Handout# 3 Introduction to Graphs

INTRODUCTION TO GRAPHS

INTRODUCTION:

Graph theory plays an important role in several areas of computer science such as:

- switching theory and logical design
- artificial intelligence
- formal languages
- computer graphics
- operating systems
- compiler writing
- information organization and retrieval.

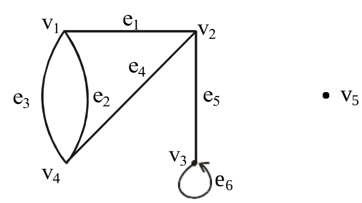
GRAPH:

A graph is a non-empty set of points called vertices and a set of line segments joining pairs of vertices called edges.

Formally, a graph G consists of two finite sets:

(i) A set V=V(G) of vertices (or points or nodes)

(ii) A set E=E(G) of edges; where each edge corresponds to a pair of vertices.

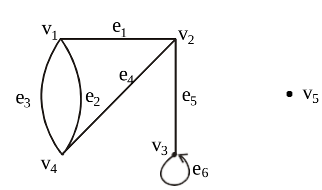


The graph G with

$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and

$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

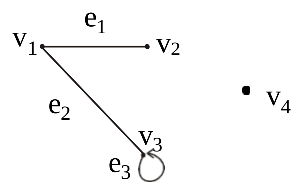
SOME TERMINOLOGY:



1. An edge connects either one or two vertices called its **endpoints** (edge e_1 connects vertices v_1 and v_2 described as $\{v_1, v_2\}$ i.e v_1 and v_2 are the endpoints of an edge e_1).
2. An edge with just one endpoint is called a **loop**. Thus a loop is an edge that connects a vertex to itself (e.g., edge e_6 makes a loop as it has only one endpoint v_3).
3. Two vertices that are connected by an edge are called **adjacent**; and a vertex that is an endpoint of a loop is said to be adjacent to itself.
4. An edge is said to be **incident** on each of its endpoints(i.e. e_1 is incident on v_1 and v_2).
5. A vertex on which no edges are incident is called **isolated** (e.g., v_5)
6. Two distinct edges with the same set of end points are said to be **parallel** (i.e. e_2 & e_3).

EXAMPLE:

Define the following graph formally by specifying its vertex set, its edge set, and a table giving the edge endpoint function.



SOLUTION:

Vertex Set = {v_1, v_2, v_3, v_4}

Edge Set = {e_1, e_2, e_3}

Edge - endpoint function is:

Edge	**Endpoint**
e_1	{v_1, v_2}
e_2	{v_1, v_3}
e_3	{v_3}

EXAMPLE:

For the graph shown below

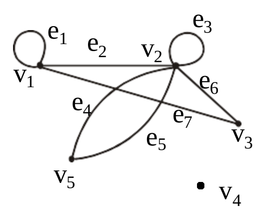
(i) find all edges that are incident on v_1;

(ii) find all vertices that are adjacent to v_3;

(iii) find all loops;

(iv) find all parallel edges;

(v) find all isolated vertices;



SOLUTION:

(i) v_1 is incident with edges e_1, e_2 and e_7

(ii) vertices adjacent to v_3 are v_1 and v_2

(iii) loops are e_1 and e_3

(iv) only edges e_4 and e_5 are parallel

(v) The only isolated vertex is v_4 in this Graph.

DRAWING PICTURE FOR A GRAPH:

Draw picture of Graph H having vertex set {v_1, v_2, v_3, v_4, v_5} and edge set {e_1, e_2, e_3, e_4} with edge endpoint function

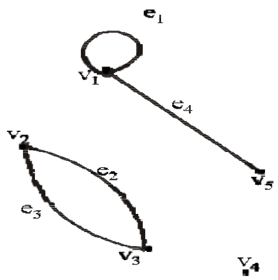
Edge	Endpoint
e_1	$\{v_1\}$
e_2	$\{v_2,v_3\}$
e_3	$\{v_2,v_3\}$
e_4	$\{v_1,v_5\}$

SOLUTION:

Given $V(H) = \{v_1, v_2, v_3, v_4, v_5\}$

and $E(H) = \{e_1, e_2, e_3, e_4\}$

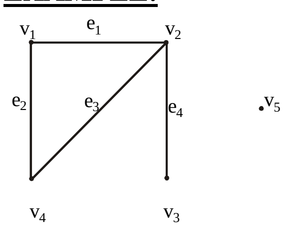
with edge endpoint function



SIMPLE GRAPH

A simple graph is a graph that does not have any loop or parallel edges.

EXAMPLE:



It is a simple graph H

$V(H) = \{v_1, v_2, v_3, v_4, v_5\}$ & $E(H) = \{e_1, e_2, e_3, e_4\}$

EXERCISE:

Draw all simple graphs with the four vertices {u, v, w, x} and two edges, one of which is {u, v}.

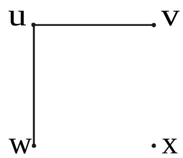
SOLUTION:

There are C(4,2) = 6 ways of choosing two vertices from 4 vertices. These edges may be listed as:

{u,v},{u,w},{u,x},{v,w}, {v,x},{w,x}

One edge of the graph is specified to be {u,v}, so any of the remaining five from this list may be chosen to be the second edge. This required graphs are:

1.



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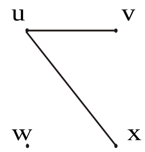
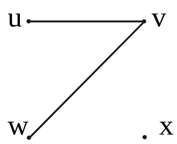


3.



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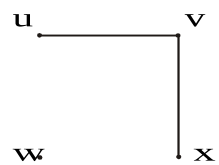

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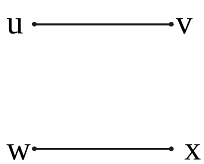


<u>DEGREE OF A VERTEX:</u>

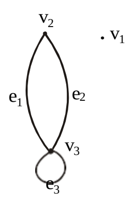
Let G be a graph and ***v*** a vertex of G. The degree of ***v***, denoted **deg(*v*),** equals the number of edges that are incident on ***v***, with an edge that is a loop counted twice.

Note:(i)The total degree of G is the sum of the degrees of all the vertices of G.

(ii) The **degree** of a loop is counted twice.

<u>EXAMPLE:</u>

For the graph shown



deg (v_1) = 0, since v_1 is isolated vertex.

deg (v_2) = 2, since v_2 is incident on e_1 and e_2 .

deg (v_3) = 4, since v_3 is incident on e_1,e_2 and the loop e_3.

Total degree of G = deg(v_1) + deg(v_2) + deg(v_3)

= 0 + 2 + 4

= 6

<u>REMARK:</u>

The total degree of G, which is 6, equals twice the number of edges of G, which is 3.

<u>THE HANDSHAKING THEOREM:</u>

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G.

Specifically, if the vertices of G are v_1, v_2, …, v_n, where n is a positive integer, then

the total degree of G $= \deg(v_1) + \deg(v_2) + \ldots + \deg(v_n)$
$= 2$. (the number of edges of G)

PROOF:

Each edge "e" of G connects its end points v_i and v_j. This edge, therefore contributes 1 to the degree of v_i and 1 to the degree of v_j.
If "e" is a loop, then it is counted twice in computing the degree of the vertex on which it is incident.
Accordingly, each edge of G contributes 2 to the total degree of G.
Thus,

the total degree of G = 2. (the number of edges of G)

COROLLARY:

The total degree of G is an even number

EXERCISE:

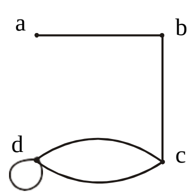
Draw a graph with the specified properties or explain why no such graph exists.
(i) Graph with four vertices of degrees 1, 2, 3 and 3
(ii) Graph with four vertices of degrees 1, 2, 3 and 4
(iii) Simple graph with four vertices of degrees 1, 2, 3 and 4

SOLUTION:

(i) Total degree of graph $= 1 + 2 + 3 + 3$
$= 9$ an odd integer
Since, the total degree of a graph is always even, hence no such graph is possible.
Note: As we know that "for any graph,the sum of the degrees of all the vertices of G equals twice the number of edges of G **or** the total degree of G is an even number".

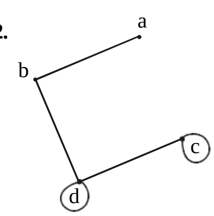
(ii) Two graphs with four vertices of degrees 1, 2, 3 & 4 are

1.



or

2.



The vertices a, b, c, d have degrees 1,2,3, and 4 respectively(i.e graph exists).

(iii) Suppose there was a simple graph with four vertices of degrees 1, 2, 3, and 4. Then the vertex of degree 4 would have to be connected by edges to four distinct vertices other than itself because of the assumption that the graph is simple (and hence has no loop or parallel edges.) This contradicts the assumption that the graph has four vertices in total. Hence there is no simple graph with four vertices of degrees 1, 2, 3, and 4, so simple graph is not possible in this case.

EXERCISE:

Suppose a graph has vertices of degrees 1, 1, 4, 4 and 6. How many edges does the graph have?

SOLUTION:

The total degree of graph = 1 + 1 + 4 + 4 + 6

= 16

Since, the total degree of graph = 2.(number of edges of graph) [by using Handshaking theorem]

$\Rightarrow$ 16 = 2.(number of edges of graph)

$\Rightarrow$ Number of edges of graph = $\frac{16}{2} = 8$

EXERCISE:

In a group of 15 people, is it possible for each person to have exactly 3 friends?

SOLUTION:

Suppose that in a group of 15 people, each person had exactly 3 friends. Then we could draw a graph representing each person by a vertex and connecting two vertices by an edge if the corresponding people were friends.

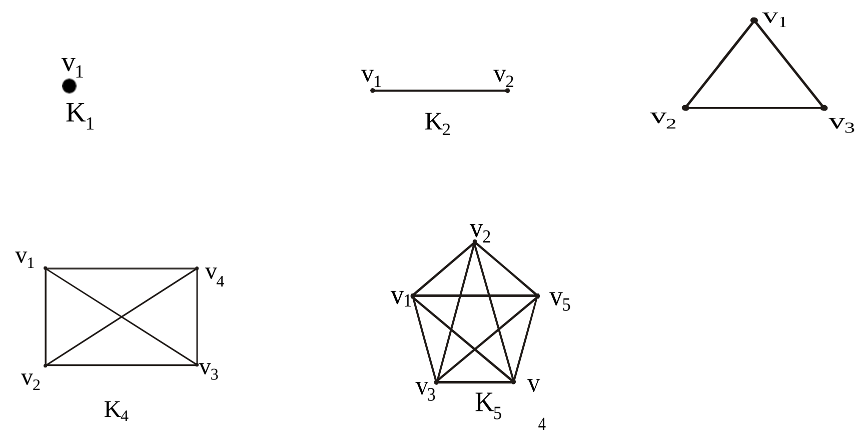
But such a graph would have 15 vertices each of degree 3, for a total degree of 45 (not even) which is not possible.

