Problem set #3

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Exercise 1. For each of the following subsets $U \subset \mathbb{C}^3$, determine whether or not U is a subspace:

(a) $U = \{(x_1, x_2, x_3) : x_1 + 2x_2 + 3x_3 = 0\}$

(b) $U = \{(x_1, x_2, x_3) : x_1 + 2x_2 + 3x_3 = 4\}$

(c) $U = \{(x_1, x_2, x_3) : x_1x_2x_3 = 0\}$

(d) $U = \{(x_1, x_2, x_3) : x_1 = 5x_3\}$

(e) $U = \{(x_1, x_2, x_3) : x_1^3 = x_3^3\}$.

Solution.

(a) Yes. The zero vector is obviously contained in U, since $0 + 2 \cdot 0 + 3 \cdot 0 = 0$.
Secondly, U is closed under addition. Given $u_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, u_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in U$, then $x_1+2y_1+3z_1 = 0$ and $x_2+2y_2+3z_2 = 0$. Clearly, $0+0 = 0$, so $(x_1+x_2)+2(y_1+y_2)+3(z_1+z_2) = 0$ and $u_1 + u_2 \in \mathbb{Z}$.
Finally, U is closed under scalar multiplication. Given $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in U$ and $c \in \mathbb{C}^3$, then $x + 2 \cdot y + 3 \cdot z = 0$. It is clear that $cx + 2 \cdot cy + 3 \cdot cz = c \cdot 0 = 0$ and $cu \in U$.

(b) No, the set does not contain $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $0 + 2 \cdot 0 + 3 \cdot 0 \neq 4$

(c) No, the set is not closed under addition. Consider the vectors $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in U$ and $u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in U$. Adding $u_1 + u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \notin U$.

(d) Yes. The zero vector is obviously contained in U, since $0 = 5 \cdot 0$.
Secondly, U is closed under addition. Given $u_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, u_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in U$, then $x_1 = 5z_1$ and $x_2 = 5z_2$. Clearly, $(x_1 + x_2) = 5(z_1 + z_2)$, so $u_1 + u_2 \in \mathbb{Z}$.
Finally, U is closed under scalar multiplication. Given $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in U$ and $c \in \mathbb{C}^3$, then $x = 5z$. It is clear that $cx = 5cz$ and $cu \in U$.

(e) No, the set is not closed under addition. Consider $u_1 = \begin{bmatrix} 1 \\ 0 \\ \frac{-1}{2}+i\frac{\sqrt{3}}{2} \end{bmatrix} \in U$ and $u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in U$. Adding $u_1 + u_2 = \begin{bmatrix} 2 \\ 0 \\ \frac{1}{2}+i\frac{\sqrt{3}}{2} \end{bmatrix} \notin U$ since $8 \neq -1$.

$\square$

Exercise 2. Let P be the $\mathbb{R}$-vector space of all polynomials with real coefficients. Show that $U = \{f \in P : f'(-1) = 3f(2)\}$ is a subspace of P. Remark: the notation $f'(x)$ means the derivative of $f(x)$, in the usual sense of calculus.

Solution. To prove U is a subspace of P, it suffices to show that U contains the 0 vector of P, and that U is closed under addition and scalar multiplication.
First, U clearly contains the 0 polynomial $f(x) = 0$, as $f'(x) = 0$ and $f'(-1) = 3f(2)$ evaluates to $0 = 0$ which is obviously true.
Secondly, U is closed under addition. Given $f_1(x), f_2(x) \in U$ then $f_1'(-1) = 3f_1(2)$ and $f_2'(-1) = 3f_2(2)$. The standard properties of derivatives allow that $\frac{dy}{dx}(f_1(x) + f_2(x)) = f_1'(x) + f_2'(x)$. It follows that

$$\begin{aligned}\frac{dy}{dx}(f_1(-1) + f_2(-1)) &= f_1'(-1) + f_2'(-1)\\ &= 3f_1(2) + 3f_2(2)\\ &= 3(f_1(2) + f_2(2))\end{aligned}$$

and therefore $f_1 + f_2 \in U$.
Finally, U is closed under scalar multiplication. Given $f(x) \in U$ and $c \in \mathbb{R}$, then $f'(-1) = 3f(2)$. By the standard properties of derivatives, $\frac{dy}{dx}cf(x) = c\frac{dy}{dx}f(x)$. It follows that

$$\begin{aligned}\frac{dy}{dx}cf(-1) = c\frac{dy}{dx}f(-1) &= cf'(-1)\\ &= c3f(2)\end{aligned}$$

and hence $cf \in U$ and the proof is complete. $\square$

Exercise 3. Prove that the only subspaces of $\mathbb{R}^1$ are the zero subspace and all of $\mathbb{R}^1$.

