To appear in *Soft Computing*, special issue on Formal Systems and Music, G. Assayag, V. Cafagna, M. Chemillier (eds.)

Periodic musical sequences and Lyndon words

Marc Chemillier

Abstract When one enumerates periodic musical structures, the computation is done up to a cyclic shift. This means that two solutions which are cyclic shifts of one another are considered the same. Lyndon words provide a powerful way to do so. We illustrate this by two examples taken from African traditional music.

Keywords Infinite periodic words, Lyndon words, traditional music from Central Africa, cyclic forms

1 Introduction

A formal system is a system of symbols together with rules to combine them into sequences of symbols which are considered as meaningful. One of the first assumptions underlying this approach is that one can recognize that two sequences of symbols are identical (for instance, among the three sequences *abc*, *abc*, *cba*, the first two are identical). A very particular type of sequence of symbols is the one obtained by combining several copies of the same sequence in such a way that this sequence is repeated endlessly, as in *abcabcabc*... Such a sequence is called "periodic".

Periodic structures are a fundamental feature of music. This fact is illustrated in classical music by well-known forms such as the chaconne. But this feature appears to be much more important in traditional music, for instance in Central Africa, where most of the repertoires are based on cyclic forms.

From a musical point of view, one can generally consider that two periodic sequences which only differ by a finite number of elements at the beginning are basically the same periodic sequence. In fact, you cannot distinguish them if you do not hear their first notes. It is possible to formalize this idea by defining an equivalence relation on periodic sequences called "conjugacy relation".

The study of periodic musical phenomena may sometimes lead to the computation of periodic structures sharing some specific property. In this case, the problem is to compute solutions which are not conjugate one with the other. There is a concept in combinatorics on words, named Lyndon words, which can help in selecting one representative among conjugacy classes. Roughly speaking, a Lyndon word is defined as a finite word which is minimal for the lexicographic order in its conjugacy class. The properties of these words have been studied in details [12], and there exists an efficient linear algorithm to compute them [8].

In this article, we show how Lyndon words can be used in the study of periodic musical structures, and we illustrate this by two examples taken from African traditional music.

2 Infinite periodic words

An *infinite word* is a function u from $\mathbf{N}$ to a given set A called the alphabet. It is said *periodic* if there exists an integer m satisfying

$$u(n + m) = u(n)$$

for any integer n. We shall say in this case that u is m-*periodic*, and we define the *period* of u as the least integer m such that u is m-periodic. In this paper, we denote by Per_A the set of infinite periodic words over A, and by $Per_A(m)$ the set of infinite periodic words with period m. They are both subsets of the set of all infinite words over A which is usually denoted by A^ω.

A *finite word* over the alphabet A is a finite sequence of elements of A. The set of all finite words over A is denoted by A^*, and we denote by $|w|$ the length of a finite word w, by $|w|_a$ the number of symbols equal to a in w, and by ε the empty word.

For each finite word u of length m, the infinite periodic word with period m, which has u as its finite prefix of length m, is denoted by u^ω using the ω-notation. The function associating u with u^ω maps bijectively A^* onto Per_A.

The cyclic shifts of a finite word are defined by introducing the permutation δ of A^* such that

$$\delta(ax) = xa, \quad a \in A, x \in A^*, \qquad \delta(\varepsilon) = \varepsilon.$$

The *cyclic shifts* of w are the words of the form $\delta^k(w)$ for any integer k. Finite words which are cyclic shifts of one another are called *conjugate*, and the conjugacy relation is an equivalence relation on A^*.

The notion of conjugacy may be extended to infinite periodic words. For any infinite word u, the translated word Tu is defined by

$$Tu(n) = u(n + 1)$$

for any integer n. We shall say that two infinite periodic words are *conjugate* if and only if one is the translated of the other, and this defines an equivalence relation on infinite periodic words. The infinite periodic words equivalent to a given one u are the translated $T^k u$ for any integer k.

