ASSIGNMENT 1.1

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1) Derangement problem

Ans.

- Let $\{1, 2, \ldots N\}$ label both letters and envelopes.
- No. of derangements is $N! \sum_{k=0}^{N} \frac{(-1)^k}{k!}$. — (1)

$\Rightarrow P(\text{No letter correct}) = \frac{(1)}{N!} = \sum_{k=0}^{k} \frac{(-1)^k}{k!}$

$\Rightarrow P(\text{at least one correct}) = 1 - \sum_{k=0}^{N} \frac{(-1)^k}{k!}$

as $N \to \infty$, $\sum_{k=0}^{N} \frac{(-1)^k}{k!} \to \frac{1}{e}$

as $N = 50$, N is large enough, so

$P(\text{at least one correct}) = 1 - \frac{1}{e} \approx 0.63$.

(2) ~~Showman~~ Monty Hall

Ans.

• 1^{st} case : \$1000 is in present ~~P~~ 1.
present 2 & present 3 (both empty).
∴ payoff = 0.

• 2^{nd} case : \$1000 is in present 2. first, we choose P1 (empty). Then he can only open present 3. on switching, he wins \$1000.

• 3^{rd} case : \$1000 is in present 3. Choose present 1. ~~He~~ can't open present 3, so present 2 open. If switched now, switch to present 3 and win \$1000.

Hence, Case 1 (prob. $\frac{1}{3}$) = \$0 + Case 2 (prob. $\frac{1}{3}$)

+ case 3 (prob. $\frac{1}{3}$) = \$1000

$\Rightarrow E(\text{win} \mid \text{switch}) = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1000 + \frac{1}{3} \cdot 1000$

$= \$666.67$.

(3) a) $P(A \cap B | C) = P(A | B \cap C)\, P(B | C)$

$P(A \cap B | C) = \dfrac{P(A \cap B \cap C)}{P(C)}$

$\dfrac{P(A \cap B \cap C)}{P(A)} = P(A | B \cap C)$, $P(B | C) = \dfrac{P(B \cap C)}{P(C)}$

$\Rightarrow P(A | B \cap C)\, P(B | C)$

$= \dfrac{P(A \cap B \cap C)}{P(B \cap C)} \times \dfrac{P(B \cap C)}{P(C)} = \dfrac{P(A \cap B \cap C)}{P(C)}$

$= P(A \cap B | C)$

True

(b) $P(A \cap B | C) = P(A | C)\, P(B | C)$

A & B independent means $\Rightarrow P(A \cap B) = P(A) P(B)$

$\dfrac{P(A \cap C)}{P(C)} \cdot \dfrac{P(B \cap C)}{P(C)} \Rightarrow \cancel{P(A)\,P}$

$\dfrac{P(A \cap B \cap C)}{P(C)}$

false

c) $P(A|D\cap B^c) > P(A|D\cap B)$ and

$P(A|D^c\cap B^c) > P(A|D^c\cap B)$,

$P(A|B) > P(A|B^c)$?

True.

(4)

a) discrete random variable.

$$P(X=k) = \frac{1}{\zeta(3)} \cdot \frac{1}{k^3}, \quad k=1,2,3,\ldots$$

$\zeta = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is finite.

