## Step 1

Consider the ordinary differential equation (ODE):

$$x'' - tx' + t^2 x = t^3$$

Subject to the mixed boundary conditions;

$$x(0) + x'(0) - x(1) + x'(1) = 3,$$

$$x(0) - x'(0) + x(1) - x'(1) = 2,$$

On the interval $[0,1]$

Employ finite difference with $h = 0.25$ to solve the equation numerically.

## Step 2

First, observe that the boundary conditions yield the following two relations by adding and subtraction one equation from the other respectively:

$$2x(0) = 5$$

$$x(0) = \frac{5}{2}$$

And

$$2x'(0) - 2x(1) + 2x'(1) = 1$$

$$x'(0) - x(1) + x'(1) = \frac{1}{2}$$

## Step 3

## Step 3

The two unknowns at the end point $t=1$ and knowing $x(0)$ suggest using the forward difference scheme.
Now, recall the forward difference method:

$$x''(t_i)=\frac{x(t_i+2h)-2x(t_i+h)+x(t_i)}{h^2}$$

And

$$x'(t_i)=\frac{x(t_i+h)-x(t_i)}{2h}.$$

Substitute the equations into the ODE and defining $t_i=ih$ for $i=0,\ldots,3$ and $x_i=x(t_i)$, then

$$\frac{x_{i+2}-2x_{i+1}+x_i}{h^2}-t\frac{x_{i+1}-x_i}{h}+t^2x_i=t^3.$$

## Step 4

Rearrange this equation and multiply both sides by $h^2$ lead to

$$x_{i+2}-(2+ih^2)x_{i+1}+(1+ih^2+i^2h^4)x_i=i^3h^5.$$

For $i=0,\ldots,3$,

$$x_2-2x_1+x_0=0,$$

$$x_3-(2+h^2)x_2+(1+h^2+h^4)x_1=h^5,$$

$$x_4-(2+2h^2)x_3+(1+2h^2+4h^4)x_2=8h^5,$$

## Step 4

Rearrange this equation and multiply both sides by $h^2$ lead to

$$x_{i+2} - \left(2 + ih^2\right)x_{i+1} + \left(1 + ih^2 + i^2h^4\right)x_i = i^3h^5.$$

For $i = 0, \ldots, 3$,

$$x_2 - 2x_1 + x_0 = 0,$$

$$x_3 - \left(2 + h^2\right)x_2 + \left(1 + h^2 + h^4\right)x_1 = h^5,$$

$$x_4 - \left(2 + 2h^2\right)x_3 + \left(1 + 2h^2 + 4h^4\right)x_2 = 8h^5,$$

$$x_5 - \left(2 + 3h^2\right)x_4 + \left(1 + 3h^2 + 9h^4\right)x_3 = 27h^5.$$

Now, the last equation can be simplified further via the boundary condition:

$$x'(0) - x(1) + x'(1) = \frac{1}{2}.$$

That is,

$$-x_4 + \frac{x_5 - x_4}{h} = 2 - x'(0)$$

$$x_5 = \left(2 - x'(0)\right)h + (1 + h)x_4$$

## Step 5

Thus, substitute this into the last equation yields;

$$-\left(1 - h + 3h^2\right)x_4 + \left(1 + 3h^2 + 9h^4\right)x_3 = 27h^5 - \left(2 - x_0'\right)h.$$

Expressed as a matrix equation,

$$\begin{pmatrix} -2 & 1 & 0 & 0 \end{pmatrix}$$

## Step 5

Thus, substitute this into the last equation yields;

$$-\left(1-h+3h^2\right)x_4+\left(1+3h^2+9h^4\right)x_3=27h^5-\left(2-x_0'\right)h.$$

Expressed as a matrix equation,

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1+h^2+h^4 & -2\left(1+\frac{h^2}{2}\right) & 1 & 0 \\ 0 & 1+2h^2+4h^4 & -2\left(1+h^2\right) & 1 \\ 0 & 0 & 1+3h^2+9h^4 & -\left(1-h+3h^2\right) \end{pmatrix}$$

.

Write the matrix equation for simplicity as $AY=B$, the solution can be found via the augmented matrix (using row-reduced echelon form):

$$[A \mid B]\rightarrow[I \mid Y].$$

Or equivalently $Y=A^{-1}B$

Implement this algorithm in Visual Basic, the solution is obtained via the shooting method, where the shooting parameter $x_0'$ is chosen such that the mixed boundary condition

## Step 6

Here, the "central" difference scheme was employed to obtain $x'(1)$:

$$x'(1)\approx x'\left(1-\frac{h}{2}\right)$$

can be found via the augmented matrix (using row-reduced echelon form):

$$[A \mid B] \to [I \mid Y].$$

Or equivalently $Y = A^{-1}B$

Implement this algorithm in Visual Basic, the solution is obtained via the shooting method, where the shooting parameter $x'_0$ is chosen such that the mixed boundary condition

## Step 6

Here, the "central" difference scheme was employed to obtain $x'(1)$ :

$$x'(1) \approx x'\left(1-\frac{h}{2}\right)$$
$$= \frac{x(1)-x(1-h)}{2h}$$

This approximation clearly introduces another layer of error.

The results are:

| | | |
|---|---|---|
| u_shooting | 4.4 | |
| h | 0.25 | |
| t | x | u(0)-x(1)+u(1) |
| 0.25 | 3.070126 | 0.785023796 |
| 0.5 | 3.640253 | |
| 0.75 | 4.234996 | |
| 1 | 4.855015 | |

Chegg