Solution. $\{[0] \in \mathbb{R}^1\}$ is obviously a subspace of $\mathbb{R}^1$, as it contains the zero vector and is clearly closed under addition and scalar multiplication. It is also trivially true that any vector space is a subspace of itself, so $\mathbb{R}^1$ is a subspace of $\mathbb{R}^1$
The rest of the proof is by contradiction. Begin by assuming that there is some set $U \subset \mathbb{R}^1$ which *is* a subspace of $\mathbb{R}^1$, but is not one of the two mentioned above. This subspace must

(a) Contain the zero vector

(b) Be closed under addition

(c) Be closed under scalar multiplication for all $c \in \mathbb{R}$

(d) Contain at least one non-zero vector, since by assumption it is not the zero subspace

Consider the implication of (c) on (d). Any non-zero vector in $\mathbb{R}^1$ is simply a real number. By scaling this vector by all $c \in \mathbb{R}$, you can yield all other vectors in $\mathbb{R}^1$. Because U is closed under scalar multiplication, it must by the above logic contain all vectors in $\mathbb{R}^1$ and therefore *is* $\mathbb{R}^1$. This contradicts our assumption, so no such U can exist and the proof is complete. $\square$

Exercise 4.

(a) Let $U = \{(x, y, y - x) \in \mathbb{R}^3\}$. Find a subspace $W \subset \mathbb{R}^3$ such that $\mathbb{R}^3 = U \oplus W$.

(b) Let $U = \{(x, x, x, y, y) \in \mathbb{R}^5\}$. Find nonzero subspaces $W_1, W_2 \subset \mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W_1 \oplus W_2$.

Solution.

(a) The solution is $W = \{(0, 0, c) \in \mathbb{R}^3\}$.
To show this, first recognize that $\mathbb{R}^3 = U + W$, as shown by the decomposition

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z + y - x \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -y + x \end{bmatrix}$$

with $\left[\begin{smallmatrix} x \\ y \\ z+y-x \end{smallmatrix}\right] \in U$ and $\left[\begin{smallmatrix} 0 \\ 0 \\ -y+x \end{smallmatrix}\right] \in W$.
Because we have *exactly two* subspaces such that $\mathbb{R}^3 = U + W$, to prove that $\mathbb{R}^3 = U \oplus W$ it suffices to show that $U \cap W = \{0\}$.
Begin by supposing that $\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] \in U \cap W$. Since $\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] \in U$, it must be true that $z = y - x$.
Since $\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] \in W$, it must be true that $x = y = 0$. Therefore, $x = y = z = 0$, or, in other words, $U \cap W = \{0\}$. Hence, $\mathbb{R}^3 = U \oplus W$.

(b) The solution is $W_1 = \{(0, a, b, 0, 0) \in \mathbb{R}^5\}, W_2 = \{(0, 0, 0, 0, c.) \in \mathbb{R}^5\}$
To show this, recognize that $\mathbb{R}^5 = U + W_1 + W_2$ as shown by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \\ x_4 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 - x_1 \\ x_3 - x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x_5 - x_4 \end{bmatrix}$$

Where $\left[\begin{smallmatrix} x_1 \\ x_1 \\ x_1 \\ x_4 \\ x_4 \end{smallmatrix}\right] \in U$, $\left[\begin{smallmatrix} 0 \\ x_2-x_1 \\ x_3-x_1 \\ 0 \\ 0 \end{smallmatrix}\right] \in W_1$, $\left[\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x_5-x_4 \end{smallmatrix}\right] \in W_2$.
Now, to prove that this sum is distinct, it suffices to show that the only decomposition of the 0 vector by vectors in U, W_1, W_2 is $0 = 0 + 0 + 0$.
Begin assuming a decomposition of the 0 vector in terms of vectors in U, W_1, W_2 exists. Due to the definitions of each subspace, ist would necessarily take the form

