Fall 2019 - Math 205
Homework 7

Due at the beginning of class on Weds. Oct. 16 (Profs. Zhang and Wu), Thurs. Oct. 17 (Profs. Coll, Weintraub, Recio-Mitter). Write your name and section number on your homework. You must show your work in order to receive full credit.

We share a philosophy about linear algebra:
we think basis-free, we write basis-free,
but when the chips are down we close the office door
and compute with matrices like fury.

Irving Kaplansky about Paul Halmos

1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -4 & 2 \\ -15 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Find a basis $\mathcal{B}$ of eigenvectors and find the matrix $[T]_{\mathcal{B}}^{\mathcal{B}}$.

Solution: (Graded)

The characteristic polynomial is

$$\det(A - \lambda I) = (-4 - \lambda)(7 - \lambda) - 2(-15) = \lambda^2 - 3\lambda + 2.$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 1$.

The nullspace of

$$A - \lambda_1 I = A - 2I = \begin{bmatrix} -6 & 2 \\ -15 & 5 \end{bmatrix}$$

is

$$S = \left\{ \left[\begin{array}{c} t \\ 3t \end{array} \right] \,\middle|\, t \in \mathbb{R} \right\}$$

and the nullspace of

$$A - \lambda_2 I = A - I = \left[\begin{array}{cc} -5 & 2 \\ -15 & 6 \end{array} \right]$$

is

$$S = \left\{ \left[\begin{array}{c} 2t \\ 5t \end{array} \right] \,\middle|\, t \in \mathbb{R} \right\}.$$

Therefore, a basis of eigenvectors is given by $\mathcal{B} = \left\{ \left[\begin{array}{c} 1 \\ 3 \end{array} \right], \left[\begin{array}{c} 2 \\ 5 \end{array} \right] \right\}$.

Note that $[T]_E^E = \left[\begin{array}{cc} -4 & 2 \\ -15 & 7 \end{array} \right]$, where E is the standard basis of $\mathbb{R}^2$. To find $[T]_{\mathcal{B}}^{\mathcal{B}}$ we just need to find the basis change matrix $P_{E\leftarrow\mathcal{B}}$ because of:

$$[T]_{\mathcal{B}}^{\mathcal{B}} = P_{\mathcal{B}\leftarrow E}\, [T]_E^E\, P_{E\leftarrow\mathcal{B}} = P_{E\leftarrow\mathcal{B}}^{-1}\, [T]_E^E\, P_{E\leftarrow\mathcal{B}}.$$

By definition, $P_{E\leftarrow\mathcal{B}} = \left[\begin{array}{cc} 1 & 2 \\ 3 & 5 \end{array} \right]$ is the matrix with the vectors of $\mathcal{B}$ as columns.

Therefore:

$$[T]_{\mathcal{B}}^{\mathcal{B}} = P_{E\leftarrow\mathcal{B}}^{-1}\, [T]_E^E\, P_{E\leftarrow\mathcal{B}} = \left[\begin{array}{cc} 1 & 2 \\ 3 & 5 \end{array} \right]^{-1} \left[\begin{array}{cc} -4 & 2 \\ -15 & 7 \end{array} \right] \left[\begin{array}{cc} 1 & 2 \\ 3 & 5 \end{array} \right] = \left[\begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \right].$$

Remark: This is of course precisely a diagonalization of the matrix $[T]_E^E$, with the diagonal entries being the eigenvalues. Diagonalization is really a special type of basis change.

Alternatively, we could find $[T]_{\mathcal{B}}^{\mathcal{B}}$ using the general formula for the matrix representing a linear transformation in terms of a pair of bases (in this cases the same basis):

$$\begin{aligned}
[T]_{\mathcal{B}}^{\mathcal{B}} &= \left[\left[T\left(\left[\begin{array}{c} 1 \\ 3 \end{array} \right] \right) \right]_{\mathcal{B}}, \left[T\left(\left[\begin{array}{c} 2 \\ 5 \end{array} \right] \right) \right]_{\mathcal{B}} \right] \\
&= \left[\left[\begin{array}{c} 2 \\ 6 \end{array} \right]_{\mathcal{B}}, \left[\begin{array}{c} 2 \\ 5 \end{array} \right]_{\mathcal{B}} \right] \\
&= \left[\begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \right].
\end{aligned}$$

2. Is the matrix

$$A = \begin{bmatrix} -9 & 4 \\ -30 & 13 \end{bmatrix}$$

similar to the matrix

$$B = \begin{bmatrix} -4 & 2 \\ -15 & 7 \end{bmatrix}$$

from exercise 1? In other words, does there exist a matrix S such that $SAS^{-1} = B$?