Hence, in a group of 15 people it is not possible for each to have exactly three friends.

COMPLETE GRAPH:

A complete graph on n vertices is a simple graph in which each vertex is connected to every other vertex and is denoted by K_n (K_n means that there are n vertices).

The following are complete graphs K_1, K_2,K_3, K_4 and K_5.



EXERCISE:

For the complete graph K_n, find

(i) the degree of each vertex

(ii) the total degrees

(iii) the number of edges

SOLUTION:

(i) Each vertex v is connected to the other (n-1) vertices in K_n; hence deg (v) = n - 1 for every v in K_n.

(ii) Each of the n vertices in K_n has degree n - 1; hence, the total degree in
K_n = (n - 1) + (n - 1) + … + (n - 1) n times
= n (n - 1)

(iii) Each pair of vertices in K_n determines an edge, and there are C(n, 2) ways of selecting two vertices out of n vertices. Hence,
Number of edges in K_n = C(n, 2)

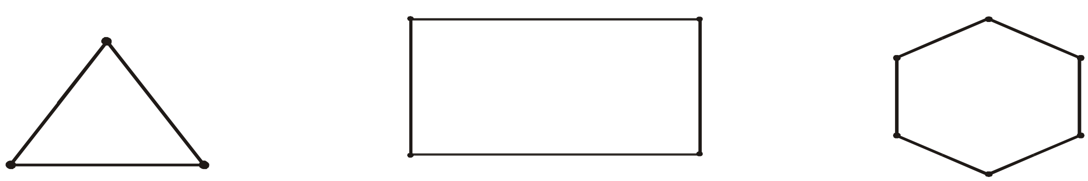
$$= \frac{n(n-1)}{2}$$

Alternatively,
The total degrees in graph K_n = 2 (number of edges in K_n)
$\Rightarrow$ n(n-1) = 2(number of edges in K_n)
$\Rightarrow$ Number of edges in K_n $= \frac{n(n-1)}{2}$

REGULAR GRAPH:
A graph G is regular of degree k or k-regular if every vertex of G has degree k.
In other words, a graph is regular if every vertex has the same degree.
Following are some regular graphs.

(i) 0-regular

(ii) 1-regular



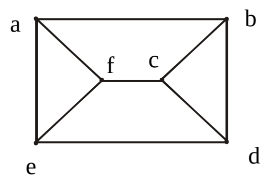
(iii) 2-regular

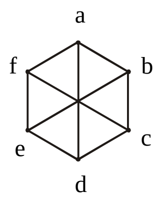
REMARK: The complete graph K_n is (n-1) regular.

EXERCISE:

Draw two 3-regular graphs with six vertices.

SOLUTION:

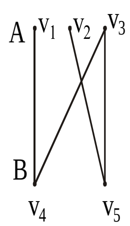


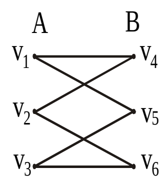


BIPARTITE GRAPH:

A bipartite graph G is a simple graph whose vertex set can be partitioned into two mutually disjoint non empty subsets A and B such that the vertices in A may be connected to vertices in B, but no vertices in A are connected to vertices in A and no vertices in B are connected to vertices in B.

The following are bipartite graphs





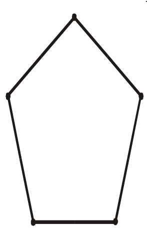
DETERMINING BIPARTITE GRAPHS:

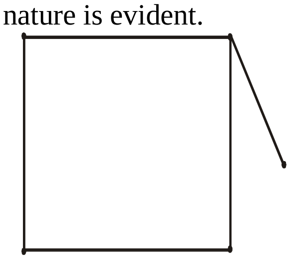
The following labeling procedure determines whether a graph is bipartite or not.

1. Label any vertex **a**
2. Label all vertices adjacent to **a** with the label **b**.
3. Label all vertices that are adjacent to a vertex just labeled **b** with label **a**.
4. Repeat steps 2 and 3 until all vertices got a distinct label (a bipartite graph) or there is a conflict i.e., a vertex is labeled with a and b (not a bipartite graph).

EXERCISE:

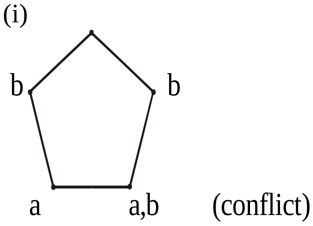
Find which of the following graphs are bipartite. Redraw the bipartite graph so that its bipartite nature is evident.





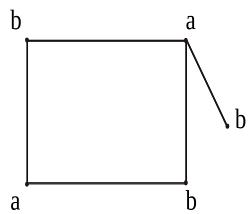
SOLUTION:

(i)

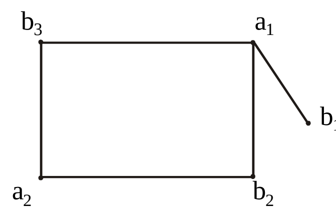


The graph is not bipartite.

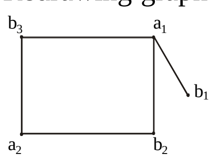
(ii)

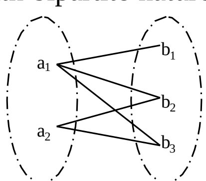


By labeling procedure, each vertex gets a distinct label. Hence the graph is bipartite. To



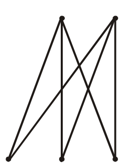
redraw the graph we mark labels a's as a_1, $a_2$and b's as b_1, b_2,
Redrawing graph with bipartite nature evident.



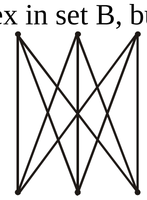


COMPLETE BIPARTITE GRAPH:

A complete bipartite graph on (m+n) vertices denoted $K_{m,n}$ is a simple graph whose vertex set can be partitioned into two mutually disjoint non empty subsets A and B containing m and n vertices respectively, such that each vertex in set A is connected (adjacent) to every vertex in set B, but the vertices within a set are not connected.



$K_{2,3}$

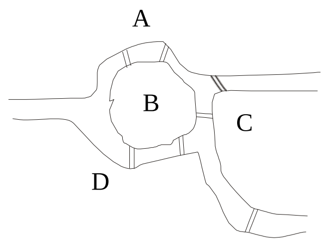


$K_{3,3}$

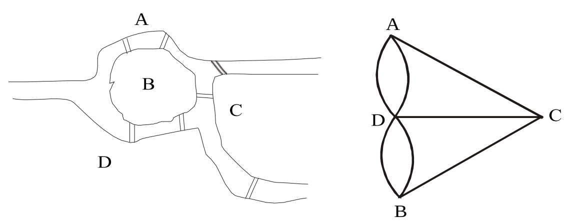
Handout# 4 Paths and Circuits

PATHS AND CIRCUITS

KONIGSBERG BRIDGES PROBLEM



It is possible for a person to take a walk around town, starting and ending at the same location and crossing each of the seven bridges exactly once?



Is it possible to find a route through the graph that starts and ends at some vertex A, B, C or D and traverses each edge exactly once?

Equivalently:

Is it possible to trace this graph, starting and ending at the same point, without ever lifting your pencil from the paper?

DEFINITIONS:

Let G be a graph and let v and w be vertices in graph G.

1. WALK

A walk from v to w is a finite alternating sequence of adjacent vertices and edges of G.

Thus a walk has the form

$$v_0e_1v_1e_2 \ldots v_{n-1}e_nv_n$$

where the v's represent vertices, the e's represent edges $v_0=v$, $v_n=w$, and for all $i = 1, 2 \ldots n$, v_{i-1} and v_i are endpoints of e_i.

The trivial walk from v to v consists of the single vertex v.

2. CLOSED WALK

A closed walk is a walk that starts and ends at the same vertex.

3. CIRCUIT

A circuit is a closed walk that does not contain a repeated edge. Thus a circuit is a walk of the form

$$v_0e_1v_1e_2 \ldots v_{n-1}\, e_n\, v_n$$

where $v_0 = v_n$ and all the e_i,s are distinct.

4. SIMPLE CIRCUIT

A simple circuit is a circuit that does not have any other repeated vertex except the first and last.
Thus a simple circuit is a walk of the form

$$v_0e_1v_1e_2 \ldots v_{n-1}e_nv_n$$

where all the e_i,s are distinct and all the v_j,s are distinct except that $v_0 = v_n$

5. PATH

A path from v to w is a walk from v to w that does not contain a repeated edge. Thus a path from v to w is a walk of the form

$$v = v_0e_1v_1e_2 \ldots v_{n-1}e_nv_n = w$$

where all the e_i,s are distinct (that is $e_i \neq e_k$ for any $i \neq k$).

6. SIMPLE PATH

A simple path from v to w is a path that does not contain a repeated vertex.
Thus a simple path is a walk of the form

$$v = v_0e_1v_1e_2 \ldots v_{n-1}e_nv_n = w$$

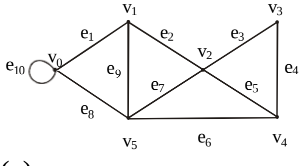
where all the e_i,s are distinct and all the v_j,s are also distinct (that is, $v_j \neq v_m$ for any $j \neq m$).

SUMMARY

	Repeated Edge	Repeated Vertex	Starts and Ends at Same Point
walk	allowed	Allowed	allowed
closed walk	allowed	Allowed	yes(means, where it starts also ends at that point)
circuit	no	Allowed	yes
simple circuit	no	first and last only	yes
path	no	Allowed	allowed
simple path	no	no	No

EXERCISE:

In the graph below, determine whether the following walks are paths, simple paths, closed walks, circuits, simple circuits, or are just walks.



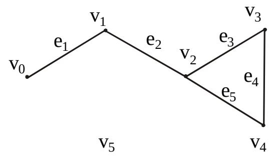
(a) $v_1e_2v_2e_3v_3e_4v_4e_5v_2e_2v_1e_1v_0$

(b) $v_1v_2v_3v_4v_5v_2$
(c) $v_4v_2v_3v_4v_5v_2v_4$
(d) $v_2v_1v_5v_2v_3v_4v_2$
(e) $v_0v_5v_2v_3v_4v_2v_1$
(f) $v_5v_4v_2v_1$

SOLUTION:

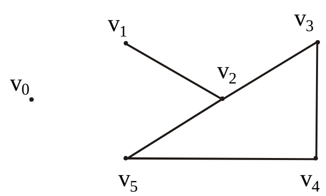
(a) $v_1e_2v_2e_3v_3e_4v_4e_5v_2e_2v_1e_1v_0$

(a)



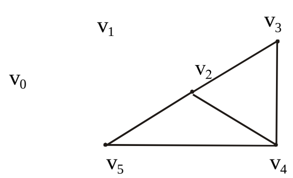
This graph starts at vertex v_1,then goes to v_2 along edge e_2,and moves continuously, at the end it goes from v_1 to v_0 along e_1.Note it that the vertex v_2 and the edge e_2 is repeated twice, and starting and ending, not at the same points.Hence The graph is just a walk.

