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Theory and Applications of Hazard Plotting for Censored Failure Data

WAYNE NELSON

General Electric Research and Development Center
Schenectady, New York

This paper presents theory and applications of a simple graphical method, called hazard plotting, for the analysis of multiply censored life data consisting of failure times of failed units intermixed with running times on unfailed units. Applications of the method are given for multiply censored data on service life of equipment, for strength data on an item with different failure modes, and for biological data multiply censored on both sides from paired comparisons. Theory for the hazard plotting method, which is based on the hazard function of a distribution, is developed from the properties of order statistics from Type II multiply censored samples.

KEY WORDS

Graphical Analysis
Censored Life Data
Competing Failure Modes
Hazard Plotting
Reliability

1. INTRODUCTION

Data plotting has long been used for display and interpretation of data because it is simple and effective. This article presents applications and theory for a simple plotting method, called hazard plotting. It was recently developed to handle multiply censored life data consisting of times to failure on failed units intermixed with running times, called censoring times, on unfailed units, as depicted in Figure 1A. Such life data are quite common and result from (i) removal of units from use before failure, (ii) loss or failure of units due to extraneous causes, and (iii) collection of data while units are still operating.

Herd [7, 8] and Johnson [9, 10] present a method for plotting multiply censored data on probability paper, but it is laborious. Hazard plotting gives the same information but with much less labor and provides useful insight on certain problems. While helpful, a knowledge of probability plotting is not essential for one to follow this presentation of hazard plotting methods and theory.

The hazard plotting method was developed to handle multiply censored life data such as shown in the first example below, and much of the description and theory is conveniently expressed with terminology from life data analysis. However, the method is also applicable to many other types of data multiply censored on the right. The second example shows such an application to strength data.

The next section provides the theoretical basis for hazard plotting papers. The following three sections contain three examples illustrating different uses of the hazard plotting method. The first example concerns life of generator field windings

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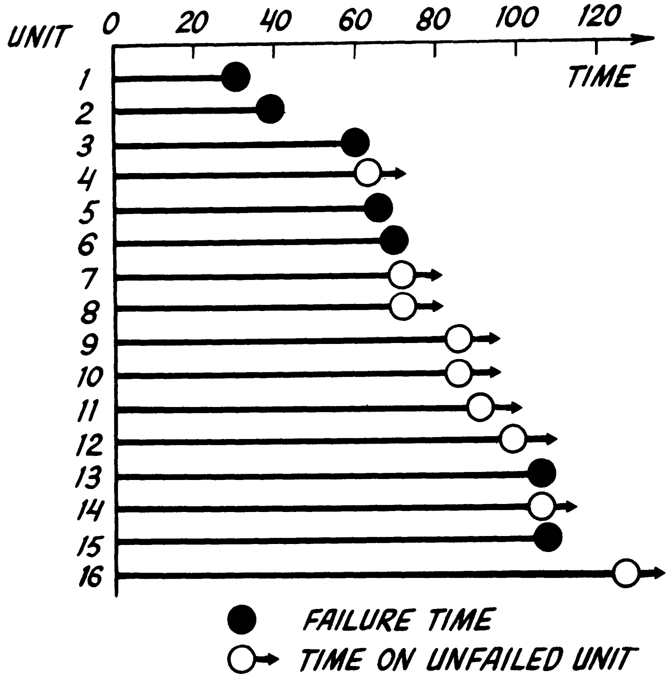

FIGURE 1A—Winding Life Data

and illustrates the basic hazard plotting method for multiply censored data. The second example concerns the breaking strength of circuit connections and illustrates the use of the method on data with different failure modes. The third example concerns a biological experiment with paired comparisons and illustrates use of the method for data multiply censored on the right and the left. The next section contains a discussion on the types of censored data and a theoretical basis for the hazard plotting positions. The final section reviews literature on methods for graphical analysis of multiply censored data.

2. THEORY FOR HAZARD PLOTTING PAPERS

Theory for the hazard plotting papers is presented in this section. First, general theory on hazard and cumulative hazard functions is presented. This theory is then applied to various theoretical distributions to obtain their hazard plotting papers. Readers interested in applications may wish to skip to the next section.

2.1 *Hazard and Cumulative Hazard Functions*

Hazard plotting is based on the concept of the hazard function of a distribution. The basic properties of hazard functions are presented here and then used below to provide a theoretical basis for hazard plotting papers. The following material is worded for convenience in terms of life distributions and time to failure. However, it applies to all distributions and other applications.

The hazard function $h(x)$ for a distribution of time x to failure is defined in terms of the cumulative distribution function $F(x)$, which is assumed to be differentiable, and its derivative, the probability density $f(x)$, by

$$h(x) = f(x)/(1 - F(x)). \tag{1}$$

The hazard function is also called the instantaneous failure rate and force of mortality. It is a measure of proneness to failure as a function of the age of units in the sense that the quantity $\Delta \cdot h(x)$ is the expected proportion of units of age x that will fail in a short time Δ from x to $x + \Delta$. For this reason, it plays a central role in life data analysis.

The cumulative hazard function $H(x)$ of a distribution is the integral of the hazard function up to time x; that is,

$$H(x) = \int_{-\infty}^{x} h(x)\, dx = -\ln (1 - F(x)). \tag{2}$$

Note that the definition of $H(x)$ applies to any distribution—not just life distributions which are bounded below. As explained below, the scales on the papers for the various theoretical distribution functions are constructed so the relationship between a theoretical cumulative hazard function $H(x)$ and time x is linear.

The relationship between the cumulative probability $F(x)$ and the cumulative hazard $H(x)$ for a distribution can be rewritten as

$$F(x) = 1 - e^{-H(x)}. \tag{3}$$

This "basic relationship" between the cumulative probability F and the cumulative hazard H can be seen on all hazard papers. The probability scale on a hazard paper is exactly the same as that on the corresponding probability paper. The cumulative hazard scale is completely equivalent to the cumulative probability scale and serves as a convenient alternative scale in the plotting of multiply censored data.

The cumulative hazard function is an alternative means of expressing a distribution. From the basic relationship, it is easy to determine the properties of cumulative hazard functions from those of cumulative distribution functions. See [16].

2.2 *Theoretical Distributions and Their Hazard Plotting Papers*

Hazard plotting papers have been developed for five commonly used theoretical distributions: the exponential, normal, lognormal, extreme value, and Weibull. The parametric form of these distributions is presented, and their hazard plotting papers are obtained from the general theory above. On hazard plotting paper for a theoretical distribution, the data and cumulative hazard scales are chosen so any such cumulative hazard function is a straight line on the paper.

Exponential distribution. The exponential cumulative distribution function is

$$F(x) = 1 - e^{-x/\theta}, \qquad x \geq 0, \tag{4}$$

where $\theta > 0$ is the mean time to failure. The hazard function is

$$h(x) = 1/\theta, \qquad x \geq 0, \tag{5}$$

and is constant over time. The cumulative hazard function is

$$H(x) = x/\theta, \qquad x \geq 0, \tag{6}$$

and is a linear function of time. This can be rewritten to express time to failure as a function of the cumulative hazard H; namely,

$$x(H) = \theta H. \tag{7}$$

This shows that time to failure x is a linear function of H that passes through the origin. Thus, exponential hazard paper is square grid paper. Its probability scale is given by the basic relationship (3). The value of θ is the time for which $H = 1$ (i.e., 100%); this fact is used to estimate θ from data plotted on exponential hazard paper.

