Problem set #4

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The letter V always denotes a vector space.

Exercise 1. Find all solutions to

$$\begin{aligned} 2x_1 - x_2 + x_3 + x_4 - x_5 &= 0 \\ x_1 + x_3 + 2x_4 &= 1 \\ x_1 - x_2 - x_4 - x_5 &= -1. \end{aligned}$$

Solution. Encode the equations as a matrix

$$\left[\begin{array}{ccccc|c} 2 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 2 & 0 & 1 \\ 1 & -1 & 0 & -1 & -1 & -1 \end{array}\right]$$

Row reduction yields

$$\left[\begin{array}{ccccc|c} 2 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 2 & 0 & 1 \\ 1 & -1 & 0 & -1 & -1 & -1 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

(See full reduction in Appendix A)
Converting back into equation form

$$\begin{aligned} x_1 + x_3 + 2x_4 &= 1 \\ x_2 + x_3 + 3x_4 &= 2 \\ x_5 &= 0 \end{aligned} \qquad \text{So,} \qquad \begin{aligned} x_1 &= 1 - x_3 - 2x_4 \\ x_2 &= 2 - x_3 - 3x_4 \\ x_3 &\text{ is free} \\ x_4 &\text{ is free} \\ x_5 &= 0 \end{aligned}$$

Therefore, the set of all solutions is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \mid x_3, x_4 \in \mathbb{R} \right\}$$

□

Exercise 2. Do the vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ span $\mathbb{R}^3$? Are they linearly independent?

Solution. To check if v_1, v_2, v_3 span $\mathbb{R}^3$, check if an arbitrary vector $v_4 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ can be written as a linear combination of v_1, v_2, v_3. In other words, solve $c_1v_1 + c_2v_2 + c_3v_3 = v_4$. Begin by writing as a matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 2 & 1 & b \\ 1 & 1 & 1 & c \end{array}\right]$$

Row reduction yields

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 2 & 1 & b \\ 1 & 1 & 1 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -a-b+2c \\ 0 & 1 & 0 & c-a \\ 0 & 0 & 1 & b-2c+2a \end{array}\right]$$

(Full reduction shown in Appendix A)
So, $c_1 = -a - b + 2c, c_2 = c - a, c_3 = b - 2c + 2a$, therefore

$$\left[\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right] = (-a - b + 2c)\left[\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right] + (c - a)\left[\begin{smallmatrix} 0 \\ 2 \\ 1 \end{smallmatrix}\right] + (b - 2c + 2a)\left[\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right]$$

and $\mathbb{R}^3 = \text{Span}\{v_1, v_1, v_3\}$.

Now, to check if v_1, v_2, v_3 are linearly independent, check if there are any nontrivial solutions to $c_1v_1 + c_2v_2 + c_3v_3 = 0$. Using the above and substituting $a = 0, b = 0, c = 0$ it is clear that the only solution occurs when $c_1 = c_2 = c_3 = 0$. So, the list is linearly independent. $\square$

Exercise 3. Suppose $v_1, v_2, v_3, v_4 \in V$, and set

$$w_1 = v_1 - v_2, \quad w_2 = v_2 - v_3, \quad w_3 = v_3 - v_4, \quad w_4 = v_4.$$

(a) Show that $\text{Span}\{v_1, v_2, v_3, v_4\} = \text{Span}\{w_1, w_2, w_3, w_4\}$.

(b) Show that v_1, v_2, v_3, v_4 are linearly independent if and only if w_1, w_2, w_3, w_4 are linearly independent.

Solution.

(a) It is readily apparent from the relationships that $v_1 = w_1 + w_2 + w_3 + w_4$, $v_2 = w_2 + w_3 + w_4$, $v_3 = w_3 + w_4$, and $v_4 = w_4$.
It follows that any item $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 \in \text{Span}\{v_1, v_2, v_3, v_4\}$ can be written as $c_1(w_1 + w_2 + w_3 + w_4) + c_2(w_2 + w_3 + w_4) + c_3(w_3 + w_4) + c_4(w_4)$, or equivalently $(c_1)w_1 + (c_1 + c_2)w_2 + (c_1 + c_2 + c_3)w_3 + (c_1 + c_2 + c_3 + c_4)w_4$, a linear combination which is clearly an element of $\text{Span}\{w_1, w_2, w_3, w_4\}$.
It is also clear that any element $d_1w_1 + d_2w_2 + d_3w_3 + d_4w_4 \in \text{Span}\{w_1, w_2, w_3, w_4\}$ can be written as $d_1(v_1 - v_2) + d_2(v_2 - v_3) + d_3(v_3 - v_4) + d_4(v_4)$, or equivalently $(d_1)v_1 + (-d_1 + d_2)v_2 + (-d_2 + d_3)v_3 + (-d_3 + d_4)v_4$, which, by similar logic to the above is a linear combination that is clearly an element of $\text{Span}\{v_1, v_2, v_3, v_4\}$.