There is an obvious relation between translated m-periodic infinite words, and the cyclic shifts of their prefix of length m. It can be written for every u^ω in Per_A and for any integer k

$$T^k(u^\omega) = \delta^k(u)^\omega.$$

3 Computation of Lyndon words

When one wants to compute periodic musical structures satisfying some specific property, two conjugate sequences are considered as the same solution. It is thus necessary to limit the computation to only one representative for each conjugacy class. The natural way to do so is to use Lyndon words, as we shall see in this section.

Let us denote by the same symbol ~ the conjugacy relations on both Per_A and A^* (this convention is justified by the fact that $u^\omega \sim v^\omega$ in Per_A if and only if $u \sim v$ in A^*, as we have seen in the previous section).

Definition 1 For any subset R of A^*, we define a *cross-section* of R for the conjugacy relation as a set T satisfying the following conditions
(1) two elements of T cannot belong to the same conjugacy class,
(2) any element of R is conjugate to at least one element of T,
(3) any element of T is conjugate to at least one element of R.
A similar definition is given for any subset R of Per_A.

Here, we make use of an ordered alphabet A. Thus one can define the traditional lexicographic order on finite words of A^*. A finite word y is a power of another word x if $y = x^k$ for some integer k. For instance $ababab = (ab)^3$ is a power of ab.

Definition 2 A *Lyndon word* is a finite word which is minimal for the lexicographic order in its conjugacy class of A^*, and which is not a power of another word.

For instance, *aaab* is the Lyndon word of the conjugacy class which contains the words *aaab*, *aaba*, *abaa*, and *baaa*. The minimal element being unique, the set of Lyndon words is a cross-section for the conjugacy relation on finite words. This can be extended to infinite periodic words, by taking w^ω for all Lyndon words w. The resulting set of infinite periodic words is a cross-section for the conjugacy relation on Per_A.

From a practical point of view, the computation of Lyndon words may be quite efficient. Jean-Pierre Duval has given an algorithm which gives the sorted list of all Lyndon words of length less than an integer n [8]. This algorithm is reproduced in Fig. 1. Let a be the smallest letter of the ordered alphabet A and M be the larger one. Furthermore, for every letter x excepted M, $s(x)$ is the letter that immediately larger than x in A. We denote by $w[1...n]$ an array of letters of dimension n.

```
w[1] ← a
i ← 1
repeat
    for j = 1 to n–i
        do w[i+j] ← w[j]
    /* at this point, w[1...i] is a Lyndon word */
    i ← n
    while i > 0 and w[i] = M
        do i ← i–1
    if i > 0 then w[i] ← s(w[i])
```

11 **until** $i = 0$

Fig. 1. Duval's algorithm for the computation of Lyndon words of length less than n

The simplicity and efficiency of this algorithm are incredible. It is far beyond the scope of this article to give a full explanation of the algorithm. It would require theoretical properties of Lyndon words which are presented in [8], and the reader is referred to it for more details. Here we only give the trace of the computation for $n = 4$, in order to show how it works.

Table 1 gives the successive values of i during the computation, the corresponding values of array w in line 6 of the algorithm, and the associated Lyndon words $w[1...i]$ which appear in lexicographic order.

i	w	Lyndon words length ≤ 4
$i = 1$	*aaaa*	*a*
$i = 4$	*aaab*	*aaab*
$i = 3$	*aaba*	*aab*
$i = 4$	*aabb*	*aabb*
$i = 2$	*abab*	*ab*
$i = 3$	*abba*	*abb*
$i = 4$	*abbb*	*abbb*
$i = 1$	*bbbb*	*b*

Table 1. Trace of Duval's algorithm for $n = 4$

4 Musical applications of Lyndon words

The idea to use Lyndon words for the computation of periodic musical structures is quite natural. The musical applications of Lyndon words fall into two categories. In the first one, the periodic structure applies to time. In the second one, the periodic structure is considered outside time, as it is the case in the study of pitch structures such as modes. The examples that we shall study in this article belong to the first category. In the same category, one can also find works such as those of Harald Fripertinger on periodic rhythmic canons [10].

