Step 1

Consider the differential equation:

$$\frac{d^2y}{d\theta^2}+\frac{y}{4}=0$$

$$y''(\theta)+\frac{y}{4}=0$$

Subject to the conditions $y(0)=0$, $y(\pi)=2$ on the interval $[0,\pi]$

The analytic solution is $y(\theta)=2\sin\left(\frac{\theta}{2}\right)$.

Step 2

(a)

Using finite difference to solve the differential equation

$$y''(\theta_i)=\frac{y(\theta_i+h)-2y(\theta_i)+y(\theta_i-h)}{h^2}$$

Substitute this equation into the 2nd order differential equation yields;

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(a)

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Substitute this equation into the 2nd order differential equation yields;

$$\frac{y(\theta_i+h)-2y(\theta_i)+y(\theta_i-h)}{h^2}+\frac{y(\theta_i)}{4}=0.$$

Solve the equation for $h=\frac{1}{4}\pi$.

Set $\theta_i=i\frac{\pi}{4}$ for $i=0,1,\ldots,4$ and $y_i=y(\theta_i)$,

$$\frac{y_2-2y_1+y_0}{h^2}+\frac{y_1}{4}=0,$$

$$\frac{y_3-2y_2+y_1}{h^2}+\frac{y_2}{4}=0,$$

$$\frac{y_4-2y_3+y_2}{h^2}+\frac{y_3}{4}=0.$$

Step 3

This is equivalent to the system of linear equation after multiplying both sides by h^2:

$$\frac{y_2 - 2y_1 + y_0}{h^2} + \frac{y_1}{4} = 0$$

$$y_2 - 2y_1 + y_0 + \frac{y_1}{4}h^2 = 0$$

$$\left(\frac{h^2}{4} - 2\right)y_1 + y_2 + 0 = 0 \qquad \left(\text{Since } y(0) = 0\right)$$

$$\left(\frac{h^2}{4} - 2\right)y_1 + y_2 = 0,$$

$$y_1 + \left(\frac{h^2}{4} - 2\right)y_2 + y_3 = 0,$$

$$y_2 + \left(\frac{h^2}{4} - 2\right)y_3 + y_4 = 0.$$

Step 4

In matrix notation, and noting that $y_0 = 0$, $y_4 = 2$,

$$y_2 + \left(\frac{h^2}{4} - 2\right) y_3 + y_4 = 0$$

$$y_2 + \left(\frac{h^2}{4} - 2\right) y_3 + 2 = 0$$

$$y_2 + \left(\frac{h^2}{4} - 2\right) y_3 = -2$$

Now, in matrix notation;

$$\begin{pmatrix} \frac{h^2}{4} - 2 & 1 & 0 \\ 1 & \frac{h^2}{4} - 2 & 1 \\ 0 & 1 & \frac{h^2}{4} - 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}.$$

Step 5

Set $\varepsilon = \frac{1}{4}h^2 - 2$.

Then, the inverse via Gaussian elimination:

$$\begin{pmatrix} -1.846 & 1 & 0 \\ 1 & -1.846 & 1 \\ 0 & 1 & -1.846 \end{pmatrix}^{-1} = \begin{pmatrix} -0.9268 & -0.7105 & -0.3849 \\ -0.7105 & -1.311 & -0.7104 \\ -0.3849 & -0.7104 & -0.9266 \end{pmatrix}$$

Whence,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -0.9268 & -0.7105 & -0.3849 \\ -0.7105 & -1.311 & -0.7104 \\ -0.3849 & -0.7104 & -0.9266 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.7698 \\ 1.4209 \\ 1.853 \end{pmatrix}.$$

Step 6

Compare with the analytic solution, the differences are:

y_exact - y_fd
-0.00445217
-0.00664159
-0.00535787

Step 7

(b)

Observe that the errors found for $h = \frac{1}{4}\pi$ are, respectively:

y_exact-y_fd	error	% error
-0.00445217	-0.00582	0.581702
-0.00664159	-0.0047	0.46963
-0.00535787	-0.0029	0.289965

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-0.00535787	-0.0029	0.289965

As much as the largest error is 0.58%, it is clear by inspection that increasing the solution by another node by setting $h = \frac{1}{5}\pi$.

So that the resultant approximate solution is $y_{fd} = (y_1, \ldots, y_4)$, the maximal error will be less than 0.5%.

The tedious process can clearly be repeated following (a).

Step 8

Step 8

In fact, by inspecting (a), and the matrix will always be tridiagonal, the 4×4 matrix can be written down immediately:

$$\begin{pmatrix} \frac{h^2}{4}-2 & 1 & 0 & 0 \\ 1 & \frac{h^2}{4}-2 & 1 & 0 \\ 0 & 1 & \frac{h^2}{4}-2 & 1 \\ 0 & 0 & 1 & \frac{h^2}{4}-2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ -2 \end{pmatrix}.$$

Then, inverting the matrix yields the finite difference solution.

Step 9

(c)

To apply the shooting method, reduce the equation to two first order ordinary differential equations (ODE) by setting $u=\frac{d}{d\theta}y$

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(c)

To apply the shooting method, reduce the equation to two first order ordinary differential equations (ODE) by setting $u = \frac{d}{d\theta} y$.

Then, the ODE becomes;

$$\frac{dy}{d\theta} = u,$$

$$\frac{du}{d\theta} = -\frac{y}{4}.$$

Try $u(0) = 1$ for the shooting estimate using the RKF method.

The computed values for $h = 0.1$ leads to maximal error ~ 15% when compared with the exact solution:

h_RKF	0.1	
t	y_RKF	% error
0	0	0
0.1	0.105171049	0.521271
0.2	0.210342099	1.067527
0.3	0.315240052	1.636379

h_RKF	0.1	
t	y_RKF	% error
0	0	0
0.1	0.105171049	0.521271
0.2	0.210342099	1.067527
0.3	0.315240052	1.636379
0.4	0.419591814	2.225315
0.5	0.523124998	2.831708
0.6	0.625568636	3.452822
0.7	0.726653884	4.085827
0.8	0.82611473	4.727805
0.9	0.923688688	5.375762
1	1.019117488	6.026641
1.1	1.112147762	6.67733
1.2	1.202531713	7.324677
1.3	1.29002777	7.965496
1.4	1.374401235	8.596586

1.5	1.455424909	9.214739
1.6	1.5328797	9.816752
1.7	1.606555216	10.39944
1.8	1.676250331	10.95965
1.9	1.741773734	11.49427
2	1.802944448	12.00025
2.1	1.85959233	12.47459
2.2	1.91155854	12.91438
2.3	1.95869598	13.31681
2.4	2.000869711	13.67915
2.5	2.037957332	13.99881
2.6	2.069849331	14.2733
2.7	2.096449404	14.50027
2.8	2.117674737	14.67753
2.9	2.133456258	14.80303
3	2.143738852	14.87489
3.1	2.148481539	14.8914

2.4	2.000869711	13.67915
2.5	2.037957332	13.99881
2.6	2.069849331	14.2733
2.7	2.096449404	14.50027
2.8	2.117674737	14.67753
2.9	2.133456258	14.80303
3	2.143738852	14.87489
3.1	2.148481539	14.8914

By changing the parameter from $h = 0.1$ to $h = 0.003$, a maximal error less than 0.5% can be achieved.