Silesian University of Technology
Faculty of Automatic Control,
Electronic and Computer Science
Major: Makro sem.4

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Numerical Methods
Laboratory Exercise

Ex.4 Numerical differentiation

Group 2, Section 4
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1. Analyze the value of maximum absolute and relative differentiation error for different functions

$$f_1(x) = 10x^5 - 4x^4 + 24x^3 - 190x^2 + 4.5x$$

$$f_2(x) = arctg(x)$$

$$f_3(x) = e^{2sin(x-2)}$$

The analysis is performed with respect to number of forward difference terms

$n_1<k$, $n_2=k$, $n_3>k$

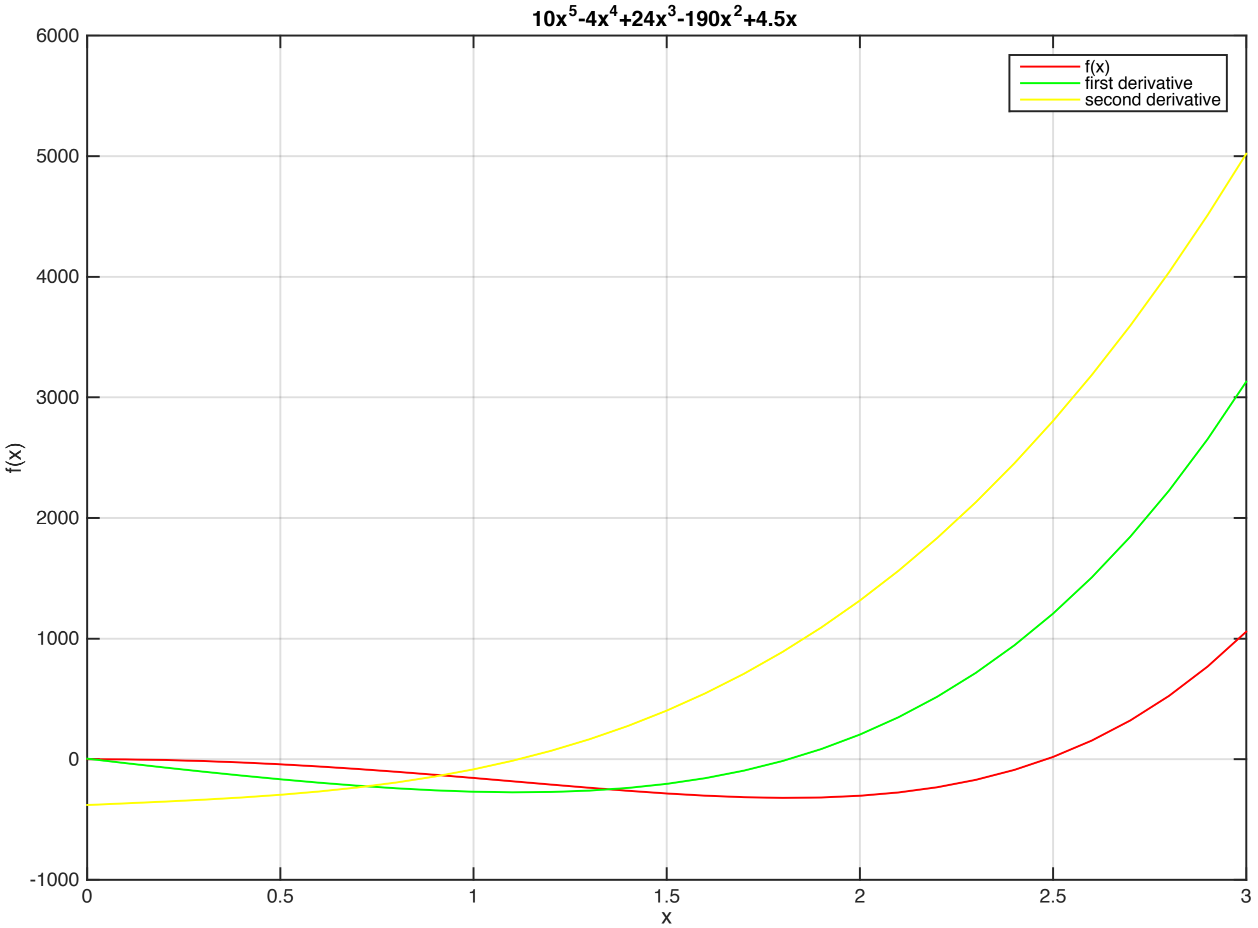
$2<5$, $5=5$, $10>5$

And the value of the step h:

