ON DISCRETE STABLE POPULATION THEORY

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SUMMARY

In studying discrete stable population theory, appeal is made to the fact that the matrix **A** of birth and survival rates has a positive maximal eigenvalue, associated with which is a positive eigenvector. Although this appeal is sometimes made without justification, in other cases it has been stated that a sufficient condition for lim $\mathbf{A}^t$ to exist is that there be a 'kernel' of at least two adjacent age groups with positive fertility rates. In this paper, those properties of **A** which influence the limiting behavior of its powers are examined in some detail; in particular, it is shown that all such matrices are irreducible, and that a necessary and sufficient condition for **A** to be primitive is that the greatest common divisor of the indices of age groups at which fertility is positive be one.

1. INTRODUCTION

In the discrete version of stable population theory, we study the familiar matrix having non-zero elements only in the first row and on the first sub-diagonal, which contain respectively fertility and survival rates for the population under consideration. It is customary to proceed by pointing out that such a matrix has a maximal characteristic (or eigen-) value, associated with which is a positive characteristic (or eigen-) vector, and then to show that this implies that the successive powers of the matrix converge to a limiting form with proportional columns, so that the effects of the initial age distribution on that at succeeding points in time will diminish as time increases, and eventually disappear altogether as the stable age distribution is approximated more and more closely.

The justification given in the literature for these conclusions varies, and is essentially of two types. Thus Keyfitz [1967] provides no proof that the matrix will have a maximal eigenvalue, mentioning only that its characteristic equation can have at most one positive root, while Leslie [1945] states of this positive root that 'excluding the rather special case when the first row... has only a single non-zero element...it will be found that the modulus of this root is greater than any of the others,' a statement which is not true (but which is close enough to the truth probably never to have caused a demographer or a biologist any difficulty). Both these authors then assume the eigenvalues of the matrix to be distinct, and establish the stable result by diagonalizing the matrix. On

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the other hand, Pollard [1966] states without proof the fact that two consecutive non-zero elements in the first row are sufficient to ensure that all powers of the matrix contain only positive elements from a certain point on, while Lopez [1961] proves more generally that this is true for products of increasingly many different matrices of the type studied so long as there is no cyclical fertility with respect to age. The latter authors then cite two theorems from matrix algebra to establish that the stable result holds: the first that such a matrix has a maximal positive eigenvalue, and the second that its powers hence approach a limit.

In this paper, we shall examine some of the properties of the matrix in question in more detail than has previously been given. The presentation will be quite formal, in the hope that it can easily be adapted for other purposes, as perhaps for use in lectures on stable population theory.

2. POPULATION PROJECTION MATRICES

We follow here the usual practice of considering only female individuals whose age is not greater than that at which reproduction is last possible, and of assuming that birth and death rates remain constant over time. It is intuitively obvious that the age restriction involves no loss of generality in the study of limiting age structure, since individuals older than the limiting reproductive age make a contribution only to age groups higher than their own; Lopez [1961] has given a formal proof that if the stable theory holds for the restricted case, then it holds for the entire age structure of the population. With these assumptions, the discrete time-homogeneous model for the growth of a population classified into p groups may be written

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t , \qquad t = 0, 1, \cdots ,$$

where $\mathbf{x}_t$ is a $p \times 1$ column vector and $\mathbf{A}$ a $p \times p$ matrix; because this equation describes the way in which population projections are calculated, we shall call the matrix $\mathbf{A}$ of the equation a population projection matrix. Various requirements for the elements of $\mathbf{A}$ have been given in the literature, but we shall find it convenient to adopt the following ones.

Definition. A $p \times p$ matrix $\mathbf{A} = (a_{ij})$ whose elements satisfy

$$a_{ij} = \begin{cases} b_j , & i = 1, \qquad j = 1, \cdots , p \\ s_i , & i = 2, \cdots , p, \qquad j = i - 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $0 < s_i \leq 1$, $0 \leq b_j$ $(j = 1, \cdots , p - 1)$, $0 < b_p$, will be called a population projection matrix.

Note that we require positive fertility only at the limiting age of reproduction.

We shall show that the Perron–Frobenius theorem applies to population projection matrices, and we shall give a necessary and sufficient condition for such a matrix to be primitive. To this end, we state the required definitions and theorems from matrix algebra, referring to the book of Gantmacher ([1959]

chapter 13) for further development and for proofs. We begin with two definitions and the basic result of Perron and Frobenius.

