Problem set #7

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Exercise 1.

(a) If $S : U \to V$ and $T : V \to W$ are linear, show that $T \circ S$ is also linear.

(b) Prove that if $T : V \to W$ is a bijective linear map, then $T^{-1} : W \to V$ is also linear.

(c) Suppose $S, T : V \to W$, $\phi : W \to U$, and $\psi : U \to V$ are linear maps. Prove that

$$\phi \circ (S + T) \circ \psi = \phi \circ S \circ \psi + \phi \circ T \circ \psi,$$

and that $\phi \circ (cS) \circ \psi = c \cdot (\phi \circ S \circ \psi)$ for any $c \in F$.

(d) Show that the image of a linear map $T : F^n \to F^m$ is equal to the span of the columns of the corresponding matrix.

(e) Show that a linear map $T : F^n \to F^m$ is injective if and only if the corresponding matrix has linearly independent columns.

Solution.

(a) First, show $T \circ S$ respects addition. Given $u_1, u_2 \in U$, then $(T \circ S)(u_1 + u_2) = T(S(u_1 + u_2)) = T(S(u_1) + S(u_2)) = T(S(u_1)) + T(S(u_2)) = (T \circ S)(u_1) + (T \circ S)(u_2)$.

Now, show $T \circ S$ respects scalar multiplication. Given $u \in U$ and $c \in F$, then $(T \circ S)(cu) = T(S(cu)) = T(cS(u)) = cT(S(u)) = c(T \circ S)(u)$.

Hence, $T \circ S$ is also linear.

(b) Consider $w, z \in W$. Because T is a bijection, there exist unique $u, v \in V$ such that $T(u) = w$ and $T(v) = z$. It follows that

$$T^{-1}(w + z) = T^{-1}(T(u) + T(v)) = T^{-1}(T(u + v)) = u + v = T^{-1}(w) + T^{-1}(z)$$

So T^{-1} respects addition. Now consider $w \in W$ and $c \in F$. Because T is a bijection, there exists a unique $v \in V$ such that $T(v) = w$. Therefore,

$$T^{-1}(cw) = T^{-1}(cT(v)) = T^{-1}(T(cv)) = cv = cT^{-1}(w)$$

So, T^{-1} respects scalar multiplication.

Therefore, T^{-1} is linear.

(c) Because composition of functions is associative and all the given functions are linear maps, it follows that
$\phi \circ (S + T) \circ \psi = \phi \circ ((S + T) \circ \psi) = \phi \circ ((S + T)(\psi)) = \phi \circ (S(\psi) + T(\psi)) = \phi(S(\psi) + T(\psi)) = \phi(S(\psi)) + \phi(T(\psi)) = \phi \circ S \circ \psi + \phi \circ T \circ \psi$
Similarly,
$\phi \circ (cS) \circ \psi = \phi \circ ((cS) \circ \psi) = \phi \circ (cS(\psi)) = \phi(cS(\psi)) = c \cdot \phi(S(\psi)) = c \cdot (\phi \circ S \circ \psi)$

(d) Given a linear map $T : F^n \to F^m$, let A be the corresponding matrix. Given $v \in \operatorname{Im}(T)$, then by definition there exists a $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in F^n$ such that $T(u) = v$. It follows that for this same u, $Au = v$.

Consider the expansion of the above matrix-vector multiplication, $u_1 A_1 + \cdots + u_n A_n = v$, where A_i is the ith column of A. This suffices to demonstrate that any vector in $\operatorname{Im}(T)$ can be written as a linear combination of the columns of A, and so the columns of A span $\operatorname{Im}(T)$. The span of the columns of A obviously does not contain anything which could not be written in the above form for some $u \in F^n$, and therefore do not contain anything not in $\operatorname{Im}(T)$. So, the image of T and the span of the columns of A are equal.

(e) Let $T : F^n \to F^m$ be a linear map and let A be the matrix corresponding to T.

T is injective $\iff \ker(T) = 0 \iff$ the only solution to $T(v) = 0$ is when $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = 0 \iff Av = 0$ has only the trivial solution $\iff$ the equation $v_1 A_1 + \cdots + v_n A_n = 0$ (where A_i is the ith column of A) has only the trivial solution $v_1 = 0, \ldots, v_n = 0$ $\iff$ the columns of A are linearly independent.

$\square$

Exercise 2. Let $T : \mathbb{C}^5 \to \mathbb{C}^4$ be the linear map corresponding to the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 2 & 0 & 2 & -1 & 3 \\ 0 & -1 & -1 & 1 & -2 \\ 1 & 1 & 2 & -1 & 3 \end{bmatrix}.$$

Find bases for the kernel and image of T.