Since every element of each span is shown to also be contained in the other, the spans can be said to be equal.

(b) The proof is by contradiction.
First, to show that v_1, v_2, v_3, v_4 being linearly independent implies that w_1, w_2, w_3, w_4 are as well, assume that v_1, v_2, v_3, v_4 are linearly independent and w_1, w_2, w_3, w_4 are not. What this means is that $d_1w_1 + d_2w_2 + d_3w_3 + d_4w_4 = 0$ has a nontrivial solution, in other words, at least one of d_1, d_2, d_3, d_4 is not 0. As shown in part (a), this statement is equivalent to $(d_1)v_1 + (-d_1 + d_2)v_2 + (-d_2 + d_3)v_3 + (-d_3 + d_4)v_4 = 0$. Because v_1, v_2, v_3, v_4 are linearly independent by assumption, the coefficients of this expression must all be equal to 0. But, that implies that $d_1 = 0, d_2 = 0, d_3 = 0, d_4 = 0$, which contradicts our assumption that w_1, w_2, w_3, w_4 are linearly dependent. Therefore, v_1, v_2, v_3, v_4 being linearly independent implies that w_1, w_2, w_3, w_4 are as well.

Second, to show that w_1, w_2, w_3, w_4 being linearly independent implies that v_1, v_2, v_3, v_4 are as well, again assume that the first list is linearly independent and that the second list *is not*. This assumption means that $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$ has a solution where at least one of c_1, c_2, c_3, c_4 is not 0. By the same reasoning as in (a) and the above, this statement is equivalent to $(c_1)w_1 + (c_1 + c_2)w_2 + (c_1 + c_2 + c_3)w_3 + (c_1 +$

$c_2 + c_3 + c_4)w_4 = 0$. Because w_1, w_2, w_3, w_4 are assumed to be linearly independent, the coefficients of this statement must all be equal to 0, but again this implies that $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$. This contradicts our assumption that v_1, v_2, v_3, v_4 are linearly dependent, and therefore w_1, w_2, w_3, w_4 being linearly independent must imply that v_1, v_2, v_3, v_4 are as well.

Together, the above show that v_1, v_2, v_3, v_4 are linearly independent if and only if w_1, w_2, w_3, w_4 are linearly independent.

□

Exercise 4. Suppose $v_1, \ldots, v_n \in V$.

(a) If $U \subset V$ is any subspace, show that

$$v_1, \ldots, v_n \in U \iff \mathrm{Span}\{v_1, \ldots, v_n\} \subset U.$$

Use this to prove that $\mathrm{Span}\{v_1, \ldots, v_n\}$ is equal to the intersection of all subspaces containing $v_1, \ldots, v_n$.

(b) Show that $w \in V$ is a linear combination of $v_1, \ldots, v_n$ if and only if

$$\mathrm{Span}\{v_1, \ldots, v_n, w\} = \mathrm{Span}\{v_1, \ldots, v_n\}.$$

Solution.

(a) First, show that $v_1, \ldots, v_n \in U \implies \mathrm{Span}\{v_1, \ldots, v_n\} \subset U$.
Since $v_1, \ldots, v_n \in U$ and U is a vector space, it follows that for *any* scalars $c_1, \ldots, c_n \in F$ then $c_1 v_1, \ldots, c_n v_n \in U$ because U is closed under scalar multiplication. Because U is closed under addition, it is also clear that adding two vectors together produces a third vector in the space, $c_1 v_1 + c_2 v_2 \in U$.
This new vector can also be added to another vector, $(c_1 v_1 + c_2 v_2) + c_3 v_3 \in U$. This can clearly be repeated a finite number of times to yield $c_1 v_1 + \cdots + c_n v_n \in U$, for any scalars $c_1, \ldots, c_n$.
The $\mathrm{Span}\{v_1, \ldots, v_n\}$ is defined as the set of all linear combinations of $v_1, \ldots v_n$, which is clearly represented by the construction above.
Therefore $v_1, \ldots, v_n \in U \implies \mathrm{Span}\{v_1, \ldots, v_n\} \subset U$

Secondly, show that $v_1, \ldots, v_n \in U \impliedby \mathrm{Span}\{v_1, \ldots, v_n\} \subset U$.
By definition, $\mathrm{Span}\{v_1, \ldots, v_n\}$ is the set of all linear combinations of the form $c_1 v_1 + \cdots + c_n v_n$. This set obviously includes combinations where some $c_i = 1$ and $c_1 = 0, \ldots, \widehat{c_i}, \ldots, c_n = 0$. These combination simply yield v_i in the span. It obviously follows that all v_i must be in the span, if i is allowed to vary from 1 to n. Because the span is a subset of U, those same vectors must obviously also be in U.
Therefore, $v_1, \ldots, v_n \in U \impliedby \mathrm{Span}\{v_1, \ldots, v_n\} \subset U$.