10^{-1}, 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}

The analysis is performed in interval {a,b} in which: $f_i' \neq 0$ and $f_i'' \neq 0$, i=1, 2

$$f_1(x) = 10x^5 - 4x^4 + 24x^3 - 190x^2 + 4.5x$$

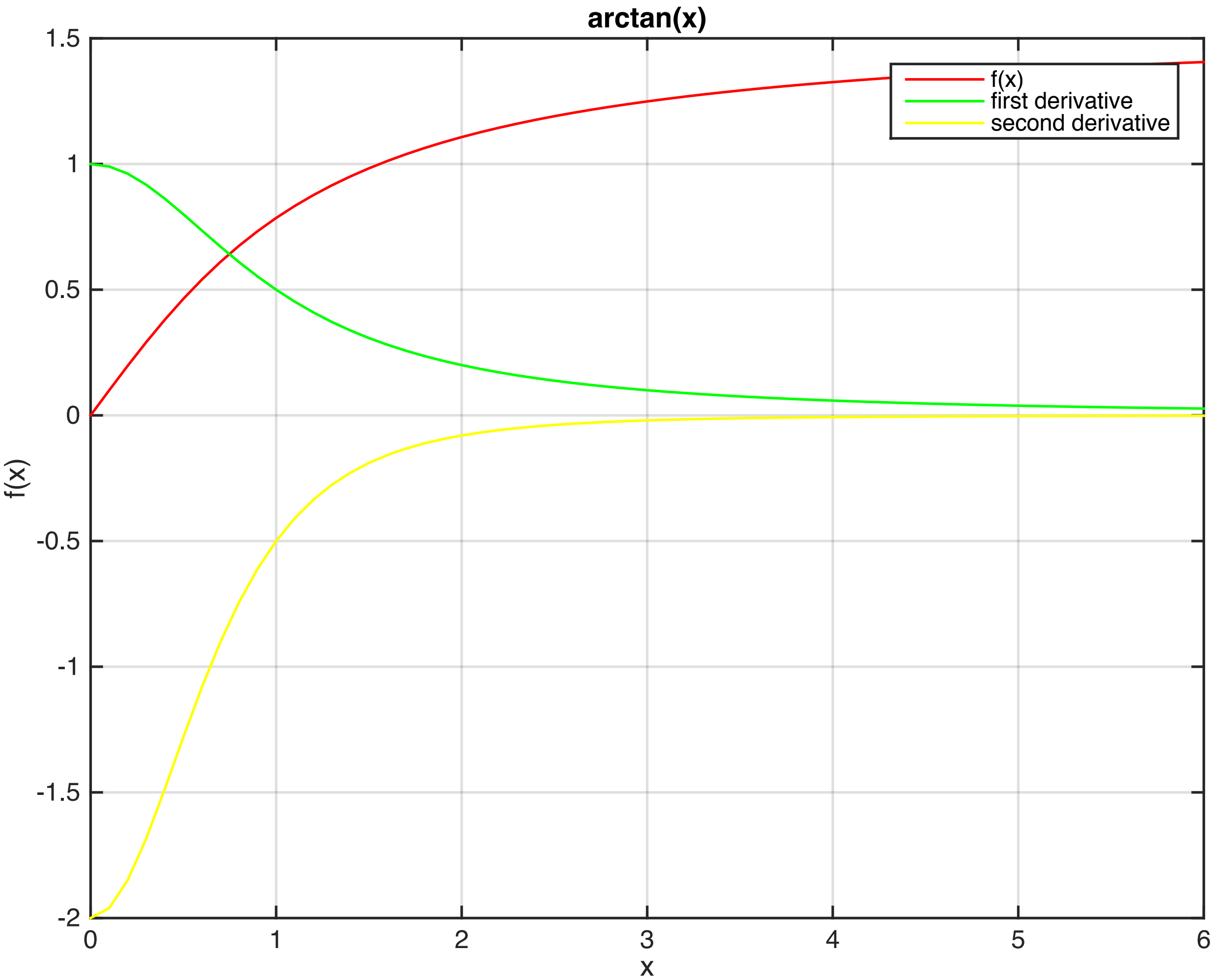


Differentiated function $f_1(x)$, n=2										
	First derivative $f'_1(x)$					Second derivative $f''_1(x)$				
h	x_0	Max absolute error	Max relative error	$f'_1(x)$ analytical	$f'_1(x)$ numerical	x_0	Max absolute error	Max relative error	$f''_1(x)$ analytical	$f''_1(x)$ numerical
10^{-1}	1	2,45	0,00909091	-269,5	-271,95	1	11,32	0,134762	-84	-95,32
10^{-3}	1	0,000216276	8,02509E-07	-269,5	-269,5	1	0,00101342	1,20645E-05	-84	-84,001
10^{-6}	1	1,59098E-08	5,90345E-11	-269,5	-269,5	1	0,0138486	0,000164865	-84	-83,9862
10^{-9}	1	7,89954E-06	2,93118E-08	-269,5	-269,5	1	28505,7	339,354	-84	28421,7
10^{-12}	1	nan	nan	-269,5	nan	1	nan	nan	-84	nan

Differentiated function $f_1(x)$, n=5										
	First derivative $f'_1(x)$					Second derivative $f''_1(x)$				
h	x_0	Max absolute error	Max relative error	$f'_1(x)$ analytical	$f'_1(x)$ numerical	x_0	Max absolute error	Max relative error	$f''_1(x)$ analytical	$f''_1(x)$ numerical
