Problem set #1

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Exercise 1. Prove there is no $x \in \mathbb{Q}$ satisfying $x^3 = 5$.

Solution. Begin by assuming that there is such an x. Write $x = a/b$ with $a, b \in \mathbb{Z}$ and $b > 0$. This implies the following

$$x^3 = 5 \implies (a/b)^3 = 5 \implies a^3 = 5b^3$$

Inspect that final equality

$$a^3 = 5b^3 \tag{1}$$

and consider how many times the prime 5 appears in the prime factorization of each side.

For any nonzero integer m, the prime 5 appears in the prime factorization of m^3 three times as much as it appears in the prime factorization of m itself. For any nonzero m, the number of times the prime 5 appears in the prime factorization of m^3 must be a multiple of three. Therefore the number of times 5 appears in the prime factorization of the left-hand side of (1) is a multiple of three. The right-hand side includes b^3, which, by the same logic, also has a prime factorization which includes a number of 5s which is a multiple of three. Consequently, $5b^3$ has a number of 5s which is *one more than* a multiple of three. This shows that the prime factorization of each side of (1) has a different frequency of the prime 5. This is a contradiction, so no x can exist to satisfy the equation. $\square$

Exercise 2. Use induction to prove

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Solution. The proof is by induction. Let $P(n)$ be the above statement. Consider the base case $n = 1$. The statement $P(1)$ asserts that

$$1^3 = \frac{1^2(1+1)^2}{4}$$

which is true. For the inductive case we assume $P(k)$ is true for some $k \in \mathbb{Z}$ and deduce that $P(k+1)$ is true. So, suppose $P(k)$ is true. This states that

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}.$$

and adding $(k+1)^3$ to both sides and simplifying results in

$$\begin{aligned}
\sum_{i=1}^{k} i^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{k^2(k^2+2k+1) + 4(k^3+3k^2+3k+1)}{4} \\
&= \frac{k^4 + 2k^3 + k^2 + 4k^3 + 12k^2 + 12k + 4}{4} \\
&= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4},
\end{aligned}$$

the expected result for $P(k+1)$. Therefore, $P(k) \implies P(k+1)$, and so $P(1), P(2), \cdots, P(n)$ are all true. $\square$

Exercise 3. Derive a formula for $\sum_{i=1}^{n} i^4$.

Solution. Consider the expansion of $(x+1)^5$

$$(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$$

and rewrite it as

$$(x+1)^5 - x^5 = 5x^4 + 10x^3 + 10x^2 + 5x + 1.$$

For every $x \in \{1, 2, 3, \cdots, n\}$ write out the resulting equalities

$$\begin{aligned}
2^5 - 1^5 &= 5*1^4 + 10*1^3 + 10*1^2 + 5*1 + 1 \\
3^5 - 2^5 &= 5*2^4 + 10*2^3 + 10*2^2 + 5*2 + 1 \\
&\cdots \\
(n-1)^5 - (n-2)^5 &= 5(n-2)^4 + 10(n-2)^3 + 10(n-2)^2 + 5(n-2) + 1 \\
(n)^5 - (n-1)^5 &= 5(n-1)^4 + 10(n-1)^3 + 10(n-1)^2 + 5(n-1) + 1 \\
(n+1)^5 - n^5 &= 5n^4 + 10n^3 + 10n^2 + 5n + 1.
\end{aligned}$$

Adding these equalities together yields

$(n+1)^5 - 1^5 = 5(1^4+2^4+\cdots+n^4) + 10(1^3+2^3+\cdots+n^3) + 10(1^2+2^2+\cdots+n^2) + 5(1+2+\cdots+n) + n.$

