Problem set #2

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Exercise 1. Compute the real and imaginary parts of

$$w = \frac{\pi + i}{5 - i} \quad \text{and} \quad z = \left(\frac{1}{1 + \frac{1}{1+i}} \right)^2 .$$

Solution. Simplify w by use of the complex conjugate of $5 - i$

$$\begin{aligned} w = \frac{\pi + i}{5 - i} &= \frac{\pi + i}{5 - i} \cdot \frac{5+1}{5+1} \\ &= \frac{5\pi + (5+\pi)i - 1}{26} \\ &= \frac{5\pi - 1}{26} + \frac{5+\pi}{26} i. \end{aligned}$$

So, $Re(w) = \frac{5\pi-1}{26}$ and $Im(w) = \frac{5+\pi}{26}$.
z is simplified similarly, first using the complex conjugate of $1 + i$,

$$\begin{aligned} z = \left(\frac{1}{1 + \frac{1}{1+i}} \right)^2 &= \left(\frac{1}{1 + \frac{1}{1+i} \cdot \frac{1-i}{1-i}} \right)^2 \\ &= \left(\frac{1}{1 + \frac{1-i}{2}} \right)^2 = \left(\frac{1}{\frac{3-i}{2}} \right)^2 = \left(\frac{2}{3-i} \right)^2 \end{aligned}$$

and then by using the complex conjugate of $3 - i$

$$\begin{aligned} z = \left(\frac{2}{3-i} \right)^2 &= \left(\frac{2}{3-i} \cdot \frac{3+i}{3+i} \right)^2 \\ &= \left(\frac{6+2i}{10} \right)^2 = \left(\frac{3}{5} + \frac{1}{5} i \right)^2 \\ &= \frac{9}{25} + \frac{6}{25} i - \frac{1}{25} = \frac{8}{25} + \frac{6}{25} i. \end{aligned}$$

So, $Re(z) = \frac{8}{25}$ and $Im(z) = \frac{6}{25}$. $\square$

Exercise 2. Let

$$z = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}.$$

(a) Write z in the form $re^{i\theta}$.

(b) What is the smallest $d \in \mathbb{Z}^+$ such that $z^d = 1$?

(c) Compute the real and imaginary parts of z^{50}.

Solution.

(a) Since $r = |z| = \sqrt{a^2 + b^2}$, $r = \sqrt{1/2 + 1/2} = 1$.
Also, because $\cos(\theta) = \frac{a}{r}$, $\theta = \arccos(\frac{-\sqrt{2}}{2}) = \frac{-\pi}{4}$.
So, $z = e^{\frac{-\pi}{4}i}$

(b) By nature of Euler's formula, $e^{\theta i} = 1 \iff \theta \in \{2\pi k : k \in \mathbb{Z}\}$. Since $z = e^{\frac{-\pi}{4}i}$, we are looking for the smallest $d \in \mathbb{Z}^+$ such that $d \cdot \frac{-\pi}{4}$ is a multiple of 2π. Simple arithmetic yields that $d = 8$.

(c) $z = e^{\frac{-\pi}{4}i}$, so $z^{50} = (e^{\frac{-\pi}{4}i})^{50}$ by simple calculation, this yields $z^{50} = e^{\frac{-25\pi}{2}i}$. Converting back into form $a + bi$ yields

$$e^{\frac{-25\pi}{2}i} = 1(\cos(\frac{-25\pi}{2}) + i\sin(\frac{-25\pi}{2})) = 0 + 1i.$$

So, $Re(z^{50}) = 0$ and $Im(z^{50}) = -1$.

$\square$

Exercise 3. Find three complex roots of the polynomial $x^3 - i$. Express your answers both in the form $re^{i\theta}$, and by giving real and imaginary parts.

Solution. If we have some solution x to $x^3 - i = 0$, then $x^3 = i \implies |x^3| = \sqrt{0^2 + 1^2} = 1 \implies |x|^3 = 1$. Hence, we can write $x = e^{i\theta}$ for some θ.
Using Euler's formula, the following is clear

$$x^3 = e^{i3\theta} = i \iff 3\theta \in \{\frac{\pi}{2} + 2\pi k : k \in \mathbb{Z}\} \iff \theta \in \{\frac{\pi}{6} + \frac{2\pi}{3}k : k \in \mathbb{Z}\}.$$