Solution:

The characteristic polynomial of

$$A = \begin{bmatrix} -9 & 4 \\ -30 & 13 \end{bmatrix}$$

is

$$\det(A - \lambda I) = (-9 - \lambda)(13 - \lambda) - 4(-30) = \lambda^2 - 4\lambda + 3.$$

The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 1$. Because similar matrices have the same eigenvalues and B has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$, A and B are *not* similar.

3. Find the eigenvalues and eigenspaces of the following matrix and determine whether the matrix is diagonalizable.

$$A = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$$

Solution: (Graded)

The characteristic polynomial of A is $(\lambda - 2)^2$. Therefore, the matrix A has only one eigenvalue $\lambda_1 = 2$, which has algebraic multiplicity 2.

The eigenspace of $\lambda_1 = 2$ is the nullspace of

$$A - \lambda_1 I = A - 2I = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}.$$

The first variable is a free variable, while the second has to be 0. Therefore the eigenspace is

$$S = \left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}.$$

Because the eigenspace is 1-dimensional, the geometric multiplicity of λ_1 is 1. However, we saw that the algebraic multiplicity is 2. Therefore, the matrix is not diagonalizable. We also say that it is defective.

4. Determine the eigenvalues and eigenspaces of the following matrix .

$$A = \begin{bmatrix} 6 & 2 & 8 \\ -2 & 1 & -4 \\ -2 & -1 & -2 \end{bmatrix}$$

Hint: You may use that the characteristic polynomial of A is $(\lambda - 1)(\lambda - 2)^2$.

Solution: (Graded)

Because the characteristic polynomial of A is $(\lambda - 1)(\lambda - 2)^2$, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$.

The nullspace of

$$A - \lambda_1 I = A - I = \begin{bmatrix} 5 & 2 & 8 \\ -2 & 0 & -4 \\ -2 & -1 & -3 \end{bmatrix}$$

is

$$S = \left\{ \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}.$$

This is the eigenspace for $\lambda_1 = 1$.

The nullspace of

$$A - \lambda_2 I = A - 2I = \begin{bmatrix} 4 & 2 & 8 \\ -2 & -1 & -4 \\ -2 & -1 & -4 \end{bmatrix}$$

is

$$S = \left\{ \begin{bmatrix} -t - 2s \\ 2t \\ s \end{bmatrix} \middle| \; t, s \in \mathbb{R} \right\}.$$

This is the eigenspace for $\lambda_2 = 2$.

5. Determine the algebraic and geometric multiplicities of the eigenvalues of the matrix A in problem 4. Is A diagonalizable? If yes give a basis of eigenvectors and the corresponding diagonalization. For the diagonalization no further computations are needed.

Solution: (Graded)

The geometric and algebraic multiplicities of $\lambda_1 = 1$ are both 1. The algebraic and geometric multiplicities of $\lambda_2 = 2$ are both 2.

The matrix is diagonalizable because the geometric and algebraic multiplicities coincide.

A basis of eigenvectors is given by

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The corresponding diagonalization is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

6. Determine the eigenvalues and eigenspaces of the following matrix .

$$A = \begin{bmatrix} 9 & -3 & 5 \\ 4 & 1 & 4 \\ -6 & 3 & -2 \end{bmatrix}$$

Hint: You may use that the characteristic polynomial of A is $(\lambda - 1)(\lambda - 3)(\lambda - 4)$.

Solution:

Because the characteristic polynomial of A is $(\lambda - 1)(\lambda - 3)(\lambda - 4)$, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$ and $\lambda_3 = 4$.

The nullspace of

$$A - \lambda_1 I = A - I = \begin{bmatrix} 8 & -3 & 5 \\ 4 & 0 & 4 \\ -6 & 3 & -3 \end{bmatrix}$$

is

$$S = \left\{ \begin{bmatrix} t \\ t \\ -t \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}.$$

This is the eigenspace for $\lambda_1 = 1$.

The nullspace of

$$A - \lambda_2 I = A - 3I = \begin{bmatrix} 6 & -3 & 5 \\ 4 & -2 & 4 \\ -6 & 3 & -5 \end{bmatrix}$$

is

$$S = \left\{ \begin{bmatrix} t \\ 2t \\ 0 \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}.$$

This is the eigenspace for $\lambda_2 = 3$.

The nullspace of

$$A - \lambda_3 I = A - 4I = \begin{bmatrix} 5 & -3 & 5 \\ 4 & -3 & 4 \\ -6 & 3 & -6 \end{bmatrix}$$

is

$$S = \left\{ \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}.$$

This is the eigenspace for $\lambda_3 = 4$.

7. Determine the algebraic and geometric multiplicities of the eigenvalues of the matrix A in problem 6. Is A diagonalizable? If yes give a basis of eigenvectors and the corresponding diagonalization. For the diagonalization no further computations are needed.