10^{-1}	1	2,6716E-12	9,91332E-15	-269,5	-269,5	1	8,5322E-11	1,01574E-12	-84	-84
10^{-3}	1	3,12866E-10	1,16091E-12	-269,5	-269,5	1	1,78565E-06	2,12577E-08	-84	-84
10^{-6}	1	2,56432E-09	9,5151E-12	-269,5	-269,5	1	0,102838	0,00122427	-84	-84,1028
10^{-9}	1	2,71539E-05	1,00757E-07	-269,5	-269,5	1	127982	1523,59	-84	127898
10^{-12}	1	0,231496	0,000858985	-269,5	-269,731	1	1,2534E+12	1,49214E+10	-84	1,2534E+12

Differentiated function $f_1(x)$, n=10										
	First derivative $f'_1(x)$					Second derivative $f''_1(x)$				
h	x_0	Max absolute error	Max relative error	$f'_1(x)$ analytical	$f'_1(x)$ numerical	x_0	Max absolute error	Max relative error	$f''_1(x)$ analytical	$f''_1(x)$ numerical
10^{-1}	1	0	0	-269,5	-269,5	1	6,90932E-10	8,22538E-12	-84	-84
10^{-3}	1	4,47363E-09	1,65998E-11	-269,5	-269,5	1	4,17087E-05	4,96533E-07	-84	-84
10^{-6}	1	3,34658E-07	1,24178E-09	-269,5	-269,5	1	5,78009	0,0688106	-84	-89,7801
10^{-9}	1	0,00072054	2,67362E-06	-269,5	-269,499	1	9,10357E+06	108376	-84	-9,10365E+06
10^{-12}	1	0,263854	0,000979051	-269,5	-269,764	1	2,78088E+12	3,31057E+10	-84	-2,78088E+12

