Problem set #6

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In all of the following exercises F denotes $\mathbb{R}$ or $\mathbb{C}$, and V and W are F-vector spaces.

Exercise 1. Suppose $T : V \to W$ is a linear map, and $v_1, \dots, v_n \in V$. For each statement give a proof or a counterexample.

(a) If $v_1, \dots, v_n$ are linearly independent, then $T(v_1), \dots, T(v_n)$ are linearly independent.

(b) If $T(v_1), \dots, T(v_n)$ are linearly independent, then $v_1, \dots, v_n$ are linearly independent.

Solution.

(a) Counterexample: If $T : V \to W$ is the trivial linear map such that for any $v \in V$, $T(v) = 0$, then the proposition is clearly false. In this instance, $T(v_1), \dots, T(v_n)$ would all be 0 and would clearly be linearly dependent, as any linear combination of them would be 0 regardless of scalars.

(b) The proof is by contradiction. Assume $T(v_1), \dots, T(v_n)$ are linearly independent and $v_1, \dots, v_n$ are linearly dependent. This would mean that there exists a nontrivial linear relation such that $c_1v_1 + \cdots + c_nv_n = 0$ with some $c_i \neq 0$. Now, compute $T(c_1v_1 + \cdots + c_nv_n)$. This is clearly equivalent to $T(0)$, which is equal to 0.

However, the linearity of T allows this to be re-written as $c_1T(v_1)+\cdots+c_nT(v_n)$. This has yielded a relation $c_1T(v_1)+\cdots+c_nT(v_n) = T(0) = 0$, in which at least one $c_i \neq 0$, demonstrating that $T(v_1), \dots, T(v_n)$ are linearly dependent. This contradicts our assumption, and therefore if $T(v_1), \dots, T(v_n)$ are linearly independent, then $v_1, \dots, v_n$ are as well.

$\square$

Exercise 2. Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear map corresponding to

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Find all solutions to

$$T(x) = \left[\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right].$$

Solution. The question is equivalent to finding all solutions to

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

which can be solved by row reducing the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccccc|c} 1 & 0 & 0 & \frac{1}{2} & 2 & 0 \\ 0 & 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & 3 & 1 \end{array}\right]$$

(See full row reduction in Appendix A) Converting into equation form

$$\begin{aligned} x_1 + \frac{1}{2}x_4 + 2x_5 &= 0 \\ x_2 - x_4 - 2x_5 &= 0 \\ x_3 + \frac{3}{2}x_4 + 3x_5 &= 1 \end{aligned} \qquad \text{So,} \qquad \begin{aligned} x_1 &= -\frac{1}{2}x_4 - 2x_5 \\ x_2 &= x_4 + 2x_5 \\ x_3 &= 1 - \frac{3}{2}x_4 - 3x_5 \\ x_4 &\text{ is free} \\ x_5 &\text{ is free} \end{aligned}$$

Therefore, the set of all solutions is

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \mid x_4, x_5 \in \mathbb{R} \right\}$$

□

Exercise 3. Show that there is a unique linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad T\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and find the corresponding 3×3 matrix.

Solution. To show that there is a unique linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ which maps $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ to the specified vectors, it suffices to show that these three vectors are a basis for $\mathbb{R}^3$. This can be shown easily by the relations

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}v_2 + \frac{1}{2}v_3 \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}v_2 - \frac{1}{2}v_3 \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = v_1 - \frac{3}{2}v_2 + \frac{1}{2}v_3$$

(Solutions found by row reduction, consult Appendix A)
Because the standard basis vectors $e_1, e_2, e_3 \in \mathbb{R}^3$ can be written in terms of v_1, v_2, v_3, these vectors span $\mathbb{R}^3$. Because this list contains three spanning vectors in a 3 dimensional space, it is a basis for $\mathbb{R}^3$, and as such a unique linear map to any three vectors can be made.

To find the corresponding matrix for the described linear map, it is necessary to find $T(e_1), T(e_2), T(e_3)$. This can be computed as follows using the linearity of the map

$$\begin{aligned} T(e_1) &= T(\frac{1}{2}v_2 + \frac{1}{2}v_3) = \frac{1}{2}T(v_2) + \frac{1}{2}T(v_3) = \frac{1}{2}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ T(e_2) &= T(\frac{1}{2}v_2 - \frac{1}{2}v_3) = \frac{1}{2}T(v_2) - \frac{1}{2}T(v_3) = \frac{1}{2}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ T(e_3) &= T(v_1 - \frac{3}{2}v_2 + \frac{1}{2}v_3) = T(v_1) - \frac{3}{2}T(v_2) + \frac{1}{2}T(v_3) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} \end{aligned}$$

Taking these three vectors as the columns of a matrix yields the matrix for T,

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$\square$

Exercise 4. Suppose V is finite dimensional with basis $v_1, \dots, v_n$. Define linear maps $T_1, \dots, T_n \in \mathrm{Hom}(V, F)$ as follows: if $v = c_1 v_1 + \cdots + c_n v_n$ then

$$\begin{aligned} T_1(v) &= c_1 \\ &\vdots \\ T_n(v) &= c_n. \end{aligned}$$

Prove that $T_1, \dots, T_n$ is a basis of $\mathrm{Hom}(V, F)$. (Do not assume that $\mathrm{Hom}(V, F)$ has dimension n. We have stated this in class, but have not yet proved it.)