Normal distribution. The normal distribution function is

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right), \qquad -\infty < x < \infty, \tag{8}$$

where μ is the distribution mean and may have any value, σ is the distribution standard deviation and is positive, and $\Phi(\cdot)$ is the standard normal cumulative distribution function. The normal hazard function is

$$h(x) = (1/\sigma)\varphi\left(\frac{x-\mu}{\sigma}\right) \Big/ \left[1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right], \qquad -\infty < x < \infty, \tag{9}$$

where $\varphi(\cdot)$ is the standard normal probability density. This failure rate is an increasing function of time. Its reciprocal for $\mu = 0$ and $\sigma = 1$ is known as Mills' ratio. The cumulative hazard function is

$$h(x) = -\ln\left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right), \qquad -\infty < x < \infty. \tag{10}$$

This can be rewritten to express time x as a function of the cumulative hazard H, namely,

$$x(H) = \mu + \sigma\Phi^{-1}(1 - e^{-H}) \tag{11}$$

where $\Phi^{-1}(\cdot)$ is the inverse of $\Phi(\cdot)$ and gives the percentiles of the standard normal distribution. By this relationship, time x is a linear function of $\Phi^{-1}(1 - e^{-H})$. Thus, on normal hazard paper, a cumulative hazard value H is located on the cumulative hazard scale at the position $\Phi^{-1}(1 - e^{-H})$ and the time scale is linear (see Figure 3). The probability scale is used to estimate the parameters μ and σ by the methods used with normal probability paper.

Lognormal distribution. The lognormal cumulative distribution function is

$$F(x) = \Phi\left(\frac{\log(x) - \mu}{\sigma}\right), \qquad x > 0, \tag{12}$$

where $\log(x)$ is the base 10 logarithm, $\Phi(\cdot)$ is the standard normal distribution function, μ is the mean logarithmic time to failure and may have any value, and σ is the standard deviation of the logarithmic time to failure and must be positive. The hazard function is

$$h(x) = (0.4343/x\sigma)\varphi\left(\frac{\log(x) - \mu}{\sigma}\right) \Big/ \left[1 - \Phi\left(\frac{\log(x) - \mu}{\sigma}\right)\right], \qquad x > 0. \tag{13}$$

This failure rate is zero at time zero, increases with time to a maximum, and then decreases back down to zero with increasing time. The cumulative hazard function is

$$H(x) = -\ln\left(1 - \Phi\left(\frac{\log(x) - \mu}{\sigma}\right)\right), \qquad x > 0. \tag{14}$$

This can be rewritten to express time x as a function of the cumulative hazard H, namely,

$$\log(x(H)) = \mu + \sigma\Phi^{-1}(1 - e^{-H}) \tag{15}$$

where $\Phi^{-1}(\ )$ is the inverse of the standard normal distribution function. By this relationship, $\log(x)$ is a linear function of $\Phi^{-1}(1 - e^{-H})$. Lognormal and normal hazard papers have the same cumulative hazard and probability scales. On lognormal paper the time scale is logarithmic, whereas it is linear on normal paper.

The probability scale is used to estimate the parameters μ and σ by the methods used with lognormal probability paper.

Extreme value distribution. The cumulative distribution function of the smallest extreme value distribution is

$$F(x) = 1 - \exp\left[-e^{(x-a)/b}\right], \qquad -\infty < x < \infty, \tag{16}$$

where a is the location parameter and may have any value and b is the scale parameter and must be positive. The parameter a is a characteristic time to failure, since it is the $100 \times (1 - e^{-1}) = 63.2$th percentile of the distribution for any value of the scale parameter. The hazard function is

$$h(x) = \frac{1}{b}\, e^{(x-a)/b}, \qquad -\infty < x < \infty. \tag{17}$$

Thus, the failure rate for the extreme value distribution increases exponentially with time. The cumulative hazard function is

$$H(x) = e^{(x-a)/b}. \qquad -\infty < x < \infty, \tag{18}$$

and is an exponential function of time. This can be rewritten to express time x as a function of the cumulative hazard H, namely,

$$x(H) = a + b \ln(H). \tag{19}$$

This shows that time to failure x is a linear function of $\ln(H)$. Thus, the extreme value hazard paper is semi-logarithmic paper. The probability scale is given by the basic relationship (3). For $H = 1$ (i.e., 100%), the corresponding time x equals a; this fact is used to estimate a from an extreme value plot. The probability scale is used to estimate the scale parameter b by the method used with extreme value probability paper.

Weibull distribution. The Weibull cumulative distribution function is

$$F(x) = 1 - e^{-(x/\alpha)^{\beta}}, \qquad x > 0, \tag{20}$$

where β is the shape parameter and α is the scale parameter and both must be positive. The scale parameter α is a characteristic time to failure, since it is the $100 \times (1 - e^{-1}) = 63.2$th percentile of the distribution for any value of the shape parameter. The hazard function is

$$h(x) = \frac{\beta}{\alpha^{\beta}}\, x^{\beta-1}, \qquad x > 0, \tag{21}$$

and is a power function of time. This failure rate increases with time for shape parameter values greater than 1 and decreases with time for values less than 1. Thus, the Weibull distribution can describe either an increasing or a decreasing failure rate according to whether the value of the shape parameter is greater or less than 1. For $\beta = 1$, the Weibull distribution is an exponential distribution and has a constant failure rate. The cumulative hazard function is

$$H(x) = (x/\alpha)^{\beta}, \qquad x > 0, \tag{22}$$

which is a power function of time x. This can be rewritten to express time as a function of the cumulative hazard, namely,

$$\log(x) = (1/\beta) \log(H) + \log(\alpha). \tag{23}$$

By this relationship, $\log(x)$ is a linear function of $\log(H)$. Thus, Weibull hazard paper is log-log graph paper. The probability scale is given by the basic relationship

(3); see Figure 1B. Weibull and extreme value hazard papers have the same cumulative hazard and probability scales. On Weibull paper the time scale is logarithmic and on extreme value paper it is linear. The slope of the straight line equals $1/\beta$; this fact is used to estimate β with the aid of the shape parameter scale from a Weibull hazard plot. For $H = 1$ (100%), the corresponding time x equals α; this fact is used to estimate α graphically.

3. The Basic Hazard Plotting Method for Multiply Censored Data

The basic hazard plotting method for graphical analysis of multiply censored data is presented in this section. The method is illustrated with service life data on field windings of sixteen generators of a certain type. These data are displayed in Figure 1A and consist of the months in service on failed windings and the months on windings still running. The running and failure times are intermixed because the units were put into service at different times. Answers to two engineering questions were sought from these data. First, do the data support the engineering conjecture that the failure rate of such windings increases with their age? If so, preventive replacement of old windings is called for. Second, what is the failure probability of a generator's windings before its next scheduled maintenance? Such information is useful to a utility in assessing the risk of failure if replacement is deferred.

3.1 *Steps in Making a Hazard Plot*

Suppose that life data on n units (16 windings here) consist of the failure times for the failed units and the running (censoring) times for the unfailed units. Order the n sample times from smallest to largest as shown in Table 1 without regard to

Table 1
Hazard Calculations

REVERSE RANK	TIME	HAZARD	CUM. HAZARD
16	31.7 *	6.25	6.25
15	39.2 *	6.67	12.92
14	57.5 *	7.14	20.06
13	65.0		
12	65.8 *	8.33	28.39
11	70.0 *	9.09	37.48
10	75.0		
9	75.0		
8	87.5		
7	88.3		
6	94.2		
5	101.7		
4	105.8 *	25.00	62.48
3	109.2		
2	110.0 *	50.00	112.48
1	130.0		

* FAILURE TIME

whether they are censoring or failure times. Label the times with reverse ranks; that is, the first time is labeled n, the second is labeled $n - 1, \cdots$, and the nth is labeled 1. The failure times are each marked to distinguish them from the censoring times, which are unmarked.

Calculate the hazard value for each failure as $100/k$, where k is its reverse rank. The hazard value is the observed conditional probability of failure, since 1 out of k units failed in passing through that age. The hazard values for the winding failures are shown in Table 1. For example, for the winding failure at 70.0 months, the reverse rank is 11 and the corresponding hazard value is $100/11 = 9.09\%$.

