Applications of Perron-Frobenius Theory to Population Dynamics

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POPULATION DEMOGRAPHY

Demography is the study of population in terms of its growth and decay, fertility and mortality, ...

John Impagliazzo, 1984

Models:

Continuous
Alfred J. Lotka [1880 -]

Discrete and Linear
P.H. Leslie, Econmetrica, (1945, 1948)
E.G. Lewis (1941)
H.Bernadelli (1941)

Single sex
(St)age distribution

Age distribution - Leslie Matrix

transition (mortality) matrix,

t_j prob of survival from j to $j+1$

$$0 \le t_j \le 1$$

$$T = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ t_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & t_{n-1} & 0 \end{pmatrix}$$

the fertility matrix

f_j no of exp newbrns indiv age j

$$0 \le f_j$$

$$F = \begin{pmatrix} f_1 & \cdots & f_{n-1} & f_n \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

projection matrix.

$$P = T + F = \begin{pmatrix} f_1 & \cdots & f_{n-1} & f_n \\ t_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & t_{n-1} & 0 \end{pmatrix}$$

$$Px = \begin{pmatrix} f_1 x_1 + \cdots f_n x_n \\ t_1 x_1 \\ \vdots \\ t_{n-1} x_{n-1} \end{pmatrix}$$

Standard model of population demography:

$$x^0 \geq 0$$

$$x^k = Px^{k-1},\ k = 1, 2, \ldots$$

“stable solution”

$$Px = rx,\ x \geq 0, r > 0$$

$$\det(\lambda I - P) = \lambda^n - (c_1 f_1 \lambda^{n-1} + \cdots c_n f_n)$$

$$c_1 = 1,\ c_i = t_1 t_2 \cdots t_{i-1} \geq 0$$

survival prob from age 1 to age $i+1$

$$1 = c_1 f_1/\lambda^{-1} + \cdots + c_n f_n/\lambda^{-n} =: p(\lambda)$$

$p(\lambda)$ is mon decr from ∞ to 0

has unique positive root r

all other roots $\leq$ in modulus

$$1 = c_1 f_1/r \cdots + c_n f_n r$$

corresp eigenvector

$$x = (c_1, c_2/\lambda, \ldots, c_n \lambda^{n-1})^t$$

$$p(1) = c_1 f_1 + \cdots + c_n f_n =: R_0$$

no. newborns from one indiv. in lifetime

net reproductive rate

$$R_0 >=< 1 \Longleftrightarrow r >=< 1$$

GENERAL CASE

$$T \geq 0, \qquad \Sigma_j t_{ij} \leq 1$$

$$t_{ij} \quad \text{prob trans } (j) \to (i)$$

$$\lim_k T^k x = 0, \forall x \Longleftrightarrow \rho(T) < 1$$

$$F \geq 0$$

$$f_{ij} \quad \text{av. new by } (j) \text{ in } (i)$$

$$P = T + F$$

VERY IMPORTANT MODEL

Population = All mathematicians

Classified by no. of papers in year k

Death = leaving the profession

t_{ij} prob that math with j papers in year k has i papers in year $k+1$

f_{ij} prob that in year k math with j papers produces Ph.D. with i papers in year $k+1$

PERRON-FROBENIUS THEORY

***P irreducible*:**

not in form by permutation similarity

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

A_{11} and A_{22} nontrivial

***P primitive* : $P^k > 0$**

P-F Theorem for irreducible matrices

Let P be an irreducible nonnegative matrix. Then

(a) The spectral radius $\rho(P)$ of P is positive and it is an algebraically simple eigenvalue of P (the *Perron root*) with corresponding left and right positive eigenvectors (the *Perron vectors*), which are unique up to scalar multiples.

(b) The spectral radius of P is the unique eigenvalue with a nonnegative eigenvector.

(c) The spectral radius of the matrix P increases (strictly), resp. decreases, if any entry of it increases, resp. decreases.