The above suffices to show that $v_1, \ldots, v_n \in U \iff \mathrm{Span}\{v_1, \ldots, v_n\} \subset U$.

Finally, using the above show that $\mathrm{Span}\{v_1, \ldots, v_n\}$ is equal to the intersection of all subspaces containing $v_1, \ldots, v_n$.
The intersections of all subspaces containing $v_1, \ldots, v_n$ would logically be equivalent to the smallest subspace which contains them, in other words, the space which contains only $v_1, \ldots, v_n$ and any vectors which can be produced by addition or scalar multiplication (notably, 0).
However, as shown above, $v_1, \ldots, v_n \in U \iff \mathrm{Span}\{v_1, \ldots, v_n\} \subset U$. Therefore, this smallest subspace would necessarily contain $\mathrm{Span}\{v_1, \ldots, v_n\}$. But, it is also clear that this subspace cannot contain any vectors which can not be expressed in terms of $v_1, \ldots, v_n$. So, it follows that the intersection of all subspaces containing $v_1, \ldots, v_n$ is equal to $\mathrm{Span}\{v_1, \ldots, v_n\}$.

(b) First, show that $w \in V$ being a linear combination of $v_1, \ldots, v_n$ implies $\text{Span}\{v_1, \ldots, v_n, w\} = \text{Span}\{v_1, \ldots, v_n\}$.
By definition, w being a linear combination of $v_1, \ldots, v_n$ means w can be decomposed as $w = a_1v_1 + \cdots + a_nv_n$ for some $a_1, \ldots, a_n \in F$. Also by definition, the span of a list of vectors is the set of all linear combinations of those vectors. So, $\text{Span}\{v_1, \ldots, v_n, w\}$ is the set of all combinations which we can denote as $c_1v_1 + \cdots + c_nv_n + dw$ where $c_1, \ldots, c_n, d \in F$. But, using the above decomposition for w, this can be rewritten as the set of all combinations of the form $c_1v_1 + \cdots + c_nv_n + d(a_1v_1 + \cdots + a_nv_n)$, which is equivalent to $(c_1 + da_1)v_1 + \cdots + (c_n + da_n)v_n$, with obviously $(c_i + da_i) \in F$. This is obviously a linear combination of $v_1, \ldots, v_n$, so it must be contained in $\text{Span}\{v_1, \ldots, v_n\}$. It trivial to show that elements in $\text{Span}\{v_1, \ldots, v_n\}$ are contained in $\text{Span}\{v_1, \ldots, v_n, w\}$, as the coefficient of w can simply be set to 0. Since each span contains all elements of the other, they are equal.
Therefore, $w \in V$ being a linear combination of $v_1, \ldots, v_n$ implies $\text{Span}\{v_1, \ldots, v_n, w\} = \text{Span}\{v_1, \ldots, v_n\}$.

Secondly, show that $\text{Span}\{v_1, \ldots, v_n, w\} = \text{Span}\{v_1, \ldots, v_n\}$ implies $w \in V$ is a linear combination of $v_1, \ldots, v_n$.
For the two spans to be equal, that means every vector in one must be able to be expressed in terms of the other. Trivially, $w \in \text{Span}\{v_1, \ldots, v_n, w\}$. Since the spans are equal, it means there must be some combination $c_1v_1 + \cdots + c_nv_n \in \text{Span}\{v_1, \ldots, v_n\}$ which is equal to w.
Therefore $\text{Span}\{v_1, \ldots, v_n, w\} = \text{Span}\{v_1, \ldots, v_n\}$ implies $w \in V$ is a linear combination of $v_1, \ldots, v_n$.

The above suffices to show that $w \in V$ is a linear combination of $v_1, \ldots, v_n$ if and only if $\text{Span}\{v_1, \ldots, v_n, w\} = \text{Span}\{v_1, \ldots, v_n\}$.