In the category of periodic structures outside time, one generally considers sequences of intervals instead of notes. It is the case in the American theory of *pitch class sets*, where the notion of "normal order" is close to the definition of Lyndon words, although not identical. A "normal order" for any pc set, e.g. any ordered set of pitches from the chromatic scale between 0 and 11 where 0 represents the note C and 11 the note B, is defined by Allen Forte [9] as one of its cyclic shifts which has

> "the least difference determined by subtracting the first integer from the last" ([9] page 4).

Since two cyclic shifts may have the same value for the difference, Forte adds a "Requirement 2" when the previous condition gives many solutions

> "If the least difference of first and last integers is the same for any two permutations, select the permutation with the least difference between first and second integers. If this is the same, select the permutation with the least difference between the first and third integers, and so on" ([9] page 4).

We observe that this second condition corresponds to the definition of a Lyndon word among the cyclic shifts of the sequence of intervals. But in fact, it is not taken into account when the first condition gives a unique permutation. Thus it follows that Forte's normal order does not always correspond to the Lyndon word. For instance, the pc set denoted as 4-10 in Forte's table has the normal order [0, 2, 3, 5], corresponding to the notes C, D, D#, F. The sequence of intervals is 2 1 2 7 (the last interval is from 5 to 12, since 0 and 12 represent the same pc corresponding to the note C). The associated Lyndon word is 1 2 7 2, but it was not chosen as a normal order in this case, because the first condition gave 2 1 2 7 as a unique solution.

4 Length decreasing functions

The general idea developed in the present article can be summarized as follows: when one wants to compute a cross-section of a given set R of infinite periodic words over the alphabet A with regard to the conjugacy relation, one first computes the Lyndon words of A^*, and then one check whether their periodic counterparts in Per_A are conjugate to sequences of R or not.

We introduce the additional notion of a "length decreasing function" associated with an auxiliary alphabet B, which can in certain cases improve the process.

Definition 3 We shall say that a function f from A^* to B^* is *length decreasing* if $f(x)$ is a word of B^* which is shorter than the corresponding word x of A^*

$$|f(x)| < |x|.$$

The idea is to replace the computation of Lyndon words of A^* by the computation of Lyndon words of B^*. This is possible when the function f is "compatible" with the conjugacy relation, in a sense that we shall make more precise. When f is also length decreasing, the computation becomes more efficient since $f(x)$ is shorter than the corresponding word x of A^*.

A mapping f from a subset R of A^* to B^* is said to be *compatible* with the conjugacy relation if and only if for any $x, y \in R$, $f(x) \sim f(y)$ is equivalent to $x \sim y$. In this case, Proposition 1 proves that in order to get a cross-section of R, it suffices to compute the Lyndon words w of B^*, and for those which are equivalent to words of $f(R)$, to choose a word in $f^{-1}(w)$.

Proposition 1 *If f is a mapping from a subset R of A^* to B^* which is compatible with the conjugacy relation, then it suffices to choose a word in $f^{-1}(w)$ for any Lyndon word w of B^* equivalent to a word of $f(R)$, to get a cross-section of R for the conjugacy relation.*

Proof: We first show that two words x and x' obtained in this way cannot belong to the same conjugacy class of R. If it were so for two words $x \in f^{-1}(w)$ and $x' \in f^{-1}(w')$, then $x \sim x'$ would imply $f(x) \sim f(x')$ according to the compatibility hypothesis, thus $w \sim w'$ and since w, w' are both Lyndon words, $w = w'$.

We now show that every class of R is represented once. In fact, for any element y of R, let c_y be the class of y. Furthermore, let w be the Lyndon word equivalent to $f(y)$ in B^*, and x the chosen word in $f^{-1}(w)$. One has $f(x) \sim f(y)$, which implies by hypothesis $x \sim y$, so that x is a representative for the class c_y.

A similar proposition may be stated for infinite periodic words, the notion of "compatible mapping" being extended in a natural way to the conjugacy relation on Per_A and Per_B.