Solution. First, note that $T_i(v_j)$ is equal to 1 when $i = j$ and 0 all other times, as is clear from the definitions of $T_1, \dots, T_n$.
Now, to show that $T_1, \dots, T_n$ are linearly independent, consider the equation $0 = a_1 T_1 + \cdots + a_n T_n$. I claim that all the a_i must be 0. This is clear from the fact that the statement must be true for all inputs of $T_1, \dots, T_n$. This can yield several relations, in particular when evaluated at $v_1, \dots, v_n$

$$\begin{aligned} 0 &= a_1 T_1(v_1) + \cdots + a_n T_n(v_1) = a_1 \\ &\vdots \\ 0 &= a_1 T_1(v_n) + \cdots + a_n T_n(v_n) = a_n \end{aligned}$$

This suffices to show that all $a_i = 0$, and so the list is linearly independent.
Now, show that the list spans $\mathrm{Hom}(V, F)$. Given $T \in \mathrm{Hom}(V, F)$, one can write $T = a_1 T_1 + \cdots + a_n T_n$ as follows:
Define $a_i = T(v_i)$.
Now, check if the two linear maps agree on the basis $v_1, \dots, v_n$ (and are therefore equivalent).

$$\begin{aligned} T(v_1) &= a_1 = a_1 T_1(v_1) + \cdots + a_n T_n(v_1) \\ &\vdots \\ T(v_n) &= a_n = a_1 T_1(v_n) + \cdots + a_n T_n(v_n) \end{aligned}$$

By construction, we have expressed any $T \in \mathrm{Hom}(V, F)$ as a linear combination of $T_1, \dots, T_n$, meaning the list spans the space. Because the list is linearly independent and spans $\mathrm{Hom}(V, F)$, it is a basis. $\square$

Exercise 5. Suppose V is finite dimensional and $U \subset V$ is a subspace. Show that any linear map $T : U \to W$ can be extended to a linear map defined on all of V. In other words, show that there is a linear map $T' : V \to W$ such that $T(u) = T'(u)$ for all $u \in U$.

Solution. Pick any basis $x_1, \dots, x_d$ for U. Extend it to a basis $x_1, \dots, x_d, y_1, \dots, y_e$ of V. Now, because a unique linear map exists which sends a basis to arbitrary vectors, for any $T : U \to W$, a corresponding $T' : V \to W$ must exist which satisfies the following:

$$\begin{aligned} T'(x_1) &= T(x_1) \\ &\vdots \\ T'(x_d) &= T(x_d) \\ T'(y_1) &= 0 \\ &\vdots \\ T'(y_e) &= 0 \end{aligned}$$

This linear map T' is clearly defined over all of V.
Furthermore, because it agrees on the basis of U with the original T, it is clear that $T(u) = T'(u)$ for all $u \in U$, as one could decompose $u = c_1 y_1 + \cdots + c_e y_e$ and evaluate as $T'(u) = c_1 T'(y_1) + \cdots + c_e T'(y_e) = c_1 T(y_1) + \cdots + c_e T(y_e) = T(u)$ $\square$

Exercise 6. Suppose V is finite dimensional and W is infinite dimensional. Show that $\mathrm{Hom}(V, W)$ is infinite dimensional.

Solution. Important: A vector space is infinite dimensional if and only if it is possible to produce a linearly independent list of arbitrary size.
Because W is infinite dimensional, it is possible to produce a linearly independent list of arbitrary length k. Choose any such list, $w_1, \dots, w_k$.
Now, choose any basis $v_1, \dots, v_n \in V$.
Now, by a result from class it is possible to construct linear maps in $\mathrm{Hom}(V, W)$ which satisfy the following maps from the basis vectors to arbitrary vectors

$$\begin{aligned} &T_1 : V \to W \text{ such that } T_1(v_1) = w_1, \dots, T_1(v_n) = w_1 \\ &\vdots \\ &T_k : V \to W \text{ such that } T_1(v_1) = w_k, \dots, T_1(v_n) = w_k \end{aligned}$$

This construction produces a list, of arbitrary length k, of linear maps $T : V \to W$. I claim this list is linearly independent. Consider $0 = a_1 T_1 + \cdots + a_k T_k$. This must be true regardless of the input for $T_1, \dots, T_k$, so in particular it must be true when evaluated at v_1. This yields $0 = a_1 T(v_1) + \cdots + a_k T_k(v_1) = a_1 w_1 + \cdots + a_k w_k$. As $w_1, \dots, w_k$ were chosen to be linearly independent, this equality implies that $a_1 = 0, \dots, a_k = 0$. The ability to produce a linearly independent list of arbitrary length in $\mathrm{Hom}(V, W)$ implies the space is infinite dimensional. $\square$