Calculate the cumulative hazard value for each failure as the sum of its hazard value and the hazard values of all preceding failure times. For example, for the failure at 70.0 months in Table 1, the cumulative hazard value of 37.48 is the hazard value 9.09 plus the previous cumulative hazard value 28.39. Cumulative hazard values can be larger than 100%. The sample cumulative hazard values provide a nonparametric estimate of the cumulative hazard function of the true distribution.

Choose the hazard paper of a theoretical distribution for time to failure. Hazard papers are available* for the exponential, Weibull, extreme value, normal, and lognormal distributions. A theoretical distribution should be chosen on the basis of engineering knowledge of the life distribution of the units. If engineering knowledge does not suggest a distribution, different distributions can be tried and one that fits the data well could then be chosen.

On the vertical axis of the chosen hazard paper, mark the time scale so as to include the range of the sample failure times. For the windings data, Weibull hazard paper was chosen, and the vertical scale was marked off from 10 to 1,000 months as shown in Figure 1B.

On the chosen hazard paper, plot each failure time vertically against its corresponding cumulative hazard value on the horizontal axis. This provides a plot of the sample cumulative hazard function. The plot of the windings failure data is shown in Figure 1B.

If the plot of the sample times to failure is reasonably straight on a hazard paper, one may conclude that the distribution adequately fits the data. Then by eye fit a straight line through the data points. This is done for the windings failure data in Figure 1B.

The line is an estimate of the relationship between age and the cumulative percentage failing, read from the horizontal probability scale. The straight line is used, as explained below and in [14, 15], to obtain information on the life distribution. If the data do not follow a reasonably straight line, then plot the data on hazard paper for some other theoretical distribution to see if a better fit can be obtained. If no theoretical distribution provides an adequate fit to the data, a nonparametric fit can be obtained by drawing a smooth curve through the data. The curve is then used in the same way as a straight line to obtain information on the distribution.

3.2 *Information from a Hazard Plot*

As shown below, a hazard plot provides estimates of the distribution parameters, the proportion of units failing by a given age, percentiles of the distribution, the behavior of the failure rate of the units as a function of their age, and conditional failure probabilities for units of any age.

* These papers are available from TEAM, Box 25, Tamworth, N. H. 03886.

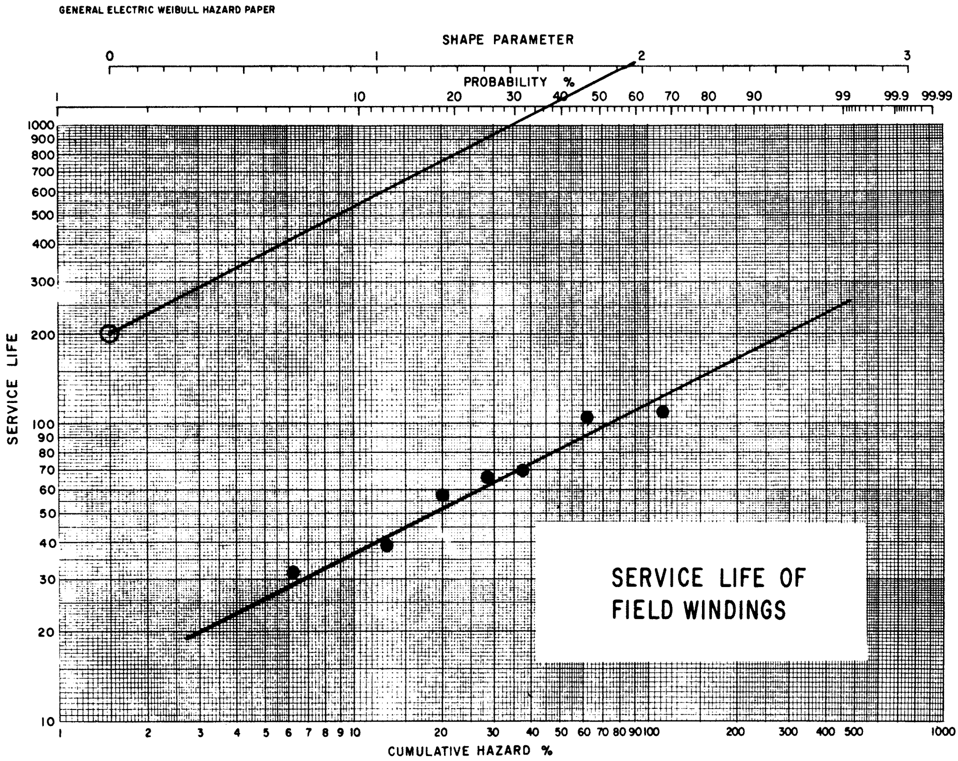

FIGURE 1B—Weibull Hazard Plot of Winding Life Data

The probability scale on a hazard paper is exactly the same as the probability scale on the corresponding probability paper. Thus, a hazard plot is interpreted the same way as a probability plot, and the probability scale on hazard paper is used in the same way as the probability scale on probability paper. The hazard scale is used only as a convenience for plotting multiply censored data. Further discussion and practical details on how to use and interpret hazard and probability plots are given in [12, 14, 15]. Some examples of graphical estimation follow.

Nature of the failure rate. For data plotted on Weibull hazard paper, the following method can be used to assess the nature of the failure rate, a question about the windings. A Weibull distribution has an increasing or decreasing failure rate according to whether the shape parameter value is greater or less than 1, and a value of 1 corresponds to a constant failure rate. Draw a straight line parallel to the plotted data so it passes both through the circled dot in the upper left hand corner of the Weibull hazard paper and through the shape parameter scale as in Figure 1B. The corresponding value on the shape scale is the estimate and is 1.94 for the winding data. The value 1.94 suggests that such windings have a failure rate that increases with age; that is, they have a wear-out pattern and should be replaced at some age. A nonparametric method based on hazard plotting for determining the nature of the failure rate with age is given in [14, 17].

Estimates of parameters, probabilities, and percentiles. For an estimate of the Weibull scale parameter, enter the hazard plot on the cumulative hazard scale at the 100% point, which is the familiar 63.2% point on the probability scale. Go directly up to the fitted line and then horizontally to the time scale to read the estimate of the scale parameter; this is 116 months in Figure 1B. Parameters for other distributions are graphically estimated on hazard papers with the aid of the probability scale by the same methods used on the corresponding probability papers.

An estimate of the population percentage failing by a given age is obtained from the fitted line with the same method used for probability paper. Enter the plot on the time scale at the given age, go horizontally to the fitted line and then up to the probability scale to read the percentage. For example, the estimate of the percentage of windings failing by 24 months is 4.3%.

An estimate of a percentile is also obtained with the probability scale by the same method used for probability paper. Enter the plot on the probability scale at the given percentage, go vertically to the fitted line and then horizontally to the time scale to read the percentile. For example, the estimate of the 50th percentile, representing a nominal life of windings, is 97 months.

Conditional failure probabilities for units of any age. Estimates of failure probabilities described above apply to new units. For units that have accumulated running time, conditional failure probabilities must be used. They provide an answer for the second engineering question concerning the conditional probability of failure of windings of a particular generator in service.

Suppose that an estimate is desired for the probability that windings on a 65 month old generator will fail within 24 months; that is, before 89 months in service. Enter the hazard plot on the time scale at the current age of the unit, 65 months. Go horizontally to the fitted line and then directly down to the cumulative hazard scale to read the corresponding cumulative hazard value, 31.5%. Similarly, obtain the cumulative hazard value, 58.5%, corresponding to the future age, 89 months. Take the difference, $58.5 - 31.5 = 27.0$, between the two cumulative hazard values. Then enter the plot on the cumulative hazard scale at this difference and go directly up to the probability scale to read the estimate of the conditional failure probability, 24%. The theoretical basis for this estimate is given in [15, 16]. Reference [15] explains how one can use such estimates to predict the number of failures in a specified period of time among a mix of units of different ages in service and also to predict when a unit should be removed from service to avoid too large a probability of failure.