In other words, $x^3 = i$ when $x \in \{\cdots, e^{-i\frac{7\pi}{6}}, e^{-i\frac{\pi}{2}}, e^{i\frac{\pi}{6}}, e^{i\frac{5\pi}{6}}, e^{i\frac{3\pi}{2}}, \cdots\}$.
Ignoring repetitions, x is one of $e^{i\frac{\pi}{6}}, e^{i\frac{5\pi}{6}}, e^{i\frac{3\pi}{2}}$. In standard form, $x = \frac{\sqrt{3}}{2} + \frac{1}{2}i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i, 0 + (-1)i$ $\square$

Exercise 4. Use power series expansions to prove Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

and then use Euler's formula to prove the trigonometric identities

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$$
$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2).$$

Hint: compute $e^{i\theta_1} \cdot e^{i\theta_2}$ in two different ways.

Solution.

(a) Consider the power series expansions for $e^x, \cos(x)$, and $\sin(x)$.

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots \\ \cos(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\ \sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots \end{aligned}$$

Now, consider the substitution $x = i\theta$ in the series for e^x and simplify using the properties of the powers of i

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \frac{1}{6!}(i\theta)^6 + \cdots \\ &= 1 + i\theta + \frac{1}{2!}(-1)\theta^2 + \frac{1}{3!}(-i)\theta^3 + \frac{1}{4!}(1)\theta^4 + \frac{1}{5!}(i)\theta^5 + \frac{1}{6!}(-1)\theta^6 + \cdots \\ &= 1 + i\theta - \frac{1}{2!}\theta^2 - \frac{1}{3!}(i)\theta^3 + \frac{1}{4!}\theta^4 + \frac{1}{5!}(i)\theta^5 - \frac{1}{6!}\theta^6 + \cdots \end{aligned}$$

Regrouping the right side of the above equality yields

$$e^{i\theta} = (1 - \frac{1}{2!}\theta^2 + + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \cdots) + i(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \cdots)$$

which is clearly equivalent to the power series expansion of $\cos(\theta) + i\sin(\theta)$. Therefore, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

(b) Consider $z_1, z_2 \in \mathbb{C}$ with $|z_1| = |z_2| = 1$. Let $z_1 = e^{i\theta_1}, z_2 = e^{i\theta_2}$ and compute $z_1 \cdot z_2$

$$z_1 \cdot z_2 = e^{i\theta_1} \cdot e^{i\theta_2} = e^{i\theta_1 + i\theta_2} = e^{i(\theta_1+\theta_2)}$$

Now, apply Euler's formula to write this final expression as

$$z_1 \cdot z_2 = e^{i(\theta_1+\theta_2)} = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2).$$

Setting this aside for a second, consider if z_1 and z_2 were expressed in the other form allowed by Euler's formula, $z_1 = \cos(\theta_1) + i\sin(\theta_1), z_2 = \cos(\theta_2) + i\sin(\theta_2)$, and compute $z_1 \cdot z_2$ a second time

$$\begin{aligned} z_1 \cdot z_2 &= (\cos(\theta_1) + i\sin(\theta_1)) \cdot (\cos(\theta_2) + i\sin(\theta_2)) \\ &= \cos(\theta_1)\cos(\theta_2) + \cos(\theta_1)i\sin(\theta_2) + \cos(\theta_2)i\sin(\theta_1) + (-1)\sin(\theta_1)\sin(\theta_2) \\ &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1)) \end{aligned}$$

Now, because obviously $z_1 \cdot z_2 = z_1 \cdot z_2$, we can use these two calculations to state that

$$\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2) = \cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)+i(\cos(\theta_1)\sin(\theta_2)+\cos(\theta_2)\sin(\theta_1)).$$

Rearrange this equality

$$\begin{aligned} \cos(\theta_1 + \theta_2) - \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) &= -i\sin(\theta_1 + \theta_2) + i(\cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1)) \\ \cos(\theta_1 + \theta_2) - \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) &= i(-\sin(\theta_1 + \theta_2) + \cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1)). \end{aligned}$$

Because the left side is exclusively real-valued and the right side is a real-valued expression multiplied by i, the two sides can *only* be equal if they are both 0. So,

$$\cos(\theta_1 + \theta_2) - \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) = 0$$

and

$$i(-\sin(\theta_1 + \theta_2) + \cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1)) = 0.$$

Simplification yields the two desired identities with ease

$$\begin{aligned}\cos(\theta_1 + \theta_2) &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\\ \sin(\theta_1 + \theta_2) &= \cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1).\end{aligned}$$

$\square$

Exercise 5. Suppose $z_1, z_2 \in \mathbb{C}$. Prove the *triangle inequality* $|z_1 + z_2| \leq |z_1| + |z_2|$.