Definition. Non-negative (positive) matrix. A $p \times q$ matrix $\mathbf{M} = (m_{ij})$ is said to be non-negative, and we write $\mathbf{M} \geq \mathbf{0}$, if each of its elements is non-negative, i.e. if $m_{ij} \geq 0$ for $i = 1, \cdots, p$ and $j = 1, \cdots, q$. If strict inequality $m_{ij} > 0$ holds for all i, j, $\mathbf{M}$ is said to be positive, and we write $\mathbf{M} > \mathbf{0}$.

(The same notation is also used to indicate that a matrix is, respectively, positive semi-definite or positive definite, concepts which do not concern us here.) An interesting subclass of square non-negative matrices is that of the stochastic matrices, which have either all row or all column sums equal to unity, and which have an important role in the study of Markov chains.

Definition. Reducible non-negative matrix. A square non-negative matrix $\mathbf{M} \geq \mathbf{0}$ is said to be reducible if there exists a permutation matrix $\mathbf{P}$ (a permutation matrix is a square matrix with exactly one element equal to one in each row and in each column, and zeros elsewhere) such that

$$\mathbf{PMP}^{-1} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} \end{bmatrix},$$

where $\mathbf{A}$ and $\mathbf{C}$ are square submatrices. If $\mathbf{M}$ is not reducible, it is said to be irreducible.

We remark that the terms 'decomposable' and 'indecomposable' are often used instead of 'reducible' and 'irreducible,' particularly in the study of stochastic matrices.

Theorem 1 (Perron–Frobenius). An irreducible non-negative matrix $\mathbf{M} \geq \mathbf{0}$ has a positive eigenvalue r which is a simple root (i.e. a root of multiplicity one) of the characteristic equation of $\mathbf{M}$, and which is not exceeded by the modulus of any other eigenvalue of $\mathbf{M}$. The eigenvector corresponding to r may be taken positive, and there are not two (linearly) independent positive eigenvectors.

We note that the positive root r is usually called the maximal root of $\mathbf{M}$; it is frequently referred to as the 'Perron root of $\mathbf{M}$,' and denoted by $p(\mathbf{M})$.

Applying these concepts to population projection matrices $\mathbf{A}$, we see at once from its definition that $\mathbf{A}$ is non-negative and square. We shall now show that $\mathbf{A}$ is irreducible, so that the Perron–Frobenius theorem applies to all population projection matrices.

Without loss of generality, we may restrict ourselves to matrices $\mathbf{A}_0$ with $b_1 = b_2 = \cdots = b_{p-1} = 0$ and only $b_p > 0$, since we can easily show that if $\mathbf{A}_0$ is irreducible, then so is a more general matrix

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{B},$$

where $\mathbf{B}$ can have non-zero elements only in the first row. Suppose in fact that this is not so. Then there exists a permutation matrix $\mathbf{P}$ such that

$$\begin{aligned} \mathbf{PAP}^{-1} &= \mathbf{P}(\mathbf{A}_0 + \mathbf{B})\mathbf{P}^{-1} = \mathbf{PA}_0\mathbf{P}^{-1} + \mathbf{PBP}^{-1} \\ &= \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{G} & \mathbf{H} \end{bmatrix}, \end{aligned}$$

where $\mathbf{F}$ and $\mathbf{H}$ are square. But $\mathbf{B} \geq 0$ implies $\mathbf{PBP}^{-1} \geq \mathbf{0}$ (since the inverse of a permutation matrix is again a permutation matrix), and $\mathbf{PA}_0\mathbf{P}^{-1}$ is not of the required form by hypothesis. Hence $\mathbf{A}$ is irreducible, since the addition of the non-negative matrix $\mathbf{PBP}^{-1}$ to a matrix not already of the required form cannot result in a sum of that form.

Furthermore, it is clear that the numerical values of the positive elements of a non-negative matrix have no bearing on its reducibility, so that we can assume all positive elements equal to unity. Hence we need show only that the $n \times n$ permutation matrix

$$\mathbf{U} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & & 1 & 0 \end{bmatrix}$$

is irreducible. But this is known to be so (for a proof, see Marcus and Minc [1964] p. 122), and we conclude that all population projection matrices $\mathbf{A}$ are irreducible.

Hence the theorem of Perron and Frobenius applies to all such matrices, and we can summarize our findings in the following theorem.

Theorem 2. A population projection matrix $\mathbf{A}$ has a positive eigenvalue r which is a simple root of the characteristic equation of $\mathbf{A}$. The eigenvector corresponding to r may be taken positive, and there are not two independent positive eigenvectors.

