Applied Numerical Analysis (7th Edition)

Chapter 3, Problem 54E | Bookmark | Show all steps: on

Problem

Repeat Exercise, but for B-spline curves. Add fictitious points at the end so the end portions are completed.

Exercise

If these four points are connected in order by straight lines, a zigzag line is created:

$(0, 0), (1, 0.3), (2, 1.7), (3, 1.5)$.

a. Using the two interior points as controls, find the cubic Bezier curve. Plot this together with the zigzag line.

b. Use this cubic equation to find interpolates at $x = 0.5$, $x = 0.75$, and $x = 2.5$. How close are these to the zigzag line?

c. If the second and third points (the control points) are moved, the Bezier curve will change. If these are moved vertically, where should they be located so that the Bezier curve passes through all of the original four points?

Step-by-step solution

Step 1 of 14

The matrix form of a cubic B-spline is

$$B_i(u)=\frac{1}{6}\begin{bmatrix}u^3 & u^2 & u & 1\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 0 & 3 & 0\\ 1 & 4 & 1 & 0\end{bmatrix}\begin{bmatrix}p_{i-1}\\ p_i\\ p_{i+1}\\ p_{i+2}\end{bmatrix},\ i=1,\ldots,(n-1).$$

where $p_i=(x_i,y_i)$ and $p_{-1}=p_{-1}=p_0$ and

Comment

Step 2 of 14

Since the problem gives 4 data points, the 3 pieces of the cubic B-spline are calculated by

$$B_i(u)=\frac{1}{6}\begin{bmatrix}u^3 & u^2 & u & 1\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 0 & 3 & 0\\ 1 & 4 & 1 & 0\end{bmatrix}\begin{bmatrix}p_{i-1}\\ p_i\\ p_{i+1}\\ p_{i+2}\end{bmatrix},\ i=1,2,3.$$

Comment

Step 3 of 14

a) To draw the graph that includes the control points (red), the zig-zag line (green) connecting the control points, and the B-Spline curve (blue), make use of the following Mathematica commands

INPUT:

```
pts={{0,0},{1,0.3},{2,1.7},{3,1.5}};
```

INPUT:

```
h=BSplineFunction[pts]
```

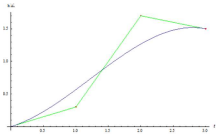
OUTPUT:

```
BSplineFunction[{{0.,1.}},"<>"]
```

INPUT:

```
Show[Graphics[{Red,Point[pts],Green,Line[pts]},Axes→True,
AxesLabel→{t,"f(t)"}],ParametricPlot[f[t],{t,0,1}]]
```

OUTPUT:



Comment

Step 4 of 14

b) To find interpolates at $x=.5, .75, 2.5$, each of the following B-spline curve equations must be solved for their corresponding u values.

$$B_1x(u)=\frac{1}{6}\begin{bmatrix}u^3 & u^2 & u & 1\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 0 & 3 & 0\\ 1 & 4 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\ 0\\ 1\\ 2\end{bmatrix}=.5$$

$$B_2x(u)=\frac{1}{6}\begin{bmatrix}u^3 & u^2 & u & 1\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 0 & 3 & 0\\ 1 & 4 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\ 1\\ 2\\ 3\end{bmatrix}=.75$$

$$B_3x(u)=\frac{1}{6}\begin{bmatrix}u^3 & u^2 & u & 1\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 0 & 3 & 0\\ 1 & 4 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\ 2\\ 3\\ 3\end{bmatrix}=2.5$$

Comment

Step 5 of 14

First, compute the first B-spline

$$B_1x(u)=\frac{1}{6}\begin{bmatrix}u^3 & u^2 & u & 1\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 0 & 3 & 0\\ 1 & 4 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\ 0\\ 1\\ 2\end{bmatrix}$$

using the Mathematica command

INPUT:

```
(1/6){{u^3,u^2,u,1}}{{-1,3,-3,1},{3,-6,3,0},{-3,0,3,0},{1,4,1,0}}.{{0},{0},{1},{2}}
```