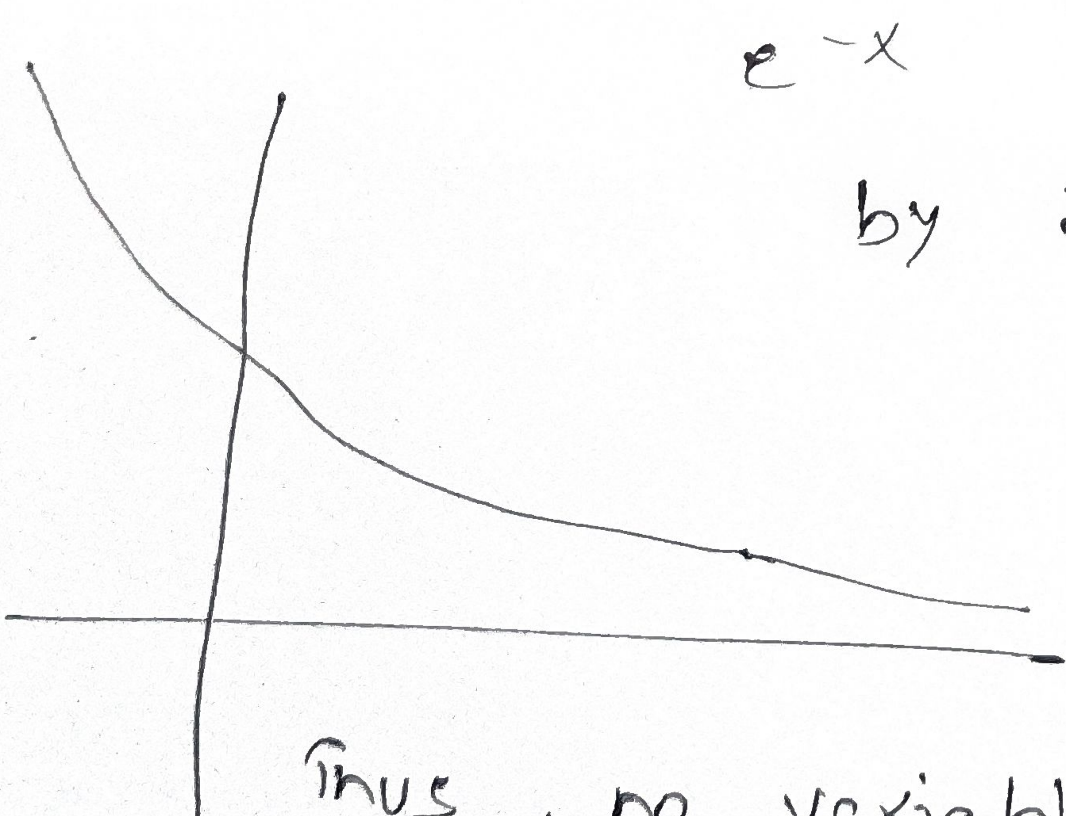
b) $f(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1 \\ 0, & x < 1 \end{cases}$

Then, $\int_1^\infty \frac{2}{x^3}\, dx = 1,$

$$E(X) = \int_1^\infty x \frac{2}{x^3} = 2\int_1^\infty \frac{dx}{x^2} = 2\left[-\frac{1}{x}\right]_1^\infty = 2.$$

$$E(X^2) = \int_1^\infty x^2 \cdot \frac{2}{x^3} = 2\int_1^\infty \frac{dx}{x} = 2\left[\ln x\right]_1^\infty = \infty.$$

c) $E(X) = 1$, $E(e^{-X}) \leq \frac{1}{3}$. e^{-x} is convex.



by Jensen's inequality,

$$E[e^{-X}] \geq e^{-E(X)}$$

$$e^{-1} \approx 0.367 > \frac{1}{3}.$$

Thus, no variable can satisfy both $E(X) = 1$ and $E(e^{-X}) < \frac{1}{3}$.