Solution. Because both sides of the inequality are non-negative, the statement is equivalent to

$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2. \tag{1}$$

Let z_1 be written in the form $a_1 + b_1 i$ and z_2 be written in the form $a_2 + b_2 i$. It follows that $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$. Using these, the above simplifies to

$$\begin{aligned}(a_1 + a_2)^2 + (b_1 + b_2)^2 &\leq (a_1^2 + b_1^2) + 2\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2} + (a_2^2 + b_2^2)\\ a_1^2 + 2a_1a_2 + a_2^2 + b_1^2 + 2b_1b_2 + b_2^2 &\leq (a_1^2 + b_1^2) + 2\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2} + (a_2^2 + b_2^2)\\ 2a_1a_2 + 2b_1b_2 &\leq 2\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}\\ a_1a_2 + b_1b_2 &\leq \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}.\end{aligned}$$

Contemplate this final inequality. The right hand side is clearly non-negative. If the left hand side is negative, the proof is obviously complete. If the left hand side is non-negative, then by the same logic employed in the first step, this inequality is equivalent to a version with both sides squared. This squaring allows further simplification as follows

$$\begin{aligned}(a_1a_2 + b_1b_2)^2 &\leq (a_1^2 + b_1^2)(a_2^2 + b_2^2)\\ a_1^2a_2^2 + 2a_1a_2b_1b_2 + b_1^2b_2^2 &\leq a_1^2a_2^2 + a_1^2b_2^2 + a_2^2b_1^2 + b_1^2b_2^2\\ 0 &\leq a_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2b_1b_2\\ 0 &\leq (a_1b_2)^2 - 2a_1a_2b_1b_2 + (a_2b_1)^2\\ 0 &\leq (a_1b_2 - a_2b_1)^2.\end{aligned}$$

This final equality is obviously true due to the nature of squares, and the proof is complete. $\square$

Exercise 6. Fix $n \in \mathbb{Z}^+$, and set

$$z_1 = e^{\frac{2\pi i}{n}}, \quad z_2 = e^{\frac{4\pi i}{n}}, \quad z_3 = e^{\frac{6\pi i}{n}}, \quad \ldots z_n = e^{\frac{2n\pi i}{n}}$$

(a) Show that the product $z_1 \cdots z_n$ is equal to ± 1. When is it 1 and when is it -1?

(b) Show that $z_1 + \cdots + z_n = 0$. Hint: think about the factorization of $x^n - 1$.

Solution.

(a) When powers of the same base are multiplied together, their product can be calculated by adding their exponents. This means that the product $z_1 \cdots z_n$ can be calculated by calculating $e^{\frac{2\pi i}{n} + \cdots + \frac{2n\pi i}{n}}$
This exponent is $\frac{2\pi i}{n}$ multiplied by the series $1 + 2 + 3 + \cdots + n$, the sum of which can be calculated by the formula $\frac{n(n-1)}{2}$.
So, with cancellation, the result of the product $z_1 \cdots z_n$ is equivalent to

$$e^{(n-1)\pi i}$$

This means the angle θ of any such product is a multiple of π, which when plugged into Euler's formula $e^{i\theta} = \cos(\theta) + i \sin(\theta)$ will yield either 1 or -1. Simple inspection of the above result yields that it will be 1 when n is odd, and -1 when n is even.

(b) Realize that $z_1 + \cdots + z_n$ is a geometric sequence where each term is increasing by a ratio of $r = e^{\frac{2\pi i}{n}}$. The formula for the sum of the first n terms of a geometric series is

$$S_n = \frac{a_1(1 - r^n)}{1 - r}.$$

If we substitute in the numbers from our sequence, this becomes

$$S_n = \frac{e^{\frac{2\pi i}{n}}(1 - (e^{\frac{2\pi i}{n}})^n)}{1 - e^{\frac{2\pi i}{n}}}.$$

With simplification,

$$\begin{aligned} S_n &= \frac{e^{\frac{2\pi i}{n}}(1 - (e^{2\pi i}))}{1 - e^{\frac{2\pi i}{n}}} \\ &= \frac{e^{\frac{2\pi i}{n}}(1 - 1)}{1 - e^{\frac{2\pi i}{n}}} \\ &= \frac{e^{\frac{2\pi i}{n}}(0)}{1 - e^{\frac{2\pi i}{n}}} = 0 \end{aligned}$$

$\square$