(b) $v_1v_2v_3v_4v_5v_2$



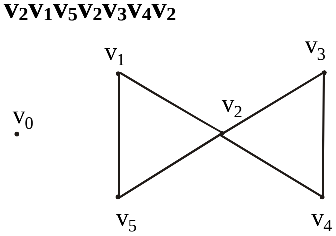
In this graph vertex v_2 is repeated twice.As no edge is repeated so the graph is a path.

(c) $v_4v_2v_3v_4v_5v_2v_4$



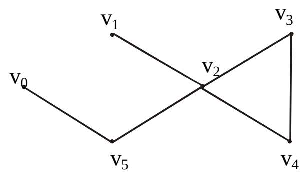
As vertices v_2 & v_4 are repeated and graph starts and ends at the same point v_4,also the edge(i.e. e_5)connecting v_2 & v_4 is repeated, so the graph is a closed walk.

(d) $v_2v_1v_5v_2v_3v_4v_2$



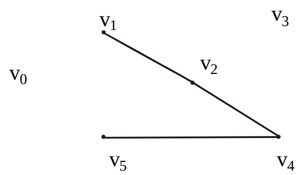
In this graph, vertex v_2 is repeated and the graph starts and end at the same vertex (i.e. at v_2) and no edge is repeated, hence the above graph is a circuit.

(e) $v_0v_5v_2v_3v_4v_2v_1$



Here vertex v_2 is repeated and no edge is repeated so the graph is a path.

(f) $v_5v_4v_2v_1$



Neither any vertex nor any edge is repeated so the graph is a simple path**.**

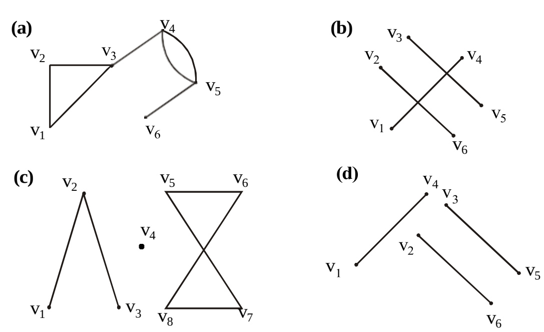
<u>CONNECTEDNESS:</u>

Let G be a graph. Two vertices *v* and *w* of G are connected if, and only if, there is a walk from *v* to *w*. The graph G is connected if, and only if, given any two vertices *v* and *w* in G, there is a walk from *v* to *w*. Symbolically:

G is connected $\Leftrightarrow$ $\forall$ vertices v, w $\in$ V (G), $\exists$ a walk from v to w:

<u>EXAMPLE:</u>

Which of the following graphs have a connectedness**?**



EULER CIRCUITS

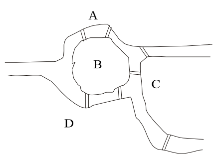
<u>DEFINITION:</u>

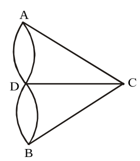
Let G be a graph. An Euler circuit for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is sequence of adjacent vertices and edges in G that starts and ends at the same vertex uses every vertex of G at least once, and used every edge of G exactly once.

THEOREM:

A graph G has an Euler circuit if, and only if, G is connected and every vertex of G has an even degree.

KONIGSBERG BRIDGES PROBLEM

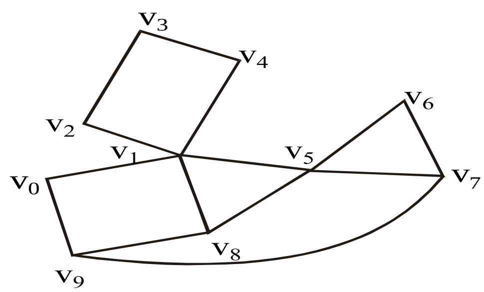




We try to solve Konigsberg bridges problem by Euler method.
Here deg(a)=3,deg(b)=3,deg(c)=3 and deg(d)=5 as the vertices have odd degree so there is no possibility of an Euler circuit.

EXERCISE:

Determine whether the following graph has an Euler circuit**.**

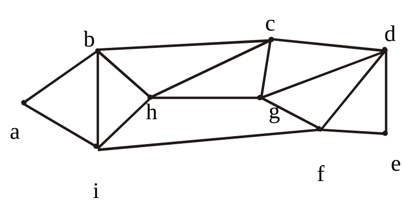


SOLUTION:

As deg (v_1) =5, an odd degree so the following graph has not an Euler circuit.

EXERCISE:

Determine whether the following graph has Euler circuit.



SOLUTION:

From above clearly deg(a)=2, deg(b)=4, deg(c)=4, deg(d)=4, deg(e)=2, deg(f)=4, deg(g)=4, deg(h)=4, deg(i)=4
Since the degree of each vertex is even, and the graph has Euler Circuit. One such circuit is:

a b c d e f g d f i h c g h b i a

EULER PATH

DEFINITION:

Let G be a graph and let v and w be two vertices of G. An Euler path from v to w is a sequence of adjacent edges and vertices that starts at v, end and w, passes through every vertex of G at least once, and traverses every edge of G exactly once.

COROLLARY

Let G be a graph and let v and w be two vertices of G. There is an Euler path from v to w if, and only if, G is connected, v and w have odd degree and all other vertices of G have even degree.

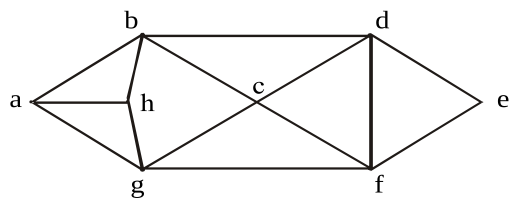
HAMILTONIAN CIRCUITS

DEFINITION:

Given a graph G, a Hamiltonian circuit for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once.

EXERCISE:

Find Hamiltonian Circuit for the following graph.



SOLUTION:

The Hamiltonian Circuit for the following graph is:

a b d e f c g h a

Another Hamiltonian Circuit for the following graph could be:

a b c d e f g h a

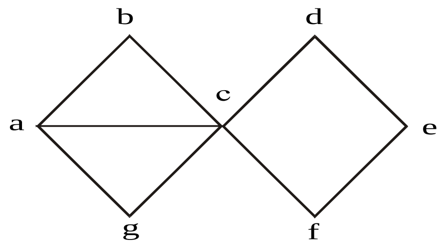
PROPOSITION:

If a graph G has a Hamiltonian circuit then G has a sub-graph H with the following properties:

1. H contains every vertex G
2. H is connected
3. H has the same number of edges as vertices
4. Every vertex of H has degree 2

EXERCISE:

Show that the following graph does not have a Hamiltonian circuit.



Here deg(c)=5,if we remove 3 edges from vertex c then deg(b)< 2 , deg(g)< 2 or deg(f)< 2,deg(d)< 2.

It means that this graph does not satisfy the desired properties as above, so the graph does not have a Hamiltonian circuit.

Handout# 5 Matrix Representation of Graphs

MATRIX REPRESENTATIONS OF GRAPHS

MATRIX:

An m × n matrix A over a set S is a rectangular array of elements of S arranged into m rows and n columns:

$$A = \begin{array}{c} \\ \\ \\ \leftarrow \text{ith row of A} \\ \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

↑

jth column of A

Briefly, it is written as:

A = [a_{ij}]$_{m\times n}$

EXAMPLE:

$$A = \begin{bmatrix} 4 & -2 & 0 & 6 \\ 2 & -3 & 1 & 9 \\ 0 & 7 & 5 & -1 \end{bmatrix}$$

A is a matrix having 3 rows and 4 columns. We call it a 3 × 4 matrix, or matrix of size 3 × 4(or we say that a matrix having an order 3 × 4).

Note it that

a_{11} = 4 (11 means 1st row and 1st column), a_{12} = -2 (12 means 1st row and 2nd column),

a_{13} = 0, a_{14} = 6

a_{21} = 2, a_{22} = -3, a_{23} = 1, a_{24} =9 etc.

SQUARE MATRIX:

A matrix for which the number of rows and columns are equal is called a square matrix. A square matrix A with m rows and n columns (size m × n) but m=n (i.e of order n × n) has the form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix}$$

Diagonal entries

Note:

The main diagonal of A consists of all the entries

$a_{11}, a_{22}, a_{33}, \ldots, a_{ii}, \ldots, a_{n\,n}$

TRANSPOSE OF A MATRIX:

The transpose of a matrix A of size m × n, is the matrix denoted by A^t of size n × m, obtained by writing the rows of A, in order, as columns.(**Or** we can say that transpose of a matrix means "write the rows instead of columns or write the columns instead of rows". Thus if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad then \;\; A^t = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

EXAMPLE:

$$A = \begin{bmatrix} 4 & -2 & 0 & 6 \\ 2 & -3 & 1 & 9 \\ 0 & 7 & 5 & -1 \end{bmatrix}$$

Then

$$A^t = \begin{bmatrix} 4 & 2 & 0 \\ -2 & -3 & 7 \\ 0 & 1 & 5 \\ 6 & 9 & -1 \end{bmatrix}$$

SYMMETRIC MATRIX:

A square matrix A = $[a_{ij}]$ of size n × n is called symmetric if, and only if, $A^t = A$
i.e., for all i, j = 1, 2, …, n, $a_{ij} = a_{ji}$

EXAMPLE:

Let $A = \begin{bmatrix} 1 & 3 & 7 \\ 5 & 2 & 9 \end{bmatrix}$, and $B = \begin{bmatrix} 4 & 2 & 0 \\ 2 & -3 & 1 \\ 0 & 1 & 5 \end{bmatrix}$

Then $A^t = \begin{bmatrix} 1 & 5 \\ 3 & 2 \\ 7 & 9 \end{bmatrix}$, and $B^t = \begin{bmatrix} 4 & 2 & 0 \\ 2 & -3 & 1 \\ 0 & 1 & 5 \end{bmatrix}$

Note that $B^t = B$, so that B is a symmetric matrix.