Proposition 2 *If f is a mapping from a subset R of Per_A to Per_B which is compatible with the conjugacy relation, then it suffices to choose a infinite periodic word in $f^{-1}(w^\omega)$ for any Lyndon word w of B^* such that w^ω is equivalent to an element of $f(R)$, to get a cross-section of R for the conjugacy relation.*

When the mapping f is length decreasing, Proposition 1 shows that it is possible to replace Lyndon words of A^* by shorter Lyndon words over the alphabet B, the computation of these being faster because of their smaller length. In the case of infinite periodic words, we can introduce a similar notion of length decreasing function. Then Proposition 2 adapts the same idea to the case of infinite periodic words.

Definition 4 We shall say that a function f from Per_A to Per_B is *length decreasing* if the period of $f(w)$ is strictly less than the period of w.

It seems that this method involving an *ad hoc* length decreasing mapping f is a general technique, which applies to different musical situations. We have encountered two examples of this type, in music formalization researches made on African traditional repertoires. The first one deals with harp melodic canons played by Nzakara people from Central African Republic. The second one deals with asymmetric rhythmic patterns which can be found in many cultures in Central Africa.

5 Two applications dealing with African traditional music

5.1 Computation of Nzakara melodic canons

The first musical example involving a mapping which is compatible with the conjugacy relation is related to ethnomusicological researches I have made on harp music from Nzakara people of Central African Republic. Each piece of poetry sung with the accompaniment of the harp relies on a short harp formula which is repeated endlessly as an *ostinato*. The traditional repertoire contains many such formulas, which were strongly related to the political organization of the former Nzakara kingdom.

Some of these formulas have a quite regular structure, the strings being plucked by pairs, one with another. The formula can thus be considered as the superimposition of two melodic lines, one made with the upper note and the other one made with the lower note of each pair. Fig. 2 shows an example of such a formula (time is on the horizontal axis, and points indicate which strings are plucked). As one can see, the upper voice is reproduced, with just a few exceptions, in the lower voice, with a delay as indicated by the two broken lines. This formula has the structure of a two-voice melodic "canon" (see [2, 5, 7] for more details, and [3, 4] for audio samples of these harp canons).

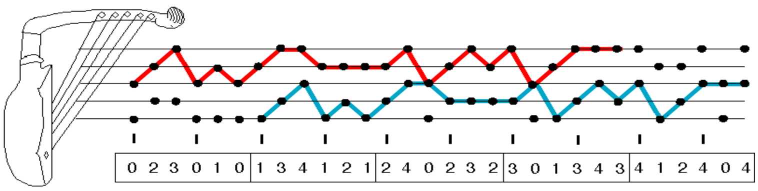

Fig. 2. A canon formula from Nzakara harp music

In order to describe more precisely the structure of these Nzakara harp canons, we consider as an alphabet the set of couples of strings plucked simultaneously, actually played by Nzakara musicians. These couples of strings are indicated in Fig. 2 by integer. One can verify that there are only five of them, numbered from 0 to 4. Thus the alphabet is equal to the set {0, 1, 2, 3, 4}.

If we look closely at the harp formula shown in Fig. 2, we observe that it has a quite regular structure. In fact, it can be factorized into five words $w = v_0v_1v_2v_3v_4$, each of them being obtained from the previous one by adding the same value to its elements, until we reach the initial word again.

v_0 = 0 2 3 0 1 0,
v_1 = 1 3 4 1 2 1,
v_2 = 2 4 0 2 3 2,
v_3 = 3 0 1 3 4 3,
v_4 = 4 1 2 4 0 4.

For instance, 1 added to each element of v_0 = 0 2 3 0 1 0 gives v_1 = 1 3 4 1 2 1, and so on. This appears to be a general construction in the Nzakara repertoire, since every harp canon is based on the same principle which consists in translating a given word several times, by adding the same value to its elements.

In order to give the appropriate mathematical treatment to this construction, we shall introduce an operation on the alphabet corresponding to the "translation" described above. This leads to identify the alphabet {0, 1, 2, 3, 4} with the finite Abelian group $\mathbf{Z}/5\mathbf{Z}$.

More generally we shall denote by G a finite Abelian group, and we study the set Per_G of infinite periodic words over G taken as an alphabet. The group operation defined on G allows one to define the difference word.