Now, solve this equality for $1^4 + 2^4 + \cdots + n^4$ (Notated as $\sum_{i=1}^{n} i^4$)

$$\begin{aligned}
\sum_{i=1}^{n} i^4 &= \frac{(n+1)^5-1}{5} - 2(1^3+2^3+\cdots+n^3) - 2(1^2+2^2+\cdots+n^2) - (1+2+\cdots+n) - \frac{n}{5} \\
&= \frac{(n+1)^5-1}{5} - \frac{n^2(n+1)^2}{2} - \frac{n(2n+1)(n+1)}{3} - \frac{n(n+1)}{2} - \frac{n}{5} \\
&= \frac{6((n+1)^5-1)}{30} - \frac{15(n^2(n+1)^2)}{30} - \frac{10(n(2n+1)(n+1))}{30} - \frac{15(n(n+1))}{30} - \frac{6n}{30} \\
&= \frac{6(n^5+5n^4+10n^3+10n^2+5n) - 15(n^4+2n^3+n^2) - 10(2n^3+3n^2+n) - 15(n^2+n) - 6n}{30} \\
&= \frac{6n^5+15n^4+10n^3-n}{30} \\
&= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.
\end{aligned}$$

Therefore, $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$. □

Exercise 4. Prove that for every $n \in \mathbb{Z}^+$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}. \tag{2}$$

Solution. The proof is by induction. Let $P(n)$ be the statement (2).

$P(1)$ asserts that $\frac{1}{1^2} \leq 2 - \frac{1}{1}$, which is true.

For the induction step, we assume some statement $P(k)$ is true and try to deduce that $P(k+1)$ is also true. In other words, we assume

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k} \tag{3}$$

is true for some k, and try to prove that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}.$$

If we add $\frac{1}{(k+1)^2}$ to both sides of (3), we obtain

$$\begin{aligned}
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(k+1)^2} &\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\
&= 2 - \frac{k^2+2k+1}{k(k+1)^2} + \frac{k}{k(k+1)^2} \\
&= 2 - \frac{k^2+k+1}{k(k+1)^2} \\
&= 2 - \frac{k^2+k}{k(k+1)^2} - \frac{1}{k(k+1)^2} \\
&< 2 - \frac{k^2+k}{k(k+1)^2} \\
&= 2 - \frac{k(k+1)}{k(k+1)(k+1)} = 2 - \frac{1}{k+1}
\end{aligned}$$

as expected. $\square$

Exercise 5. Let

$$\alpha = \frac{1+\sqrt{5}}{2} \qquad \beta = \frac{1-\sqrt{5}}{2}$$

be the two roots of the quadratic equation $x^2 - x - 1 = 0$. If $n \geq 1$, prove that the n^{th} Fibonacci number satisfies

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{4}$$

Solution. The proof is by strong induction. Let $P(n)$ be statement (4).

$P(1)$ asserts that $f_1 = 1 = \frac{\alpha-\beta}{\alpha-\beta}$, which is true.

For the induction step, we assume $P(1), P(2), \cdots, P(k)$ are true and try to deduce that $P(k+1)$ is also true. In other words, we assume

$$f_1 = \frac{\alpha-\beta}{\alpha-\beta}, f_2 = \frac{\alpha^2-\beta^2}{\alpha-\beta}, f_3 = \frac{\alpha^3-\beta^3}{\alpha-\beta}, \cdots, f_k = \frac{\alpha^k-\beta^k}{\alpha-\beta}$$

Before continuing, we must establish some basic facts about the relationships between α and β

$$\alpha + \beta = \frac{1+\sqrt{5}+1-\sqrt{5}}{2} = 1$$
$$\alpha^{-1} = \frac{1}{\alpha} = \frac{1}{\frac{1+\sqrt{5}}{2}} = \frac{-1+\sqrt{5}}{2} = -\beta.$$

$\beta^{-1} = -\alpha$ can be established by the same logic.

Now we can show the following

$$\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \\
&= \frac{\alpha^k-\beta^k}{\alpha-\beta} + \frac{\alpha^{k-1}-\beta^{k-1}}{\alpha-\beta} \\
&= \frac{\alpha^k-\beta^k}{\alpha-\beta} + \frac{\alpha^k\alpha^{-1}-\beta^k\beta^{-1}}{\alpha-\beta} \\
&= \frac{\alpha^k-\beta^k}{\alpha-\beta} + \frac{\alpha^k(-\beta)-\beta^k(-\alpha)}{\alpha-\beta} \\
&= \frac{\alpha^k(1-\beta)-\beta^k(1-\alpha)}{\alpha-\beta} \\
&= \frac{\alpha^k(\alpha)-\beta^k(\beta)}{\alpha-\beta} \\
&= \frac{\alpha^{k+1-\beta^{k+1}}}{\alpha-\beta}
\end{aligned}$$

as expected. $\square$

Exercise 6. Suppose $n \geq 0$ is an integer.