MATRIX MULTIPLICATION:

Suppose A and B are two matrices such that the number of columns of A is equal to the number of rows of B, say A is an m × p matrix and B is a p × n matrix. Then the product of A and B, written AB, is the $m \times n$ matrix whose ijth entry is obtained by multiplying the elements of the ith row of A by the corresponding elements of the jth column of B and then adding;

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{mi} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & & & & \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & & & & \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & & & & \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix}$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

REMARK:

If the number of columns of A is not equal to the number of rows of B, then the product AB is not defined.

EXAMPLE:

Find the product AB and BA of the matrices

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{bmatrix}$$

SOLUTION:

Size of A is 2×2 and of B is 2×3, the product AB is defined as a 2×3 matrix. But BA is not defined, because no. of columns of B = 3 ≠ 2 = no. of rows of A.

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(2)+(3)(3) & (1)(0)+(3)(6) & (1)(-4)+(3)(6) \\ (2)(2)+(-1)(3) & (2)(0)+(-1)(-2) & (2)(-4)+(-2)(6) \end{bmatrix} = \begin{bmatrix} 11 & -6 & 14 \\ 1 & 2 & -14 \end{bmatrix}$$

EXERCISE:

Find AA^t and A^tA, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

SOLUTION:

A^t is obtained from A by rewriting the rows of A as columns:

i.e $A^t = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$

Now

$\begin{bmatrix} 1 & 3 \end{bmatrix}$

$$AA^t = \begin{bmatrix} & & \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} & \\ 2 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1+4+0 & 3-2+0 \\ 3-2+0 & 9+1+16 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 26 \end{bmatrix}$$

and

$$A^t A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+9 & 2-3 & 0+12 \\ 2-3 & 4+1 & 0-4 \\ 0+12 & 0-4 & 0+16 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -1 & 12 \\ -1 & 5 & -4 \\ 12 & -4 & 16 \end{bmatrix}$$

ADJACENCY MATRIX OF A GRAPH:

Let G be a graph with ordered vertices $v_1, v_2, \ldots, v_n$.The adjacency matrix of G is the matrix $A = [a_{ij}]$ over the set of non-negative integers such that

a_{ij} = the number of edges connecting v_i and v_j for all i, j = 1, 2, …, n.

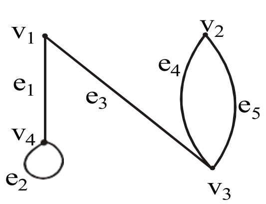
OR

The adjancy matrix say $A= [a_{ij}]$ is also defined as

$$a_{ij} = \begin{cases} 1 & if\ \{v_i, v_j\}\ is\ an\ edge\ of\ G \\ 0 & otherwise \end{cases}$$

EXAMPLE:

A graph with it's adjacency matrix is shown.



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Note that the nonzero entries along the main diagonal of A indicate the presence of loops and entries larger than 1 correspond to parallel edges.
Also note A is a symmetric matrix.

EXERCISE:

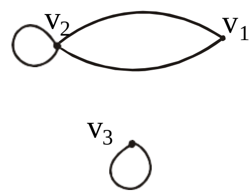
Find a graph that have the following adjacency matrix.

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

SOLUTION:

Let the three vertices of the graph be named v_1, v_2 and v_3. We label the adjacency matrix across the top and down the left side with these vertices and draw the graph accordingly(as from v_1 to v_2 there is a value "2",it means that two parallel edges between v_1 and v_2 and same condition occurs between v_2 and v_1 and the value "1" represent the loops of v_2 and v_3).

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

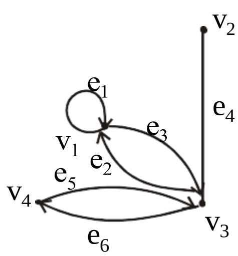


DIRECTED GRAPH:

A directed graph or digraph, consists of two finite sets: a set V(G) of vertices and a set D(G) of directed edges, where each edge is associated with an ordered pair of vertices called its end points.

If edge e is associated with the pair (v, w) of vertices, then e is said to be the directed edge from v to w and is represented by drawing an arrow from v to w.

EXAMPLE OF A DIGRAPH:



ADJACENCY MATRIX OF A DIRECTED GRAPH:

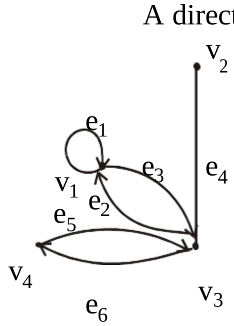
Let G be a graph with ordered vertices $v_1, v_2, \ldots, v_n$.

The adjacency matrix of G is the matrix $A = [a_{ij}]$ over the set of non-negative integers such that

a_{ij} = the number of arrows from v_i to v_j for all i, j = 1, 2, …, n.

EXAMPLE:

A directed graph with its adjacency matrix is shown



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

is the adjacency matrix

EXERCISE:

Find directed graph that has the adjacency matrix

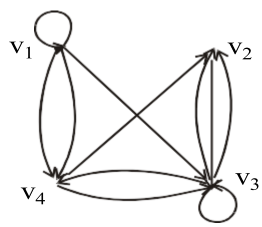
$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

SOLUTION:

The 4×4 adjacency matrix shows that the graph has 4 vertices say v_1, v_2, v_3 and v_4 labeled across the top and down the left side of the matrix.

$$A = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

A corresponding directed graph is



It means that a loop exists from v_1 and v_3, two arrows go from v_1 to v_4 and two from v_3 and v_2 and one arrow go from $v_1$to v_3, v_2 to v_3, v_3 to v_4, v_4 to v_2 and v_3.

THEOREM

If G is a graph with vertices $v_1, v_2, \ldots, v_m$ and A is the adjacency matrix of G, then for each positive integer n,

the ijth entry of A^n = the number of walks of length n from v_i to v_j

for all integers i,j = 1, 2, …, n

PROBLEM:

Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

be the adjacency matrix of a graph G with vertices v_1, v_2, and v_3. Find

(a) the number of walks of length 2 from v_2 to v_3

(b) the number of walks of length 3 from v_1 to v_3

Draw graph G and find the walks by visual inspection for (a)

SOLUTION:

(a)

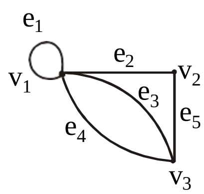
$$A^2 = AA = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 5 \end{bmatrix} \longrightarrow$$ it shows the entry (2,3) from v_2 to v_3

Hence, number of walks of length 2(means “multiply matrix A two times”) from v_2 to v_3 = the entry at (2,3) of A^2 = 2

(b) $A^3 = AA^2 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 6 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 15 & 9 & 15 \\ 9 & 5 & 8 \\ 15 & 8 & 8 \end{bmatrix}$ → it shows the entry (1,3) from v_1 to v_3

Hence, number of walks of length 3 from v_1 to v_3 = the entry at (1,3) of $A^3 = 15$
Walks from v_2 to v_3 by visual inspection of graph is

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$



so in part (a)two Walks of length 2 from v_2 to v_3 are
(i) $v_2\ e_2\ v_1\ e_3\ v_3$ (by using the above theorem).
(ii) $v_2\ e_2\ v_1\ e_4\ v_3$

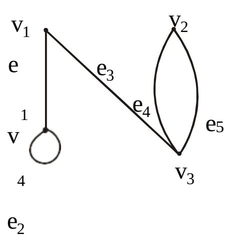
INCIDENCE MATRIX OF A SIMPLE GRAPH:

Let G be a graph with vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_n$. The incidence matrix of G is the matrix $M = [m_{ij}]$ of size $n \times m$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if the vertex } v_i \text{ is incident on the edge } e_j \\ 0 & \text{otherwise} \end{cases}$$

EXAMPLE:

A graph with its incidence matrix is shown.



$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

REMARK:

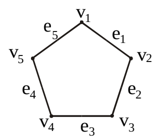
In the incidence matrix

1. Multiple edges are represented by columns with identical entries (in this matrix e_4 & e_5 are multiple edges).
2. Loops are represented using a column with exactly one entry equal to 1, corresponding to the vertex that is incident with this loop and other zeros (here e_2 is only a loop).

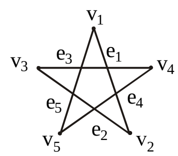
Handout# 6   Isomorphism of graphs

ISOMORPHISM OF GRAPHS

Here we have a graph



Which can also be defined as



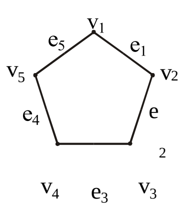
Its vertices and edges can be written as:

$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{e_1, e_2, e_3, e_4, e_5\}$

Edge endpoint function is:

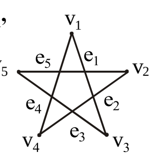
Edge	Endpoints
E_1	$\{v_1,v_2\}$
E_2	$\{v_2,v_3\}$
E_3	$\{v_3,v_4\}$
E_4	$\{v_4,v_5\}$
E_5	$\{v_5,v_1\}$

G



Another graph G' is

G'



Edge endpoint function of G is:

Edge	Endpoints
e_1	$\{v_1,v_2\}$
e_2	$\{v_2,v_3\}$
e_3	$\{v_3,v_4\}$
e_4	$\{v_4,v_5\}$
e_5	$\{v_5,v_1\}$

Edge endpoint function of G' is:

Edge	Endpoints
e_1	$\{v_1,v_3\}$
e_2	$\{v_2,v_4\}$
e_3	$\{v_3,v_5\}$
e_4	$\{v_1,v_4\}$
e_5	$\{v_2,v_5\}$

Two graphs (G and G') that are the same except for the labeling of their vertices are not considered different.

GRAPHS OF EDGE POINT FUNCTIONS

Edge point function of G is:

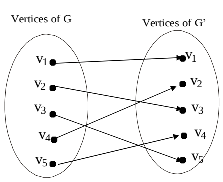
Edge	Endpoints
e_1	$\{v_1,v_2\}$
e_2	$\{v_2,v_3\}$
e_3	$\{v_3,v_4\}$
e_4	$\{v_4,v_5\}$
e_5	$\{v_5,v_1\}$

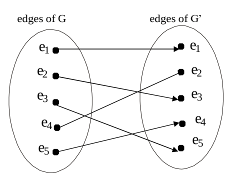
Edge point function of G' is:

Edge	Endpoints
e_1	$\{v_1,v_3\}$
e_2	$\{v_2,v_4\}$
e_3	$\{v_3,v_5\}$
e_4	$\{v_1,v_4\}$
e_5	$\{v_2,v_5\}$

Note it that the graphs G and G' are looking different because in G the end points of e_1 are v_1,v_2 but in G' are v_1, v_3 etc.

