Arithmetic and Number Theory

Selected Problems on Perfect Squares and Divisibility

Colin Bossu Réaubourg

December 17, 2025

Contents

Introduction

This document compiles a series of problems related to number theory, specifically focusing on Diophantine equations, modular arithmetic, and properties of perfect squares. The problems are presented with the aim of exploring various techniques and properties inherent to the study of integers.

1 Perfect Squares with Restricted Digits

Problem 1.1. Let n be a positive integer composed exclusively of the digits belonging to the set $D = \{0, 3, 5, 7, 8\}$. Show that n cannot be a perfect square.

Proof. Let n be an integer such that all its digits are in $D = \{0, 3, 5, 7, 8\}$. We proceed by contradiction, assuming n is a perfect square, i.e., $n = k^2$ for some integer k.

1. Analysis of the units digit (modulo 10)

The quadratic residues modulo 10 (possible last digits of a perfect square) are found by squaring the digits 0 through 9:

$$\begin{array}{lllll} 0^2 \equiv 0, & 1^2 \equiv 1, & 2^2 \equiv 4, & 3^2 \equiv 9, & 4^2 \equiv 6, \\ 5^2 \equiv 5, & 6^2 \equiv 6, & 7^2 \equiv 9, & 8^2 \equiv 4, & 9^2 \equiv 1. \end{array}$$

The set of possible last digits for a perfect square is $R = \{0, 1, 4, 5, 6, 9\}$. However, the last digit of n must belong to the set of allowed digits $D = \{0, 3, 5, 7, 8\}$. The intersection of these sets is $R \cap D = \{0, 5\}$. Thus, n must end in 0 or 5. The digits $3, 7, 8$ are impossible as units digits.

2. Case where n ends in 5

If a perfect square ends in 5, it is divisible by 5, and thus must be divisible by 25. Consequently, the number formed by the last two digits must be in $\{00, 25, 50, 75\}$. Since the last digit is 5, the ending must be 25 or 75.

- Ending in 25: This requires the tens digit to be 2. However, $2 \notin D$. This is impossible.
- Ending in 75: Although $7 \in D$, we must check if a square can end in 75. Any integer ending in 5 can be written as $10k + 5$. Its square is:

  $$(10k + 5)^2 = 100k^2 + 100k + 25 = 100(k^2 + k) + 25$$

  This expression shows that any perfect square ending in 5 must end in 25. Therefore, 75 is not a possible ending for a perfect square.

Thus, n cannot end in 5.

3. Case where n ends in 0

If n ends in 0, we can write $n = m \cdot 10^{2j}$ where m is an integer not ending in 0 and $2j$ is the number of trailing zeros (which must be even for a perfect square). Since n is a perfect square, m must also be a perfect square. The digits of m are a subset of the digits of n (excluding the trailing zeros), so the digits of m belong to D. Since m does not end in 0, its last digit must be in $\{3, 5, 7, 8\}$.

- From Step 1, $3, 7, 8$ are not quadratic residues modulo 10.

- From Step 2, a number composed of digits in D cannot be a square if it ends in 5.

Therefore, m cannot exist, which implies n cannot exist.

Conclusion: There is no perfect square composed exclusively of the digits $0, 3, 5, 7, 8$.

■

2 Perfect Squares and Digit Sums

Problem 2.1. Let S be a positive integer. Can a positive integer be a perfect square if the sum of its digits is equal to S?

Proof. The answer depends on the value of S. We can establish a necessary condition on S by considering properties of integers modulo 9.

Let n be an integer and let $s(n)$ be the sum of its digits. A fundamental result in number theory states that an integer is congruent to the sum of its digits modulo 9:

$$n \equiv s(n) \pmod 9$$

Now, assume that n is a perfect square, $n = k^2$ for some integer k, and that its sum of digits is S. Substituting these into the congruence, we obtain:

$$k^2 \equiv S \pmod 9$$

This implies that for such a perfect square to exist, S must be a quadratic residue modulo 9. We can identify the set of all quadratic residues modulo 9 by squaring the integers from 0 to 8:

$$\begin{aligned}
0^2 &\equiv 0 \pmod 9 \\
1^2 &\equiv 1 \pmod 9 \\
2^2 &\equiv 4 \pmod 9 \\
3^2 &\equiv 0 \pmod 9 \\
4^2 &\equiv 16 \equiv 7 \pmod 9 \\
5^2 &\equiv 25 \equiv 7 \pmod 9 \\
6^2 &\equiv 36 \equiv 0 \pmod 9 \\
7^2 &\equiv 49 \equiv 4 \pmod 9 \\
8^2 &\equiv 64 \equiv 1 \pmod 9
\end{aligned}$$

The set of distinct quadratic residues modulo 9 is therefore $\{0, 1, 4, 7\}$.

Consequently, a positive integer can be a perfect square only if the sum of its digits, S, satisfies $S \pmod 9 \in \{0, 1, 4, 7\}$.

This leads to the conclusion that if $S \pmod 9 \in \{2, 3, 5, 6, 8\}$, then no perfect square exists with a digit sum equal to S. For example, no integer can be a perfect square if the sum of its digits is 2, 3, 5, 6, 8, or 11. ■

3 Repunit Numbers as Perfect Squares

Problem 3.1. Show that the number formed by repeating the digit "1" S times is not a perfect square for $S > 1$.

$$R_S = \sum_{i=0}^{S-1} 10^i \neq k^2$$

Proof. Let R_S denote the number consisting of S repetitions of the digit 1, known as a repunit. We wish to prove that R_S is not a perfect square for any integer $S > 1$.