3.3 *The Basic Assumption Underlying Hazard Plotting*

The hazard plotting method, like all other methods for analyzing censored failure data, rests on a basic assumption. It is assumed that the censoring is random; that is, if the unfailed units were to run to failure, their failure times would be statistically independent of their censoring times. Said another way, the life distribution of the units censored at a particular age must be the same as the conditional life distribution of the units that run beyond that age. If the assumption is not satisfied, the hazard plotting method may fail to give a valid estimate of the distribution. For example, it is not satisfied if units are removed from service when they look like they are about to fail.

4. Hazard Plotting Analysis of Data with Different Failure Modes

Failure data may include an identification of the mode or cause of each failure. Under certain circumstances, the hazard plotting method can then be used to plot the distributions for individual failure modes or combinations of them (see [17] for various examples of life data analysis). This use of hazard plotting is not limited to life data but can also be used for certain types of strength data as in the following example. These data appear in [12] and are used here with the kind permission of Mr. James R. King, who recognized that the methods in [17] apply to these data.

4.1 *The Method for Making Hazard Plots for Different Failure Modes*

The data shown in Table 2 are a complete sample of breaking strengths of 23 wire connections. The wires are bonded at one end to a semiconductor wafer and at the other end to a terminal post. The type of break is recorded with each strength value as B or W. B denotes a Bond lift, which is a failure of the bond, and W denotes a Wire break, which is a failure of the wire. The engineering problem was to determine from the data how to improve such connections to meet a specification that no more than 1% of the breaking strengths be below 500 milligrams.

Hazard calculations for the complete data are shown in Table 2 in the column labeled "Both", and the plotted data are shown in Figure 2A on normal hazard paper. The plot suggests that the true percentage of connections with strength below 500 milligrams is greater than 1%. The following analyses indicate what improvements are needed.

The plot indicates that the 3150 value and the two 0 values of breaking strength are out of line with the others and are thus suspect. The suspect 3150 value will be discussed in detail later. The zero values are bond failures and can be regarded as bonds that were not made. This indicates that the first step should be to find and eliminate the cause of such bonds.

A plot of the data without the two zero values indicates that the specification

TABLE 2

Hazard Calculations for Connection Strength Data

Breaking Strength	Type of Break	Reverse Rank	Hazard Value	Cumulative Hazard Value Both	W Only	B Only
0	B	23	4.35	4.35		4.35
0	B	22	4.55	8.90		8.90
550	B	21	4.76	13.66		13.66
750	W	20	5.00	18.66	5.00	
950	B	19	5.26	23.92		18.92
950	W	18	5.56	29.48	10.56	
1150	W	17	5.88	35.36	16.44	
1150	B	16	6.25	41.61		25.17
1150	B	15	6.67	48.28		31.84
1150	W	14	7.14	55.42	23.58	
1150	W	13	7.69	63.11	31.27	
1250	B	12	8.33	71.44		40.17
1250	B	11	9.09	80.53		49.26
1350	W	10	10.00	90.53	41.27	
1450	B	9	11.11	101.64		60.37
1450	B	8	12.50	114.14		72.87
1450	W	7	14.29	128.43	55.56	
1550	B	6	16.67	145.10		89.54
1550	W	5	20.00	165.10	75.56	
1550	W	4	25.00	190.10	100.56	
1850	W	3	33.33	223.43	133.89	
2050	B	2	50.00	273.43		139.54
3150	B	1	100.00	373.43		239.54

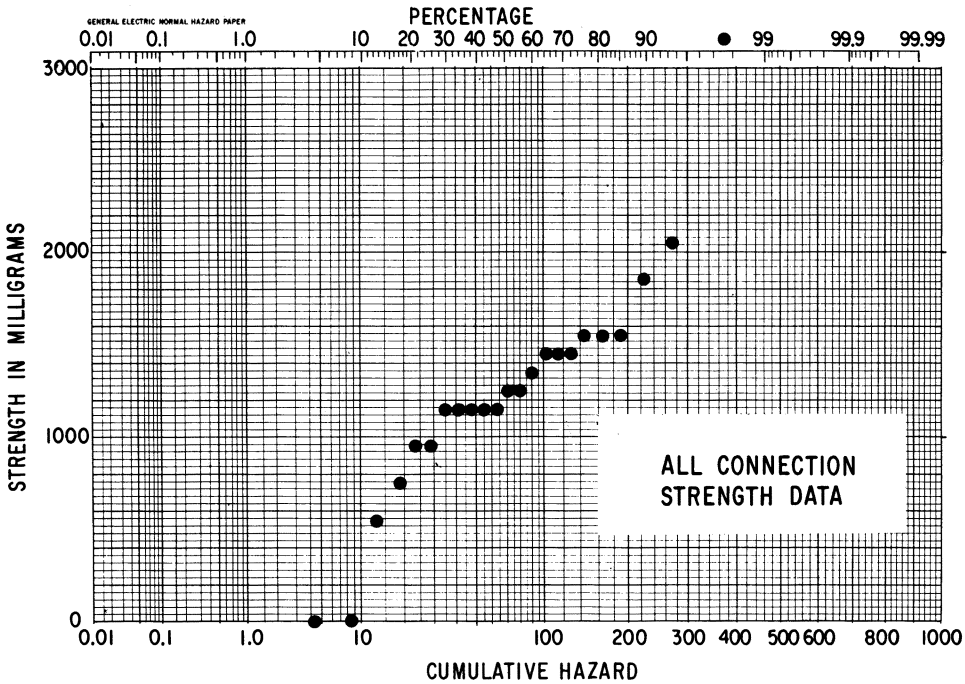

FIGURE 2A—Normal Plot of All Connection Data

will still not be met, when the no bonding problem is eliminated. Thus, it is necessary to look separately at the distribution of wire strength and the distribution of bond strength to determine which needs improvement.

Hazard plotting provides plots for the separate distributions of wire strength and of bond strength. The hazard plotting positions for the wire strengths are shown in Table 2 in the column labeled "*W* Only". They are obtained by treating the bond strengths as censoring values. The wire strengths are plotted against their cumulative hazard values in Figure 2B. This plot of wire strength indicates that about 1% of the wire is below the 500 milligram specification. Thus, the distribution of wire strength is marginal and should be improved by the use of stronger wire.

This analysis is based on the reasoning that each connection has a wire strength and a bond strength, but only the smaller of the two strengths is observed when the connection is tested. For the purpose of analyzing wire strengths, any connection with a bond lift can be regarded as one that was not stressed high enough to produce a wire break. Thus, an observed bond strength is a censoring value and the wire strength for that connection is above it. The hazard plotting method can then be used for such multiply censored data. A connection is regarded as a series system that is as strong as its weakest link.

The hazard plotting positions for the bond strengths are shown in Table 2 in the column labeled "*B* Only". They are obtained similarly by treating the wire strengths as censoring values. The hazard plot of the bond strengths is shown in Figure 2C. This plot of bond strength indicates that 2 or 3% of the bond strengths are below the 500 milligram specification. Thus, the distribution of bond strength is also marginal and should be improved.