We remark that we can now assert that the matrices $\mathbf{A}$ have exactly one positive root, since on applying Descartes' rule of signs to the characteristic equation $|\lambda\mathbf{I} - \mathbf{A}| = 0$ we find that the number of positive roots (equal to the number of changes of sign) of the equation is at most one, and the Perron–Frobenius theorem tells us there is at least one. Hence the remaining roots of $\mathbf{A}$ are negative or complex.

The question next arises as to whether the maximal root exceeds the modulus of all other roots, or if it is merely at least as large as their moduli; we shall see that the stable theory holds strictly only in the former case. To answer this question, we shall need the following definitions and theorems.

Definition. Index of imprimitivity. The index of imprimitivity h of an irreducible non-negative matrix $\mathbf{M}$ is the algebraic multiplicity of its eigenvalues of modulus r; i.e. h is the number of eigenvalues λ_i of $\mathbf{M}$ such that $|\lambda_i| = r$.

Definition. Primitive matrix. An irreducible non-negative matrix $\mathbf{M}$ is said to be primitive if the index of imprimitivity $h = 1$.

In the study of stochastic matrices, the terms 'aperiodic' and 'periodic' are used instead of 'primitive' and 'imprimitive,' respectively, and a stochastic matrix which is irreducible (indecomposable) and primitive (aperiodic) is said to be 'regular' or 'ergodic' (more precisely, the corresponding Markov chain is said to be a regular or an ergodic chain).

Theorem 3. The index of imprimitivity h of an $n \times n$ irreducible non-negative matrix **M** is the greatest common divisor of the differences $\{n - n_1, n_1 - n_2, \cdots, n_{s-1} - n_s\}$, where

$$|\lambda \mathbf{I} - \mathbf{M}| = a_0\lambda^n + a_1\lambda^{n_1} + a_2\lambda^{n_2} + \cdots + a_s\lambda^{n_s}.$$

Finally, although we shall not use it, we state the following theorem because it is on this that the discussions of Pollard and Lopez on discrete stable theory rely.

Theorem 4. A square non-negative matrix **M** is primitive if and only if there exists an integer q such that $\mathbf{M}^q > \mathbf{0}$.

We now investigate the conditions under which this root will exceed the modulus of any other root, i.e. the conditions which suffice to ensure that **A** be primitive. To do this, we calculate the characteristic polynomial $|\lambda \mathbf{I} - \mathbf{A}|$ of **A** and then apply Theorem 3. Expanding the determinant by minors of the first row (or by some other method) we find easily that

$$|\lambda \mathbf{I} - \mathbf{A}| = \lambda^p - b_1\lambda^{p-1} - s_1b_2\lambda^{p-2} - \cdots - \prod_{i=1}^{k-1} s_i b_k \lambda^{p-k} - \cdots - s_1 \cdots s_{p-1}b_p .$$

In order that the greatest common divisor of the differences of successive exponents appearing in this equation be one, it is obviously sufficient (since $s_j > 0$, $j = 1, \cdots, p - 1$, by definition) that any two adjacent b_j be positive, i.e. that there exist an index i ($i = 1, \cdots, p$) such that $b_i > 0$ and $b_{i+1} > 0$ (we reduce $i + 1$ modulo p, if necessary, thus regarding b_p and b_1 as adjacent). For applications of the theory to human populations, adopting any of the customary age groupings, this is the most obvious condition to apply since it holds in every case likely to be met in practice. It is well known that it is not, however, a necessary condition, and another sufficient condition is that the differences between successive subscripts of the positive b_j be relatively prime (thus when $p = 5$, $b_2 > 0$ and $b_5 > 0$ suffice to assure primitivity, as do $b_3 > 0$ and $b_5 > 0$).

To find a necessary and sufficient condition for primitivity, we consider again the characteristic equation of **A** given above, and we suppose that there are $m(\leq p)$ groups for which fertility rates b_j are positive, so that $b_j > 0$ for $j = k_1, k_2, \cdots, k_{m-1}, p$, where $k_1 < k_2 < \cdots < k_{m-1} < p$. The exponents of λ appearing in the characteristic equation are then

$$p, p - k_1, p - k_2, \cdots, p - k_{m-1}, 0$$

and their differences are

$$p - (p - k_1) = k_1, k_2 - k_1, \cdots, k_{m-1} - k_{m-2}, p - k_{m-1} .$$

Hence **A** is primitive if and only if the greatest common divisor of the integers $\{k_1, k_2 - k_1, k_3 - k_2, \cdots, k_{m-1} - k_{m-2}, p - k_{m-1}\}$ is one. This condition unfortunately does not have an immediate intuitive appeal, and it is natural to look for an equivalent condition of less cumbersome appearance. Now Lopez [1961] has shown that a sufficient condition for the primitivity of **A** is that the greatest common divisor of the age groups at which fertility is positive be one, and this condition is in fact also necessary. I am indebted to J. E. Delany for the following proof that this is so.