Solution. Begin by row reducing A to obtain the following matrix in reduced row echelon form (full row reduction can be found in Appendix A)

$$A' = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To find a basis for the kernel, find the solution set of $Av = 0$. This is trivial to find using A'.

$$\begin{aligned} x_1 + x3 + x_5 &= 0 \\ x_2 + x_3 + x_5 &= 0 \\ x_4 - x_5 &= 0 \end{aligned} \qquad \text{So,} \qquad \begin{aligned} x_1 &= -x_3 - x_5 \\ x_2 &= -x_3 - x_5 \\ x_3 &\text{ is free} \\ x_4 &= x_5 \\ x_5 &\text{ is free} \end{aligned}$$

Therefore, the solution set to $Av = 0$ is

$$\left\{ x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \mid x_3, x_5 \in \mathbb{C} \right\}$$

I claim that $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ is a basis for $\ker(T)$. Indeed, it is clear from the solution above that they can generate all of $\ker(T)$ and therefore span it. Furthermore, as neither is a multiple of the other they are linearly independent. Therefore, they are a basis.

To find a basis for the image of T, assume we are given an arbitrary vector $u \in \mathrm{Im}(T)$, and consider the result of finding a v such that $T(v) = u$ by row reducing $[A|u]$. Because A' has free variables, there would clearly be more than one solution. But, it is also clear that a solution exists even if we take the free variables to be zero, meaning that those columns are not necessary to find a solution.

Therefore, consider only the columns of A which correspond to the pivot columns in A'. I claim that the vectors corresponding to those columns, $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, are a basis for $\mathrm{Im}(T)$. Indeed, they were already shown to span the image of T above, and linear independence is clear from the row reduction

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

As they are linearly independent and span, they form a basis for $\mathrm{Im}(T)$. $\square$

Exercise 3. Find a linear map $T : \mathbb{R}^5 \to \mathbb{R}^2$ with kernel

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} : \begin{array}{c} x_1 = 3x_2 \\ x_3 = x_4 = x_5 \end{array} \right\},$$

or show that no such T exists.

Solution. No such T exists. The proof is by contradiction. Assume a T with $\ker(T)$ as described exists. The space of this kernel can also be written in form

$$\left\{ \begin{bmatrix} a \\ 3a \\ b \\ b \\ b \end{bmatrix} \in \mathbb{R}^5 \right\}$$

This can clearly be spanned by $v_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, since any vector in the space could be written $av_1 + bv_2$. As neither vector is a multiple of the other, this list is clearly linearly independent. This list is therefore a basis for $\ker(T)$ and thus $\dim \ker(T) = 2$.
But, it is known that $\dim \mathbb{R}^5 = \dim \ker(T) + \dim \mathrm{Im}(T)$. Substituting in the obvious and subtracting yields $\dim \mathrm{Im}(T) = 5 - 2 = 3$. This is impossible, since $\mathrm{Im}(T) \subset \mathbb{R}^2 \implies \dim \mathrm{Im}(T) \leq 2$. $\square$

Exercise 4. Let $T : P_3 \to P_2$ be the derivative, defined by $T(f) = f'$.

(a) Compute the matrix of T with respect to the bases $1, x, x^2, x^3 \in P_3$ and $x^2, x, 1 \in P_2$. (Note the order! The matrix depends on the order in which you list the basis elements.)

(b) Find bases of P_3 and P_2 such that the matrix of T is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Solution.

(a) Computation of the transformation the chosen basis vectors of P_3 in terms of the chosen basis for P_2 yields

$$\begin{aligned} T(1) &= 0x^2 + 0x + 0 \cdot 1 \\ T(x) &= 0x^2 + 0x + 1 \cdot 1 \\ T(x^2) &= 0x^2 + 2x + 0 \cdot 1 \\ T(x^3) &= 3x^2 + 0x + 0 \cdot 1 \end{aligned}$$

Taking the coefficients of these equations as the columns of a matrix A yields

$$A = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(b) Choose the bases $x^3, x^2, x, 1 \in P_3$ and $3x^2, 2x, 1 \in P_2$. Both are obviously bases of their respective spaces, because they are polynomials of different degree and are therefore linearly independent lists of length $\dim P_3$ and $\dim P_2$ respectively.