Buts G' is very similar to G ,if the vertices and edges of G' are relabeled by the function shown below, then G' becomes same as G:





It shows that if there is one-one correspondence between the vertices of G and G', then also one-one correspondence between the edges of G and G'.

ISOMORPHIC GRAPHS:

Let G and G' be graphs with vertex sets V(G) and V(G') and edge sets E(G) and E(G'), respectively.

G is isomorphic to G' if, and only if, there exist one-to-one correspondences g: $V(G) \rightarrow V(G')$ and h: $E(G) \rightarrow E(G')$ that preserve the edge-endpoint functions of G and G' in the sense that for all $v \in V(G)$ and $e \in E(G)$.

v is an endpoint of e $\Leftrightarrow$ g(v) is an endpoint of h(e).

EQUIVALENCE RELATION:

Graph isomorphism is an equivalence relation on the set of graphs.

1. Graphs isomorphism is Reflexive (It means that the graph should be isomorphic to itself).

2. Graphs isomorphism is Symmetric (It means that if G is isomorphic to G' then G' is also isomorphic to G).

3. Graphs isomorphism is Transitive (It means that if G is isomorphic to G' and G' is isomorphic to G'', then G is isomorphic to G'').

ISOMORPHIC INVARIANT:

A property P is called an isomorphic invariant if, and only if, given any graphs G and G', if G has property P and G' is isomorphic to G, then G' has property P.

THEOREM OF ISOMORPHIC INVARIANT:

Each of the following properties is an invariant for graph isomorphism, where n, m and k are all non-negative integers, if the graph:

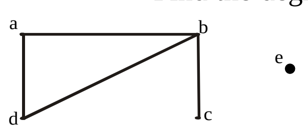
1. has n vertices.
2. has m edges.
3. has a vertex of degree k.
4. has m vertices of degree k.
5. has a circuit of length k.
6. has a simple circuit of length k.
7. has m simple circuits of length k.
8. is connected.
9. has an Euler circuit.
10. has a Hamiltonian circuit.

DEGREE SEQUENCE:

The degree sequence of a graph is the list of the degrees of its vertices in non-increasing order.

EXAMPLE:

Find the degree sequence of the following graph.



SOLUTION:

Degree of a = 2, Degree of b = 3, Degree of c = 1,
Degree of d = 2, Degree of e = 0

By definition, degree of the vertices of a given graph should be in decreasing (non-increasing) order.

Therefore Degree sequence is: 3, 2, 2, 1, 0

GRAPH ISOMORPHISM FOR SIMPLE GRAPHS:

If G and G' are simple graphs (means the "graphs which have no loops or parallel edges") then G is isomorphic to G' if, and only if, there exists a one-to-one correspondence (1-1 and onto function) g from the vertex set V (G) of G to the vertex set V (G') of G' that preserves the edge-endpoint functions of G and G' in the sense that for all vertices u and v of G,

{u, v} is an edge in G ⇔ {g(u), g(v)} is an edge in G'.

OR

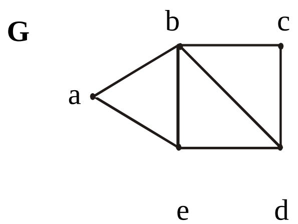
You can say that with the property of one-one correspondence, u and v are adjacent in graph G ⇔ if g (u) and g (v) are adjacent in G'.

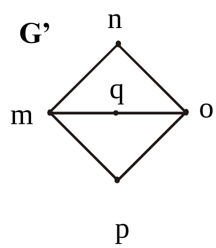
Note:

It should be noted that unfortunately, there is no efficient method for checking that whether two graphs are isomorphic(methods are there but take so much time in calculations).Despite that there is a simple condition. Two graphs are isomorphic if they have the same number of vertices(as there is a 1-1 correspondence between the vertices of both the graphs) and the same number of edges(also vertices should have the same degree.

EXERCISE:

Determine whether the graph G and G' given below are isomorphic.





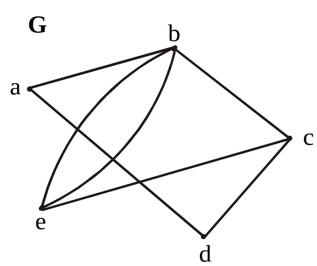
SOLUTION:

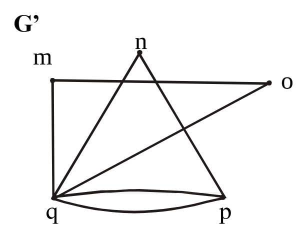
As both the graphs have the same number of vertices. But the graph G has 7 edges and the graph G' has only 6 edges. Therefore the two graphs are not isomorphic.

Note: As the edges of both the graphs G and G' are not same then how the one-one correspondence is possible ,that the reason the graphs G and G' are not isomorphic.

EXERCISE:

Determine whether the graph G and G' given below are isomorphic.



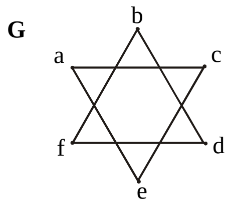


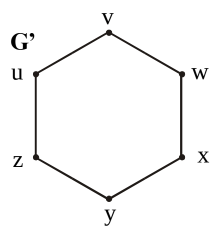
SOLUTION:

Both the graphs have 5 vertices and 7 edges. The vertex q of G' has degree 5. However G does not have any vertex of degree 5 (so one-one correspondence is not possible). Hence, the two graphs are not isomorphic.

EXERCISE:

Determine whether the graph G and G' given below are isomorphic.



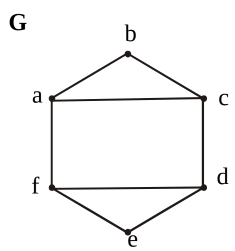


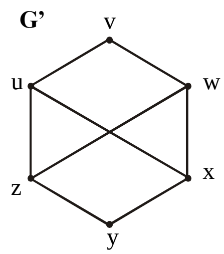
SOLUTION:

Clearly the vertices of both the graphs G and G' have the same degree (i.e "2") and having the same number of vertices and edges but isomorphism is not possible. As the graph G' is a connected graph but the graph G is not connected due to have two components (eca and bdf). Therefore the two graphs are non isomorphic.

EXERCISE:

Determine whether the graph G and G' given below are isomorphic.





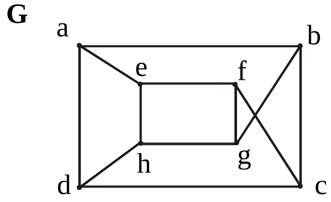
SOLUTION:

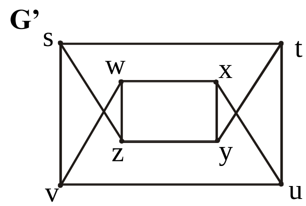
Clearly G has six vertices, G' also has six vertices. And the graph G has two simple circuits of length 3; one is abca and the other is defd. But G' does not have any simple circuit of length 3(as one simple circuit in G' is uxwv of length 4).Therefore the two graphs are non-isomorphic.

Note: A simple circuit is a circuit that does not have any other repeated vertex except the first and last.

EXERCISE:

Determine whether the graph G and G' given below are isomorphic.



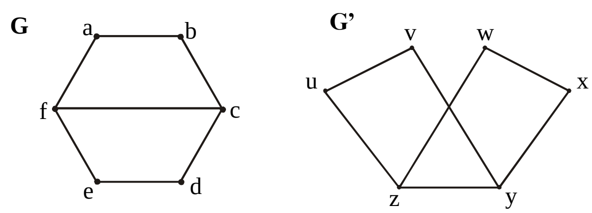


SOLUTION:

Both the graph G and G’ have 8 vertices and 12 edges and both are also called regular graph(as each vertex has degree 3).The graph G has two simple circuits of length 5; abcfea(i.e starts and ends at a) and cdhgfc(i.e starts and ends at c). But G’ does not have any simple circuit of length 5 (it has simple circuit tyxut,vwxuv of length 4 etc). Therefore the two graphs are non-isomorphic.

EXERCISE:

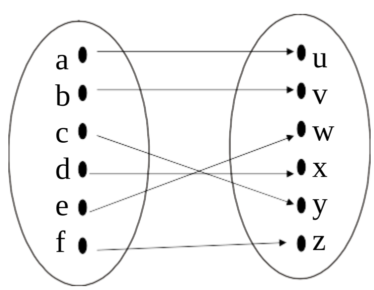
Determine whether the graph G and G’ given below are isomorphic.



SOLUTION:

We note that all the isomorphism invariants seems to be true.
We shall prove that the graphs G and G’ are isomorphic.
Here G has four vertices of degree “2” and two vertices of degree“3”. Similar case in G’. Also G and G’ have circuits of length 4.As a is adjacent to b and f in graph G.In graph G’ u is adjacent to v and z. And as a and u has degree 2 so both are mapped. And b mapped with v, f mapped with z(as both have the same degree also a is adjacent to f and u is to z), and as we moves further we get the 1-1 correspondence.

Define a function f: V(G) $\rightarrow$V(G’) as follows.



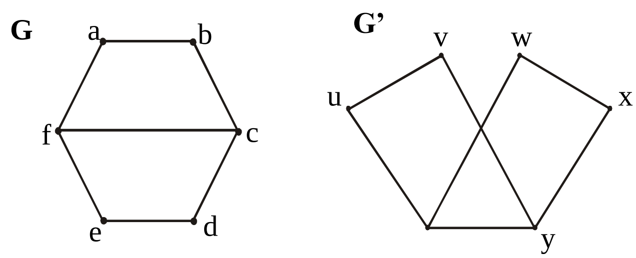
Clearly the above function is one and onto that is a bijective mapping. Note that I write the above mapping by keeping in mind the invariants of isomorphism as well as the fact that the mapping should preserve edge end point function. Also you should note that the mapping is not unique.

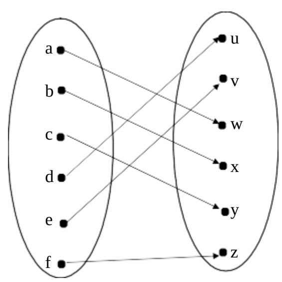
f is clearly a bijective function. The fact that f preserves the edge endpoint functions of G and G’ is shown below.