OUTPUT:

$$\left\{\left\{\frac{1}{6}(1+3u+3u^2-u^3)\right\}\right\}$$

Comment

Step 6 of 14

Next, give the following command to type the equation into Mathematica to solve for u when $x=.5$

INPUT:

```
B1x[u_]:=1/6*(1+3u+3u^2-u^3)
```

INPUT:

```
Solve[B1x[u]==.5,u]
```

OUTPUT:

$$\{\{u\to -1.1451\},\{u\to 0.476024\},\{u\to 3.66908\}\}$$

Since the parameter u for a B-Spline must be in the interval $[0,1]$, choose $u=.476024$

Comment

Step 7 of 14

Next, give the following command to type the equation into Mathematica to solve for u when $x=.75$

INPUT:

```
Solve[B1[u]==.75,u]
```

OUTPUT:

$$\{\{u\to -1.31335\},\{u\to 0.747311\},\{u\to 3.56604\}\}$$

Since the parameter u for a B-Spline must be in the interval $[0,1]$, choose $u=.747311$

Comment

Step 8 of 14

Next, compute the third B-spline

$$B_3(u)=\frac{1}{6}\begin{bmatrix}u^3 & u^2 & u & 1\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 0 & 3 & 0\\ 1 & 4 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\ 2\\ 3\\ 3\end{bmatrix}$$

using the Mathematica command

INPUT:

$$\begin{pmatrix} 1 & 4 & 1 & 0 \end{pmatrix}$$

OUTPUT:

$$\{\{\frac{1}{6}(1-3u+3u^2+2u^3+3(1+3u+3u^2-3u^3)+2(4-6u^2+3u^3))\}\}$$

Comment

Step 9 of 14

Next, give the following command to type the equation into Mathematica to solve for u when $x = 2.5$

INPUT:

Solve[B3x[u] $\hat{S}$ 2.5,u]

OUTPUT:

$\{\{u \to -2.66908\}, \{u \to 0.523976\}, \{u \to 2.1451\}\}$

Since the parameter u for a B-Spline must be in the interval $[0,1]$ choose $u = 0.523976$

Comment

Step 10 of 14

To interpolate at $u = .476024$ and $u = .747311$ for the first B-Spline piece, substitute the values of u into

$$B_{3Y}(u) = \frac{1}{6}\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ .3 \\ 1.7 \end{bmatrix}$$

using the following Mathematica commands. For $u = .476024$.

INPUT:

$$(1/6)\{\{.476024\hat{}3, .476024\hat{}2, .476024, 1\}\}.\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}.\{\{0\},\{0\},\{.3\},\{1.7\}\}$$

OUTPUT:

$\{\{0.169776\}\}$

Comment

Step 11 of 14

Now give the following commands to retrieve the interpolate at $u = .747311$.

INPUT:

$$(1/6)\{\{0.747311\hat{}3, 0.747311\hat{}2, 0.747311, 1\}\}.\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}.\{\{0\},\{0\},\{.3\},\{1.7\}\}$$

OUTPUT:

$\{\{0.301515\}\}$

Comment

Step 12 of 14

To interpolate at $u = 0.523976$ for the third B-Spline piece, substitute the values of u into

$$B_{3Y}(u) = \frac{1}{6}\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} .3 \\ 1.7 \\ 1.5 \\ 1.5 \end{bmatrix}$$

using the following Mathematica commands.

INPUT:

$$(1/6)\{\{0.523976\hat{}3, 0.523976\hat{}2, 0.523976, 1\}\}.\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}.\{\{.3\},\{1.7\},\{1.5\},\{1.5\}\}$$

OUTPUT:

$\{\{1.57124\}\}$

Comment

Step 13 of 14

c) It is hard to specify exactly how far vertically a B-Spline curve control point should be moved so that the curve passes through all the original points. However, it can be generally said that moving a control point down will pull the curve down and pulling a control point up will pull the curve up.

Comment

Step 14 of 14

Being that the B-Spline curve lies above the first interior control point and below the second interior control point, move the first interior control point down and move the second interior point up until the curve passes through the original points.

Comment