The wire strength and bond strength plots indicate that the 3150 value is spurious for the following reasons. If one draws a straight line through the bulk of bond

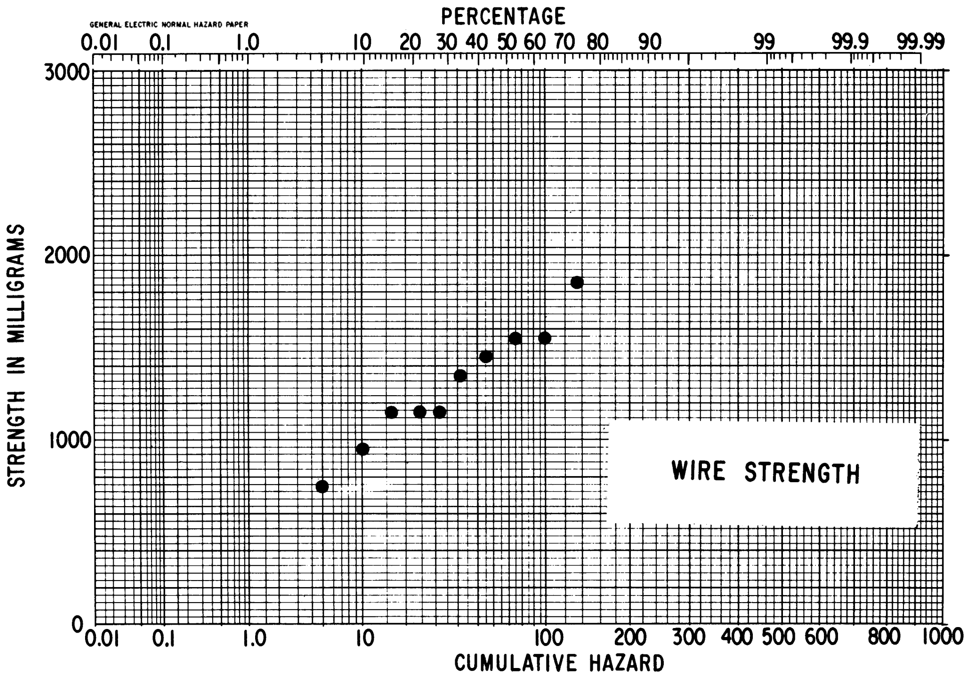

FIGURE 2B—Normal Plot of Wire Strength

strength data in Figure 2C, ignoring the 3150 value, one can see that the 3150 value is at roughly the 99.9% point of the fitted distribution. Similarly, if one draws a straight line through the wire strength data in Figure 2B, one can see that the 3150 value is at roughly the 99.9% point of the fitted distribution. Thus, for that particular connection, the bond strength was roughly at the 99.9% point and its wire strength above that. The unlikeliness of two such high strengths and previous erratic experience with the method of measurement clearly indicate that the value is spurious and should be discarded. No assignable cause for it could be found.

4.2 *The Basic Assumptions Underlying Analysis of Different Failure Modes*

Treating observed values of certain failure modes as censoring values is valid if there are "independent competing failure modes." That is, it is assumed that each unit under observation has a potential observed value for each failure mode, that these values are statistically independent, that the value actually observed is the smallest of them, and that the mode is identified. Failure modes compete in the sense that the failure goes to the mode with the smallest value. In some applications, there may also be censoring; then a unit's smallest value is not observed. The independence assumption can be stated in a weaker form. It is sufficient that the observations from modes treated as censoring values be statistically independent of those treated as failures, but within each of the two groups the values for the different modes need not be independent.

A theoretical basis for analysis of such data, when these assumptions hold, is briefly presented in [17]. The theory is given in terms of the cumulative hazard functions of the different modes and provides an insight into such problems.

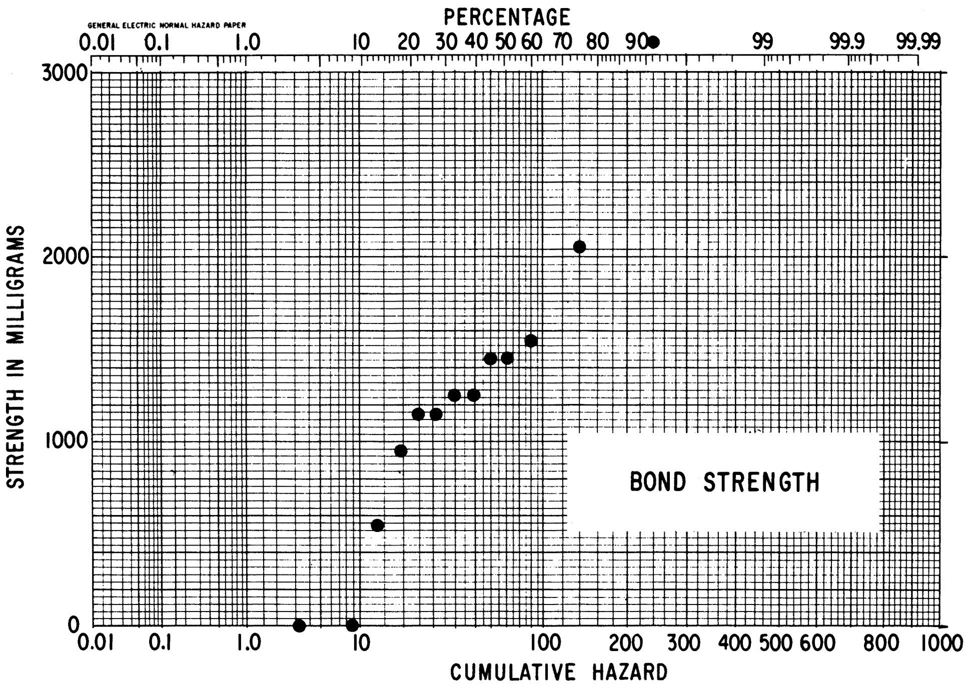

FIGURE 2C—Normal Plot of Bond Strength

5. HAZARD PLOTTING ANALYSIS OF DATA MULTIPLY CENSORED ON BOTH SIDES

The hazard plotting method is applied about to data multiply censored on the right. The method is also applicable to data multiply censored on the left. When such data values are multiplied by minus one, they are reversed and are multiply censored on the right. Then a hazard plot of them can be made. However, the values that should be at the high end of the probability scale were reversed to the low end and vice versa. To compensate for this reversal, it is necessary to reverse the probability scale, that is, to replace a fraction F by its complement $1 - F$. As is shown in the following example, data multiply censored on both sides can be plotted with a combination of the methods for data multiply censored on the right and on the left, when the values censored on the left are all below the values censored on the right.

The method is illustrated with data given by Sampford and Taylor [20]. These data, which are shown in Table 3A, come from a randomized pairs design for an experimental evaluation of a treatment to prolong time to death from a poison. The data are the logarithms of time to death for 17 pairs of rats. One rat of each pair was selected at random to receive the treatment and the other served as a control. Sampford and Taylor analyzed the differences of the paired logarithmic survival times. Observation on each rat was terminated after 960 minutes. At that time three rats were still alive and thus had logarithmic survival times greater than the censoring value log (960) = 2.98. Two of these rats were from the control group. The third was from the treatment group and was not paired with either of the other two survivors. Consequently, two of the differences for these pairs are censored on the left and one is censored on the right. In this situation one is assured that all differences censored on the left are below those censored on the right. In

TABLE 3A
Plotting Positions of Differences

Log Time			Reverse		Cumulative
Treatment	Control	Difference	Order No.	hazard	Hazard
2.01	2.84	-0.83	17	5.88	
2.19	2.76	-0.57	16	6.25	
2.34	2.83	-0.49	15	6.67	(33.63)
2.73	2.98 +	-0.25 -	14	7.14	
2.80	2.98 +	-0.18 -	13	7.69	
2.61	2.73	-0.12	12	8.33	41.96
2.51	2.62	-0.11	11	9.09	51.05
2.65	2.70	-0.05	10	10.00	61.05
2.72	2.76	-0.04	9	11.11	72.16
2.79	2.82	-0.03	8	12.50	84.66
2.90	2.79	0.11	7	14.29	98.95
2.78	2.64	0.14	6	16.67	115.62
2.78	2.48	0.30	5	20.00	135.62
2.98 +	2.68	0.30 +	4		
2.97	2.64	0.33	3	33.33	168.95
2.74	2.31	0.43	2	50.00	218.95
2.96	2.51	0.45	1	100.00	318.95

Table 3A, pluses and minuses are used to denote differences censored on the right and left, respectively. Assuming that the differences of the logarithmic times are normally distributed, Sampford and Taylor tested the hypothesis of no treatment effect by using an approximate t statistic obtained from the maximum likelihood estimate of the mean of the logarithmic differences and its standard error. A hazard plot of these data permits a graphical assessment of the normality assumption and simple graphical estimates of the mean and standard deviation.