Edges of G	Edges of G’
{a, b}	{u, v} = {g(a), g(b)}
{b, c}	{v, y} = {g(b), g(c)}
{c, d}	{y, x} = {g(c), g(d)}
{d, e}	{x, w} = {g(d), g(e)}
{e, f}	{w, z} = {g(e), g(f)}
{a, f}	{u, z} = {g(a), g(f)}
{c, f}	{y, z} = {g(c), g(f)}

ALTERNATIVE SOLUTION:

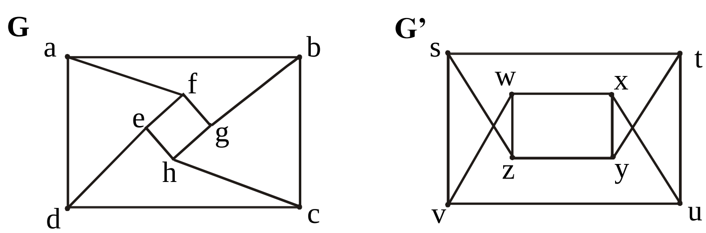
We shall prove that the graphs G and G’ are isomorphic. Define a function f: V(G) →V(G’) as follows.





EXERCISE:

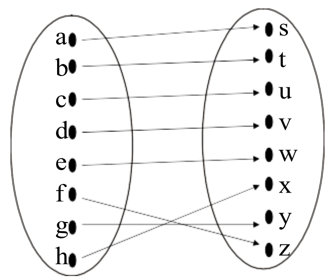
Determine whether the graph G and G’ given below are isomorphic.



SOLUTION:

We shall prove that the graphs G and G’ are isomorphic.
Clearly the isomorphism invariants seems to be true between G and G’.

Define a function f: V(G) →V(G') as follows.



f is clearly a bijective function(as it satisfies conditions the one-one and onto function clearly). The fact that f preserves the edge endpoint functions of G and G' is shown below.

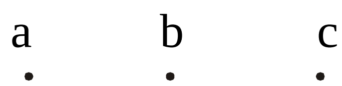
Edges of G	**Edges of G'**
{a, b}	{s, t} = {f(a), f(b)}
{b, c}	{t, u} = {f(b), f(c)}
{c, d}	{u, v} = {f(c), f(d)}
{a,d}	{s, v} = {f(a), f(d)}
{a, f}	{s, z} = {f(a), f(f)}
{b, g}	{t, y} = {f(b), f(g)}
{c, h}	{u, x} = {f(c), f(h)}
{d, e}	{v, w} = {f(d), f(e)}
{e, f}	{w, z} = {f(e), f(f)}
{f, g}	{z, y} = {f(f), f(g)}
{g, h}	{y, x} = {f(g), f(h)}
{h, e}	{x, w} = {f(h), f(e)}

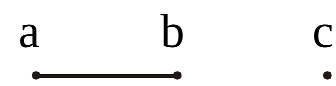
EXERCISE:

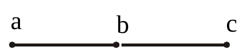
Find all non isomorphic simple graphs with three vertices.

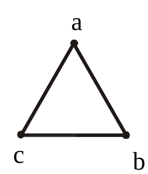
SOLUTION:

There are four simple graphs with three vertices as given below(which are non-isomorphic simple graphs).







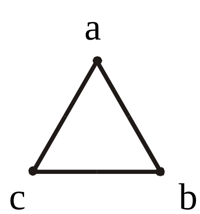


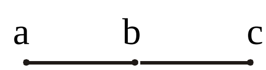
EXERCISE:

Find all non isomorphic simple connected graphs with three vertices.

SOLUTION:

There are two simple connected graphs with three vertices as given below(which are non-isomorphic connected simple graphs).



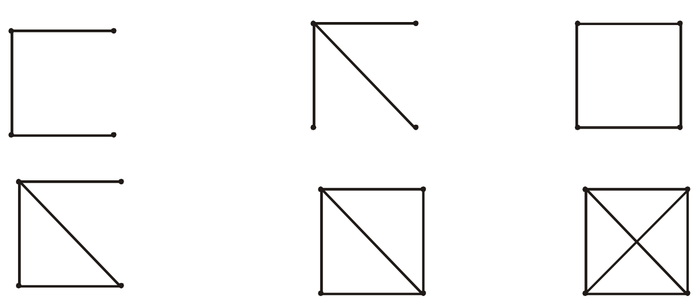


EXERCISE:

Find all non isomorphic simple connected graphs with four vertices.

SOLUTION:

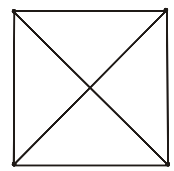
There are six simple connected graphs with four vertices as given below.



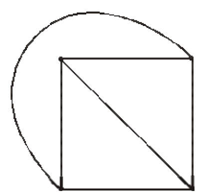
Handout# 7 Planar graphs

PLANAR GRAPHS
GRAPH COLORING

In this handouts, we will study that whether any graph can be drawn in the plane (means "a flat surface") without crossing any edges.



It is a graph on 4 vertices and written as K_4. Each vertex is connected to every other vertex.
Note it that here edges are crossed. Also the above graph can also be drawn as

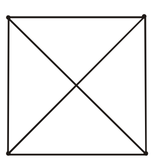


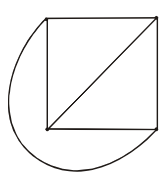
<u>In this graph, note it that each vertex is connected to every other vertex, but no edge is crossed.</u>

Note: The graphs shown above are complete graphs with four vertices (denoted by K_4).

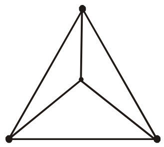
<u>DEFINITION:</u>
A graph is called planar if it can be drawn in the plane without any edge crossed (crossing means the intersection of lines). Such a drawing is called a plane drawing of the graph.
OR
You can say that a graph is called planar in which the graph crossing number is "0".

<u>EXAMPLES:</u>





The graphs given above are planar .In the first figure edges are crossed but it can be redrawn in second figure where edges are not crossed, so called planar.

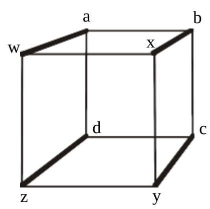


It is also a graph on 4 vertices (written as K_4) with no edge crossed, hence called planar.

Note: The graphs given above are also complete graphs (except second; are those where each vertex is connected to every other vertex) on 4 vertices and is written as K_4 **.**
Note: Complete graphs are planar only for $n \leq 4$.

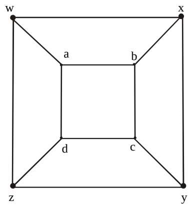
<u>EXAMPLE:</u>

Show that the graph below is planar.



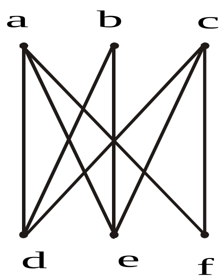
<u>SOLUTION:</u>

This graph has 8 vertices and 12 edges, and is called **3-cube** and is denoted $\mathbf{Q_3}$.
The above representation includes many "edge crossing." A plane drawing of the graph in which no two edges cross is possible and shown below.



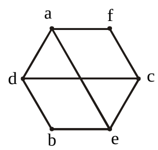
<u>EXERCISE:</u>

Determine whether the graph below is planar. If so, draw it so that no edges cross.

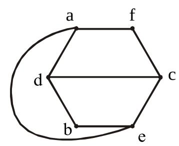


<u>SOLUTION:</u>

The graph given above is bipartite graph denoted by K_3. It also has a circuit afcebda. This graph can be re-drawn as



or

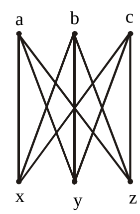


Hence the given graph is planar

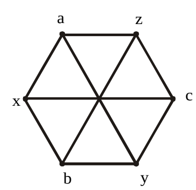
<u>THEOREM:</u>

Show that $K_{3,3}$ is not planar.

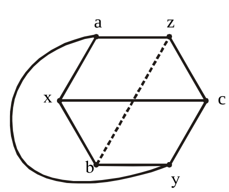
<u>PROOF:</u>



Clearly it is a complete bipartite graph (means bipartite graph, but the vertices within a set are not connected) denoted by $\mathbf{K_{3,3}}$**.** Now $\mathbf{K_{3,3}}$ can be re-drawn as



We re-draw the edge ***ay*** so that it does not cross any other edge like that.



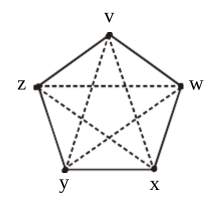
Note it that ***bz*** cannot be drawn without crossings. Hence, $K_{3,3}$ is not planar.
Similary if ay can be drawn inside(i.e drawn with crossing) and bz drawn outside, then same result exits.

<u>THEOREM:</u>

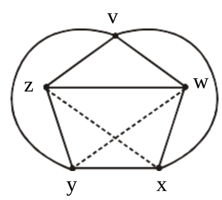
Show that K_5 is non-planar.

PROOF:

Graph K_5 (means a "complete graph" in which every vertex is connected to every other vertex) can be drawn as



To show that K_5 is non-planar, it can be re-drawn as

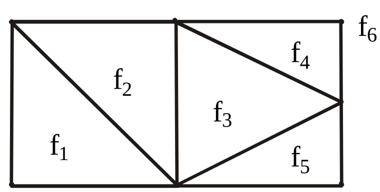


But still edges wy and zx contain the lines which crossed each other. Hence called non-planar.

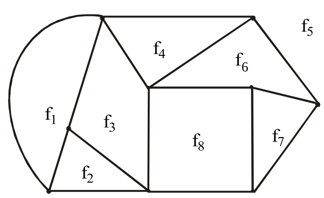
DEFINITION:

A plane drawing of a planar graph divides the plane into regions, including an unbounded region, called **faces**.

The unbounded region is called the **infinite face**.



Here we have 6 faces,7 vertices and 10 edges.f_6 is the unbounded region or called the infinite face because f_6 is outside of the graph.



In this graph, it has 8 faces,9 vertices and 14 edges. Here f_5 is the infinite face or unbounded region.

EULER'S FORMULA

THEOREM:

Let G be a connected planar simple graph with *e* edges and *v* vertices. Let *f* be the number of faces in a plane drawing of G. Then $\boldsymbol{f = e - v + 2}$

EXERCISE:

Suppose that a connected planar simple graph has 30 edges. If a plane drawing of this graph has 20 faces, how many vertices does this graph have?