***Fundamental Theorem of Demography*:**
Let P be the projection matrix of a standard population model $x^k = P^k x^0$, $k = 0, 1, \ldots$. Suppose that P is primitive with spectral radius $\rho(P) = r$ and has left and right Perron vectors v^t and u resp. normalized so that $v^t u = 1$. Then ($\lim_{k\to\infty}(P/r)^k = uv^t$)

$$\lim_{k\to\infty} x^0/r^k = (v^t x^0)u.$$

Consequently, if $|x^k|$ denotes the total population at time k then

$$\lim_{k\to\infty} |x^k| = \begin{cases} 0 & \text{if } r < 1, \\ |(v^t x^0)u| & \text{if } r = 1, \\ \infty & \text{if } r > 1. \end{cases}$$

NET REPRODUCTIVE RATE R_0

$$\rho(T) < 1$$

$$I + T + T^2 + \cdots = (I - T)^{-1}$$

dist of newborns from init dist over lifetime

$$Fx + FTx + FT^2x + \cdots = F(I - T)^{-1}x$$

$$Q := F(I - T)^{-1}$$

Next Generation matrix

$$R_0 := \rho(Q)$$

Model:

q_{ij} **lifetime Ph.D's born with i papers to math born with j papers**

***Stability and comparison theorem*: Assume that a projection matrix $P = T + F$ is irreducible with T and F nonzero. Denote the growth rate $\rho(P)$ by r and the net reproductive rate $\rho(Q)$ by R_0. Then**

$$R_0 > 0,$$

$$\rho(T + F/R_0) = 1,$$

and one of the following holds:

$$1 < r < R_0,$$
$$r = 1 = R_0,$$
$$0 < R_0 < r < 1.$$

cf. Stein-Rosenberg

SOME PROOFS

$R_0 > 0$ follows from

Proposition: Let T and F be nonnegative matrices with $\rho(T) < 1$ and $F \neq 0$. Suppose $T + F$ is irreducible and $Q = F(I-T)^{-1}$. Then, after a permutation similarity,

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & 0 \end{pmatrix},$$

where Q_{11} is a nontrivial irreducible nonnegative matrix, Q_{12} is a nonnegative matrix every column of which has a positive entry, and the 0 rows of Q correspond to the 0 rows of F, if any.

S (1984), Szyld (1985)

$$\boldsymbol{y^t \gneq 0, \quad R_0 y^t = y^t Q = y^t F(I-T)^{-1}}$$

$$\boldsymbol{y^t(T + F/R_0) = y^t}$$

Hence

$$\boldsymbol{\rho(T + F/R_0) = 1}$$

Case $\boldsymbol{R_0 > 1}$

$$\boldsymbol{T + F/R_0 \lneq T + F \lneq R_0 T + F}$$

Hence

$$\boldsymbol{1 = \rho(T + F/R_0) < \rho(T + F) = r}$$

$$\boldsymbol{r < \rho(R_0 T + F) = R_0}$$

INTUITIVE CONTENT

Suppose

$$r = \rho(P) = \rho(T + F) > 1$$

$$P^r x_0 \to \infty$$

Aim: **Stationary Population**

Birth/death control

$$T + F \longrightarrow (T + F)/r$$

Birth control $\uparrow \downarrow$

$$T + F \longrightarrow T + F/R_0$$

Intuitive: Second more radical

$$R_0 > r$$

A GENERALIZATION

Achieving a given growth rate s by scaling F

THEOREM: Let P, T and F satisfy the previous conditions. For $s > \rho(T)$ define

$$q(s) = \rho(F(I - T/s)^{-1})/s.$$

Then $q(s) > 0$. Let $P(s) = T + F/q(s)$. Then its growth rate, $\rho(P(s))$, is s, and its net reproductive rate is

$$R_0(s) = R_0/q(s).$$

Further, one of the following holds:

$$1 = s = R_0(s),$$
$$1 < s < R_0(s),$$
$$0 < R_0(s) < s < 1.$$

Previous Theorem: $\quad s = r, \;\; q(s) = 1$