$$f_2(x) = arctg(x)$$

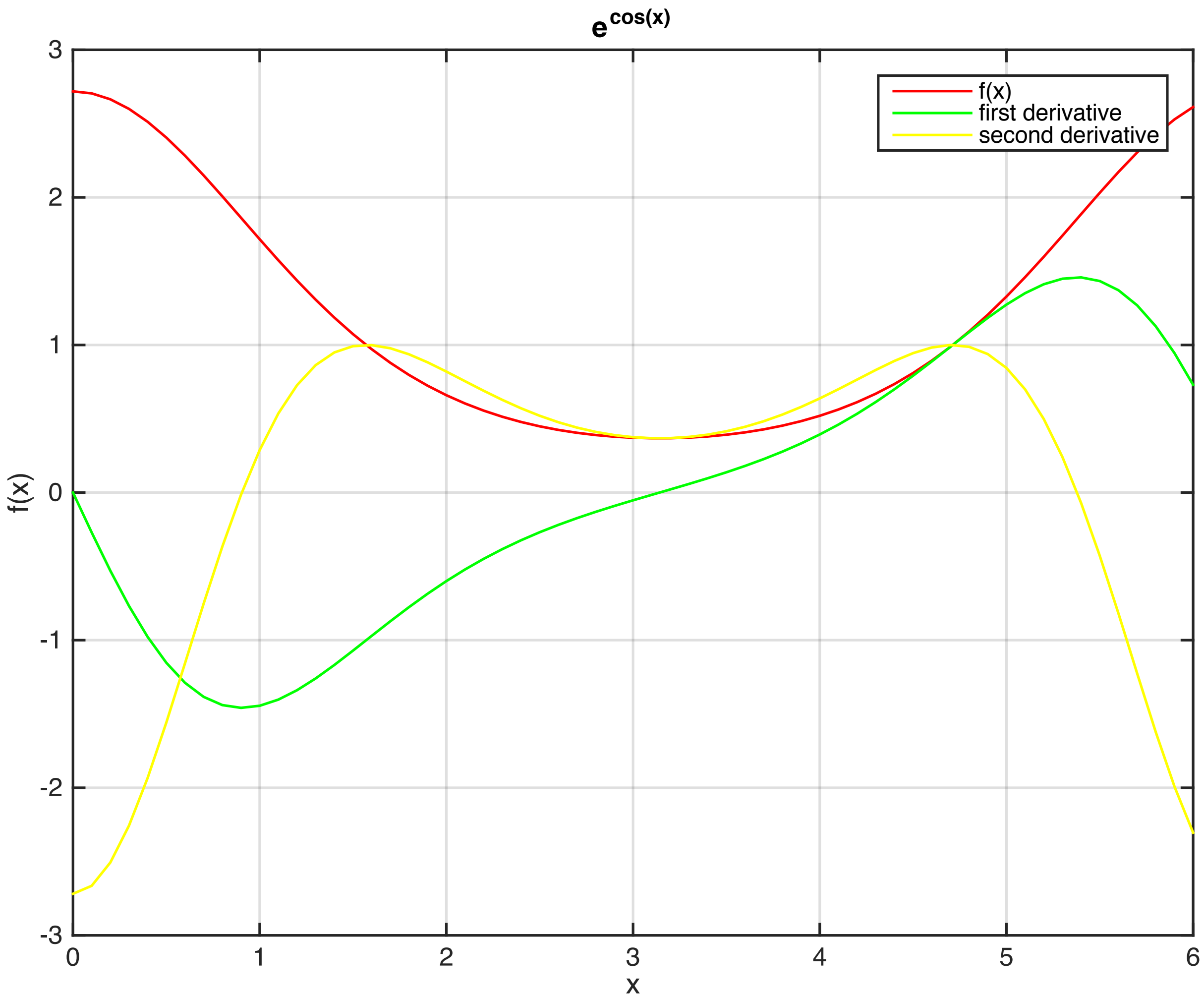

arctan(x)
f(x)
first derivative
second derivative
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
x
f(x)

