system with augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & 7 & 2 \\ 4 & 14 & 4 \end{array}\right]$$

which reduces to $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ with solution $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so Yes, $u_2 \in W$ and $[u_2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(4) Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 10 \end{bmatrix} \right\}. \Rightarrow P_{E \leftarrow B} = \begin{pmatrix} 2 & 3 \\ 7 & 10 \end{pmatrix}$$

$\mathcal{B}$ is a basis of $\mathbb{R}^2$.

where $E = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is the Standard basis for $\mathbb{R}^2$.

(a) (4 points) If $[\mathrm{w}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, find w.

$$[W]_E = P_{E \leftarrow B} [W]_B = \begin{pmatrix} 2 & 3 \\ 7 & 10 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 28 \\ 95 \end{pmatrix}$$

$$\boxed{\text{Alternatively, } 5\begin{pmatrix} 2 \\ 7 \end{pmatrix} + 6\begin{pmatrix} 3 \\ 10 \end{pmatrix} = \begin{pmatrix} 28 \\ 95 \end{pmatrix}}$$

(b) (4 points) If $\mathrm{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find $[\mathrm{v}]_{\mathcal{B}}$.

$$[V]_B = P_{B \leftarrow E} [V]_E = \left(P_{E \leftarrow B}\right)^{-1} [V]_E = \begin{pmatrix} 2 & 3 \\ 7 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 & 3 \\ 7 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$\boxed{\text{Alternatively, } \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 7 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 10 \end{pmatrix} \Rightarrow \alpha = -4, \beta = 3}$$

(c) (8 points) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation $T(\mathrm{x}) = A\mathrm{x}$, where A is the matrix

$$A = \begin{bmatrix} 11 & 13 \\ 17 & 19 \end{bmatrix}.$$

Find $[T]_{\mathcal{B}}$. (You may leave your answer expressed as a product of matrices and their inverses.)

$$[T]_B = P_{B \leftarrow E} [T]_E P_{E \leftarrow B}$$

$$= \begin{pmatrix} 2 & 3 \\ 7 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 11 & 13 \\ 17 & 19 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 7 & 10 \end{pmatrix}$$

Problem 5

A has the characteristic polynomial $-(\lambda-1)(\lambda-2)(\lambda-4)$, so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 4$.

- For $\lambda_1 = 1$, we solve $(A - \lambda_1 I)\mathbf{v} = \mathbf{0}$.

$$A - \lambda_1 I = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{P_{12}} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{A_{12}(-2)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$\xrightarrow{P_{23}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -4 & 0 \end{bmatrix} \xrightarrow{A_{23}(4)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, $v_3 = t$ is a free variable, we may obtain $v_2 = 0$ and $v_1 = -t$. Hence,

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, the eigenspace E_1 is spanned by $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

- For $\lambda_2 = 2$, we solve $(A - \lambda_2 I)\mathbf{v} = \mathbf{0}$.

$$A - \lambda_2 I = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{A_{12}(-1)} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{A_{23}(-1)} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, $v_3 = s$ is a free variable, we may obtain $v_2 = s$ and $v_1 = -2s$. Hence,

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, the eigenspace E_2 is spanned by $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

- For $\lambda_3 = 4$, we solve $(A - \lambda_3 I)\mathbf{v} = \mathbf{0}$.

$$A - \lambda_3 I = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{M_1(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 1 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{A_{12}(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{bmatrix}$$
$$\xrightarrow{M_2(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{A_{23}(-1)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, $v_3 = r$ is a free variable, we may obtain $v_2 = 3r$ and $v_1 = 2r$. Hence,

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = r \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

Therefore, the eigenspace E_3 is spanned by $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

In summary, letting

$$P = \begin{bmatrix} -1 & -2 & 2 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix},$$

we have $P^{-1}AP = D$.

(6) (36 points) Solve each of the following differential equations/initial value problems:

(a) $y'''' + 2y''' - 5y'' - 6y' = 0$.