The desired matrix follows from the calculations

$$\begin{aligned} T(x^3) &= 1 \cdot 3x^2 + 0 \cdot 2x + 0 \cdot 1 \\ T(x^2) &= 0 \cdot 3x^2 + 1 \cdot 2x + 0 \cdot 1 \\ T(x) &= 0 \cdot 3x^2 + 0 \cdot 2x + 1 \cdot 1 \\ T(1) &= 0 \cdot 3x^2 + 0 \cdot 2x + 0 \cdot 1 \end{aligned}$$

and forming the matrix with columns corresponding to the coefficients

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\square$

Exercise 5. Suppose V and W are finite dimensional. Prove or give a counterexample:

(a) If $\dim(V) \leq \dim(W)$ then there is an injective linear map $V \to W$.

(b) If $\dim(V) \geq \dim(W)$ then there is a surjective linear map $V \to W$.

Solution.

(a) Pick bases $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_m \in W$. It is known that a linear map exists which sends a basis to any arbitrary vectors. Consider $T : V \to W$, the linear map which satisfies

$$\begin{aligned} T(v_1) &= w_1 \\ &\vdots \\ T(v_n) &= w_n \end{aligned}$$

The map can be described this way because it is given that $n \leq m$.

I claim this map is injective. Indeed, if one considers the equation $\dim V = \dim \ker(T) + \dim \operatorname{Im}(T)$, it clearly follows that $\dim V \geq \dim \operatorname{Im}(T)$. However, the as $w_1, ..., w_n$ were part of a basis of W, they must be linearly independent. The existence of a linearly independent list of length $n = \dim V$ now implies that $\dim \operatorname{Im}(T) \geq n = \dim V$. Therefore, $\dim V \geq \dim \operatorname{Im}(T) \geq \dim V$, and so $\dim V = \dim \operatorname{Im}(T)$. Now, subtraction within the original equation yields $\dim \ker(T) = 0$, so $\ker(T) = 0$ and the constructed linear map T is injective.

(b) Pick bases $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_m \in W$. It is known that a linear map exists which sends a basis to any arbitrary vectors. Consider $T : V \to W$, the linear map which satisfies

$$\begin{aligned} T(v_1) &= w_1 \\ &\vdots \\ T(v_m) &= w_m \\ T(v_{m+1}) &= 0 \\ &\vdots \\ T(v_n) &= 0 \end{aligned}$$

The map can be described this way because it is given that $n \geq m$.

Clearly, the construction places each of $w_1, \ldots, w_m \in \operatorname{Im}(T)$, so it must also contain their span. Therefore, $\operatorname{Span}\{w_1, \ldots, w_m\} \subset \operatorname{Im}(T) \subset W$. But, because $w_1, \ldots, w_m$ is a basis for W and therefore span it, this statement can be rewritten as $W \subset \operatorname{Im}(T) \subset W$. Therefore, equality must hold throughout and T is surjective.

□

Exercise 6. Suppose $T : \mathbb{R}^n \to \mathbb{R}$ is a nonzero linear map, and $v \in \mathbb{R}^n$ satisfies $T(v) \neq 0$. Show that $\mathbb{R}^n = \ker(T) \oplus \mathbb{R}v$.

Solution. To show $\mathbb{R}^n = \ker(T) \oplus \mathbb{R}v$, we must first show that $\mathbb{R}^n = \ker(T) + \mathbb{R}v$.

To show this, fix any $u \in \mathbb{R}^n$. Define a constant $c \in \mathbb{R}$ as $c = \frac{T(u)}{T(v)}$. I claim that u can be written as an element in $\mathbb{R}v$ and element in $\ker(T)$ as follows

$$u = cv + k$$

Surely, cv is a constant times v and is obviously in $\mathbb{R}v$. Is $k \in \ker(T)$? Check by solving for $k = u - cv$ and then evaluating

$$T(k) = T(u - cv) = T(u) - cT(v) = T(u) - \frac{T(u)}{T(v)}T(v) = T(u) - T(u) = 0$$

So k is in the kernel of T.
Thus, any element in $\mathbb{R}^n$ can be written as a sum of elements in the two subspaces, and $\mathbb{R}^n = \ker(T) + \mathbb{R}v$. To check that the sum is direct, it now suffices to check if $\ker(T) \cap \mathbb{R}v = 0$.

To show this, pick any element $w \in \ker(T) \cap \mathbb{R}v$. By the definition of kernel, $T(w) = 0$. And, $w \in \mathbb{R}v \implies w$ can be written in the form $w = cv$ for some $c \in \mathbb{R}$. So, $T(cv) = cT(v) = 0$. However, by the problem statement, $T(v) \neq 0$. So, $c = 0$ and the only element in the intersection is the vector 0.
Thus, $R^n = \ker(T) \oplus \mathbb{R}v$.

□