Differentiated function $f_2(x)$, n=2										
	First derivative $f'_1(x)$					Second derivative $f''_1(x)$				
h	x_0	Max absolute error	Max relative error	$f'_1(x)$ analytical	$f'_1(x)$ numerical	x_0	Max absolute error	Max relative error	$f''_1(x)$ analytical	$f''_1(x)$ numerical
10^{-1}	1	0,00163737	0,00327474	0,5	0,498363	1	0,00214953	0,00429905	-0,5	-0,49785
10^{-3}	1	1,66666E-07	3,33333E-07	0,5	0,5	1	3,48282E-09	6,9665E-09	-0,5	-0,5
10^{-6}	1	6,9889E-11	1,39778E-10	0,5	0,5	1	4,44503E-05	8,89006E-05	-0,5	-0,500044
10^{-9}	1	9,68813E-08	1,93763E-07	0,5	0,5	1	332,567	665,134	-0,5	-333,067
10^{-12}	1	9,99614E-05	0,000199923	0,5	0,5001	1	3,33067E+08	6,66134E+08	-0,5	-3,33067E+08

Differentiated function $f_s(x)$, n=5										
	First derivative $f'_1(x)$					Second derivative $f''_1(x)$				
h	x_0	Max absolute error	Max relative error	$f'_1(x)$ analytical	$f'_1(x)$ numerical	x_0	Max absolute error	Max relative error	$f''_1(x)$ analytical	$f''_1(x)$ numerical
10^{-1}	1	1,1361E-05	2,27221E_05	0,5	0,500011	1	0,00018454	0,000369081	-0,5	-0,500185
10^{-3}	1	6,58196E-13	1,31639E-12	0,5	0,5	1	3,97256E-09	7,94511E-09	-0,5	-0,5
10^{-6}	1	1,14298E-10	2,8596E-10	0,5	0,5	1	0,00090549	0,00181098	-0,5	-0,500905
10^{-9}	1	3,20776E-07	6,41553E-07	0,5	0,5	1	915,434	1830,87	-0,5	-915,934
10^{-12}	1	0,000462634	0,000925269	0,5	0,500463	1	2,7478E+09	5,4956E+09	-0,5	-2,7478E+09

Differentiated function $f_2(x)$, n=10										
	First derivative $f'_1(x)$					Second derivative $f''_1(x)$				
h	x_0	Max absolute error	Max relative error	$f'_1(x)$ analytical	$f'_1(x)$ numerical	x_0	Max absolute error	Max relative error	$f''_1(x)$ analytical	$f''_1(x)$ numerical
10^{-1}	1	3,16959E-08	6,33918E-08	0,5	0,5	1	1,48381E-06	2,96762E-06	-0,5	-0,499999
10^{-3}	1	7,20973E-12	1,44195E-11	0,5	0,5	1	6,55817E-08	1,31163E-07	-0,5	-0,5
10^{-6}	1	3,33165E-09	6,66331E-09	0,5	0,5	1	0,0290444	0,0580887	-0,5	-0,529044
10^{-9}	1	3,65422E-06	7,30844E-06	0,5	0,499996	1	44628,1	89256,1	-0,5	44627,6
10^{-12}	1	0,00205417	0,00410835	0,5	0,502054	1	2,16593E+09	4,33185E+09	-0,5	-2,16593E+09

$$f_3(x) = e^{cos(x)}$$



e^cos(x)
f(x)
first derivative
second derivative
3
2
1
0
-1
-2
-3
0
1
2
3
4
5
6
x
f(x)

Differentiated function $f_4(x)$, n=2										
	First derivative $f'_1(x)$					Second derivative $f''_1(x)$				
h	x_0	Max absolute error	Max relative error	$f'_1(x)$ analytical	$f'_1(x)$ numerical	x_0	Max absolute error	Max relative error	$f''_1(x)$ analytical	$f''_1(x)$ numerical
10^{-1}	1	0,00779954	0,00539982	-1,44441	-1,45221	1	0,051993	0,180542	0,287983	0,339976
10^{-3}	1	9,19591E-07	6,36657E-07	-1,44441	-1,44441	1	5,05431E-06	1,75507E-05	0,287983	0,287988
10^{-6}	1	3,86403E-10	2,67517E-10	-1,44441	-1,44441	1	0,00132384	0,00459694	0,287983	0,28666
10^{-9}	1	5,78595E-07	4,00576E-07	-1,44441	-1,44441	1	1998,11	6938,29	0,287983	1998,4
10^{-12}	1	0,000104606	7,24218E-05	-1,44441	-1,4451	1	8,88178E+08	3,08413E+09	0,287983	8,88178E+08