We shall need the following theorem, for a proof of which we refer to Birkhoff and MacLane ([1953] pp. 18–20).

Theorem 5. The greatest common divisor of n integers $x_1, x_2, \cdots, x_n$, not all equal to zero, is the smallest positive integer of the form $\sum_{i=1}^{n} a_i x_i$, where $a_1, \cdots, a_n$ are integers.

With this theorem, it is easy to prove

Theorem 6. A population projection matrix A is primitive if and only if the greatest common divisor of the subscripts j of the positive b_j is one.

In fact, these subscripts are $k_1, k_2, \cdots, k_{m-1}, p$. Then the differences $y_1 = k_1, y_2 = k_2 - k_1, \cdots, y_m = p - k_{m-1}$ are clearly each linear combinations of the k_i, and conversely the k_i can be expressed uniquely as linear combinations $k_1 = y_1, k_2 = y_1 + y_2, \cdots, k_m = y_{m-1} + y_m$ of the y_i. Now let d_1 be the greatest common divisor of the k_i, and d_2 that of the y_i. By Theorem 5, d_1 is the smallest positive linear combination of the k_i with integer coefficients, and d_2 that of the y_i. But each k_i can be expressed uniquely as a linear combination, with integral coefficients, of the y_i, and so d_1 can be represented as a linear combination $\Sigma\, a_i y_i$ of the y_i. Hence $d_2 \leq d_1$. Similarly, $d_1 \leq d_2$, and so $d_1 = d_2$, and the theorem follows (as the proof indicates, this is a special case of a more general result).

The condition of Theorem 6 would seem to aid one's intuitive grasp of the situation since, as Lopez [1961] has pointed out, it may be expressed loosely as 'fertility is not a periodic event with respect to age.' It is important to note, though, that this statment must be construed very strictly, since our intuitive notions of periodicity may not correspond with those of the theorem. For example, if p is odd a matrix with only b_p, b_{p-2}, b_{p-4}, etc. positive is primitive, while if p is even the matrix is imprimitive with index two. Thus, writing x for non-zero elements, the matrix with first row

$$0\ 0\ x\ 0\ x\ 0\ x$$

is primitive, while that with first row

$$0\ x\ 0\ x\ 0\ x$$

is not, and similarly a first row of the form

$$0\ x\ 0\ 0\ x\ 0\ 0\ x$$

provides primitivity, while

$$0\ 0\ x\ 0\ 0\ x\ 0\ 0\ x$$

does not. It seems that, speaking loosely, we should tend to regard each of these patterns as exhibiting periodic fertility with respect to age; here, as elsewhere, it is necessary to verify intuitive results by some other method.

An interesting example of a case in which the requirement of Theorem 6 is not met is provided by the insect population considered by Bernardelli and referred to frequently by others (e.g. see Leslie [1945] or Lopez [1961]). Here

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & b \\ s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{bmatrix}$$

so that the matrix is clearly not primitive, and the maximal root is not larger than the modulus of all other roots. In fact, all roots of this matrix are of the same modulus.

3. DISCRETE STABLE POPULATION THEORY

As we mentioned earlier, if a given population projection matrix is primitive, then appeal can be made to the fact that the powers $\mathbf{A}^t$ approach a limiting form; more precisely, we have

Theorem 7. Let $\mathbf{A}$ be a primitive population projection matrix with maximal eigenvalue r and associated positive eigenvector $\mathbf{z}$. Then $\lim_{t\to\infty} \mathbf{A}^t/r^t = \mathbf{H}$ exists, where $\mathbf{H}$ is a matrix whose columns are positive multiples of $\mathbf{z}$.

To establish that this limit exists, it has been customary in a part of the demographic literature to assume that the remaining eigenvalues of $\mathbf{A}$ are distinct, a condition sufficient to ensure that $\mathbf{A}$ be diagonalizable, i.e. that $\mathbf{A}$ be similar over the complex numbers to a diagonal matrix. That the condition that a primitive $\mathbf{A}$ be diagonalizable is not necessary in order for $\mathbf{A}^t/r^t$ to converge is well known, both in the mathematical and in the demographic literature, as we have pointed out earlier. The reader is referred to Gantmacher ([1959] p. 81) and to Karlin ([1966] pp. 479–80) for two proofs of Theorem 7 for a general primitive matrix.

