Problem set #8

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Due **WEDNESDAY** April 4, 2018

In all of the following exercises F denotes $\mathbb{R}$ or $\mathbb{C}$, and V and W are F-vector spaces. Let P_n be the space of polynomials with real coefficients of degree $\leq n$.

Exercise 1. Suppose X and Y are sets, and $f : X \to Y$ and $g : Y \to X$ are functions.

(a) If $g(f(x)) = x$ for all $x \in X$, does it follow that f is injective? What about surjective?

(b) If $f(g(y)) = y$ for all $y \in Y$, does it follows that f is injective? What about surjective?

(Give proofs or counterexamples for all claims.)

Solution.

(a) It follows that f is injective. Given a x_1, x_2 where $f(x_1) = f(x_2)$, x_1 must be equal to x_2. Indeed, considering the given relation $g(f(x)) = x$, it is clear that $g(f(x_1)) = x_1$ and $g(f(x_1)) = x_2$. But, as $f(x_1) = f(x_2)$, $g(f(x_1)) = g(f(x_2))$ This clearly implies $x_1 = x_2$, so f is injective.

It does not follow that f needs to be surjective. Consider the situation in which X is the vector space $\mathbb{R}^1$, Y is the vector space $\mathbb{R}^2$, f is the linear map given by the matrix $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and g is the linear map given by the matrix $\left[\begin{smallmatrix}1 & 0\end{smallmatrix}\right]$. It is clear that $g \circ f = [1]$, and so $g(f(x)) = [1]x = x$, satisfying the requirement. But, clearly $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \in \mathbb{R}^2$ cannot be written as $f(x)$ for some x. Thus, f need not be surjective.

(b) It does not follow that f needs to be injective. Consider the situation in which X is the vector space $\mathbb{R}^2$, Y is the vector space $\mathbb{R}^1$, f is the linear map given by the matrix $\left[\begin{smallmatrix}1 & 0\end{smallmatrix}\right]$ and g is the linear map given by the matrix $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. Clearly, $f \circ g = [1]$, and so $f(g(y)) = [1]y = y$, satisfying the requirement. But, given $v_1 = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], v_2 = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, clearly $f(v_1) = [1] = f(v_2)$, but $v_1 \neq v_2$, so f is not injective.

It follows that f is surjective. This is obvious from the statement, given any $y \in Y$, it is easy to exhibit a $x \in X$ such that $f(x) = y$. Namely, that x is $g(y)$. So, f is surjective.

$\square$

Exercise 2. Suppose $T \in \mathrm{Hom}(V, W)$ is a linear map, and $\phi : F^n \to V$ and $\psi : F^m \to W$ are isomorphisms. Set $A = \psi^{-1} \circ T \circ \phi \in \mathrm{Hom}(F^n, F^m)$.

(a) If $x_1, \ldots, x_s$ is a basis for $\ker(A)$, show that $\phi(x_1), \ldots, \phi(x_s)$ is a basis for $\ker(T)$.

(b) If $y_1, \ldots, y_t$ is a basis for $\mathrm{Im}(A)$, show that $\psi(y_1), \ldots, \psi(y_t)$ is a basis for $\mathrm{Im}(T)$.

Solution.

(a) To show $\phi(x_1), \ldots, \phi(x_s)$ is a basis for $\ker(T)$, we must check linear independence and that it spans the kernel.

Linear independence is simple. Consider the equation $c_1\phi(x_1) + \cdots + c_s\phi(x_s) = 0$. The linearity of ϕ allows this to be rewritten as $\phi(c_1x_1 + \cdots + c_sx_s) = 0$. Because ϕ is bijective, it admits an inverse ϕ^{-1}. Computing the inverse of both sides yields $c_1x_1+\cdots+c_sx_s = 0$. But, $x_1, \ldots, x_s$ is a basis for $\ker(A)$, and in particular is linearly independent, so it is known that the only solution to $c_1x_1 + \cdots + c_sx_s = 0$ is with $c_1 = 0, \ldots, c_s = 0$. This implies the original equation had only the trivial solution, so $\phi(x_1), \ldots, \phi(x_s)$ are linearly independent.

To show span, now consider that given any $v \in \ker(T)$, one can now establish a unique $x \in F^n$ such that $T(\phi(x)) = 0$ by defining $x = \phi^{-1}(v)$. I claim that $v = \phi(x)$ can be written as a linear combination of $\phi(x_1), \ldots, \phi(x_s)$. Indeed, computing ψ^{-1} of both sides of $T(\phi(x)) = 0$ yields $\psi^{-1}(T(\phi(x))) = Ax = 0$. This means x is in $ker(A)$, and as such can be decomposed as $x = c_1x_1 + \cdots + c_sx_s$ for some scalars $c_1, \ldots, c_s$. Computing ϕ of both sides of this decomposition and simplifying yields $\phi(x) = c_1\phi(x_1) + \cdots + c_s\phi(x_s)$.