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \\ x_4 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2' \\ x_3' \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x_5'' \end{bmatrix}$$

with $\left[\begin{smallmatrix} x_1 \\ x_1 \\ x_1 \\ x_4 \\ x_4 \end{smallmatrix}\right] \in U$, $\left[\begin{smallmatrix} 0 \\ x_2' \\ x_3' \\ 0 \\ 0 \end{smallmatrix}\right] \in W_1$, and $\left[\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x_5'' \end{smallmatrix}\right] \in W_2$. From this, five relationships are apparent.

$$\begin{aligned}
0 &= x_1 + 0 + 0 \\
0 &= x1 + x_2' + 0 \\
0 &= x_1 + x_3' + 0 \\
0 &= x_4 + 0 + 0 \\
0 &= x_4 + 0 + x_5''.
\end{aligned}$$

Simple arithmetic shows that $x_1 = x_4 = x_2' = x_3' = x_5'' = 0$, so the only decomposition of the zero vector by vectors in these subspaces is when those vectors are themselves the zero vector. Because the zero vector can be uniquely decomposed, $\mathbb{R}^5 = U \oplus W_1 \oplus W_2$.

$\square$

Exercise 5. Let V be an F-vector space with subspaces $U_1, U_2 \subset V$.

(a) If there is a subspace $W \subset V$ such that $V = U_1 \oplus W$ and $V = U_2 \oplus W$, does it follow that $U_1 = U_2$? Prove or provide a counterexample.

(b) If $U_1 \cup U_2$ is a subspace, does it follow that either $U_1 \subset U_2$ or $U_2 \subset U_1$? Prove or provide a counterexample.

Solution.

(a) Counterexample: Let $W = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \in \mathbb{R}^3 \right\} \subset \mathbb{R}^3$.

Consider $U_1 = \left\{ \begin{bmatrix} 0 \\ c \\ c \end{bmatrix} \in \mathbb{R}^3 \right\}$. It can be shown that $\mathbb{R}^3 = W + U_1$ since $\mathbb{R}^3$ can be decomposed by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y - z \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ z \\ z \end{bmatrix}$$

with $\left[\begin{smallmatrix} x \\ y-z \\ 0 \end{smallmatrix}\right] \in W$ and $\left[\begin{smallmatrix} 0 \\ z \\ z \end{smallmatrix}\right] \in U_1$.
Because we have *exactly two* subspaces such that $\mathbb{R}^3 = U_1 + W$, to prove that $\mathbb{R}^3 = U_1 \oplus W$ it suffices to show that $U_1 \cap W = \{0\}$.
Any vector $v = \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] \in U_1 \cap W$ would have to be in both U_1 and W, and therefore must satisfy $x = 0, z = 0$, and $y = z$. Hence, $x = y = z = 0$, therefore $U_1 \cap W = \{0\}$ and $\mathbb{R}^3 = U_1 \oplus W$.

Consider $U_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \in \mathbb{R}^3 \right\}$. It can be shown that $\mathbb{R}^3 = W + U_1$ since $\mathbb{R}^3$ can be decomposed by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$$

with $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \in W$ and $\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \in U_2$.

Because we have *exactly two* subspaces such that $\mathbb{R}^3 = U_2 + W$, to prove that $\mathbb{R}^3 = U_2 \oplus W$ it suffices to show that $U_2 \cap W = \{0\}$.

Any vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in U_2 \cap W$ would have to be in both U_2 and W, and therefore must satisfy $x = 0, y = 0$, and $z = 0$. Hence, $x = y = z = 0$, therefore $U_2 \cap W = \{0\}$ and $\mathbb{R}^3 = U_2 \oplus W$.

Obviously, $U_1 \neq U_2$ even though both can be directly summed with W to yield $\mathbb{R}^3$.

(b) The proof is by contradiction. Assume $U_1 \cup U_2$ is a vector space such that $U_1 \not\subset U_2$ and $U_2 \not\subset U_1$.

Given arbitrary vectors $u_1 \in U_1$ and $u_2 \in U_2$ then it is obvious both $u_1, u_2 \in U_1 \cup U_2$. Since by our assumption this union is a vector space, it must be closed under addition. This means that $u_1 + u_2 \in U_1 \cup U_2$, in other words, $u_1 + u_2$ is in *either* U_1 *or* U_2.

