Solution 8

Problem 1

Determine whether the following matrices are diagonalizable or not.

1. $A = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$.

2. $B = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$.

3. $C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

4. $D = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

Solution:

1. The characteristic polynomial is $p(\lambda) = (1 - \lambda)(2 - \lambda)$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$. Since they are distinct, there must be two linearly independent eigenvectors, so A is diagonalizable.

2. The characteristic polynomial is $p(\lambda) = (2 - \lambda)^2$. The eigenvalues are $\lambda_1 = \lambda_2 = 2$. However, there is only one linearly independent eigenvector, so B is not diagonalizable.

3. Since C is diagonal, it is certainly diagonalizable.

4. The characteristic polynomial is $p(\lambda) = \lambda(\lambda - 4)$. The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 4$. Since they are distinct, there must be two linearly independent eigenvectors, so D is diagonalizable.

Problem 2

Determine whether the following matrices are diagonalizable or not.

1. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 0 \end{bmatrix}$.

2. $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Solution:

1. The characteristic polynomial is $p(\lambda) = (2-\lambda)(\lambda^2 - 12)$. The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = \sqrt{12}$ and $\lambda_3 = -\sqrt{12}$. Since they are distinct, there must be three linearly independent eigenvectors, so A is diagonalizable.

2. The characteristic polynomial is $p(\lambda) = (\lambda - 1)^2(\lambda + 1)$. The eigenvalues are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -1$. There exist three linearly independent eigenvectors, so B is diagonalizable.

Problem 3

Determine whether $A = \begin{bmatrix} -1 & 2 & 2 \\ -4 & 5 & 2 \\ -4 & 2 & 5 \end{bmatrix}$ is diagonalizable or not. The characteristic polynomial is $p(\lambda) = (3-\lambda)^3$.

Solution: The eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 3$. However, there are only two linearly independent eigenvectors, so A is not diagonalizable.

Problem 4

Diagonalize the following matrices:

1. $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$.

2. $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

3. $C = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$.

4. $D = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$.

Solution:

1. The characteristic polynomial is $p(\lambda) = (1-\lambda)(2-\lambda)$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$. Correspondingly, the bases of eigenspaces are $\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}, \left\{ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Hence, letting $S = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, we have $S^{-1}AS = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

2. The characteristic polynomial is $p(\lambda) = \lambda(\lambda - 2)$. The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$. Correspondingly, the the bases of eigenspaces are $\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \left\{ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Hence, letting $S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, we have $S^{-1}BS = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

3. The characteristic polynomial is $p(\lambda) = (\lambda+2)(\lambda-2)$. The eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 2$. Correspondingly, the bases of eigenspaces are $\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \left\{ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Hence, letting $S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, we have $S^{-1}CS = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$.

4. The characteristic polynomial is $p(\lambda) = (1-\lambda)(\lambda-2)^2$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 2$. Correspondingly, the bases of eigenspaces are $\left\{ \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \left\{ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Hence, letting $S = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, we have $S^{-1}DS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Problem 5

Assume A is an invertible matrix.

1. Prove that 0 is not an eigenvalue of A.
2. Assume λ is an eigenvalue of A. Show that λ^{-1} is an eigenvalue of A^{-1}.

Solution:

1. We prove by contradiction. Assume 0 is an eigenvalue of A. Then $\det(\lambda I - A) = \det(-A) = (-1)^n \det(A) = 0$. Hence, $\det(A) = 0$, which means A is not invertible. This contradicts our assumption.

2. λ is an eigenvalue of A, so there exists a vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Then multiplying $\lambda^{-1}A^{-1}$ on both sides, we have $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$. Hence, we know λ^{-1} is an eigenvalue of A^{-1}. In particular, they have the same eigenspace.

Remark 0.1. *Part (1) can also be proved as follows: If* 0 *is an eigenvalue, then there exists a nonzero vector $\vec{v}$ such that $A\vec{v} = 0\vec{v} = \vec{0}$. Hence, the homogeneous equation $A\boldsymbol{x} = 0$ has a nonzero solution, which yields that A must be non-invertible. This contradicts our assumption.*

Problem 6

Prove that $\mathrm{e}^x \sin(x)$ and $\mathrm{e}^x \cos(x)$ are linearly independent.