(a) Using l'Hôpital's rule and induction, prove that

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

You may assume without proof that the formula is true when $n = 0$.

(b) Using induction to prove that

$$\int_0^\infty x^n e^{-x}\, dx = n!$$

(hint: use (a) and integration by parts).

Solution.

(a) The proof is by induction. Let $P(n)$ be the statement $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$. It is given that $P(0)$ is true. For the induction step, we must assume some statement $P(k)$ is true, and try to deduce $P(k+1)$.

In other words, we assume

$$\lim_{x\to\infty} \frac{x^k}{e^x} = 0$$

for some k, and try to prove that

$$\lim_{x\to\infty} \frac{x^{k+1}}{e^x} = 0$$

is also true.

Using l'Hôpital's rule and the properties of limits the following can be ascertained

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^{k+1}}{e^x} &= \lim_{x\to\infty} \frac{\frac{d}{dx}(x^{k+1})}{\frac{d}{dx}(e^x)} \\
&= \lim_{x\to\infty} \frac{(k+1)x^k}{e^x} \\
&= (k+1) \lim_{x\to\infty} \frac{x^k}{e^x} \\
&= (k+1) * 0 = 0.
\end{aligned}$$

This is as expected and the proof is complete.

(b) The proof is by induction. Let $P(n)$ be the statement $\int_0^\infty x^n e^{-x}\, dx = n!$. $P(0)$ asserts that

$$\int_0^\infty x^0 e^{-x}\, dx = 0!$$

Evaluating this integral

$$\begin{aligned}
\int_0^\infty x^0 e^{-x}\, dx &= \lim_{a\to\infty} \int_0^a (1)e^{-x}\, dx \\
&= \lim_{a\to\infty} (-e^{-a} - -e^0) \\
&= \frac{-1}{\infty} + 1 = 0 + 1 = 1 = 0!
\end{aligned}$$

proves that $P(0)$ is true.

For the induction step, we must assume some statement $P(k)$ is true, and try to deduce $P(k+1)$. In other words, we assume

$$\int_0^\infty x^k e^{-x}\, dx = k!$$

for some k, and try to prove that

$$\int_0^\infty x^{k+1} e^{-x}\, dx = (k+1)!$$

is also true.

Evaluate the integral in statement $P(k+1)$ using integration by parts and the limit established in (a)

$$\begin{aligned}
\int_0^\infty x^{k+1} e^{-x}\, dx &= \lim_{a\to\infty} \int_0^a x^{k+1} e^{-x}\, dx \\
&= \lim_{a\to\infty} \left[(x^{k+1}(-e^{-x}))\Big|_0^a - \int_0^a -e^{-x}(k+1)x^k\, dx \right] \\
&= \lim_{a\to\infty} \left[(x^{k+1}(-e^{-x}))\Big|_0^a \right] - \lim_{a\to\infty} \left[\int_0^a -e^{-x}(k+1)x^k\, dx \right] \\
&= \lim_{a\to\infty} \left[(x^{k+1}(-e^{-x}))\Big|_0^a \right] + (k+1) \int_0^\infty e^{-x} x^k\, dx \\
&= \lim_{a\to\infty} \left[(x^{k+1}(-e^{-x}))\Big|_0^a \right] + (k+1)k! \\
&= \lim_{a\to\infty} \left[a^{k+1}(-e^{-a}) - 0^{k+1}(-e^0) \right] + (k+1)k! \\
&= 0 + (k+1)k! \\
&= (k+1)!
\end{aligned}$$

and receive the expected result.

$\square$