$\square$

Exercise 5. For each of the following, find a proof or a counterexample:

(a) Adding a vector to a linearly independent list results in a linearly independent list.

(b) Deleting a vector from a linearly independent list results in a linearly independent list.

(c) Adding a vector to a linearly dependent list results in a linearly dependent list.

(d) Deleting a vector from a linearly dependent list results in a linearly dependent list.

Solution.

(a) Counterexample: The list $v_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ is linearly independent, since $c_1v_1 = 0$ has only the trivial solution. Adding $v_2 = \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ to this list creates a linearly *dependent* list, since $2v_1 - v_2 = 0$.

(b) The proof is by contradiction. Assume there is a linearly independent list $v_1, \ldots, v_n$, and a list $v_1, \ldots, \widehat{v_i}, \ldots, v_n$ which is linearly *dependent*. After reordering, we can say that the removed element was the last, so the second list is denoted as $v_1, \ldots, v_{n-1}$. By definition, this second list would have to have a solution to $c_1v_1 + \cdots + c_nv_{n-1} = 0$ such that at least one of $c_1, \ldots, c_{n-1} \neq 0$. However, it is obvious that adding $0 \cdot v_n$ to this linear combination would not change that it is equal to 0. This would yield a nontrivial linear relation for the first list $v_1, \ldots, v_n$. This is a contradiction of our assumption and as such removing a vector from a linearly independent list must yield another linearly independent list.

(c) By definition, a list $v_1, \ldots, v_n$ is linearly dependent if there exists a solution to $c_1v_1 + \cdots + c_nv_n = 0$ such that at least one of $c_1, \ldots, c_n \neq 0$. Adding an arbitrary vector w to this list does not change this relationship, as it is clear that $c_1v_1 + \cdots + c_nv_n + 0 \cdot w$

would still be equal to 0 when using the same $c_1, \ldots, c_n$ as the shorter list. Therefore, adding a vector to a linearly dependent list can be shown to yield another linearly dependent list.

(d) Counterexample: The list $v_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], v_2 = \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ is linearly dependent, since $2v_1 - v_2 = 0$. Deleting v_2 from this list creates a linearly *independent* list, since the only solution to $c_1 v_1 = 0$ is obviously $c_1 = 0$.

□

Exercise 6. Let P_3 be the vector space of all polynomials with real coefficients and degree ≤ 3. Do the polynomials

$$f_1(x) = x^3 + 2x, \quad f_2(x) = x^2 + x + 1, \quad f_3(x) = x^3 + 5, \quad f_4(x) = x^2 + 3x - 4$$

span P_3?

Solution. To check if the polynomials $f_1(x), f_2(x), f_3(x), f_4(x)$ span P_3 check if an arbitrary polynomial ax^3+bx^2+cx+d can be written as a linear combination of $f_1(x), f_2(x), f_3(x), f_4(x)$. In other words, solve $x_1 \left[\begin{smallmatrix}0\\2\\0\\1\end{smallmatrix}\right] + x_2 \left[\begin{smallmatrix}1\\1\\1\\0\end{smallmatrix}\right] + x_3 \left[\begin{smallmatrix}5\\0\\0\\1\end{smallmatrix}\right] + x_4 \left[\begin{smallmatrix}-4\\3\\1\\0\end{smallmatrix}\right] = \left[\begin{smallmatrix}d\\c\\b\\a\end{smallmatrix}\right]$.
Begin by writing as a matrix

$$\left[\begin{array}{cccc|c} 0 & 1 & 5 & -4 & d \\ 2 & 1 & 0 & 3 & c \\ 0 & 1 & 0 & 1 & b \\ 1 & 0 & 1 & 0 & a \end{array}\right]$$

Partial row reduction yields

$$\left[\begin{array}{cccc|c} 0 & 1 & 5 & -4 & d \\ 2 & 1 & 0 & 3 & c \\ 0 & 1 & 0 & 1 & b \\ 1 & 0 & 1 & 0 & a \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a \\ 0 & 1 & 0 & 1 & b \\ 0 & 0 & 1 & -1 & \frac{2a+b-c}{2} \\ 0 & 0 & 0 & 0 & -10a - 7b + 5c + 2d \end{array}\right]$$

(See full reduction in Appendix A)
Inspecting the final row, it is clear that any solution would have the restriction $-10a - 7b + 5c + 2d = 0$. This means that an arbitrary polynomial $ax^3 + bx^2 + cx + d$ can not be written as a linear combination of $f_1(x), f_2(x), f_3(x), f_4(x)$. Therefore, $f_1(x), f_2(x), f_3(x), f_4(x)$ do not span P_3. □