Solution: We compute the Wronskian $W = \begin{vmatrix} \mathrm{e}^x \sin(x) & \mathrm{e}^x \cos(x) \\ \mathrm{e}^x \sin(x) + \mathrm{e}^x \cos(x) & \mathrm{e}^x \cos(x) - \mathrm{e}^x \sin(x) \end{vmatrix} = -\mathrm{e}^x \neq 0$. Hence, they are linearly independent.

Problem 7

Find the general solution $y(x)$ to the equation $y'' - 6y' + 9y = 0$.

Solution: The characteristic equation is $r^2 - 6r + 9 = 0$ which has repeated root $r_1 = r_2 = 3$. Hence, the general solution is $y(x) = C_1\mathrm{e}^{3x} + C_2x\mathrm{e}^{3x}$.

Problem 8

Find the general solution $y(x)$ to the equation $y'' - 2y' + 2y = 0$.

Solution: The characteristic equation is $r^2 - 2r + 2 = 0$ which has complex roots $r_1 = 2 + 2\mathrm{i}$ and $r_2 = 2 - 2\mathrm{i}$. Hence, the general solution is $y(x) = C_1\mathrm{e}^{2x}\sin(2x) + C_2\mathrm{e}^{2x}\cos(2x)$.

Problem 9

Find the general solution $y(x)$ to the equation $2y'' + 3y' - 2y = 0$.

Solution: The characteristic equation is $2r^2 + 3r - 2 = 0$ which has distinct real roots $r_1 = -2$ and $r_2 = \dfrac{1}{2}$. Hence, the general solution is $y(x) = C_1\mathrm{e}^{-2x} + C_2\mathrm{e}^{\frac{1}{2}x}$.

Problem 10

1. Find the general solution $y(x)$ to the equation $y'' + 6y' + 9y = 0$.
2. Find a particular solution $y_p(x)$ that has the form $y_p(x) = Dx^2\mathrm{e}^{-3x}$ for some constant D to the equation $y'' + 6y' + 9y = 2\mathrm{e}^{-3x}$.
3. Find the general solution $y(x)$ to the equation $y'' + 6y' + 9y = 2\mathrm{e}^{-3x}$.

Solution:

1. The characteristic equation is $r^2+6r+9 = 0$ whose roots are $r_1 = r_2 = -3$. Hence, the solution is

$$y(x) = C_1\mathrm{e}^{-3x} + C_2x\mathrm{e}^{-3x}.$$

2. We know

$$\begin{aligned} y_p'(x) &= 2Dx\mathrm{e}^{-3x} - 3Dx^2\mathrm{e}^{-3x}, \\ y_p''(x) &= 2D\mathrm{e}^{-3x} - 12Dx\mathrm{e}^{-3x} + 9Dx^2\mathrm{e}^{-3x}. \end{aligned}$$

Insert above into the non-homogeneous equation, we have

$$\begin{aligned} y_p'' + 6y_p' + 9y_p &= \Big(2D\mathrm{e}^{-3x} - 12Dx\mathrm{e}^{-3x} + 9Dx^2\mathrm{e}^{-3x}\Big) + 6\Big(2Dx\mathrm{e}^{-3x} - 3Dx^2\mathrm{e}^{-3x}\Big) + 9\Big(Dx^2\mathrm{e}^{-3x}\Big) \\ &= 2D\mathrm{e}^{-3x} = 2\mathrm{e}^{-3x}. \end{aligned}$$

Hence, we have $D = 1$. Therefore, a particular solution is

$$y_p(x) = x^2\mathrm{e}^{-3x}.$$

3. Hence, the general solution is

$$y(x) = C_1\mathrm{e}^{-3x} + C_2t\mathrm{e}^{-3x} + x^2\mathrm{e}^{-3x}.$$