Problem set #10

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Exercise 1. Find the eigenvalues of $A = \begin{bmatrix} 7 & 4 & 4 \\ 0 & -1 & 0 \\ -8 & -4 & -5 \end{bmatrix} \in M_3(\mathbb{C})$, and find a basis for each eigenspace. Is A is diagonalizable? If so, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

Solution. Calculate the characteristic polynomial of A,

$$\begin{aligned} C_A &= \det(xI_3 - A) \\ &= \det(x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 7 & 4 & 4 \\ 0 & -1 & 0 \\ -8 & -4 & -5 \end{bmatrix}) \\ &= \det \begin{bmatrix} x-7 & -4 & -4 \\ 0 & x+1 & 0 \\ 8 & 4 & x+5 \end{bmatrix} \\ &= (x+1) \cdot \det \begin{bmatrix} x-7 & -4 \\ 8 & x+5 \end{bmatrix} \\ &= (x+1)(x^2 - 2x - 3) = (x+1)^2(x-3) \end{aligned}$$

Therefore, the eigenvalues of A are $\lambda = -1, 3$.
Now, compute $E_{-1} = \ker(-1I_3 - A)$ by row reduction:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 7 & 4 & 4 \\ 0 & -1 & 0 \\ -8 & -4 & -5 \end{bmatrix} = \begin{bmatrix} -8 & -4 & -4 \\ 0 & 0 & 0 \\ 8 & 4 & 4 \end{bmatrix} \mapsto \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $E_{-1} = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$.
Now, compute $E_3 = \ker(3I_3 - A)$ by row reduction:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 7 & 4 & 4 \\ 0 & -1 & 0 \\ -8 & -4 & -5 \end{bmatrix} = \begin{bmatrix} -4 & -4 & -4 \\ 0 & 4 & 0 \\ 8 & 4 & 8 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $E_3 = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

I claim the eigenvectors $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{C}^3$. Indeed, constructing a matrix and row reducing yields

$$\begin{bmatrix} -1 & -1 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, A has a basis of eigenvectors and is diagonalizable. To construct a diagonalization, let C be the above matrix, $\begin{bmatrix} -1 & -1 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ and let $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. A result from class demonstrates that $A = CDC^{-1}$ $\square$

Exercise 2. Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 & 2 \\ -5 & 5 & 3 \\ -4 & 1 & 6 \end{bmatrix} \in M_3(\mathbb{C})$, and find a basis for each eigenspace. Is A is diagonalizable? If so, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

Solution. Calculate the characteristic polynomial of A,

$$\begin{aligned}
C_A &= \det(xI_3 - A) \\
&= \det(x \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 & 1 & 2 \\ -5 & 5 & 3 \\ -4 & 1 & 6 \end{smallmatrix}\right]) \\
&= \det \left[\begin{smallmatrix} x & -1 & -2 \\ 5 & x-5 & -3 \\ 4 & -1 & x-6 \end{smallmatrix}\right] \\
&= x \cdot \det \left[\begin{smallmatrix} x-5 & -3 \\ -1 & x-6 \end{smallmatrix}\right] - (-1) \cdot \det \left[\begin{smallmatrix} 5 & -3 \\ 4 & x-6 \end{smallmatrix}\right] + (-2) \cdot \det \left[\begin{smallmatrix} 5 & x-5 \\ 4 & -1 \end{smallmatrix}\right] \\
&= (x^3 - 11x^2 + 27x) + (5x - 18) + (8x - 30) \\
&= x^3 - 11x^2 + 40x - 48 = (x-4)^2(x-3)
\end{aligned}$$

Therefore, the eigenvalues of A are $\lambda = 3, 4$.
Now, compute $E_3 = \ker(3I_3 - A)$ by row reduction:

$$\left[\begin{smallmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 & 1 & 2 \\ -5 & 5 & 3 \\ -4 & 1 & 6 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 & -1 & -2 \\ 5 & -2 & -3 \\ 4 & -1 & -3 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{smallmatrix}\right]$$

So, $E_3 = \text{Span}\left\{\left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]\right\}$.
Now, compute $E_4 = \ker(4I_3 - A)$ by row reduction:

$$\left[\begin{smallmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 & 1 & 2 \\ -5 & 5 & 3 \\ -4 & 1 & 6 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 4 & -1 & -2 \\ 5 & -1 & -3 \\ 4 & -1 & -2 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{smallmatrix}\right]$$

So, $E_4 = \text{Span}\left\{\left[\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}\right]\right\}$.

Hence, A does not have enough eigenvectors to form a basis for C^3 out of them, as it only has two eigenvectors (ignoring scaling).

$\square$

Exercise 3.

(a) Suppose we have an $n \times n$ matrix

$$P = \begin{pmatrix} p_{11}(x) & \cdots & p_{1n}(x) \\ \vdots & & \vdots \\ p_{n1}(x) & \cdots & p_{nn}(x) \end{pmatrix}$$

whose entries are polynomials $p_{ij}(x)$ with coefficients in F. Prove that the degree of $\det(P)$ is at most the sum of the degrees of all of its entries.

(b) Prove that the characteristic polynomial of a matrix $A \in M_n(F)$ is monic of degree n.

Solution.

(a) The proof is by induction on n.

If $n = 1$, then directly from the definition, $\det(P) = p_{11}(x)$, which clearly has degree equal to the sum of the degree of its entries. Thus, the claim is true.

For the induction step, assume that the statement is true for some $k-1$ and try to induce that it is true for k. Consider the calculation of $\det(P)$ by expansion by minors on the first row. This turns the determinant into a sum (with alternating signs) of expressions of the form $p_{1i}(\det(P_{1i}))$. As multiplication of exponents of the same base adds in the degree, each of these terms has a degree equal to $\deg(p_{ij}) + \deg(\det(P_{1i}))$. Clearly, the degree of this determinant is the *maximum* degree of these terms. However, we know that P_{1i} is a $(k-1) \times (k-1)$ matrix, so its degree is no more than the sum of its entries. Therefore the degree of the determinant of this smaller matrix is at most the sum of the degree of its entries. This means that each term of the expansion has a degree which is no greater than the degree of the term plus the degree of the terms in the minor determinant. Each of these will be smaller than the sum of all the degrees of the entries of P, so it follows that $\det(P)$ has a degree $\leq$ the sum of the degree of all of its entries.

(b) Again, the proof is by induction on n.

If $n = 1$, let $A = \begin{pmatrix} a \end{pmatrix}$. Clearly, $C_A = \det \begin{pmatrix} x-a \end{pmatrix} = x - a$, which is clearly a monic polynomial of degree n.

Now, assume the statement is true for some $k-1$ and try to induce that it is true for $n = k$. Consider how one would calculate the characteristic polynomial for a matrix $A \in M_k(F)$, $C_A = \det(xI_k - A)$.
Consider the expansion by minors along the first row of this determinant. The first term of this expansion would be $(x - a_{11}) \det(xI_{k-1} - A_{11})$, a degree one monic polynomial times the characteristic polynomial of the matrix A_{11}, which is size $(k-1) \times (k-1)$. Therefore, by the induction hypothesis, $C_{A_{11}}$ is a monic polynomial of degree $k-1$. Thus, it is clear that the first term of this expansion is a monic polynomial of degree k. All other terms of the expansion by minors will be a constant times a characteristic polynomial of a $(k-1) \times (k-1)$ matrix, and as such have degree $k-1$, and will never change the degree or leading coefficient of the characteristic polynomial.

Therefore, the characteristic polynomial for a $n \times n$ matrix is a monic polynomial of degree n.

□

Exercise 4. The *trace* of a square matrix $A = (a_{ij}) \in M_n(F)$ is defined as the sum of its diagonal entries: $\text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}$. Prove that $\text{trace}(AB) = \text{trace}(BA)$. Use this to prove that any two similar matrices have the same trace.

Solution. If both BA and AB are defined, then both A and B are square. Thus, given $A, B \in M_n(F)$, recall that $BA_{ij} = \sum_{k=1}^{n} B_{ik} A_{kj}$. Now, consider the diagonal entries of BA, so that $i = j$. In these cases, it is clear that $BA_{ii} = \sum_{k=1}^{n} B_{ik} A_{ki}$.
From this, it becomes clear that $\text{trace}(BA) = \sum_{i=1}^{n} \sum_{k=1}^{n} B_{ik} A_{ki}$
However, addition and multiplication are commutative, so the above can be re-written as $\sum_{k=1}^{n} \sum_{i=1}^{n} A_{ki} B_{ik}$. This summation is clearly equivalent to the sum of all the diagonal entries of AB, and is such equal to the trace of AB.
Hence, $\text{trace}(BA) = \text{trace}(AB)$.

Now, given two similar matrices $A, B \in M_n(F)$, then there exists an invertible matrix $C \in M_n(F)$ such that $A = CBC^{-1}$. This, along with the associativity of matrix multiplication, implies

$$\text{trace}(A) = \text{trace}((CB)(C^{-1})) = \text{trace}((C^{-1})(CB)) = \text{trace}(C^{-1}CB) = \text{trace}(B)$$

□

Exercise 5. Suppose $A \in M_n(F)$.

(a) Show that there is some nonzero polynomial $f(x)$ of degree $\deg(f) \leq n^2$ such that $f(A) = 0$. Hint: use the fact that $M_n(F)$ is a vector space of dimension n^2.

(b) Among all nonzero monic polynomials $f(x) \in F[x]$ with $f(A) = 0$, let $m_A(x)$ be the one of smallest degree. The polynomial $m_A(x)$ is the *minimal polynomial* of A. Use the division algorithm to show that for any polynomial $f(x)$,

$$f(A) = 0 \iff f(x) \text{ is a multiple of } m_A(x).$$

Solution.

(a) Consider the general form of a polynomial of degree n^2, $f(x) = c_{n^2}x^{n^2} + c_{n^2-1}x^{n^2-1} + \cdots + c_1x^1 + c_0x^0$.

The claim is that given $A \in M_n(F)$, there exist a nonzero polynomial of the above form such that $f(A) = 0$, in other words, $c_{n^2}A^{n^2} + c_{n^2-1}A^{n^2-1} + \cdots + c_1A^1 + c_0A^0 = 0$ for some scalars c_i (not all 0). This is the same as asking if the list $A^{n^2}, A^{n^2-1}, \ldots, A^1, A^0$ is linearly dependent. This list consists of n^2+1 vectors in a vector space $M_n(F)$ which has dimension n^2, so they must be linearly dependent and such scalars must exist.

(b) First, suppose $f(A) = 0$. As $m_A(x) \neq 0$, there exists unique polynomials $q(x)$ and $r(x)$ such that $f(x) = q(x)m_A(x) + r(x)$, with $\deg(r) < \deg(m_A)$. I claim that $r(x) = 0$, or in other words, $f(x)$ is a multiple of $m_A(x)$. Indeed, evaluating at A,

$$\begin{aligned} f(A) &= q(A)m_A(A) + r(A) \\ 0 &= q(A) \cdot 0 + r(A) \\ 0 &= r(A) \end{aligned}$$

However, m_A was chosen to be the smallest nonzero polynomial such that $m_A(A) = 0$. As such, $\deg(r) < \deg(m_A)$, along with the above, now implies $r(x) = 0$ as the only option for $r(x)$. Therefore, $f(x) = q(x)m_A(x)$ is a multiple of $m_A(x)$.

Now, assume $f(x)$ is a multiple of $m_A(x)$. This means there exists a $q(x)$ such that $f(x) = q(x)m_A(x)$. Now, evaluate at A to find $f(A) = q(A)m_A(A) = q(A) \cdot 0 = 0$.

Therefore, $f(A) = 0 \iff f(x)$ is a multiple of $m_A(x)$.

$\square$

Exercise 6. Find the minimal and characteristic polynomials of $\begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}$ and $\begin{bmatrix} -13 & 5 & 5 \\ -15 & 7 & 5 \\ -15 & 5 & 7 \end{bmatrix}$, and for each matrix verify that the characteristic polynomial is a multiple of the minimal polynomial. Remark: The *Cayley-Hamilton theorem* asserts that the characteristic polynomial is always a multiple of the minimal polynomial.