(In operator notation, this is the differential equation $(D^4+2D^3-5D^2-6D)y = 0$. Note that $D^4 + 2D^3 - 5D^2 - 6D = D(D+1)(D-2)(D+3)$.)

The auxiliary equation is
$\lambda(\lambda+1)(\lambda-2)(\lambda+3)=0$.

The solutions are
$\lambda=0$, $\lambda=-1$, $\lambda=2$, $\lambda=-3$.

Therefore, the solution is
$y = C_1 + C_2 e^{-x} + C_3 e^{2x} + C_4 e^{-3x}$,
where C_1, C_2, C_3, C_4 are constants.

(b) $y'''' - 13y''' + 60y'' - 122y' + 64y = 0$.

(In operator notation, this is the differential equation $(D^4-13D^3+60D^2-112D+64)y = 0$. Note that $D^4 - 13D^3 + 60D^2 - 112D + 64 = (D-1)(D-4)^3$.)

The auxiliary equation is
$(\lambda-1)(\lambda-4)^3=0$.

The solutions are $\lambda=1$, $\lambda=4$.

Therefore, the solution of the differential equation is

$y = C_1 e^{x} + C_2 e^{4x} + C_3 x e^{4x} + C_4 x^2 e^{4x}$,

where C_1, C_2, C_3, C_4 are real constants

(c) $y'' - 4y' + 29y = 0$.
(In operator notation, this is the differential equation $(D^2 - 4D + 29)y = 0$.)

The auxiliary equation is
$\lambda^2 - 4\lambda + 29 = (\lambda - 2)^2 + 5^2 = 0$.
The solutions are $\lambda = 2 \pm 5i$.
Therefore, the solution is

$$y = C_1 e^{2x} \cos(5x) + C_2 e^{2x} \sin(5x),$$

where C_1 and C_2 are real constants

(d) $y'' - 4y' + 3y = 0, \quad y(0) = 3,\ y'(0) = 7$.
(In operator notation, this is the differential equation $(D^2 - 4D + 3)y = 0$. Note that $(D^2 - 4D + 3) = (D - 1)(D - 3)$.)

It is very easy to find the solution
$y = C_1 e^x + C_2 e^{3x}$ Let us
compute the derivative We have
$y' = C_1 e^x + 3C_2 e^{3x}$ Now let
us use the initial conditions.

$$3 = y(0) = C_1 + C_2$$
$$7 = y'(0) = C_1 + 3C_2$$

So $C_1 = 1$ $C_2 = 2$ Therefore,
$y = e^x + 2e^{3x}$

(e) $y'' + 4y' + 3y = 60e^{2x}$.
(In operator notation, this is the differential equation $(D^2 + 4D + 3)y = 60e^{2x}$. Note that $(D^2 + 4D + 3) = (D + 1)(D + 3)$.)

The complementary solution is

$y_c(x) = C_1 e^{-x} + C_2 e^{-3x}$

Suppose that $y_p(x) = Ae^{2x}$ is a particular solution. Then

$y_p'' + 4y_p' + 3y_p = 15Ae^{2x} = 60e^{2x}$

So $A = 4$. Therefore, the general solution is

$y(x) = C_1 e^{-x} + C_2 e^{-3x} + 4e^{2x}$

(f) $y'' - 4y = 12e^{2x}$.
(In operator notation, this is the differential equation $(D^2 - 4)y = 12e^{2x}$. Note that $(D^2 - 4) = (D - 2)(D + 2)$.)

The complementary solution is

$y_c(x) = C_1 e^{2x} + C_2 e^{-2x}$

Suppose that $y_p(x) = Axe^{2x}$ is a particular solution. Then

$y_p''(x) = 4Axe^{2x} + 4Ae^{2x}$

$y_p''(x) - 4y_p = 4Ae^{2x} = 12e^{2x}$

So $A = 3$. Therefore,

$y(x) = C_1 e^{2x} + C_2 e^{-2x} + 3xe^{2x}$