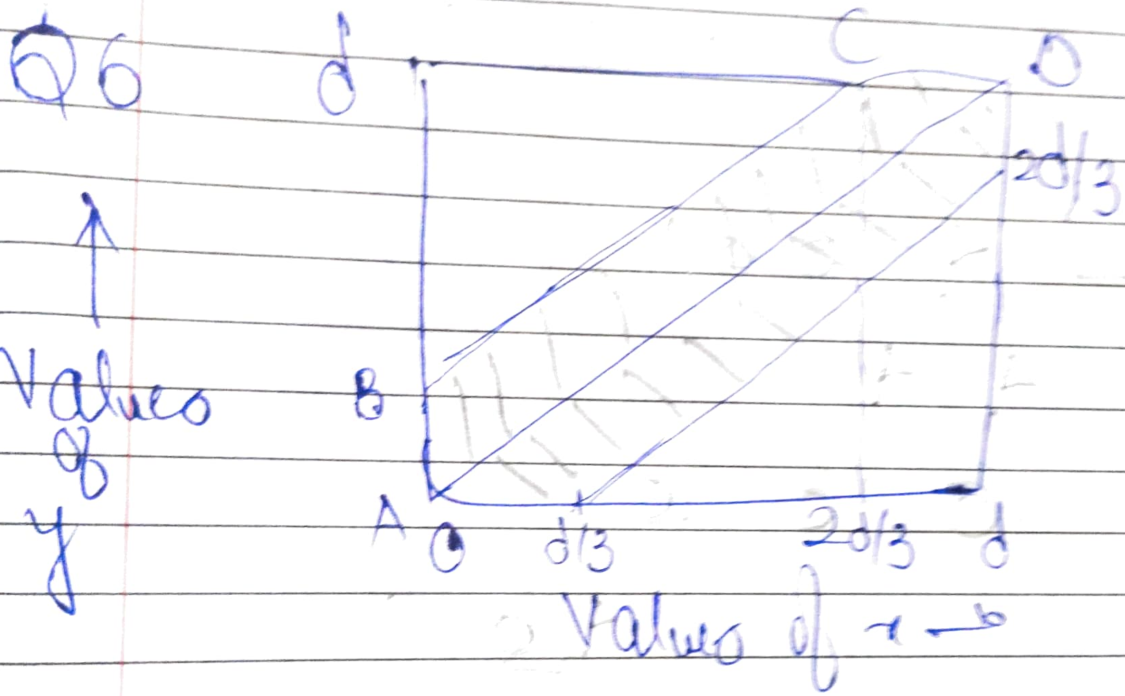
d (values

$|x-y| \leq \dfrac{d}{3}$

$x + \dfrac{d}{3} = y \quad x > y \Rightarrow y > x$

$x > y$

Q6



$|x-y| < \dfrac{d}{3}$

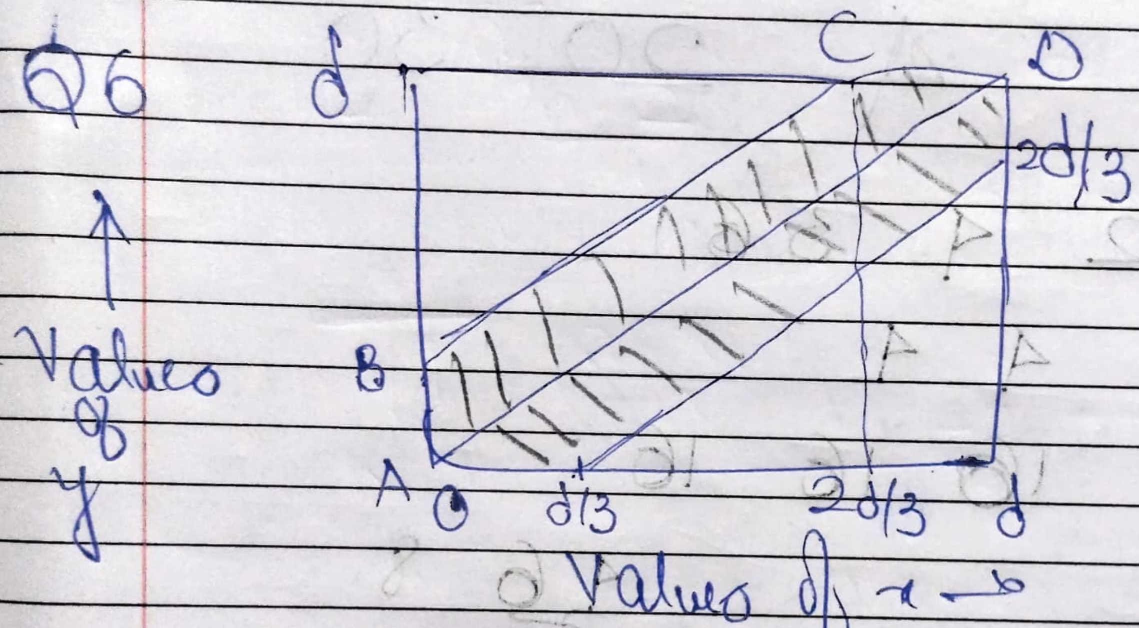
$\boxed{\text{Case I}} \rightarrow x \geq y$

