QUERIES AND NOTES

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265 NOTE: **A Generalization of the Logistic Law of Growth**

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SUMMARY

A generalization of the logistic function in which the maximum population size is allowed to vary is described.

Ever since Verhulst [1838, 1845, 1847] introduced the logistic law of growth, much work has been done on application, criticism, and generalization. For examples, see Pearl and Reed [1920], Robertson [1923], Lotka ([1924] 1st Edn.; [1956] 2nd Edn.), Yule [1925], Hotelling [1927], Pearl [1930], Will [1930], Wilson and Puffer [1933], Kostitzin ([1939], pp. 93–101), Nair [1954], Nelder [1961a, b], and Lefkovitch [1966].

A new generalization has been found especially useful by the authors, and is reported here.

Suppose x is the size of an appropriate population at time t. We postulate, as many have done, that the rate of growth $\dot{x}$ of the population is given by

$$\dot{x} = \beta x\left(\frac{\phi(k) - \phi(x)}{\phi(k)}\right). \tag{1}$$

That is, the rate of population growth is assumed to be directly proportional (β = proportionality constant, or 'intrinsic growth coefficient') to the population x and also jointly proportional to a factor which represents the proportion of 'available spaces'. The number of available spaces is determined by a monotonically increasing function $\phi(\cdot)$ of the population size. Here $\phi(\cdot)$ denotes the process of operating on *any* argument $(\cdot)$ with the operator ϕ. If we take $\phi(\cdot)$ to be the identity function (i.e., the function which maps any variable x into itself), so that $\phi(x) = x$, then (1) leads to the ordinary logistic law of growth.

This assumption is tantamount to assuming that the number of empty spaces is precisely proportional to the maximum population size less the number of individuals in the population. This is patently unrealistic. A simple but appealing choice for $\phi(\cdot)$ is the power function $(\cdot)^m$, where $m > 0$. Thus (1) becomes

$$\dot{x} = \beta k^{-m} x(k^m - x^m) . \tag{2}$$

Integration using the Bernoulli equation gives the solution

$$x = \frac{k}{[1 + (k^m x_0^{-m} - 1)e^{-\beta m t}]^{1/m}} . \tag{3}$$

Another approach to the solution has been given by Nelder [1961a]. He notes that $\dot{x}^m/x^m = m\dot{x}/x$ so that (2) becomes the ordinary logistic differential equation in x^m. When $m = 1$, we get the ordinary logistic equation.

One of the criticisms most often leveled at the logistic law concerns the constant k which is the maximum population size according to the basic assumption (1). It seems unreasonable to assume that the maximum population supportable is constant over any long period of time. We will suppose that k, rather than being constant, is a function of time. In other words, the maximum population k which may be accommodated is itself a monotonically increasing function of time. This assumption implies that the maximum population supportable depends upon technological advances in such things as housing and food sources. The spatial requirements per individual become less and less with such advances. Empirically, many such advances have been described adequately by sigmoid growth functions. A need for such an assumption has been indicated for human populations by the fact that extrapolation of the ordinary logistic equation has usually under-estimated the actual population size. An attractive choice for $k(t)$ is given by equation (3). Thus, we may assume that x, the population size, is increasing toward a maximum k which itself is increasing at a slower rate B toward an ultimate population maximum K. So, we may write

$$k(t) = \frac{K}{(1 + \alpha e^{-Bmt})^{1/m}} . \tag{4}$$

Then integration[1] of (2) yields the generalized logistic law of growth:

$$x = K\left[1 + \left(K^m x_0^{-m} - 1 - \frac{\alpha\beta}{\beta - B}\right)e^{-\beta m t} + \frac{\alpha\beta}{\beta - B} e^{-Bmt}\right]^{1/m} . \tag{5}$$

If $m = 1$ and either $\alpha = 0$ or $B \to \infty$ then (5) reduces to the ordinary logistic equation. The case $m = 1$, $B = 0$ leads to the logistic equation but with modified constants.

To illustrate the use of (5) we have fitted the model by least squares to the U. S. Census data 1790 to 1960 (Long [1968]) by use of a FORTRAN program

[1] Details of the integration are available in an unpublished appendix obtainable from the authors.

written by one of us (J. L. S.). We have for the estimated[2] population $\hat{x}$:

$$\hat{x} = \hat{K}[1 + (6.514, 44)(10^{16})e^{-1.799,27t} + (2.940, 09)(10^{12})e^{-0.820,30t}]^{-0.154,64}, \quad (6)$$

where t = (year -1790)/10, and where the least squares estimate $\hat{K}_1$ of the maximum population size is 1,719 million persons. It may be more appropriate (cf. Hotelling [1927]) to choose $\hat{K}$ in such a manner that the fitted curve goes exactly through the latest (1960) census point. This leads to a somewhat higher estimate of the population maximum. In this case $\hat{K}_2$ is 1,751 million persons. Neither estimate should be taken too seriously due to the extreme degree of extrapolation. We may expect somewhat more accuracy in predicting the U. S. population for 1970. This estimate, using K = 1,751, yields an estimated population of 202.7 million persons. For the Orwellian 1984 the predicted population is 242.1 million persons.

We find the estimated index to be given by $\hat{m} = 6.467$, and the two estimated growth constants to be given by $\hat{\beta} = 0.02782$ and $B = 0.01268$ reciprocal years.

Asymptotic confidence regions are easily obtained but are omitted here for sake of brevity.

The example above of U. S. population growth has been chosen for illustrative purposes only. The failure of earlier deterministic models (such as the ordinary logistic) to provide useful predictions of future populations is notorious. The present effort, although more general than the previously studied models, may not necessarily be more successful. However, the mental discipline required to develop various models, and to compare model to data, may be expected to lead eventually to a better understanding of the growth phenomenon. The authors hope that the present paper will make a minor contribution toward that end.

UNE GENERALISATION DE LA LOI DE CROISSANCE LOGISTIQUE

RESUME

On décrit une généralisation de la fonction logistique dans laquelle la taille maxima de la population peut varier.

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[2] We acknowledge with thanks the help of Mr. Chester Kolodziej in carrying out some of the computations.

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266 NOTE: A Method for Obtaining Initial Estimates of the Parameters in Exponential Curve Fitting

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SUMMARY

This note describes a technique for obtaining the initial estimates of the parameters in exponential curve fitting. The method obtains the initial estimates by a least square 'peeling-off' technique. The method is particularly well adapted to decay-type data where the exponentials are well separated on the time scale.

1. INTRODUCTION

This note presents a reasonable computer-oriented technique for obtaining the initial estimates of the parameters in exponential curve fitting. The methoP arrives at the initial estimates by a least square 'peeling-off' technique. The 'peeling-off' procedure is particularly well adapted to decay-type data. The data, however, must be reasonably accurate if one is to obtain convergence with any