A procedure is shown in Table 3A for obtaining cumulative hazard plotting positions for the differences greater than the largest difference censored on the left, -0.18. For the five differences with reverse ranks 17 through 13, the cumulative hazard values cannot be assigned at this point. However, these differences are below the rest of the sample and the total of their hazard values consequently is 33.63. The cumulative hazard values for the other differences can then be obtained in the usual manner starting from 33.63 as shown in Table 3A. These differences are plotted as dots in Figure 3.

A similar procedure is shown in Table 3B for obtaining the cumulative hazard plotting positions for the differences less than the smallest difference censored on the right, 0.30. The data are first reversed. Then, as above, the cumulative hazard values are obtained for the differences greater than the largest difference censored on the left, -0.30. In Table 3B, the cumulative hazard value H' for each observation is converted into a corresponding cumulative probability F' by the basic relationship $F' = 1 - e^{-H'}$. A probability complement $F = 1 - F'$ is calculated for each observation to compensate for the reversal of the data. These F values correspond to a cumulative hazard value $H = -\ln(1 - F)$ as shown in Table 3B. The differences, shown as crosses in Figure 3, are plotted against these cumulative hazard values. The relationshop between corresponding H and H' values is $H = -\ln(1 - e^{-H'})$. This is an Einstein function and is tabulated in [21].

The preceding calculations provide two sets of hazard plotting positions for the data values between the one value censored on the left and the smallest one censored on the right. The small differences between these two plotting positions

TABLE 3B

Plotting Positions of Reversed Differences

Negative Difference $x'=-x$	Reverse Order No.	hazard	Cumulative Hazard H'	Cumulative Probability $F'=1-e^{-H'}$	Plotting Positions: Probability Complement $F=1-F'$	Plotting Positions: Cumulative Hazard $H=-\ln(1-F)$
-0.45	17	5.88				
-0.43	16	6.25	(25.94)			
-0.33	15	6.67				
-0.30	14	7.14				
-0.30	13	7.69	33.63	28.56	71.44	125.32
-0.14	12	8.33	41.96	34.27	65.73	107.09
-0.11	11	9.09	51.05	39.98	60.02	91.68
0.03	10	10.00	61.05	45.69	54.31	78.32
0.04	9	11.11	72.16	51.40	48.60	66.55
0.05	8	12.50	84.66	57.11	42.89	56.01
0.11	7	14.29	98.95	62.82	37.18	46.48
0.12	6	16.67	115.62	68.53	31.47	37.79
0.18 +	5					
0.25 +	4					
0.49	3	33.33	148.95	77.45	22.55	25.55
0.57	2	50.00	198.95	86.32	13.68	14.71
0.83	1	100.00	298.95	94.97	5.03	5.16

can be ignored for practical purposes. The small differences arise because expected hazard plotting positions were used, as is shown later.

The plot shown in Figure 3 indicates that the normality assumption for the analysis appears to be reasonable. The line fitted to the plot corresponds to the maximum likelihood estimates $\hat{\mu} = -0.055$ and $\hat{\sigma} = 0.402$ of the mean and standard deviation of the distribution of differences, which were obtained by Sampford and Taylor. Such a line would ordinarily be fitted to the data by eye.

For multiply censored data where the values censored on the left are not all

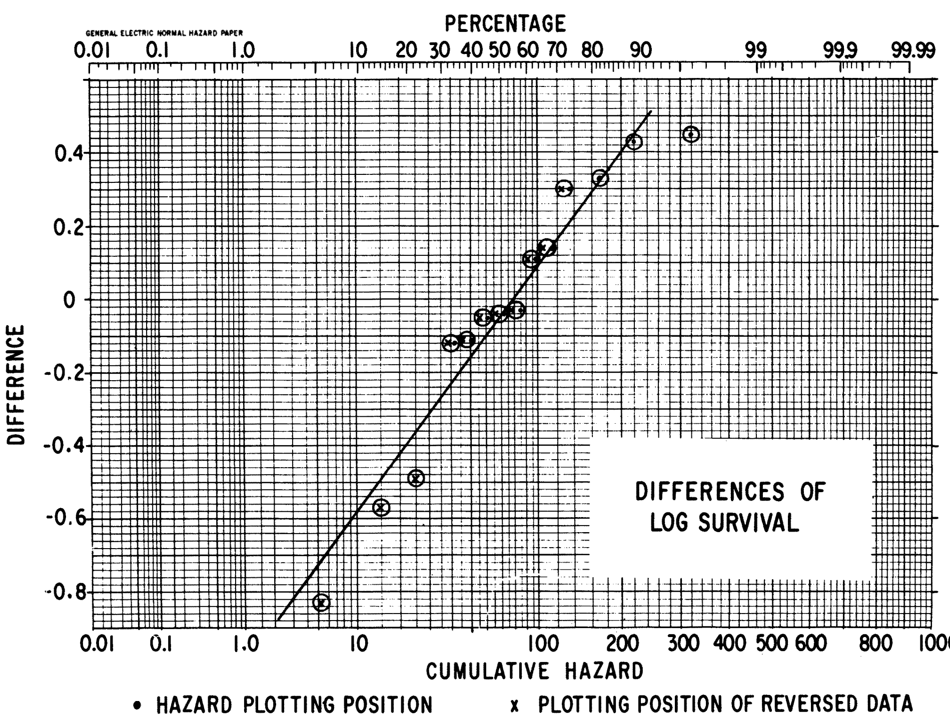

FIGURE 3—Normal Plot of Log Differences

below the values censored on the right, this procedure can be used to plot the observations which are larger than the largest value censored on the left and the observations which are smaller than the smallest value censored on the right. The remaining observations cannot easily be plotted.

6. Theory on Hazard Plotting Positions

This section presents a theoretical basis for the hazard plotting positions. The first part of the section describes various types of censoring and the second part shows the validity of the plotting positions for Type II censoring.

6.1 *Types of Censored Data*

Different types of censoring on the right can arise in failure data, and the type of censoring determines the appropriate method of analysis. The major types of censoring are presented here as background for the second part of this section on the theoretical basis for the hazard plotting positions.

It is first useful to differentiate between planned and unplanned censoring. Planned censoring occurs when data are collected so that the number of failures or the censoring times or some combination of them are completely pre-chosen. Planned censoring arises in the collection of some life test and field data. Unplanned censoring is characteristic of data with competing failure modes and certain field data.

Planned Censoring. There are two basic types of planned censoring, called Type I and Type II censoring. These involve time censoring and failure censoring, respectively. They are a result of pre-planned removal of unfailed units for testing or from analysis of the data before all units have failed.

If all units have the same pre-planned censoring time, the resulting data are called *singly time censored* (Type I) and consist of the times to failure and the number of censored units. In this type of censoring, which is depicted in Figure 4, the common censoring time is pre-chosen and the number of failures is random.

If units are put on test together and unfailed units are removed from test together when a pre-chosen number of failures occur, regardless how long that takes, the data are called *singly failure censored* (Type II). In this type of planned censoring, which is depicted in Figure 4, the number of failures is pre-chosen and the censoring time is random.

Note that, for both the Type I and the Type II censoring described above, there is a single censoring time for all unfailed units. Moreover, all failures occur before that censoring time. Such data are called singly censored to distinguish them from multiply censored data with different censoring times.

If units have different pre-planned censoring times, the resulting multiply censored data are called *progressively time censored* (Type I). In this type of planned censoring, which is depicted in Failure 4, a different censoring time is pre-chosen for each unit and the number of failures is random.

If units are put on test together and pre-chosen numbers of unfailed units are randomly chosen and removed from test immediately after certain pre-specified numbers of failures have occurred, the resulting multiply censored data are called *progressively failure censored* (Type II). In this type of planned censoring, which is depicted in Figure 4, the ranks and number of failures are pre-chosen and the censoring times are random.