However, if, as by our assumption, $U_1 \not\subset U_2$ and $U_2 \not\subset U_1$, then there would necessarily be at least one vector $v_1 \in U_1$ such that $v_1 \notin U_2$ and at least one vector $v_2 \in U_2$ such that $v_2 \notin U_1$. Obviously, $v_1, v_2 \in U_1 \cup U_2$. Now, according to the above, $v_1 + v_2 \in U_1 \cup U_2$, meaning $v_1 + v_2$ must be in either U_1 or in U_2. But, this vector $v_1 + v_2$ is not in either. If it was in U_1, then simple vector addition of $-v_1$ would also mean $v_2 \in U_1$, contradicting our assumption. Similar logic would yield the contradiction $v_1 \in U_2$ if the sum was in U_2. Since this sum is not in either U_1 or U_2, it cannot be in $U_1 \cup U_2$. This is a contradiction of our assumption.

Therefore $U_1 \cup U_2 \implies$ either $U_1 \subset U_2$ or $U_2 \subset U_1$.

$\square$

Exercise 6. Let P be the space of all polynomials with real coefficients. We say that $f \in P$ is *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$. Show that

$$U_0 = \{f \in V : f \text{ is even}\}, \quad U_1 = \{f \in V : f \text{ is odd}\}$$

are subspaces of P, and that $P = U_0 \oplus U_1$.

Solution. First, to show that U_0 and U_1 are subspaces of P, we must show that they contain the zero vector, are closed under addition, and are closed under scalar multiplication.

Let us begin by checking if both U_0 and U_1 contain 0.
It is clear that U_0 contains the zero polynomial, $f(x) = 0$, as $f(-x) = 0 = f(x)$ is obviously true. Similarly, U_1 contains the zero polynomial, as $f(-x) = 0 = -f(x)$.

Now, confirm that U_0 and U_1 are closed under addition.
Given $f_1, f_2 \in U_0$, then $f_1(-x) = f_1(x)$ and $f_2(-x) = f_2(x)$. Clearly $f_1(-x) + f_2(-x) = f_1(x) + f_2(x)$, so $f_1 + f_2 \in U_0$. Similarly, given $f_1, f_2 \in U_1$, then $f_1(-x) = -f_1(x)$ and $f_2(-x) = -f_2(x)$. Adding these polynomials yields $f_1(-x) + f_2(-x) = -f_1(x) + -f_2(x) = -(f_1(x) + f_2(x))$ as expected, so $f_1 + f_2 \in U_1$.

Now, confirm that U_0 and U_1 are closed under scalar multiplication.
Given $f \in U_0$ and $c \in \mathbb{R}$, then $f(-x) = f(x)$. It is clear that $cf(-x) = cf(x)$ and hence $cf \in U_0$. Similarly, $f \in U_1$ and $c \in \mathbb{R}$, then $f(-x) = -f(x)$. It is clear that $cf(-x) = c \cdot -f(x) = -cf(x)$ and hence $cf \in U_1$.

Therefore, U_0 and U_1 are subspaces of P.

Secondly, to show that $P = U_0 \oplus U_1$, we must demonstrate first that $P = U_0 + U_1$. First, recognize what "even" and "odd" generally mean in terms of polynomials. Even polynomials consist strictly of terms with even powers, as can be clearly seen by the fact that -1 raised to an even power is 1. Similarly, odd polynomials consist strictly of terms with odd powers, as shown by the fact that -1 raised to any odd power is -1.
Any polynomial $f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in P$ can be written as either

$$f = (a_n x^n + +a_{n-2}x^{n-2} + \cdots + a_0) + (a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_1 x)$$

where $(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_0) \in U_0$ and $(a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_1 x) \in U_1$ if n is even *or*

$$f = (a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_0) + (a_n x^n + a_{n-2}x^{n-2} + \cdots + a_1 x)$$

where $(a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_0) \in U_0$ and $(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_1 x) \in U_1$ if n is odd. Therefore, $P = U_0 + U_1$.

Now, because we have *exactly two* subspaces such that $P = U_0 + U_1$, to prove that $P = U_0 \oplus U_1$ it suffices to show that $U_0 \cap U_1 = \{0\}$.
Given any $f \in U_0 \cap U_1$, it must be true that both $f(-x) = f(x)$ and $f(-x) = -f(x)$. Therefore, $f(x) = -f(x)$. This is only possible if $f(x) = 0$. Therefore, $U_0 \cap U_1 = \{0\}$ and $P = U_0 + U_1$. $\square$