SOLUTION:

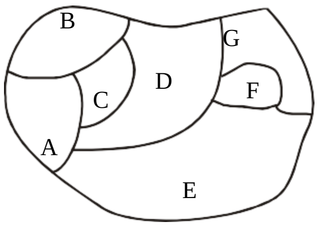
Given that $e = 30$, and $f = 20$. Substituting these values in the Euler's Formula $f = e - v + 2$, we get

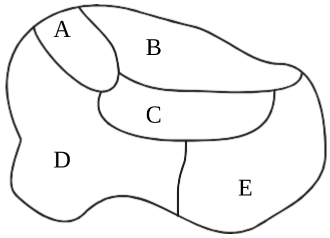
$$20 = 30 - v + 2$$

Hence,

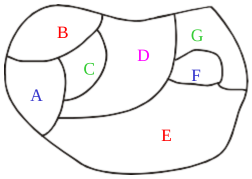
$$v = 30 - 20 + 2 = 12$$

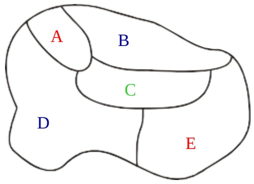
GRAPH COLORING





We also have to face many problems in the form of maps (maps like the parts of the world), which have generated many results in graph theory. Note it that in any graph, many regions are there, but two adjacent regions can't have the same color. And we have to choose a small number of color whenever possible.
Given two graphs above, our problem is to determine the least number of colors that can be used to color the map so that no adjacent regions have the same color.
In the first map given above, 4 colors are necessary, but three colors are not enough. In the second graph, 3 colors are necessary but 2 colors are not enough.



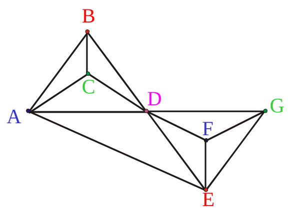


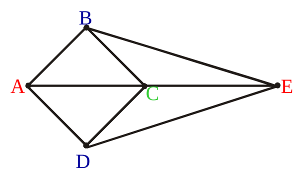
As in the 1st graph, four colors (red, pink, green, blue) are used like that adjacent regions not have the same color. In 2nd graph, three colors (red, blue, green) are used in the same manner.

HOW TO DRAW A GRAPH FROM A MAP:

1. Each map in the plane can be represented by a graph.
2. Each region is represented by a vertex (in 1st map as there are 7 regions, so 7 vertices are used in drawing a graph, similarly we can see 2nd map).

3. If the regions connected by these vertices have the common border, then edge connect two vertices.
4. Two regions that touch at only one point are not adjacent.
So apply these rules, we have (first graph drawn from first map given above, second graph from second map).



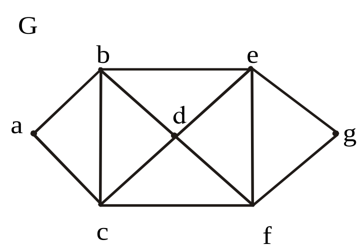


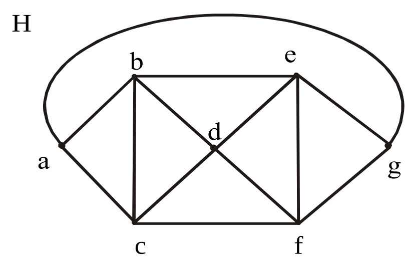
DEFINITION:

1. A **coloring** of a **simple graph** is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.
2. The **chromatic number** of a graph is the least (minimum) number of colors for coloring of this graph.

EXAMPLE:

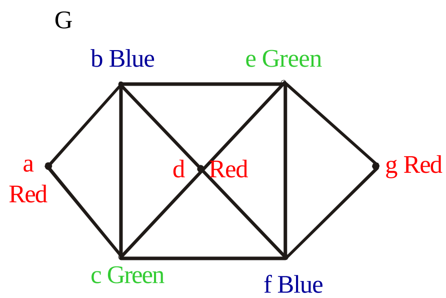
What is the chromatic number of the graphs G and H shown below?

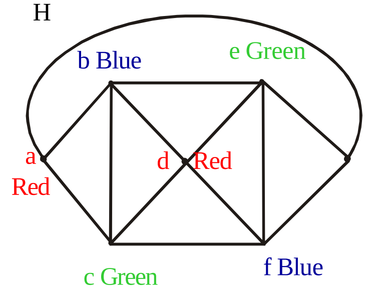




SOLUTION:

Clearly the chromatic number of G is 3 and chromatic number of H is 4(by using the above definition).
In graph G,
As vertices a, b and c are adjacent to each other so assigned different colors. So we assign red color to vertex a, blue to b and green to vertex c. Then no more colors we choose (due to above definition).Now vertex d must be colored red because it is adjacent to vertex b(with blue color) and c(with green color). And e must be colored green because it is adjacent to vertex b(blue color) and vertex d(red color). And f must be colored blue as it is adjacent to red and green color. At last,vertex g must be colored red as it is adjacent to green and blue color.
Same process is used in Graph H.



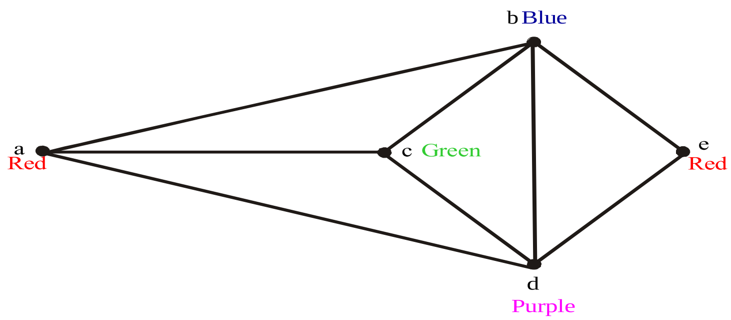


THE FOUR COLOR THEOREM:
The chromatic number of a simple planar graph is no greater than four.
APPLICATION OF GRAPH COLORING

EXAMPLE:
Suppose that a chemist wishes to store five chemicals *a, b, c, d* and *e* in various areas of a warehouse. Some of these chemicals react violently when in contact, and so must be kept in separate areas. In the following table, an asterisk indicates those pairs of chemicals that must be separated. How many areas are needed?

	a	*b*	*c*	*d*	*e*
a	–	*	*	*	–
b	*	–	*	*	*
c	*	*	–	*	–
d	*	*	–	–	*
e	–	*	*	*	–

SOLUTION:
We draw a graph whose vertices correspond to the five chemicals, with two vertices adjacent whenever the corresponding chemicals are to be kept apart.



Clearly the chromatic number is 4 and so four areas are needed.

Handout# 8 Trees

TREES

APPLICATION AREAS:

Trees are used to solve problems in a wide variety of disciplines. In computer science trees are employed to

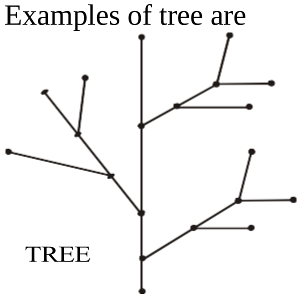
1) construct efficient algorithms for locating items in a list.
2) construct networks with the least expensive set of telephone lines linking distributed computers.
3) construct efficient codes for storing and transmitting data.
4) model procedures that are carried out using a sequence of decisions, which are valuable in the study of sorting algorithms.

TREE:

A tree is a connected graph that does not contain any non-trivial circuit. (i.e. it is circuit-free).

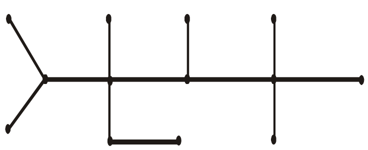
A trivial circuit is one that consists of a single vertex.

Examples of tree are



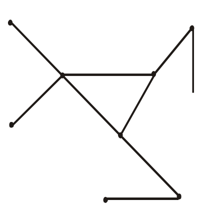
TREE

TREE

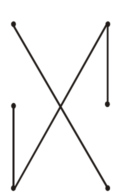


TREE

EXAMPLES OF NON TREES



(a) Graph with a circuit



(b) Disconnected graph

(c) Graph with a circuit

In graph (a), there exists circuit, so not a tree.
In graph (b), there exists no connectedness, so not a tree.
In graph (c), there exists a circuit (also due to loop), so not a tree (because trees have to be a circuit free).

SOME SPECIAL TREES

1. TRIVIAL TREE:

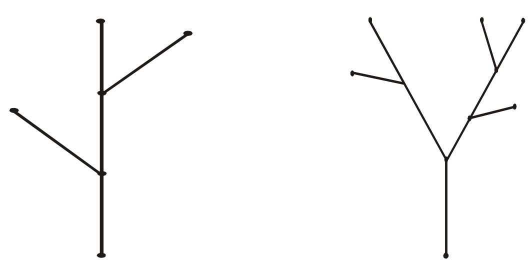
A graph that consists of a single vertex is called a trivial tree or degenerate tree.

2. EMPTY TREE

A tree that does not have any vertices or edges is called an empty tree.

3. FOREST

A graph is called a forest if, and only if, it is circuit-free.
OR "Any non-connected graph that contains no circuit is called a forest."
Hence, it clears that the connected components of a forest are trees.



A forest

As in both the graphs above, there exists no circuit, so called forest.

PROPERTIES OF TREES:

1. A tree with n vertices has n - 1 edges (where $n \geq 0$).
2. Any connected graph with n vertices and n - 1 edges is a tree.
3. A tree has no non-trivial circuit; but if one new edge (but no new vertex) is added to it, then the resulting graph has exactly one non-trivial circuit.
4. A tree is connected, but if any edge is deleted from it, then the resulting graph is not connected.

5. Any tree that has more than one vertex has at least two vertices of degree 1.
6. A graph is a tree iff there is a unique path between any two of its vertices.

EXERCISE:

Explain why graphs with the given specification do not exist.

1. Tree, twelve vertices, fifteen edges.
2. Tree, five vertices, total degree 10.

SOLUTION:

1. Any tree with 12 vertices will have 12 - 1 = 11 edges, not 15.
2. Any tree with 5 vertices will have 5 - 1 = 4 edges.

Since, total degree of graph = 2 (No. of edges)
= 2(4) = 8

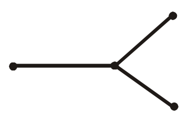
Hence, a tree with 5 vertices would have a total degree 8, not 10.

EXERCISE:

Find all non-isomorphic trees with four vertices.

SOLUTION:

Any tree with four vertices has (4-1=3) three edges. Thus, the total degree of a tree with 4 vertices must be 6 [by using total degree=2(total number of edges)].
Also, every tree with more than one vertex has at least two vertices of degree 1, so the only possible combinations of degrees for the vertices of the trees are 1, 1, 1, 3 and 1, 1, 2, 2.
The corresponding trees (clearly non-isomorphic, by definition) are



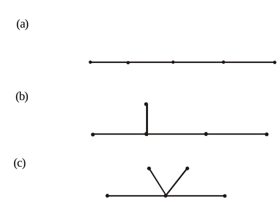
and

EXERCISE:

Find all non-isomorphic trees with five vertices.

SOLUTION:

There are three non-isomorphic trees with five vertices as shown (where every tree with five vertices has 5-1=4 edges).