More general methods of planned censoring are based on more than one sample.

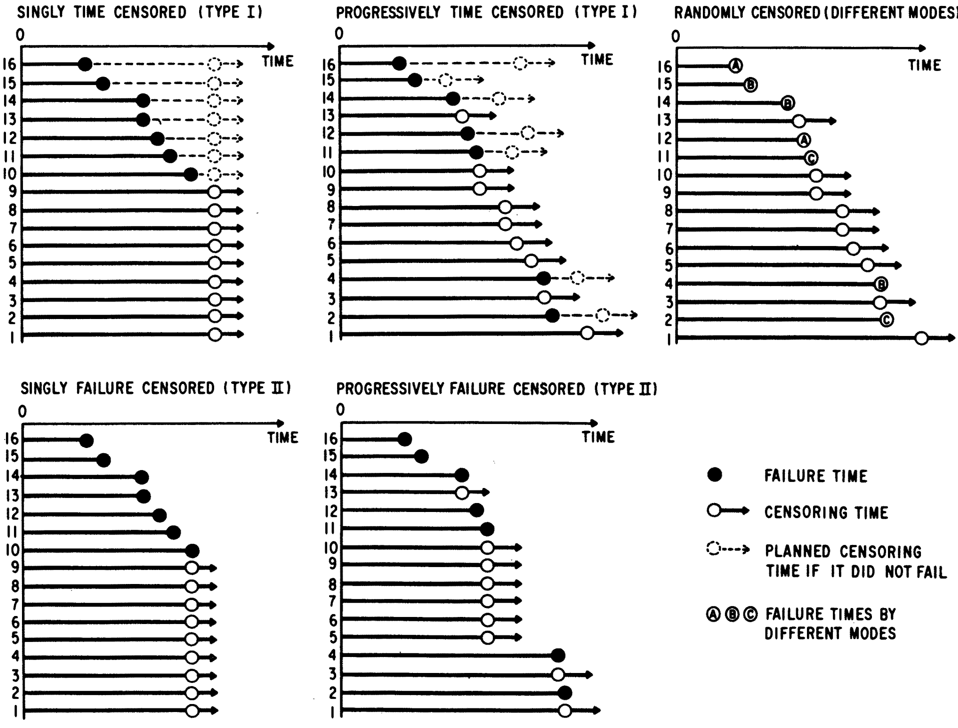

FIGURE 4—Types of Censored Data

With such so-called multisample methods, separate samples are put on test and each is censored with one of the types of censoring given above.

If several samples of units are put on test and each sample is singly or progressively time censored with different pre-determined censoring times, then the data are called *multisample time censored* (Type I). Such data arise, if groups of units are put on test periodically and if the data are analyzed before all units have failed. Then each group has a different censoring time equal to its time on test. The pooled sample is progressively time censored, as defined previously.

If several samples of units are put on test and each sample is singly or progressively failure censored, then the data are *multisample failure censored* (Type II). For example, life testing of certain types of fluorescent bulbs consists of putting on test a daily sample of six bulbs from production and removing the survivors when the fourth failure is observed. The different samples may have different sizes and different censoring patterns. The pooled sample is not progressively failure censored because the ranks of the failures in it are not pre-chosen but depend on the way the observations fall in the different samples.

Unplanned Censoring. Two types of unplanned censoring arise in practice. For one type, the censoring times are determined by chance causes, and for the other, there is no method of data collection in a statistical sense.

Suppose that each unit in the sample can be regarded as potentially having both a random censoring time and a failure time which are statistically independent. Also, suppose that only the smaller of the two such times is observed. Then the data are called *randomly* (*time*) *censored.* For example, if units in use are destroyed accidently, then the unplanned times of destruction may be regarded as random censoring times that are statistically independent of the failure times.

Data with independent competing failure modes as shown in Figure 4 may be regarded as randomly censored. Then each unit is regarded as having a time to failure for each mode of failure and the time to failure for a unit is the smallest of those times. If the failure times for a unit are statistically independent for the different modes and if each unit's mode of failure can be identified by a post mortem, then the failure times for a specific mode may be regarded as observations and the times for other modes may be regarded as random censoring times. Units that may be regarded as series or weakest link systems of components with independent times to failure yield such data. In the example on wire connection strength, the data for either one of the failure modes are randomly censored by the other mode.

For some unplanned multiply censored data, there is no method of data collection in a statistical sense. Little can be done for formal statistical analysis of such data. This is so because any such analysis must be based on and be appropriate to the method used in collecting and censoring the data. The information contained in such unplanned data has to be extracted by a subjective graphical method like hazard plotting or formal analytical methods which are not strictly applicable but are nevertheless useful.

6.2 *Hazard Plotting Positions*

The hazard plotting positions are first motivated below by an analogy with probability plotting positions and then are derived from the theoretical properties of order statistics of progressively failure (Type II) censored samples.

Analogy with Probability Plotting Positions. The scales on probability paper are chosen so a theoretical cumulative distribution function is a straight line. Similarly, on hazard paper, the scales are chosen so a theoretical cumulative hazard function is a straight line. In plotting on probability paper, one plots the sample cumulative distribution function, which approximates the true function. Similarly, in plotting on hazard paper, one plots the sample cumulative hazard function, which approximates the true function. The basis for hazard plotting is further motivated by the following comparison with probability plotting.

For probability plotting of a complete sample of n failure times, each has a corresponding probability of $1/n$. The sample histogram of the failure times, each with probability $1/n$, approximates the probability density of the true distribution. Also, the sample cumulative distribution function, based on the sum of the probabilities, approximates the true one, which is the integral of the probability density. The sample cumulative distribution function is plotted because it smooths the data better than the sample histogram.

Similarly, for hazard plotting of a multiply censored sample of failure times, each has a corresponding conditional failure probability of $1/k$, where k is its reverse rank and $1/k$ is the observed proportion failing among the k units that passed through that age. The histogram of failure times and their conditional failure probabilities approximates the true hazard function. Also, the sample cumulative hazard function, based on the sum of these conditional probabilities, approximates the true one, which is the integral of the hazard function. The sample cumulative hazard function is used because it smooths the data better than the hazard function histogram.

Because of the basic relationship between the cumulative distribution and hazard functions, the plots of the sample cumulative hazard and distribution functions are identical. In fact, the cumulative probability scale on any hazard paper is exactly the same as the one on the corresponding probability paper.

For a complete sample, the sample cumulative distribution function $\hat{F}(x)$, which is the proportion of the sample less than or equal to x, is a nonparametric unbiased estimator for the true cumulative distribution function $F(x)$. It is shown in [16] that there is no corresponding nonparametric unbiased estimator for a cumulative hazard function for a complete sample and hence none for the more general case of a multiply censored sample.

Derivation of Hazard Plotting Positions. A theoretical basis for the reasonableness of the hazard plotting positions is presented next. It is based on the properties of order statistics of Type II progressively censored samples. Motivation for this is given by the following result on probability plotting.

For probability plotting of a complete or singly censored sample of size n from a continuous distribution, a commonly used plotting position for the ith order statistic is $100i/(n + 1)$. This probability plotting position corresponds to the expected value of the probability below the ith order statistic. That is, if $X_1 < X_2 < \cdots < X_n$ are the order statistics of a random sample of size n from a continuous cumulative distribution function $F(\cdot)$, then the probability, $F_i = F(X_i)$, below the ith order statistic is a random quantity and has expectation $EF_i = i/(n + 1)$. A corresponding result is derived below for the cumulative hazard plotting positions for Type II progressively censored samples; namely, the cumulative hazard plotting positions given above are the expected cumulative hazard values.