Differentiated function $f_4(x)$, n=5										
	First derivative $f'_1(x)$					Second derivative $f''_1(x)$				
h	x_0	Max absolute error	Max relative error	$f'_1(x)$ analytical	$f'_1(x)$ numerical	x_0	Max absolute error	Max relative error	$f''_1(x)$ analytical	$f''_1(x)$ numerical
10^{-1}	1	4,77817E-05	3,30805E-05	-1,44441	-1,44436	1	0,000729135	0,00253187	0,287983	0,287254
10^{-3}	1	2,22511E-12	1,5405E-12	-1,44441	-1,44441	1	1,47778E-08	5,13149E-08	0,287983	0,287983
10^{-6}	1	1,28198E-09	8,8755E-10	-1,44441	-1,44441	1	0,00365531	0,0126928	0,287983	0,284328
10^{-9}	1	2,19952E-06	1,52278E-06	-1,44441	-1,44441	1	9328,05	32390,9	0,287983	9328,34
10^{-12}	1	0,00110381	0,000764194	-1,44441	-1,44551	1	6,4948E+09	2,25527E+10	0,287983	6,4948E+09

Differentiated function $f_4(x)$, n=10										
	First derivative $f'_1(x)$					Second derivative $f''_1(x)$				
h	x_0	Max absolute error	Max relative error	$f'_1(x)$ analytical	$f'_1(x)$ numerical	x_0	Max absolute error	Max relative error	$f''_1(x)$ analytical	$f''_1(x)$ numerical
10^{-1}	1	7,335173E-08	5,08981E+08	-1,44441	-1,44441	1	1,64086E-06	5,69776E-06	0,287983	0,287982
10^{-3}	1	2,90474E-11	2,01103E-11	-1,44441	-1,44441	1	2,64194E-07	9,17395E-07	0,287983	0,287983
10^{-6}	1	1,46402E-08	1,01358E-08	-1,44441	-1,44441	1	0,179827	0,624434	0,287983	0,46781
10^{-9}	1	2,74582E-06	1,901E-06	-1,44441	-1,44444	1	76613,5	266034	0,287983	-76613,2
10^{-12}	1	0,00349387	0,0024189	-1,44441	-1,4479	1	3,51707E+09	1,22128E+10	0,287983	-3,51707E+09

