Problem set #9

Brian Ward

Due **WEDNESDAY** April 18, 2018

Exercise 1. Directly from the definition $\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = ad - bc$ of the 2×2 determinant, prove the following.

(a) $\det(A) = \det(A^T)$

(b) $\det(AB) = \det(A)\det(B)$

(c) A is invertible if and only if $\det(A) \neq 0$, in which case $A^{-1} = \frac{1}{\det(A)}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$.

Solution.

(a) Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Clearly, $A^T = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$. Now, computing from the definition, it is clear that $\det(A) = ad - bc = ad - cb = \det(A^T)$.

(b) Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

From the definition, $\det(A) = ad - cb, \quad det(B) = eh - fg$.

Computing their product yields $AB = \left(\begin{smallmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{smallmatrix}\right)$.

But now, from the definition,

$$\begin{aligned}
\det(AB) &= (ae + bg)(cf + dh) - (ce + dg)(af + bh) \\
&= (aecf + bgcf + aedh + bgdh) - (ceaf + dgaf + cebh + dgbh) \\
&= (bgcf + aedh) - (dgaf + cebh) \\
&= adeh - cbeh - adfg + cbfg \\
&= (ab - cd)(eh - fg) \\
&= \det(A)\det(B)
\end{aligned}$$

(c) The statement is obviously equivalent to the statement that A is *not* invertible if and only if $\det(A) = 0$.

A is not invertible $\iff$ The columns of A are linearly dependent $\iff$ The columns of A, $\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right), \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)$ are multiples of each other $\iff$ There is some scalar s for which $a = sc$ and $b = sd$ $\iff$ $ad - bc = \frac{ab}{s} - \frac{ab}{s} = 0$ $\iff$ $\det(A) = 0$.

Now, to show that whenever the determinant of A is not 0, $B = \frac{1}{\det(A)}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ is the inverse of A, consider the calculation BA

$$\begin{aligned}
BA &= \frac{1}{\det(A)}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \\
&= \frac{1}{\det(A)}\left(\begin{smallmatrix} ad-bc & -bd+bd \\ -ac+ca & -cb+ad \end{smallmatrix}\right) \\
&= \frac{1}{\det(A)}\begin{pmatrix} \det(A) & 0 \\ 0 & \det(A) \end{pmatrix} \\
&= \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)
\end{aligned}$$

This demonstrates $BA = I$, and now a result from class states that $B = A^{-1}$.

$\square$

Exercise 2.

(a) If $A \in M_n(F)$ and c is a scalar, how are $\det(cA)$ and $\det(A)$ related?

(b) Suppose $A \in M_{m\times n}(F)$. Prove that A is surjective if and only if A^T is injective.

(c) Assuming $A \in M_n(F)$ is invertible, prove that $(A^{-1})^T = (A^T)^{-1}$, and that $\det(A^{-1}) = \det(A)^{-1}$.

Solution.

(a) Because the determinant is linear in every column, one can pull the scalar c out of each of the n columns, to yield $\det(cA) = c^n \det(A)$.

(b) Important: A can only be surjective if $n \geq m$. It follows that

A is surjective $\iff$ The columns of A span F^m $\iff$ The rows of A^T span F^m $\iff$ The dimension of the row span of A^T is m $\iff$ The dimension of the row span of the reduced row echelon form of A^T is m $\iff$ The reduced row echelon form of A^T has m pivots $\iff$ The reduced row echelon form of A^T has a pivot in every column $\iff$ The columns of A^T are linearly independent $\iff$ A^T is injective.

(c) By definition, $A^{-1}A = I$. As $I^T = I$, it follows that $(A^{-1}A)^T = I$. But, $(BA)^T = B^T A^T$. So, $I = (A^{-1}A)^T = A^T(A^{-1})^T$. However, a result from class says that if $AB = I$ for any square matrices A, B, then A is invertible with inverse $A^{-1} = B$. So, the above expression clearly implies that A^T has inverse $(A^T)^{-1} = (A^{-1})^T$.

Again, it follows from the definition that $A^{-1}A = I$. From this, and the facts that $\det(I) = 1$ and $\det(AB) = \det(A)\det(B)$, it follows that

$$\begin{aligned} \det(A^{-1}A) &= \det(I) = 1 \\ \det(A^{-1})\det(A) &= 1 \\ \det(A^{-1}) &= \frac{1}{\det(A)} \\ \det(A^{-1}) &= \det(A)^{-1} \end{aligned}$$

$\square$

Exercise 3. Compute the determinants

$$\det\begin{bmatrix} 0 & 1 & 2 \\ 2 & 6 & -1 \\ 3 & 0 & 4 \end{bmatrix} \qquad \text{and} \qquad \det\begin{bmatrix} 2 & 0 & 2 & -4 \\ 12 & 6 & 6 & 1 \\ 0 & -1 & 4 & 5 \\ 3 & 2 & 3 & 2 \end{bmatrix}.$$

Solution. A combination of elementary operations and expansion by minors yields

$$\det\begin{bmatrix} 0 & 1 & 2 \\ 2 & 6 & -1 \\ 3 & 0 & 4 \end{bmatrix} = -47 \qquad \text{and} \qquad \det\begin{bmatrix} 2 & 0 & 2 & -4 \\ 12 & 6 & 6 & 1 \\ 0 & -1 & 4 & 5 \\ 3 & 2 & 3 & 2 \end{bmatrix} = -232$$

Full computation shown in Appendix A. $\square$

Exercise 4. Prove or find counterexamples:

$$\operatorname{rank}(BA) \leq \operatorname{rank}(B), \quad \operatorname{rank}(CB) \leq \operatorname{rank}(B)$$

for any matrices and A, B, and C for which the products are defined.