Definition 5 We define the *difference word* Du of an infinite word u of Per_G by

$$Du(n) = u(n + 1) - u(n)$$

for any integer n.

In the case of Nzakara harp canons, the infinite periodic word u corresponding to the formula given Fig. 2 as period $m = 30$. The fact that this formula is based on a word v_0 followed by its successive translations v_1, v_2, v_3, v_4 can be expressed by the fact that *the period of Du divides the period of u*. Actually, one can verify that Du has period $r = 6$. The following proposition is taken from [7]. It has a converse part which is restricted to a very special case, and the interesting fact is that Nzakara harp formulas precisely fall into this case.

Proposition 3 *If two infinite periodic words u and v are conjugate, then Du and Dv are conjugate. Conversely, if Du and Dv are conjugate, and the period of Dv divides the period of v in such a way that the quotient of the two periods is equal to card*(G)*, then u and v are conjugate.*

For every Nzakara harp formula having the same structure as the one reproduced in Fig. 2, the quotient of the periods of u and Du is always equal to 5, which is the cardinal of the group $G = \mathbf{Z}/5\mathbf{Z}$. Let R be the subset of Per_G containing infinite periodic words such that their period is equal to the product of card(G) and the period of their difference word. We define a mapping f as the restriction of the operator D to the set R. Proposition 3 may be rewritten as

Proposition 4 *The mapping f from the subset R of Per_G to Per_G, associating to each infinite periodic word its difference word, is a length decreasing mapping compatible with the conjugacy relation.*

Thus we want to compute a cross-section of the set $R_m = R \cap Per_G(m)$ of infinite words u with period $m = \text{card}(G) \times r$, where r is the period of Du. Then $f(R_m)$ is included in $Per_G(r)$, and $f(R_m) = S_r$, where S_r is the subset of infinite words v of $Per_G(r)$ satisfying the following condition

(5.1) the sum of the elements occurring in the prefix of length r of v is equal to an element of order card(G)

(see [7] for details). Note that if this condition is true, then it is true for any conjugate infinite word. In the Nzakara case, the sum can be any non-zero element of G, since card(G) = 5 is prime.

Proposition 4 shows that for each Lyndon word of length r such that the corresponding infinite periodic word v satisfies condition (5.1), we can choose a word u in $f^{-1}(v)$. Then the set of corresponding infinite periodic words with period m is a cross-section of R_m for the conjugacy relation.

Thus Proposition 4 gives an efficient algorithm to compute a cross-section of R_m. In fact, words in $f(R_m) = S_r$ have a much shorter period than those in R_m. For instance, in the example above, $m = 30$ and $r = 6$, so that the computation of Lyndon words of length 6 is much faster than the computation of those of length 30.

5.2 The rhythmic oddity property

Many musical traditions all over the world have asymmetric rhythmic patterns based on two different durations of two and three units (these patterns are sometimes called "aksak rhythm", such as the Turkish pattern 2 2 2 3). In central Africa, there is a particular type of asymmetric patterns, satisfying a property called "rhythmic oddity property" discovered by Simha Arom [1].

Let us consider the Aka pygmies pattern represented as a circle on Fig. 3. The property asserts that one cannot break the circle into two parts of equal length whatever the chosen breaking point. There is always one unit lacking on one side.

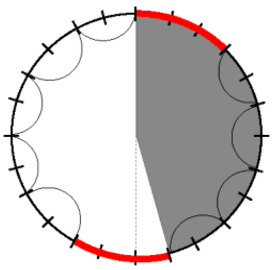

Fig. 3. Aka pygmies pattern 3 2 2 2 2 3 2 2 2 2 2
with no breaking point giving two parts of equal length

The asymmetry of the pattern is to some extent intrinsic, in the sense that there exists no breaking point giving two parts of equal length. Note that the oddity property requires that the circle is divided into an even number of units. We have described in [6] an algorithm for enumerating all the patterns satisfying the rhythmic oddity property.