If it is assumed that a random observation X is from a continuous cumulative distribution function $F(\cdot)$, then the random probability below X is $F = F(X)$ and has a uniform distribution on the unit interval. Under this assumption, the cumulative hazard function $H(\cdot)$ is continuous, and it is easy to show that the corresponding random cumulative hazard value, $H = H(X) = -\ln(1 - F(X))$ has a standard exponential distribution, i.e., with mean equal to one. Thus, the cumulative hazard values for the observations in a random sample are a random sample from a standard exponential distribution, and the corresponding ordered cumulative hazard values $H_1 = H(X_1), \cdots, H_n = H(X_n)$ are order statistics from a standard exponential distribution.

Terminology and notation for progressively failure (Type II) censored samples are needed. The reverse rank of a failure in such a sample is defined as its rank counting backwards in the entire ordered sample of failure and censoring times. If there are M preselected failures in such a sample of size n, their preselected reverse ranks are denoted by $r(1) > r(2) > \cdots > r(M)$ and can have integer values from 1 through n. By definition $r(1) = n$. The number of censoring times immediately after the failure with reverse rank $r(m - 1)$ is $r(m - 1) - r(m) - 1$. The mth failure time with reverse rank $r(m)$ occurs among the $r(m)$ units running at that age. For example, the progressively failure censored sample depicted in Figure 4 has $M = 7$ failures with reverse ranks $r(1) = 16$, $r(2) = 15$, $r(3) = 14$, $r(4) = 12$, $r(5) = 11$, $r(6) = 4$, and $r(7) = 2$.

The following results are based on an easily verified property of order statistics in a progressively failure censored sample from a standard exponential distribution; namely, the successive differences $H_{r(1)} - 0$, $H_{r(2)} - H_{r(1)}$, $\cdots$, $H_{r(M)} - H_{r(M-1)}$, which correspond to the hazard values, are statistically independent and exponentially distributed where $E(H_{r(m)} - H_{r(m-1)}) = 1/r(m)$ for $m = 1, \cdots, M$. Thus, the expected cumulative hazard value for the mth ordered failure is

$$EH_{r(m)} = E(H_{r(1)} - 0) + E(H_{r(2)} - H_{r(1)}) + \cdots + E(H_{r(m)} - H_{r(m-1)})$$
$$= \frac{1}{r(1)} + \frac{1}{r(2)} + \cdots + \frac{1}{r(m)}. \tag{24}$$

This shows that the hazard plotting position for a failure is its expected cumulative hazard value. (The expectations of order statistics of a progressively failure censored sample from an exponential distribution are given by Roberts [19].) The hazard plotting positions thus provide a reasonable nonparametric estimate of the true cumulative hazard function. Also, they are convenient to work with. In comparison, the probability plotting positions given by Herd [7, 8] and Johnson [9, 10] are the expected cumulative probability values, $EF_{r(m)}$, for the order statistics from progressively failure censored samples.

The previous argument for the plotting positions can be extended to a pooled sample of a number of progressively failure censored samples from the same distribution. The sample sizes and censoring may differ from sample to sample. Such data were termed multisample progressively failure censored. The pooled sample is not progressively failure censored, since its censoring is not prespecified but is random and depends on the way the observations fall in the different samples. For such a pooled sample, the conditional expected cumulative hazard values for the order statistics are those given above and are conditional on the observed censoring pattern in the pooled sample.

The derivation of the hazard plotting positions applies only to progressively failure (Type II) censored samples (and hence to complete samples). It is expected that a similar result holds for randomly censored samples. A lemma of Barlow and Proschan [2] would imply that the hazard plotting positions are the expected cumulative hazard values also for progressively time (Type I) censored samples. In a private cummunication, Proschan has confirmed that the lemma does not hold under the assumed progressive (Type I) censoring. The hazard plotting positions would be the expected cumulative hazard values for censoring schemes under which their lemma holds.

It appears that the plotting positions are satisfactory for all types of multiply censored samples. A number of hazard plots of various types of multiply censored Monte Carlo data seem to bear this out [14]. Alternative cumulative hazard plotting positions are presented in [16, 18] and may be better suited to progressively time (Type I) censored data. The Time-Sharing computer program PRPLOT [18] provides data plots printed by teletype, which are based on such modified hazard plotting positions. For most practical purposes, the differences between the various plotting positions appear to be negligible in comparison with the inherent variability in sample data.

7. Literature on Graphical Analysis of Multiply Censored Data

This section contains a short review of the literature on graphical analysis of multiply censored data. Reviews of the literature on analytical methods for such data and of competing failure modes are given in [16, 17]. Papers on methods for analysis of censored data are listed in the bibliographies [3, 6, 13]. However, most of those papers treat singly censored data. Only papers on multiply censored data are mentioned below.

Two basic types of methods for graphical analysis of multiply censored data have appeared in the literature. These are the probability plotting method and the hazard plotting method. The probability plotting methods employ a nonparametric estimate of the cumulative distribution function which is plotted on the probability paper for some parametric distribution. The hazard plotting method employs a nonparametric estimate of the cumulative hazard function which is plotted on the hazard paper for some parametric distribution. A short review of literature on these methods follows.

For analysis of human mortality data, which are usually multiply censored, the actuarial method was developed two hundred years ago to provide an estimate of the cumulative distribution function. A detailed description of this method, its history, and an extensive bibliography emphasizing medical applications are given by Chiang [4]. This method requires grouping of the data by partitioning the range of the data into intervals and provides estimates of the cumulative proportion below the endpoints of the intervals. Then the proportions are plotted against the endpoints to obtain a plot of the cumulative distribution function.

For estimating a cumulative distribution function, Kaplan and Meier [11] give the product limit method, which is the actuarial method in the limit as all intervals go to zero length. They show that the method provides the nonparametric maximum likelihood estimate. The individual failure times can be plotted against the corresponding cumulative probabilities. Herd [7, 8] implicitly gives a similar method for plotting individual failure times. His plotting positions are the expected cumulative probabilities corresponding to the order statistics in a multiply failure (Type II) censored sample. Johnson [9, 10] presents those same plotting positions from a different viewpoint and advocates coverting them into the corresponding median plotting positions. Abbott [1] presents a method for plotting multisample time censored data. It is applicable if units are put on test and removed in groups. For each time when units are censored, the estimate of the proportion failing by that age is the proportion failing among those that were scheduled to be on test at least that length of time. This method may give an estimate of the cumulative distribution function that decreases in the upper tail. Cohen [5] given a method based on grouping the data so the interval endpoints are at the censoring times. The method provides an estimate of the cumulative proportion failing by those times.

The hazard plotting method was developed by the author [14] for analysis of multiply censored field data like those in the first example, which was one of the first applications of the method. Subsequent publications present applications to failure prediction [15], applications to analysis of data with different failure modes [17], and a Time-Sharing computer program, PRPLOT, that produces plots at the teletype terminal [18]. Additional applications are given in those references and in [12]. Ernest Bianco told the author that he had previously developed Weibull hazard plotting and paper, and they appear in his unpublished and now unavailable "Reliametrics Notes". This author subsequently located material by Lalit A. Sarin, an associate of Bianco, on the use of Weibull hazard plotting to obtain an estimate for the Weibull renewal function with a method like the one in this paper. Shooman [22] presents a histogram method for making a sample hazard function. The integral of it is identical to the sample cumulative hazard function obtained with the hazard plotting method. Of course, a plot of the sample cumulative hazard function is more informative than the hazard function, as it smooths the data better and thus provides more accurate graphical estimates of the distribution.

The hazard and probability plotting methods for multiply censored data yield plots that are essentially identical for all practical purposes. Indeed, the probability scales on probability and hazard papers for a distribution are identical. However, it is more convenient to plot multiply censored data against the cumulative hazard scale. The advantages of hazard plotting over the other methods are that the hazard plotting positions are much easier to calculate and that the hazard function is often a more informative way to look at a distribution. For example, hazard plotting makes it easy to see how to handle the second example, which concerns independent competing failure modes.

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