$\Rightarrow (i)\ y \geq x - \dfrac{d}{3}$

$\boxed{\text{Case II}}$

$x \leq y$

Q6



$|x-y| \leq \frac{d}{3}$

$\boxed{\text{Case I}} \rightarrow x \geq y$

$\Rightarrow$ (i) $y \geq x - \frac{d}{3}$

$\boxed{\text{Case II}} \rightarrow y \geq x$

$x \geq y - \frac{d}{3}$

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Shaded area $= 2 \times ar(ABCD)$

$= 2 \times \frac{1}{2} \times \left(\frac{d}{3} \times \frac{1}{\sqrt{2}}\right) \left(\sqrt{2}d - 2 \times \frac{d}{3\sqrt{2}}\right)$

$= \frac{5}{9} d^2$

$\therefore ar(\text{Rectangle}) = d^2$

$\boxed{\text{Probability} = \frac{5}{9}}$

Q7. The current holder will choose from n remaining ~~habitant~~ inhabitants.

[Case-I]

(a)

~~For the first~~

At a particular time the ~~for~~ person who wants to tell the rumor will not choose himself and the originator.

$\therefore$ P[above written line happens] $= \frac{n-1}{n}$

$\therefore$ For that to happen "r" time $= \left(\frac{n-1}{n}\right)^r$

[$\because$ They are independent]

(b) Start

Originator → Person 1

n people remaining — n−1 people remaining

~~Pf~~

E_x :- Event that person "x" will tell the rumor to someone who doesn't know it.

$P(E_1) = \frac{n-1}{n}$

$P(E_2) = \frac{n-2}{n}$

$$p(\text{no repeat in first r steps}) = \prod_{k=1}^{r} \left(\frac{n-k}{n}\right)$$

(∵) Because events are independent

Case-II

(a) Probability that originator is not there in group of "N" people $= \frac{{}^{n-1}C_N}{{}^{n}C_N}$

Probability $= \dfrac{\dfrac{(n-1)!}{N!\,(n-1-N)!}}{\dfrac{n!}{N!\,(n-N)!}}$

$$= \frac{n-N}{n}$$

$$\boxed{P\begin{pmatrix}\text{no return} \\ \text{in } r \text{ steps}\end{pmatrix} = \left(\frac{n-N}{n}\right)^r}$$

(b) Copying the same notations from $\boxed{\text{case-I}}$ part (b):

$$P(E_1) = {}^{n-1}C_N$$

$$P(E_2) = {}^{n-1-N}C_N$$

$$P(E_3) = {}^{n-1-2N}C_N$$

$$\therefore \boxed{\text{Total probability} = \prod_{k=1}^{r} {}^{n-1-(k-1)N}C_N}$$

Q8. As A_1, A_2 etc. are independent.

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} P(A_i^c)$$

Their complements will also be independent.

~~$A' \cap B' = 1 - (A \cup B)$~~

~~$= 1 - A - B + A \cap B$~~

$$\boxed{\begin{aligned} P(A' \cap B') &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - P(A) - P(B) + P(A)\,P(B) \\ &= [1 - P(A)][1 - P(B)] \\ &= P(A')\,.\,P(B') \end{aligned}}$$

$$\therefore P\left(\bigcap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} P(A_i^c) = \prod_{i=1}^{n} [1 - P(A_i)]$$

We know for real x: $1 - x \leq e^{-x}$

$$\therefore \boxed{P\left(\bigcap_{i=1}^{n} A_i^c\right) \leq \prod_{i=1}^{n} e^{-\sum_{i=1}^{n} P(A_i)}}$$

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Q5 $M = \max\{draw_1, draw_2, \ldots, draw_n\}$

For $k = 1, 2, \ldots, N$

$$P(M \leq k) = \left(\frac{k}{N}\right)^n$$

$$P(M = k) = P(M \leq k) - P(M \leq k-1)$$

$$= \left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n$$

$$E(M) = \sum_{k=1}^{N} P(M \geq k) = \sum_{k=1}^{N} \left[1 - P(M \leq k-1)\right]$$

$$= \sum_{k=1}^{N} \left[1 - \left(\frac{k-1}{N}\right)^n\right]$$

$$\therefore \boxed{E(M) = N - \frac{1}{N^n} \sum_{k=1}^{N} (k-1)^n}$$

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Solution → to show: convolution of two distribution functions F and G is also a distribution function.