In this section, the alphabet is $A = \{2, 3\}$. For any finite word u in A^*, we define the *height* of u, denoted by $h(u)$, as the sum of its symbols.

Definition 6 A finite word w of A^* satisfies the *rhythmic oddity property* if and only if
(1) $h(w)$ is even, and
(2) no cyclic shift of w can be factorized into words uv such that $h(u) = h(v)$.

The notion of rhythmic oddity has received different mathematical treatments. Rachel Hall has proposed a generalization to periodic rhythms formed from notes with arbitrary durations (not restricted to 2- or 3-unit notes) [11]. Godfried Toussaint has studied new methods for rhythm classification based on measures of rhythmic oddity and off-beatness [13].

Our construction for the computation of words satisfying the rhythmic oddity property (see [6]) relies on two functions a and b from $A^* \times A^*$ into itself defined by

$$a(u, v) = (3u, 3v), \qquad b(u, v) = (v, 2u).$$

Considering the set B^* of finite words over the alphabet $B = \{a, b\}$, we identify the concatenation of words with the composition of functions. Thus any word α of B^* is identified with a function from $A^* \times A^*$ into itself. We proved the following characterization.

Proposition 5 *A word w satisfies the rhythmic oddity property if and only if there exists a unique word $\alpha \in B^*$ with $|\alpha|_b$ being odd, such that $w = uv$ or $w = vu$ with $(u, v) = \alpha(\varepsilon, \varepsilon)$.*

Let S, S' be subsets of A^* defined by

$$S = \{uv, \exists\alpha \text{ unique} \in B^*, |\alpha|_b \text{ odd}, (u, v) = \alpha(\varepsilon, \varepsilon)\},$$
$$S' = \{vu, \exists\alpha \text{ unique} \in B^*, |\alpha|_b \text{ odd}, (u, v) = \alpha(\varepsilon, \varepsilon)\}.$$

Proposition 5 implies that the set R of words of A^* satisfying the rhythmic oddity property is equal to

$$R = S \cup S'.$$

Note that words of S' are cyclic shifts of words of S.

We define a mapping f from S to B^* associating to each word $w = uv$ the corresponding word $\alpha = f(w)$ of B^*. This is possible since α is unique by definition of S. It can be established that for any $w, w' \in S$, $f(w')$ is a cyclic shift of $f(w)$ if and only if w' is a cyclic shift of w. Furthermore, f is length decreasing because the number of 3 in w is equal to twice the number of a in α, and the number of 2 is equal to the number of b. Then one has the following result.

Proposition 6 *The mapping f from the subset S of A^* to B^*, associating to each word $w = uv$ of S the corresponding word α such that $(u, v) = \alpha(\varepsilon, \varepsilon)$, is a length decreasing mapping compatible with the conjugacy relation.*

Thus we want to compute a cross-section of the set $R = S \cup S'$ of words of A^* satisfying the rhythmic oddity property. One can notice that $\alpha(\varepsilon, \varepsilon) = \beta(\varepsilon, \varepsilon)$ implies $\alpha = \beta$, which means that as soon as α exists satisfying Proposition 5, then α is unique. This implies that f is surjective from S onto the subset of all words of B^* with an odd number of b. Furthermore, if $f(w) = f(w')$, then $(u, v) = (u', v')$ where $w = uv$ and $w' = u'v'$, thus $w = w'$. This means that f is injective, so that f is a bijection from S to the set of words of B^* with an odd number of b.

Proposition 6 shows that for each Lyndon word α of B^* with an odd number of b, we can take the word $f^{-1}(\alpha) = uv$ such that $(u, v) = \alpha(\varepsilon, \varepsilon)$ to get a cross-section of S for the conjugacy relation. But since every word of S' is a cyclic shift of a word of S, it follows that the resulting subset is also a cross-section of R for the conjugacy relation.

As in the previous musical example, f is length decreasing. It follows that we obtain an efficient algorithm to compute a cross-section of R. In fact, words in $f(S)$ have a shorter length than those in S. For instance, considering

$$w = 3333332, \qquad f(w) = aaab,$$

one has $|w| = 7$, whereas $|f(w)| = 4$.