2. Source code:

```
#include <iostream>
#include <cmath>

using std::cout;
using std::cin;
using std::endl;

double function(double x)
{
    //return exp(cos(x));
    return atan(x);
    //return 10*pow(x,5)-4*pow(x,4)+24*pow(x,3)-190*pow(x,2)+4.5*x;
}


double function_derivative1(double x)
{
    //return (sin(x)*(-exp(cos(x))));
    return 1/(pow(x,2)+1);
    //return 50*pow(x,4)-16*pow(x,3)+72*pow(x,2)-380*x+4.5;
}

double function_derivative2(double x)
{
    //return (exp(cos(x)))*((sin(x)*sin(x))-cos(x));
    return -2/pow((1+pow(x,2)),2);
    //return 200*pow(x,3)-48*pow(x,2)+144*x-380;
}


double* difference_operators(int n, double h, double x0)
{
    double* x;
    double** operators;
    double* terms;



    x = new double[n+2];

    for (int i = 0; i < n+2; i++)
    {
        x[i] = x0 + i*h;
        cout<<"x"<<i<<":"<<x[i]<<endl;
    }

    operators = new double*[n+1];

    for(int i = 0; i < n+1; i++)
    {
        operators[i] = new double[n-i+1];
    }


    for(int i = 0; i < n+1; i++)
    {
        operators[0][i] = function(x[i+1])-function(x[i]);
    }


    for (int i = 1; i < n+1; i++)
    {
        for (int j = 0; j < n-i+1; j++)
        {
            operators[i][j] = operators[i-1][j+1]-operators[i-1][j];
        }
```

```
    }

    terms = new double[n+1];

    for (int i = 0; i < n+1; i++)
    {
        terms[i] = operators[i][0];
    }


    for (int i = 0; i < n+1; i++)
    {
        delete []operators[i];
    }

    delete []operators;


    delete []x;

    return terms;
}


double derivative1(double n, double h, double x0)
{
    double* terms;
    double value;
    double absolute_error=0;
    double relative_error=0;


    cout<<"-----------FIRST DERIVATIVE-----------"<<endl;
        value = 0;
        terms = difference_operators(n, h, x0);

    for (int i = 0; i < n; i++)
    {
        value += pow(-1,i)/(double)(i+1)*terms[i];
    }

        value = 1/h*value;


    cout<<"f'(x0) analytical: "<<function_derivative1(x0)<<endl;
    absolute_error=fabs((function_derivative1)(x0)-value);
    relative_error=fabs(absolute_error)/(fabs((function_derivative1)(x0)));

    cout<<"Absolute error "<<absolute_error<<endl;
    cout<<"Relative error "<<relative_error<<endl;

    delete []terms;

    cout<<"f'(x0) numerical: "<<value<<endl;
    cout<<endl;
    return value;
}

double derivative2(double n, double h, double x0){
    double* terms;
    double value=0;
    double absolute_error=0;
    double relative_error=0;
    cout<<"-----------SECOND DERIVATIVE-----------"<<endl;
    double* b;
    double* a;

    a = new double[n];
    a[0] = 1;

    for (int i = 1; i < n; i++)
    {
```

```
        b = new double[i+1];

        for (int j = 0; j < i+1; j++)
        {
            b[j] = pow(-1,j)/(j+1);
        }

        a[i] = 0;

        for (int j = 0; j <= i; j++)
        {
            a[i] += b[j]*b[i-j];
        }
        delete []b;
    }

    terms = difference_operators(n+1, h, x0);

    for (int i = 0; i < n; i++)
    {
        value += a[i]*terms[i+1];
    }

    value = 1/(h*h)*value;

    cout<<"f''(x0) analytical: "<<function_derivative2(x0)<<endl;
    absolute_error=fabs((function_derivative2)(x0)-value);
    relative_error=fabs(absolute_error)/(fabs((function_derivative2)(x0)));

    cout<<"Absolute error "<<absolute_error<<endl;
    cout<<"Relative error "<<relative_error<<endl;

    delete []terms;
    delete []a;
    cout<<"f''(x0) numerical: "<<value<<endl;
    return value;
}

int main()
{
    double n, x0, h;

    cout<<"Give point x0: ";
    cin>>x0;

    cout<<endl<<"Give steps n: ";
    cin>>n;

    cout<<endl<<"Give fixed step h: ";
    cin>>h;
    cout<<endl;

    derivative1(n, h, x0);
    derivative2(n, h, x0);


    return 0;
}
```

3. Conclusions:

The overall (real) error of the method is as follows:

$$R^{(1)}_{n,h}(x) \approx \frac{\Delta^{n+1} f(x)}{n!h}$$

We can notice that in case of first derivative the overall error should decrease with increase a difference step *h* and with increase of difference operators *n*. Unfortunately together with increase the step and difference operators, values of the difference terms $\Delta^i f(x)$also increases which affects of increase the error. To obtain the best result in given point, the iterative method need to be used to get the most optimal *h.*
The error changes significantly when it comes to find second derivative. The algorithm returned the unusefull result for each function for very small step h= 10^{-9}, 10^{-12}. The error is also inaccurate, but acceptable when the step is too big h=0.1.
From testing tables above we can noticed that error for first and second derivative is the smallest when the step is between 10^{-3}, 10^{-6} .