Because $\phi(x_1), \ldots, \phi(x_s)$ are linearly independent and span $\ker(T)$, they form a basis for $\ker(T)$.

(b) To show that $\psi(y_1), \ldots, \psi(y_t)$ is a basis for $\mathrm{Im}(T)$, we must check linear independence and that it spans the image.

To check linear independence, consider the equation $c_1\psi(y_1) + \cdots + c_t\psi(y_t) = 0$. As ψ is a isomorphism, it admits an inverse ψ^{-1}. Computing the inverse of both sides yields $\psi^{-1}(c_1\psi(y_1)+\cdots+c_t\psi(y_t)) = c_1\psi^{-1}(\psi(y_1))+\cdots+c_t\psi^{-1}(\psi(y_t)) = c_1y_1+\cdots+c_ty_t = 0$. The linear independence of $y_1, \ldots, y_t$ now implies $c_1 = 0, \ldots, c_t = 0$, so $\psi(y_1), \ldots, \psi(y_t)$ are linearly independent.

To check span, consider $w \in \mathrm{Im}(T)$. This means there exists some $v \in V$ such that $T(v) = w$. Define $y \in F^m$ as $y = \psi^{-1}(w)$. Clearly, $\psi(y) \in \mathrm{Im}(T)$. I claim that $w = \psi(y)$ can be written as a linear combination of $\psi(y_1), \ldots, \psi(y_t)$. Indeed, as $\psi(y) = T(v)$ for some $v \in V$, define $x \in F^n$ as $x = \phi(v)$. Now, rewrite the above as $\psi(y) = T(\phi(x))$. Compute ψ^{-1} of both sides to yield $y = \psi^{-1}(T(\phi(x))) = Ax$. This demonstrates y is in the image of A, and as such can be decomposed $y = c_1y_1 + \cdots + c_ty_t$. Computing ψ of both sides of this decomposition and simplifying yields $\psi(y) = \psi(y_1), \ldots, \psi(y_t)$.

Because $\psi(y_1), \ldots, \psi(y_t)$ are linearly independent and span $\mathrm{Im}(T)$, they form a basis.

□

Exercise 3. Consider the linear map $T : P_2 \to \mathbb{R}^3$ defined by

$$T(p) = \begin{pmatrix} p(1) \\ p(2) \\ p(-1) \end{pmatrix}.$$

Compute the matrix of T with respect to the bases $1, x, x^2 \in P_2$ and $e_1, e_2, e_3 \in \mathbb{R}^3$, and find bases for the kernel and image of T.

Solution. Computing the transformation of the given basis of P_2 in terms of the given basis for $\mathbb{R}^3$ yields

$$T(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = e_1 + e_2 + e_3$$
$$T(x) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = e_1 + 2e_2 - e_3$$
$$T(x^2) = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = e_1 + 4e_2 + e_3$$

Taking the coefficients of these equations as the columns of a matrix A yields the matrix of T with respect to the chosen bases

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \end{bmatrix}$$

To compute the bases of the kernel and image of T, first find A', the reduced row echelon form of A (Row reduction shown in appendix A)

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because there is a pivot in every row and column of A', the columns of A form a basis for $\mathbb{R}^3$. Thus, $\text{Im}(T) = \mathbb{R}^3$, and T is surjective. Because P_2 and $\mathbb{R}^3$ have the same dimension, we can furthermore conclude T is bijective. As T is injective, $\ker(T) = 0$, so 0 serves a basis for the kernel. As $\text{Im}(T) = \mathbb{R}^3$, take the standard basis vectors for $\mathbb{R}^3$, e_1, e_2, e_3 as a basis for $\text{Im}(T)$ (though one could obviously also use the columns of A). $\square$

Exercise 4. Define a linear map $T : P_3 \to P_3$ by

$$T(p) = p(0)x^3 + p(1)x^2 + p(1)x + p(0).$$

Compute the matrix of T with respect to the basis $1, x, x^2, x^3 \in P_3$, and find bases for the kernel and image of T.