In [6], we published a table giving the numbers of patterns satisfying the rhythmic oddity property, depending on the numbers n_2, n_3 of durations of two and three units respectively. We used a model of the problem as a constraint satisfaction problem designed by Charlotte Truchet. Louis-Martin Rousseau has implemented this model in an ILOG solver. The maximal value that we obtained was 4389, corresponding to $n_2 = 17$, $n_3 = 12$. Using Duval's algorithm implemented in Common Lisp, the computation of a bigger table up to $n_2 = 21$, $n_3 = 16$ took only about 15 minutes, and it gave the following new values.

	17	19	21
12	4389	7084	10966
14	14421	25300	42288
16	43263	82225	148005

Table 2 Additional values for the number of odd rhythms ($n_2 \geq 17$, $n_3 \geq$ 12)

6 Conclusion

We have seen in this paper that Lyndon words provide a natural tool for enumerating periodic structures. When a musical structure is periodic, there is generally no criteria for distinguishing two different cyclic shifts, that is to say two elements belonging to the same conjugacy class. Thus one needs a tool for associating a unique representative to each conjugacy class. This tool is provided by Lyndon words, which can be computed efficiently thanks to Duval's algorithm..

Furthermore, when a musical structure is not only periodic, but also satisfies some additional specific property, it follows that one has to make a selection among the Lyndon words (rejecting those which do not represent conjugacy classes satisfying the additional property). It appears that in some cases discussed in this paper, an efficient way to do so is to introduce a function associated with an auxiliary alphabet, which is compatible with the conjugacy relation. Therefore the enumeration of the representatives of the conjugacy classes can be done on the auxiliary alphabet. If the function is also length decreasing, then this is much faster because of the smaller length of the Lyndon words on the auxiliary alphabet. This increases the efficiency of the enumeration process. It is probably a general framework that applies to many different musical situations, and which could be illustrated by examples taken from the formalization of other kind of periodic musical structures.

References

1. Arom S (1991) African Polyphony and Polyrhythm. Cambridge Univ Press

2. Chemillier M (1995) La musique de la harpe, in: É. de Dampierre (Ed.), Une esthétique perdue. Presses de l'École normale supérieure, Paris, 99-208

3. Chemillier M, de Dampierre É (1996) Central African Republic. Music of the former Bandia courts. Recordings, texts and photographs by M. Chemillier & É. de Dampierre, Paris, CNRS/Musée de l'Homme, Le Chant du Monde, CNR 2741009

4. Chemillier M (1999) three notices from the CD Song of the river. Harps of Central Africa. Paris, Cité de la musique, CM001

5. Chemillier M (2002) Ethnomusicology, Ethnomathematics. The Logic Underlying Orally Transmitted Artistic Practices, in: G. Assayag, H.G. Feichtinger, J.F. Rodrigues (eds), Mathematics and Music. Diderot Forum, European Mathematical Society, Berlin, Springer Verlag, 161-183

6. Chemillier M, Truchet C (2003) Computation of words satisfying the rhythmic oddity property (after Simha Arom's work). Information Processing Letters **86:5** 255-261

7. Chemillier M (2003) Synchronization of musical words. Theoretical Computer Science **310:** 35-60

8. Duval JP (1988) Génération d'une section des classes de conjugaison et arbre des mots de Lyndon de longueur bornée. Theoretical Computer Science **60:** 255-283

9. Forte A (1973) The Structure of Atonal Music. Yale University Press, NY

10. Fripertinger H (2001) Enumeration of non-isomorphic canons. Tatra Mt. Math. Publ. **23:** 47 - 57

11. Hall R, Klingsberg P (2004) Arom's Rhythmic Oddity Property, Rhythmic Tiling Canons, and Burnside's Lemma, submitted abstract

12. Lothaire M (1983) Combinatorics on Words. Addison-Wesley, Reading, MA

13. Toussaint G (2003) Classification and Phylogenetic Analysis of African Ternary Rhythm Timelines. Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science, Granada, Spain, 25-36