Solution:

All eigenvalues have algebraic multiplicity 1 and geometric multiplicity 1.

The matrix is diagonalizable because the geometric and algebraic multiplicities coincide.

A basis of eigenvectors is given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

The corresponding diagonalization is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

8. Find the eigenvalues and eigenspaces of the following matrix.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Is the matrix diagonalizable? If the answer is yes, find a basis of eigenvectors and the corresponding diagonalization.

Solution: (Graded)

The characteristic polynomial is

$$\det(A - \lambda I) = (1 - \lambda)(1 - \lambda) + 1 = \lambda^2 - 2\lambda + 2.$$

The eigenvalues are $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$.

The nullspace of

$$A - \lambda_1 I = A - (1 + i)I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$$

is

$$S = \left\{ \begin{bmatrix} it \\ t \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}.$$

This is the eigenspace of $\lambda_1 = 1 + i$.

The eigenspace of $\lambda_2 = 1 - i$ will be conjugate to the eigenspace of $\lambda_1 = 1 + i$, because the eigenvalues are conjugate to each other. Complex eigenvalues of real-valued matrices always come in conjugate pairs.

Thus the eigenspace of $\lambda_2 = 1 - i$ is

$$S = \left\{ \begin{bmatrix} -it \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

Therefore, basis of eigenvectors is given by $\mathcal{B} = \left\{ \begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$.

The corresponding diagonalization is

$$\begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}.$$

9. Find the eigenvalues and a basis of eigenvectors of the following matrix.

$$\begin{bmatrix} 7 & -5 \\ 13 & 8 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 17 \end{bmatrix} \begin{bmatrix} 7 & -5 \\ 13 & 8 \end{bmatrix}^{-1}$$

Solution:

We can read off the eigenvalues $\lambda_1 = 6$ and $\lambda_2 = 17$, as well as a basis of eigenvectors

$$\mathcal{B} = \left\{ \begin{bmatrix} 7 \\ 13 \end{bmatrix}, \begin{bmatrix} -5 \\ 8 \end{bmatrix} \right\}.$$

10. Use diagonalization to compute

$$\begin{bmatrix} 13 & -42 \\ 4 & -13 \end{bmatrix}^{999}.$$

Solution:

The characteristic polynomial is

$$\det(A - \lambda I) = (13 - \lambda)(-13 - \lambda) + (-42)4 = \lambda^2 - 1.$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$.

The nullspace of

$$A - \lambda_1 I = A - I = \begin{bmatrix} 12 & -42 \\ 4 & -14 \end{bmatrix}$$

is

$$S = \left\{ \left[\begin{array}{c} 7t \\ 2t \end{array} \right] \;\middle|\; t \in \mathbb{R} \right\}$$

and the nullspace of

$$A - \lambda_2 I = A + I = \left[\begin{array}{rr} 14 & -42 \\ 4 & -12 \end{array} \right]$$

is

$$S = \left\{ \left[\begin{array}{c} 3t \\ t \end{array} \right] \;\middle|\; t \in \mathbb{R} \right\}.$$

Therefore, a basis of eigenvectors is given by $\mathcal{B} = \left\{ \left[\begin{array}{c} 7 \\ 2 \end{array} \right], \left[\begin{array}{c} 3 \\ 1 \end{array} \right] \right\}$.

The corresponding diagonalization is

$$\left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array} \right] = \left[\begin{array}{rr} 7 & 3 \\ 2 & 1 \end{array} \right]^{-1} \left[\begin{array}{rr} 13 & -42 \\ 4 & -13 \end{array} \right] \left[\begin{array}{rr} 7 & 3 \\ 2 & 1 \end{array} \right].$$

Using this we can compute:

$$\begin{aligned}
\left[\begin{array}{rr} 13 & -42 \\ 4 & -13 \end{array} \right]^{999} &= \left[\begin{array}{rr} 7 & 3 \\ 2 & 1 \end{array} \right] \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array} \right]^{999} \left[\begin{array}{rr} 7 & 3 \\ 2 & 1 \end{array} \right]^{-1} \\
&= \left[\begin{array}{rr} 7 & 3 \\ 2 & 1 \end{array} \right] \left[\begin{array}{cc} 1^{999} & 0 \\ 0 & (-1)^{999} \end{array} \right] \left[\begin{array}{rr} 7 & 3 \\ 2 & 1 \end{array} \right]^{-1} \\
&= \left[\begin{array}{rr} 7 & 3 \\ 2 & 1 \end{array} \right] \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array} \right] \left[\begin{array}{rr} 7 & 3 \\ 2 & 1 \end{array} \right]^{-1} \\
&= \left[\begin{array}{rr} 13 & -42 \\ 4 & -13 \end{array} \right]
\end{aligned}$$