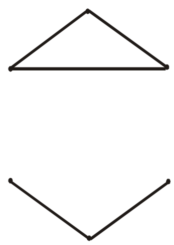
In part (a), tree has 2 vertices of degree ‘1’ and 3 vertices of degree ‘2’.
In part (b), 3 vertices have degree ‘1’, 1 has degree ‘2’and 1 vertex has degree ‘3’.
In part (c), possible combinations of degree are 1, 1, 1, 1, 4.

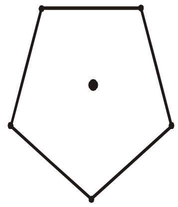
EXERCISE:

Draw a graph with six vertices, five edges that is not a tree.

SOLUTION:

Two such graphs are:





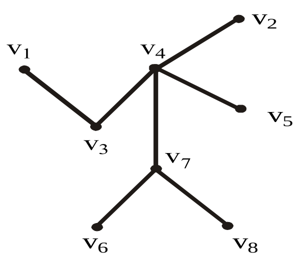
First graph is not a tree; because it is not connected also there exists a circuit.
Similarly, second graph not a tree.

DEFINITION:

A vertex of degree 1 in a tree is called a ***terminal vertex*** or a leaf and a vertex of degree greater than 1 in a tree is called an ***internal vertex*** or a branch vertex.

EXAMPLE:

The terminal vertices of the tree are v_1, v_2, v_5, v_6 and v_8 and internal vertices are v_3, v_4, v_7.



ROOTED TREE:

A **rooted tree** is a tree in which one vertex is distinguished from the others and is called the ***root.***

The **level** of a vertex is the number of edges along the unique path between it and the root.

The **height** of a rooted tree is the maximum level to any vertex of the tree.

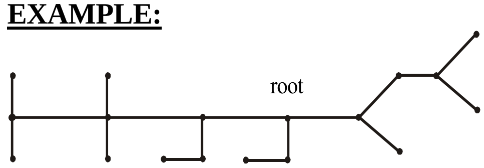
The **children** of any internal vertex v are all those vertices that are adjacent to v and are one level farther away from the root than v.

If **w** is a **child** of **v**, then **v** is called the **parent** of **w**.

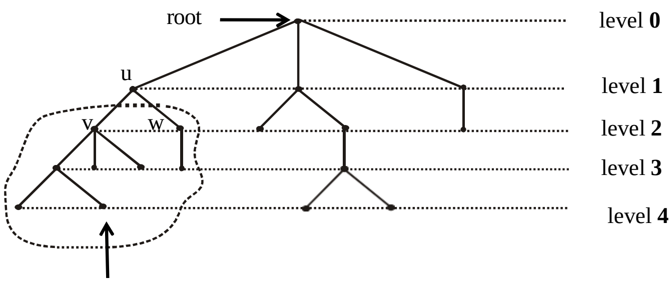
Two vertices that are both children of the same parent are called **siblings**.

Given vertices v and w, if v lies on the unique path between w and the root, then v is an **ancestor** of w and w is a **descendant** of v.

EXAMPLE:



We redraw the tree as and see what the relations are



Vertices in enclosed region
are descendants of u, which
is an ancestor of each

v is a child of u
u is the parent of v
v and w are sublings
height = 4

EXERCISE:

Consider the rooted tree shown below with root v_0

a. What is the level of v_8?

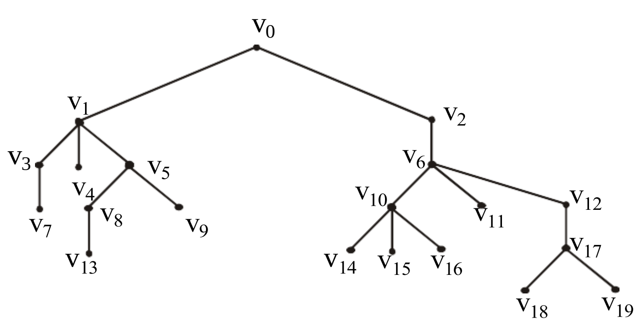
b. What is the level of v_0?

c. What is the height of this tree?

d. What are the children of v_{10}?

e. What are the siblings of v_1?

f. What are the descendants of v_{12}?



SOLUTION:

As we know that "Level means

the total number of edges along the unique path between it and the root".

(a). As v_0 is the root so the level of v_8 (from the root v_0 along the unique path) is 3,because it covers the 3 edges.

(b).The level of v_0 is 0(as no edge cover from v_0 to v_0).

(c).The height of this tree is 5.

Note: As levels are 0, 1, 2, 3, 4, 5 but to find height we have to take the maximum level.

(d).The children of v_{10} are v_{14}, v_{15} and v_{16} .

(e).The siblings of v_1 are v_3 , v_4 , and v_5.

(f).The descendants of v_{12} are v_{17}, v_{18}, and v_{19}.

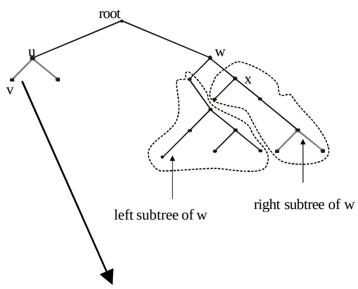
BINARY TREE

A ***binary tree*** is a rooted tree in which every internal vertex has at most two children.

Every child in a binary tree is designated either a left child or a right child (but not both).

A ***full binary tree*** is a binary tree in which each internal vertex has exactly two children.

EXAMPLE:



v is the left child of u.

THEOREMS:

1. If k is a positive integer and T is a full binary tree with k internal vertices, then T has a total of 2k + 1 vertices and has k + 1 terminal vertices.

2. If T is a binary tree that has t terminal vertices and height h, then $t \le 2^h$

Equivalently,

$$\log_2 t \le h$$

Note: The maximum number of terminal vertices of a binary tree of height h is 2^h.

EXERCISE:

Explain why graphs with the given specification do not exist.

1. full binary tree, nine vertices, five internal vertices.
2. binary tree, height 4, eighteen terminal vertices.

SOLUTION:

1. Any full binary tree with five internal vertices has six terminal vertices, for a total of eleven vertices (according to 2(5) +1=11), not nine vertices in all.

OR

As total vertices = 2k+1=9

. k=4(internal vertices)

but given internal vertices=5 ,which is a contradiction.

Thus there is no full binary tree with the given properties.

2. Any binary tree of height 4 has at most 2^4 = 16 terminal vertices.

Hence, there is no binary tree that has height 4 and eighteen terminal vertices.

EXERCISE:

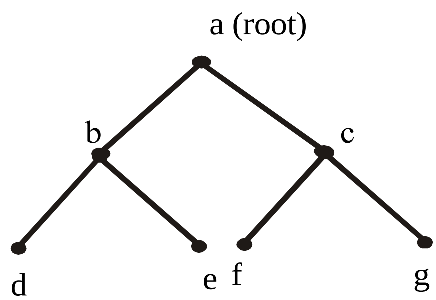
Draw a full binary tree with seven vertices.

SOLUTION:

Total vertices= 2k + 1 = 7 (by using the above theorem)

$\Rightarrow$ k = 3

Hence, total number of internal vertices (i.e. a vertex of degree greater than 1)=k=3
and total number of terminal vertices(i.e. a vertex of degree 1 in a tree) =k+1=3+1=4
Hence, a full binary tree with seven vertices is

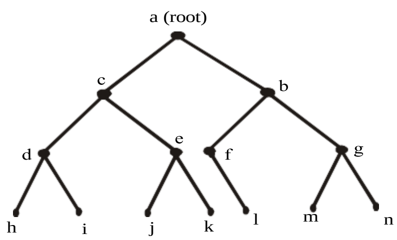


EXERCISE:

Draw a binary tree with height 3 and having seven terminal vertices.

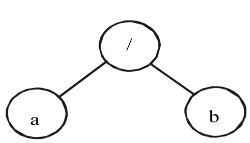
SOLUTION:

Given height=h=3
Any binary tree with height 3 has atmost 2^3=8 terminal vertices.
But here terminal vertices are 7
and Internal vertices=k=6 so binary tree exists and is as fellows:



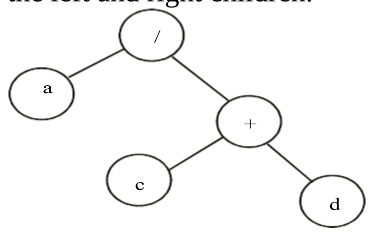
REPRESENTATION OF ALGEBRAIC EXPRESSIONS BY BINARY TREES

Binary trees are specially used in computer science to represent algebraic expression with Arbitrary nesting of balanced parentheses.



Binary tree for a/b

The above figure represents the expression a/b. Here the operator (/) is the root and b are the left and right children.



Binary tree for a/(c+d)

The second figure represents the expression a/(c+d).Here the operator (/) is the root. Here the terminal vertices are variables (here a, c and d), and the internal vertices are arithmetic operators (+ and /).

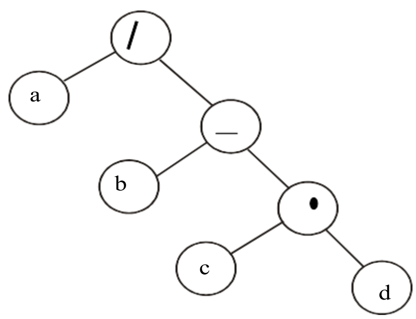
EXERCISE:

Draw a binary tree to represent the following expression

a/(b-c.d)

SOLUTION:

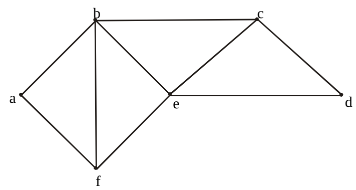
Note that the internal vertices are arithmetic operators, the terminal vertices are variables and the operator at each vertex acts on its left and right sub trees in left-right order.



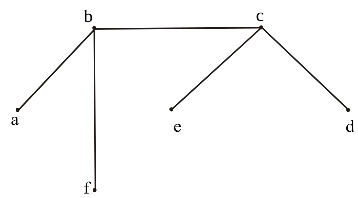
Handout# 9 Spanning Trees

SPANNING TREES:

Suppose it is required to develop a system of roads between six major cities.
A survey of the area revealed that only the roads shown in the graph could be constructed.



For economic reasons, it is desired to construct the least possible number of roads to connect the six cities.
One such set of roads is



Note that the subgraph representing these roads is a tree, it is connected & circuit-free (six vertices and five edges)

SPANNING TREE:

A spanning tree for a graph G is a subgraph of G that contains every vertex of G and is a tree.

REMARK:

1. Every connected graph has a spanning tree.
2. A graph may have more than one spanning trees.
3. Any two spanning trees for a graph have the same number of edges.
4. If a graph is a tree, then its only spanning tree is itself.

EXERCISE:

Find a spanning tree for the graph below: