# Problem set #5

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In all of the following exercises $F$ denotes either $\mathbb{R}$ or $\mathbb{C}$, and $V$ is an $F$-vector space. Let $P_n$ be the space of polynomials with real coefficients of degree $\leq n$.

**Exercise 1.** Are the vectors

$$v_1 = (1, -1, 2, 1), \quad v_2 = (1, -2, 0, 2) \quad v_3 = (0, -1, 1, -3), \quad v_4 = (1, -1, 2, -3)$$

in $\mathbb{R}^4$ linearly independent? Do they span $\mathbb{R}^4$? Do they form a basis?

**Solution.** To check both if the vectors are linearly independent and if they span $\mathbb{R}^4$, encode them as column vectors of a matrix

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & -2 & -1 & -1 \\ 2 & 0 & 1 & 2 \\ 1 & 2 & -3 & -3 \end{bmatrix}$$

Row reduction yields

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & -2 & -1 & -1 \\ 2 & 0 & 1 & 2 \\ 1 & 2 & -3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(See full row reduction in Appendix A)
Because the reduced row echelon form matrix has a pivot in every row *and* every column, it is both linearly independent and spans $\mathbb{R}^4$. Therefore, it is also a basis. □

**Exercise 2.**

(a) Find a basis for the subspace

$$U = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \begin{array}{c} x_1 + x_2 = 0 \\ x_3 - x_4 = 0 \end{array} \right\},$$

extend it to a basis of $\mathbb{R}^4$, and find a subspace $W \subset \mathbb{R}^4$ such that $\mathbb{R}^4 = U \oplus W$.

(b) Find a basis for the subspace

$$U = \{f(x) \in P_5 : f(2) = 0\},$$

extend it to a basis of $P_5$, and find a subspace $W \subset P_5$ such that $P_5 = U \oplus W$.

**Solution.**

(a) The vectors $u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \in U$ form a basis for $U$.

They clearly span $U$, since any vector in $U$ can be written in the form $\begin{bmatrix} a \\ -a \\ b \\ b \end{bmatrix}$ with $a, b \in \mathbb{R}$ and this can obviously be expressed by $au_1 + bu_2$. They are also linearly independent, as the only solution to $c_1u_1 + c_2u_2 = 0$ is clearly $c_1 = 0, c_2 = 0$. Therefore they form a basis.

This list can be extended to a basis for $\mathbb{R}^4$ with the addition of the vectors $w_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^4$. This can be shown by encoding those four vectors as the columns of a matrix and row reducing.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Because the reduced row echelon form of the matrix has a pivot in every row and every column, it is a basis for $\mathbb{R}^4$.

Let $W \subset \mathbb{R}^4$ be $\text{Span}\{w_1, w_2\}$. I claim that $\mathbb{R}^4 = U \oplus W$. As shown above the vectors $u_1, u_2, w_1, w_2$ span $\mathbb{R}^4$, so it is clear that $\mathbb{R}^4 = U + W$, as any vector in $\mathbb{R}^4$ can be written as sums of the vectors in the two spaces.

Now it suffices to show that $U \cap W = 0$. Suppose $v \in U \cap W$. It follows that $v \in U$, so $v = a_1u_1 + a_2u_2$ for some $a_1, a_2 \in \mathbb{R}$. But, $v \in W$, so $v = b_1w_1 + b_2w_2$ for some $b_1, b_2 \in \mathbb{R}$. Subtraction yields

$$0 = v - v = a_1u_1 + a_2u_2 - b_1w_1 - b_2w_2$$

but, because $u_1, u_2, w_1, w_2$ form a basis for $\mathbb{R}^4$ and must be linearly independent, the coefficients of the above expression must all be 0. This means that $v = 0$. Therefore, $\mathbb{R}^4 = U \oplus W$.

(b) The polynomials $f_1 = x^5 - 32$, $f_2 = x^4 - 16$, $f_3 = x^3 - 8$, $f_4 = x^2 - 4$, $f_5 = x - 2$ are a basis for $U$.

These can be shown to span $U$ because all polynomials in $P_5$ take the form $a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. This means all polynomials such that $f(2) = 0$ would have the relation $a_5(32) + a_4(16) + a_3(8) + a_2(4) + a_1(2) + a_0 = 0$. By rewriting this as $a_0 = -a_5(32) - a_4(16) - a_3(8) - a_2(4) - a_1(2)$, we can get a standard form for a vector in $U$, namely $a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + (-a_5(32) - a_4(16) - a_3(8) - a_2(4) - a_1(2))$. This standard form can clearly be expressed as $a_5f_1 + a_4f_2 + a_3f_3 + a_2f_4 + a_1f_5$, and as such $f_1, f_2, f_3, f_4, f_5$ span $U$.

These vectors are also linearly independent, as they are polynomials of different degree and as such none can be expressed as a combination of the others.

Therefore, they form a basis for $U$.

This list can be extended to a basis for $P^5$ by adding the polynomial $f_6 = 1$. To show this, encode all the polynomials as columns of a matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -32 & -16 & -8 & -4 & -2 & 1 \end{bmatrix}$$

Row reduction obviously yields

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -32 & -16 & -8 & -4 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Because the reduced row echelon form of the matrix has a pivot in every row and every column, it is a basis for $P^5$.

Now, let $W = \text{Span}\{1\}$ be a subspace of $P^5$. I claim that $P_5 = U \oplus W$. As shown above the polynomials $f_1, f_2, f_3, f_4, f_5, f_6$ span $P^5$, so it is clear that $P^5 = U + W$, as any polynomial in $P^5$ can be written as sums of the vectors in the two spaces.

It now suffices to show that $U \cap W = 0$. Suppose $v \in U \cap W$. It follows that $v \in U$, so $v = a_5 f_1 + a_4 f_2 + a_3 f_3 + a_2 f_4 + a_1 f_5$ for some $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}$. But, $v \in W$, so $v = b_1 f_6$ for some $b_1 \in \mathbb{R}$. Subtraction yields

$$0 = v - v = a_5 f_1 + a_4 f_2 + a_3 f_3 + a_2 f_4 + a_1 f_5 - b_1 f_6$$

but, because $f_1, f_2, f_3, f_4, f_5, f_6$ form a basis for $P^5$ and must be linearly independent, the coefficients of the above expression must all be 0. This means that $v = 0$. Therefore, $P^5 = U \oplus W$.

$\square$

**Exercise 3.** Suppose $U$ and $W$ are 4-dimensional subspaces of $\mathbb{C}^6$. Show that one can find two vectors in $U \cap W$, neither of which is a scalar multiple of the other.

**Solution.** Consider the vector space $U+W$. Since this space is the sum of two 4-dimensional spaces, it cannot have fewer than 4 dimensions even in the trivial case where $U = W$. However, it is also a subspace of $\mathbb{C}^6$, so it can not have more than 6 dimensions.
It is true that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

However, it is clear that $\dim(U) = \dim(W) = 4$ from the problem statement. And, as stated above, $\dim(U+W)$ is either $4, 5$, or 6. It follows from subtraction that $\dim(U \cap W)$ must be either 4, 3, or 2. Because $\dim(U \cap W) \geq 2$, at least two vectors must exist in the basis for $U \cap W$. These vectors would be linearly independent and thus not multiples of each other, so the proof is complete. $\square$

**Exercise 4.** Suppose $f \in P_n$ has degree $n$. Prove that the polynomials

$$f, f^{(1)}, f^{(2)}, \dots, f^{(n)}$$

form a basis of $P_n$. Here the notation $f^{(k)}$ means the $k^{\text{th}}$ derivative of $f$.

**Solution.** Consider the general form of a polynomial.. It follows that $f$ can be written as $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$. Using this general form and the product rule we can establish descriptions of $f, f^{(1)}, f^{(2)}, \dots, f^{(n)}$ as follows

$$\begin{aligned}
f &= a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 \\
f^{(1)} &= a_n(n)x^{n-1} + a_{n-1}(n-1)x^{n-2} + a_{n-2}(n-2)x^{n-3} + \cdots + a_2(2)x + a_1 + 0 \\
f^{(2)} &= a_n(n)(n-1)x^{n-2} + a_{n-1}(n-1)(n-2)x^{n-3} + a_{n-2}(n-2)(n-3)x^{n-4} + \cdots + a_2(2) + 0 + 0 \\
&\vdots \\
f^{(n-1)} &= a_n(n!)x + a_{n-1}((n-1)!) \\
f^{(n)} &= a_n(n!)
\end{aligned}$$

These polynomials can be encoded as the columns of a matrix as follows

$$\begin{bmatrix}
a_n & 0 & 0 & \cdots & 0 & 0 \\
a_{n-1} & a_n(n) & 0 & \cdots & 0 & 0 \\
a_{n-2} & a_{n-1}(n-1) & a_n(n)(n-1) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_2 & a_3(3) & a_4(12) & \cdots & 0 & 0 \\
a_1 & a_2(2) & a_3(6) & \cdots & a_n(n!) & 0 \\
a_0 & a_1 & a_2(2) & \cdots & a_{n-1}((n-1)!) & a_n(n!)
\end{bmatrix}$$

Because it is given that $a_n$ is nonzero, entries on the diagonal are always nonzero. It follows easily that this matrix can be row reduced algorithmically to the $(n+1)$x$(n+1)$ identity matrix by scaling the first row and using it to cancel the elements in the first column of every row, then scaling the second row and using it to cancel every element in the second column of each row, and so on until complete.

Because the reduced row echelon form of the matrix is the identity matrix, and subsequently has a pivot in every row and column, the vectors are both linearly independent and span $P_n$. As a result, $f, f^{(1)}, f^{(2)}, \dots, f^{(n)}$ form a basis for $P_n$. $\square$

**Exercise 5.** Suppose $U_1, \dots, U_n$ are subspaces of a finite dimensional vector space $V$. Prove that

$$\dim(U_1 + \cdots + U_n) \leq \dim(U_1) + \cdots + \dim(U_n).$$

**Solution.** Pick any basis for each of $U_1, \dots, U_n$. A spanning set for $U_1 + \cdots + U_n$ can be constructed by concatenating those bases, since this space is finite dimensional and any element in it can surely be constructed by linear combinations of the vectors that generate $U_1, \dots, U_n$. Two cases exist. Either this spanning set is linearly independent, and therefor a basis for $U_1 + \cdots + U_n$, or it is not.

If the spanning set is already a basis, then clearly $\dim(U_1 + \cdots + U_n) = \dim(U_1) + \cdots + \dim(U_n)$ as the length of this basis is simply the sum of the lengths of the bases for

$U_1, \ldots, U_n$.
If the spanning set *is not* a basis, it is possible to remove elements from it to form a basis. This eventual basis would obviously have fewer elements than the total number of basis vectors for $U_1, \ldots, U_n$, so $\dim(U_1 + \cdots + U_n) < \dim(U_1) + \cdots + \dim(U_n)$ in this case.
It is thus clear that $\dim(U_1 + \cdots + U_n) \leq \dim(U_1) + \cdots + \dim(U_n)$. $\square$

**Exercise 6.**

(a) Show that $V$ is infinite dimensional if and only if it satisfies the following property: for every integer $k > 0$ one can find a list $v_1, \ldots, v_k \in V$ of $k$ linearly independent vectors.

(b) Show that the vector space

$$\mathbb{R}^\infty = \{\text{all sequences } (a_1, a_2, a_3, \ldots) \text{ of real numbers}\}$$

is infinite dimensional.

**Solution.**

(a) If one can find a list of linearly independent vectors $v_1, \ldots, v_k \in V$ for any $k > 0$, then it follows that one cannot find a finite spanning list, because the length of any spanning list is bounded below by the length of any linearly independent list. This means any spanning list for $V$ would have to be longer than $k$ *for any* positive integer $k$. Therefore, no finite spanning set exists and the vector space is infinite dimensional.

Conversely, if $V$ is infinite dimensional, then it is possible to construct a list of linearly independent vectors $v_1, \ldots, v_k \in V$ for any $k > 0$. To do so, begin with any linearly independent list $v_1, \ldots, v_n$. This list is finite, and as such cannot span $V$. Therefore, there exists some vector, call it $v_{n+1}$, which is not in $\text{Span}\{v_1, \ldots, v_n\}$. Adding this element to the original list, yields a larger list of vectors, $v_1, \ldots, v_n, v_{n+1}$, which is still linearly independent as no element is in the span of the preceding elements. Because no finite list will ever span $V$, this process can be repeated an arbitrary number of times, allowing the creation of a list of linearly independent vectors $v_1, \ldots, v_k \in V$ for any $k > 0$.

Therefore, $V$ is infinite dimensional if and only if it is possible to find a list $v_1, \ldots, v_k \in V$ of $k$ linearly independent vectors for every integer $k > 0$.

(b) The proof is by contradiction. Assume $\mathbb{R}^\infty$ is finite dimensional and has dimension $n$. Let $e_i \in \mathbb{R}^\infty$ be the $i$th standard basis vector, in other words, the sequence in which the $i$th term is 1 and all other terms are 0. The list $(e_1, e_2, \ldots, e_n, e_{n+1})$ is an obviously linearly independent list of length $n + 1$. The length of any linearly independent list is bounded above by the length any spanning list, in particular, the length of any basis. Therefore, the existence of this list contradicts our assumption that $\mathbb{R}^\infty$ has finite dimension $n$.

$\square$