When Theorem 7 holds, then for any non-negative vector $\mathbf{u} \geq \mathbf{0}$, we shall have

$$\begin{aligned} \mathbf{y} = \mathbf{H}\mathbf{u} &= (m_1\mathbf{z}, \cdots, m_p\mathbf{z})\mathbf{u} \\ &= \mathbf{z} \sum_{j=1}^{p} m_j u_j , \end{aligned}$$

so that the relative values of the components y_i of $\mathbf{y}$ are independent of $\mathbf{u}$, i.e.

$$\begin{aligned} y_i \Big/ \sum_{k=1}^{p} y_k &= z_i \sum_j m_j u_j \Big/ \sum_k \left(z_k \sum_j m_j u_j\right) \\ &= z_i \Big/ \sum_{k=1}^{p} z_k . \end{aligned}$$

Hence in particular

$$\begin{aligned} \lim_{t\to\infty} \mathbf{x}_t/r^t &= \lim_{t\to\infty} (\mathbf{A}^t/r^t)\mathbf{x}_0 \\ &= \mathbf{H}\mathbf{x}_0 = k\xi \end{aligned}$$

and the age distribution vector $\xi_t = \mathbf{x}_t(\Sigma_i\, x_{t,i})^{-1}$ approaches a limiting value ξ regardless of the initial population $\mathbf{x}_0$, and depending only on the matrix $\mathbf{A}$

(A determines $\lim_{t\to\infty} \mathbf{x}_t$ to within a scalar k which depends on both $\mathbf{x}_0$ and A). The limiting vector ξ is called the stable age distribution or stable population vector corresponding to **A**, and the fact that it is independent of $\mathbf{x}_0$ is known as the ergodic property for population growth (the latter term is of course borrowed from other fields than demography).

We summarize the preceding discussion of primitivity and stability in

Theorem 8. Suppose the linear recursive model $\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t$, $t = 0, 1, \cdots$ holds, where **A** is a primitive population projection matrix and $\mathbf{x}_0$ is given. Then $\lim_{t\to\infty} \mathbf{x}_t/r^t = k\xi$ exists, where ξ is independent of $\mathbf{x}_0$ and satisfies $\sum_{i=1}^{p} \xi_i = 1$, and k is a constant which depends on both **A** and $\mathbf{x}_0$.

It can be shown (Gantmacher [1959] p. 81) that if **A** is imprimitive, then $\lim \mathbf{A}^t/r^t$ will not exist, and hence $\mathbf{x}_t$ will not approach the stable distribution corresponding to **A** as a limiting form independently of $\mathbf{x}_0$. Thus the last two theorems are really 'if and only if' rather than just 'if' statements. In fact $\mathbf{A}^t$ (and hence $\mathbf{x}_t$) will behave periodically if **A** is imprimitive, with period of course equal to the index of imprimitivity of **A**. It is important to realize that there is nonetheless in the imprimitive case a stable vector, in the sense of a vector satisfying $\mathbf{A}\xi = r\xi$ with $\Sigma\, \xi_i = 1$, and that the limiting behavior of $\mathbf{x}_t$ can be calculated from the elements of ξ, essentially independently of $\mathbf{x}_0$. In this sense, the distinction between primitive and imprimitive population projection matrices is a bit misleading, since the changes in the formulation of the theorems for the imprimitive case are really rather trivial. We defer discussion of this matter to a later paper, in which we shall examine the imprimitive, or periodic, case in some detail, using methods different from those employed here.

A PROPOS DE LA THEORIE DE LA STABILITE DES POPULATIONS DISCRETES

RESUME

En étudiant la théorie de la stabilité des populations discrètes, on fait appel au fait que la matrice **A** des taux de naissance et de survie a une valeur propre maximale positive, associée à ce que l'on peut appeler un vecteur propre positif. Bien que cela soit fait parfois sans justification, on peut montrer dans d'autres cas qu'une condition suffisante pour qu'il existe une limite pour $\mathbf{A}^t$ est qu'il existe un 'noyau' d'au moins deux groupes d'âges adjacents ayant des taux positifs de fertilité.

Dans ce travail, les propriétés de la matrice **A** qui conditionnent le comportement asymptotique de ses puissances sont examinées en détail; on montre en particulier que ces matrices sont toutes irréductibles et qu'une condition suffisante pour que **A** soit primitive est que le plus grand commun diviseur des indices des groupes d'âge pour lesquels la fertilité est positive soit un.

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