Solution. Pick $A \in M_{n\times l}(F), B \in M_{m\times n}(F), C \in M_{p\times m}(F)$. Clearly, $A : F^l \to F^n, B : F^n \to F^m$, and $C : F^m \to F^p$.

Consider BA as the composition $B(Av)$ for $v \in F^l$. Clearly $Av \in F^n$, and $\mathrm{Im}(A) \subset F^n$, so $\mathrm{Im}(BA) \subset \mathrm{Im}(B)$. From this it follows obviously that $\mathrm{rank}(BA) \leq \mathrm{rank}(B)$.

Now, consider CB as a composition $C(Bv)$ for $v \in F^n$. Clearly, the only part of the domain of C we are concerned with is the subspace $\mathrm{Im}(B) \subset F^m$. So, consider $C' : \mathrm{Im}(B) \to F^p$, which is the linear map C restricted on its domain to be the subspace $\mathrm{Im}(B)$. The fundamental theorem of linear algebra now clearly states $\mathrm{rank}(B) = \dim\ker(C') + \mathrm{rank}(C')$, and from this we can conclude $\mathrm{rank}(C') \leq \mathrm{rank}(B)$. However, clearly the image of C' and the image of CB are the same, so this now implies $\mathrm{rank}(CB) \leq \mathrm{rank}(B)$.

□

Exercise 5. Show that

$$\det \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} = (z-x)(z-y)(y-x).$$

Solution. This follows directly from calculation by elementary row operations and expansion by minors

$$\begin{aligned}
\det \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} &= \det \begin{bmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{bmatrix} \\
&= 1 \cdot \det \begin{bmatrix} y-x & y^2-x^2 \\ z-x & z^2-x^2 \end{bmatrix} \\
&= (y-x)(z^2-x^2) - (y^2-x^2)(z-x) \\
&= (y-x)(z-x)(z+x) - (y-x)(z-x)(y+x) \\
&= (y-x)(z-x)((z+x)-(y+x)) \\
&= (z-x)(z-y)(y-x)
\end{aligned}$$

□

Exercise 6. Suppose V is finite dimensional and $T : V \to V$ is a linear map. If A is the matrix of T with respect to some basis $e_1, \dots, e_n \in V$, and B is the matrix of T with respect to another basis $f_1, \dots, f_n \in V$, show that there is an invertible matrix C with $B = CAC^{-1}$, and deduce from this that $\det(A) = \det(B)$.

Solution. The basis $e_1, \dots, e_n$ determines an isomorphism $\alpha : F^n \to V$. Similarly, the basis $f_1, \dots, f_n$ determines an isomorphism $\beta : F^n \to V$.

By the definition of the matrix of a linear map, $A = \alpha^{-1} \circ T \circ \alpha : F^n \to F^n$ and $B = \beta^{-1} \circ T \circ \beta : F^n \to F^n$.

Construct a new matrix as follows: Take e_1 and decompose it in terms of $f_1, \dots, f_n$. Make the coefficients of this decomposition the first column of a matrix. Repeat this for $e_2, \dots, e_n$ until an $n \times n$ matrix is constructed. This matrix, C, maps from elements written in terms of $e_1, \dots, e_n$ to elements written in terms of $f_1, \dots, f_n$.

Clearly, this map is equivalent to mapping $F^n \to V$ with respect to the first basis $e_1, \dots, e_n$, performing no operation, then mapping $V \to F^n$ with respect to the second basis $f_1, \dots, f_n$. Using the notation established above, this is the same as $C = \beta^{-1} \circ \alpha$.

Now, C is clearly invertible, as α and β are invertible, and so their composition is as well. It follows that $C^{-1} = \alpha^{-1} \circ \beta$, as is clear from the calculations $CC^{-1} = \beta^{-1} \circ \alpha \circ \alpha^{-1} \circ \beta = I$ and $C^{-1}C = \alpha^{-1} \circ \beta \circ \beta^{-1} \circ \alpha = I$.

Now, calculate

$$\begin{aligned} CAC^{-1} &= (\beta^{-1} \circ \alpha) \circ (\alpha^{-1} \circ T \circ \alpha) \circ (\alpha^{-1} \circ \beta) \\ &= \beta^{-1} \circ (\alpha \circ \alpha^{-1}) \circ T \circ (\alpha \circ \alpha^{-1}) \circ \beta \\ &= \beta^{-1} \circ T \circ \beta \\ &= B \end{aligned}$$

As desired.

Now, considering $B = CAC^{-1}$, it follows that $\det(B) = \det(CAC^{-1})$. But, as $\det(M_1 M_2) = \det(M_1)\det(M_2)$ for any M_1, M_2 for which the determinant is defined, the above is clearly equivalent to $\det(B) = \det(C)\det(A)\det(C^{-1})$. As shown earlier in this homework, $\det(C^{-1}) = \det(C)^{-1}$, so this equation simplifies to $\det(B) = \frac{\det(C)}{\det(C)} \det(A) = \det(A)$, as desired. $\square$