Solution.

(a) First consider $A = \begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}$

The characteristic polynomial $C_A(x) = \det(xI_2 - A) = \det \begin{bmatrix} x-4 & -2 \\ 3 & x-5 \end{bmatrix} = (x-4)(x-5) + 6 = x^2 - 9x + 26$. To find the minimal polynomial, consider A^1 and $A^0 = I$, and check if they are linearly independent. Indeed they are, as shown by the simple calculation

$$c_1 \begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0 \implies \begin{bmatrix} 4c_1+c_2 & 2c_1 \\ -3c_1 & 5c_1+c_2 \end{bmatrix} = 0 \implies c_1 = c_2 = 0$$

So, the minimal polynomial cannot have degree 1.

Now, consider A^2, A^1 and A^0. These are linearly dependent, as show by

$$c_1 \begin{bmatrix} 10 & 18 \\ -27 & 19 \end{bmatrix} + c_2 \begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0 \implies \begin{bmatrix} 10c_1+4c_2+c_3 & 18c_1+2c_2 \\ -27c_1-3c_2 & 19c_1+5c_2+c_3 \end{bmatrix} = 0$$

This system of equations has a less obvious solution, but calculating it by row reduction (Shown in Appendix A as always) yields $c_1 = \frac{1}{26}c_3, c_2 = -\frac{9}{26}$ with c_3 being free. Picking $c_3 = 26$ so that $c_1 = 1$, we get $A^2 - 9A^1 + 26A^0$, and therefore $m_A(x) = x^2 - 9x + 26$.

Clearly, $C_A = m_A$ in this case, so in particular C_A is a multiple of m_A.

(b) Now, consider $A = \begin{bmatrix} -13 & 5 & 5 \\ -15 & 7 & 5 \\ -15 & 5 & 7 \end{bmatrix}$

The characteristic polynomial $C_A(x) = \det(xI_2 - A) = \det \begin{bmatrix} x+13 & -5 & -5 \\ 15 & x-7 & -5 \\ 15 & -5 & x-7 \end{bmatrix} = (x-2)(x^2 + x - 6)$.

To find the minimal polynomial, consider A^1 and $A^0 = I$, and check if they are linearly independent. Indeed they are, as shown by the simple calculation

$$c_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} -13 & 5 & 5 \\ -15 & 7 & 5 \\ -15 & 5 & 7 \end{bmatrix} = 0 \implies \begin{bmatrix} c_1 - c_2 13 & c_2 5 & c_2 5 \\ -c_2 15 & c_1 + c_2 7 & c_2 5 \\ -c_2 15 & c_2 5 & c_1 + c_2 7 \end{bmatrix} = 0 \implies c_1 = c_2 = 0$$

So, now consider A^0, A^1, and A^2. They are linearly dependent, as shown by

$$c_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} -13 & 5 & 5 \\ -15 & 7 & 5 \\ -15 & 5 & 7 \end{bmatrix} + c_3 \begin{bmatrix} 19 & -5 & -5 \\ 15 & -1 & -5 \\ 15 & -5 & -1 \end{bmatrix} = 0 \implies \begin{bmatrix} c_1 - c_2 13 + c_3 19 & c_2 5 - c_3 5 & c_2 5 - c_3 5 \\ -c_2 15 + c_3 15 & c_1 + c_2 7 - c_3 1 & c_2 5 - c_3 5 \\ -c_2 15 + c_3 15 & c_2 5 - c_3 5 & c_1 + c_2 7 - c_3 1 \end{bmatrix} = 0$$

Solving for c_1, c_2, c_3 using row reduction (Shown in Appendix A) yields $c_1 = -6c_3, c_2 = c_3$, with c_3 free. Choosing $c_3 = 1$ (so that $m_A(x)$ is monic) yields $A^2 + A^1 - 6A^0$, so $m_A(x) = x^2 + x - 6$.

From this, $C_A(x) = (x-2)(x^2 + x - 6)$ is obviously a multiple of $m_A(x)$.

$\square$