Defination :-

$$(F * G)(x) = \int_{-\infty}^{\infty} F(x-y)\, dG(y)$$

if F and G correspond to independent random variables X and Y, then $F * G$ is the distribution function of $X+Y$

Proof that $F \times G$ is a distribution function is firstly

① Let $x_1 \le x_2$ then $F(x_1 - y) \le F(x_2 - y)$ for all y, since F is non-decreasing

$$(F \times G)(x) = \int_{-\infty}^{\infty} F(x_1 - y)\, dG(y) \le \int_{-\infty}^{\infty} F(x_2 - y)\, dG(y)$$
$$= F \times G(x_2).$$

means that it is Non-decreasing.

② F is Right continuous, $F(x-y)$ is right continuous in x for each y. and this integral passess this property.

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(3) As $x \to -\infty$ $F(x-y) \to 0$ for all y

so $(F \times G)(x) \to 0$

As $x \to \infty$ $F(x-y) \to 1$ for all y

so $(F \times G)(x) \to 1$

so, $F \times G$ satisfies all properties of a distribution function.

(10)

solution → CDF $F(x) = P(X \leq x)$

where ~~x~~ X is non-negative random variable

To Proof :- $E(X) = \int_0^\infty (1 - F(x))\, dx.$

proof :-

$$X(\omega) = \int_0^{X(\omega)} 1\, dx = \int_0^\infty I_{[0, X(\omega))}(x)\, dx.$$

$$E(X) = \int_\Omega X(\omega)\, dP(\omega) = \int_\Omega \int_0^\infty I_{[0, X(\omega))}(x)\, d\omega\, dP(\omega)$$

swapping Integrals using fubini's theorem

$$E(X) = \int_0^\infty \int_\Omega I_{[0, X(\omega))}(x)\, dP(\omega)\, dx = \int_0^\infty P(X > x)\, dx = \int_0^\infty (1 - F(x))\, dx.$$

Thus $\underline{E(X) = \int_0^\infty (1 - F(x))\, dx}$

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(i) $\mathbb{E}\, e^{ux} = \int_{-\infty}^{\infty} e^{ux} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(ux - \frac{(x-\mu)^2}{2\sigma^2}\right) dx$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}(x-(\mu+u\sigma^2))^2 + u\mu + \frac{1}{2}u^2\sigma^2\right) dx$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}(x-(\mu+u\sigma^2))^2\right) e^{u\mu+\frac{1}{2}u^2\sigma^2}\, dx$

$= \frac{1}{\sigma\sqrt{2\pi}} e^{\mu u + \frac{1}{2}u^2\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-(\mu+u\sigma^2))^2} dx$

$= \frac{1}{\sigma\sqrt{2\pi}} e^{\mu u + \frac{1}{2}u^2\sigma^2} \cdot \sigma\sqrt{2\pi}$

$= e^{\mu u + \frac{1}{2}u^2\sigma^2}$

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(ii) $\mathbb{E}[\psi(X)] = e^{\mu u + \frac{1}{2}u^2\sigma^2}$

$\psi(\mathbb{E}X) = \psi(\mu) = e^{\mu u}$

$\frac{1}{2}u^2\sigma^2 \geq 0$

$\Rightarrow \mu u + \frac{1}{2}u^2\sigma^2 \geq \mu u$

$\Rightarrow e^{\mu u + \frac{1}{2}u^2\sigma^2} \geq e^{\mu u}$

$\Rightarrow \mathbb{E}[\psi(X)] \geq \psi[\mathbb{E}X]$