Solution. Compute the transformation of the given basis of P_3 in terms of the given basis for P_3. This yields

$$\begin{aligned} T(1) &= 1x^3 + 1x^2 + 1x + 1 = 1(1) + 1(x) + 1(x^2) + 1(x^3) \\ T(x) &= 0x^3 + 1x^2 + 1x + 0 = 0(1) + 1(x) + 1(x^2) + 0(x^3) \\ T(x^2) &= 0x^3 + 1x^2 + 1x + 0 = 0(1) + 1(x) + 1(x^2) + 0(x^3) \\ T(x^3) &= 0x^3 + 1x^2 + 1x + 0 = 0(1) + 1(x) + 1(x^2) + 0(x^3) \end{aligned}$$

Taking the coefficients of these equations as the columns of a matrix A yields the matrix of T with respect to the chosen bases

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

To compute the bases of the kernel and image of T, first find A', the reduced row echelon form of A (Row reduction shown in appendix A)

$$A' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Using this, it is easy to find that $ker(A') = ker(A)$ is the set of all solutions to

$$\begin{aligned} x_1 &= 0 \\ x_2 + x_3 + x_4 &= 0 \end{aligned} \qquad \text{So,} \qquad \begin{aligned} x_1 &= 0 \\ x_2 &= -x_3 - x_4 \\ x_3 &\text{ is free} \\ x_4 &\text{ is free} \end{aligned}$$

In other words, the set of all solutions to $Ax = 0$ is

$$\left\{x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \mid x_3, x_4 \in \mathbb{R}\right\}$$

Thus, $\ker(T) = \text{Span}\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$. These two vectors are clearly linearly independent, and so form a basis for $\ker(T)$.

To find a basis for $\text{Im}(T)$, we begin with the knowledge that $\text{Im}(T)$ is spanned by the columns of A. However, the third and fourth columns of A are linear combinations of the first and second (hence, they are free variables in A'). As such, they can be removed without changing the span. Thus, $\text{Im}(T) = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}$. These are clearly linearly independent, and so form a basis for $\text{Im}(T)$.

□

Exercise 5. Compute the inverse of $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, and factor A as a product of elementary matrices. Using your calculation of A^{-1}, solve

$$Ax = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad Ax = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad Ax = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Solution. Row reduction yields

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

and the factorization

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Both are computed in full in appendix A)
Now, using A^{-1}, it is trivial to compute the following

$$Ax = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \implies x = A^{-1}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 1 \\ -2 \end{bmatrix}$$

$$Ax = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \implies x = A^{-1}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ -2 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \implies x = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

□

Exercise 6. Suppose $f(x) \in P_{99}$. Show that there is a polynomial $g(x) \in P_{100}$ such that

$$5g''(x) + 3g'(x) = f(x),$$

and that $g(x)$ unique up to addition of a constant polynomial.

Solution. Consider the linear map $T : P_{100} \to P_{99}$ satisfying $T(p) = 3\frac{d}{dx}(p) + 5\frac{d^2}{dx^2}(p)$. Consider the basis $v_1 = 1, v_2 = x, v_3 = x^2, \ldots, v_{101} = x^{100} \in P_{100}$ and $u_1 = 1, u_2 = x, u_3 =$

$x^2, \ldots, u_{100} = x^{99} \in P_{99}$. It is possible to compute a matrix for the linear map T by computing

$$\begin{aligned}
T(v_1) &= 0 = 0u_1 + \cdots + 0u_{100} \\
T(v_2) &= 3 = 3u_1 + 0u_2 + \cdots + 0u_{100} \\
T(v_3) &= 10 + 6x = 10u_1 + 6u_2 + 0u_3 + \cdots + 0u_{100} \\
&\vdots \\
T(v_i) &= 3\frac{d}{dx}(v_i) + 5\frac{d^2}{dx^2}(v_i) \\
&\vdots \\
T(v_{101}) &= 300x^{99} + 49500x^{98} = 0u_1 + \cdots + 0u_{98} + 49500u_{99} + 300u_{100}
\end{aligned}$$

and taking the coefficients of each equation as the columns of a matrix, $A \in M_{100x101}(\mathbb{R})$. Contemplate how this matrix would row reduce. As the first row has one nonzero entry, and the following rows all have two, it is clear that the result of this process is the reduced row echelon form of A, A', taking the form $[0|I_{100}]$. As A' has a pivot in every row, T is surjective. Thus, for any $f \in P_{99}$, there is a polynomial $g \in P_{100}$ such that $T(g) = 5g'' + 3g' = f$. Furthermore, uniqueness of g up to the addition of a constant polynomial follows from the fact that $A' = [0|I_{100}]$ demonstrates that the columns of A *after* the first column are linearly independent, so they uniquely determine a solution when considered on their own.

$\square$