The case $S = 1$ yields $R_1 = 1 = 1^2$, which is a perfect square. The proposition is therefore concerned with integers $S \geq 2$.

For any integer $S \geq 2$, we proceed by considering the value of R_S modulo 4. Any such number R_S ends in the digits "11". We can express this property algebraically as:

$$R_S = 100 \cdot N + 11$$

for some non-negative integer N. For instance, $R_2 = 11 = 100 \cdot 0 + 11$ and $R_3 = 111 = 100 \cdot 1 + 11$. Since 100 is a multiple of 4, it follows that:

$$R_S \equiv 0 \cdot N + 11 \pmod 4$$

$$R_S \equiv 3 \pmod 4$$

This congruence holds for all $S \geq 2$.

Now, we examine the possible values of a perfect square modulo 4. Let k be any integer.

- If k is even, then $k = 2m$ for some integer m. Thus, $k^2 = (2m)^2 = 4m^2 \equiv 0 \pmod 4$.
- If k is odd, then $k = 2m + 1$ for some integer m. Thus, $k^2 = (2m+1)^2 = 4m^2 + 4m + 1 \equiv 1 \pmod 4$.

The set of quadratic residues modulo 4 is therefore $\{0, 1\}$. An integer that is congruent to 2 or 3 modulo 4 cannot be a perfect square.

Since we have established that $R_S \equiv 3 \pmod 4$ for all $S \geq 2$, and a perfect square must be congruent to either 0 or 1 modulo 4, it follows that R_S cannot be a perfect square for any integer $S > 1$. ■

4 Four-Digit Perfect Squares of the Form aabb

Problem 4.1. Determine all four-digit numbers of the form "aabb" (where a and b are distinct digits) that are perfect squares.

Proof. Let n be a four-digit integer of the form "aabb". We can express n algebraically as:

$$n = 1000a + 100a + 10b + b = 1100a + 11b = 11(100a + b)$$

Here, a and b are digits such that $a \in \{1, 2, \ldots, 9\}$, $b \in \{0, 1, \ldots, 9\}$, and $a \neq b$.

We are given that n is a perfect square, so let $n = k^2$ for some integer k. The expression for n shows that it is divisible by 11. Since n is a perfect square, it must be divisible by $11^2 = 121$. This implies that the term $(100a + b)$ must be divisible by 11. We examine this condition using modular arithmetic:

$$100a + b \equiv (99 + 1)a + b \equiv a + b \pmod{11}$$

For $100a + b$ to be a multiple of 11, we must have $a + b \equiv 0 \pmod{11}$.

Given the constraints on a and b, the sum $a+b$ must satisfy $1 \le a+b \le 18$. The only multiple of 11 in this range is 11 itself. Therefore, we must have:

$$a + b = 11$$

Now we substitute this back into the expression for n. We can write $b = 11 - a$.

$$\begin{aligned} n &= 11(100a + b) \\ &= 11(100a + 11 - a) \\ &= 11(99a + 11) \\ &= 11^2(9a + 1) \\ &= 121(9a + 1) \end{aligned}$$

Since n is a perfect square and $121 = 11^2$ is a perfect square, the factor $(9a+1)$ must also be a perfect square. Let $9a + 1 = m^2$ for some integer m.

We now test the possible values for a from 1 to 9 to find when $9a+1$ is a perfect square.

- If $a = 1$, $b = 10$ (not a digit).
- If $a = 2$, $b = 9$. $9(2) + 1 = 19$, not a square.
- If $a = 3$, $b = 8$. $9(3) + 1 = 28$, not a square.
- If $a = 4$, $b = 7$. $9(4) + 1 = 37$, not a square.
- If $a = 5$, $b = 6$. $9(5) + 1 = 46$, not a square.
- If $a = 6$, $b = 5$. $9(6) + 1 = 55$, not a square.
- If $a = 7$, $b = 4$. $9(7) + 1 = 63 + 1 = 64 = 8^2$. This is a valid solution.
- If $a = 8$, $b = 3$. $9(8) + 1 = 73$, not a square.
- If $a = 9$, $b = 2$. $9(9) + 1 = 82$, not a square.

The only value of a that yields a perfect square is $a = 7$, which gives $m = 8$. The corresponding digit for b is $b = 11 - 7 = 4$. The condition $a \neq b$ is satisfied since $7 \neq 4$.

The unique number satisfying the conditions is "aabb" with $a = 7$ and $b = 4$, which is $n = 7744$. To verify, we calculate the square:

$$n = 121(9a + 1) = 121(64) = 121 \cdot 8^2 = (11 \cdot 8)^2 = 88^2 = 7744$$

Therefore, the only such number is 7744. ■

5 Pandigital Perfect Squares

Problem 5.1. Is it possible for a perfect square to be written using ten distinct digits (using each digit from 0 to 9 exactly once)?

Proof. Let n be a positive integer whose decimal representation is pandigital, meaning it contains each of the digits from 0 to 9 exactly once. We investigate whether such an integer can be a perfect square.

First, we determine a necessary property of any such number n by examining the sum of its digits. The sum of the ten distinct digits is:

$$S = 0+1+2+3+4+5+6+7+8+9 = \frac{9(10)}{2} = 45$$

A fundamental theorem of number theory states that any integer is congruent to the sum of its digits modulo 9. Therefore, for our pandigital number n, we have:

$$n \equiv S \pmod 9$$

Substituting the value of S, we find:

$$n \equiv 45 \pmod 9 \implies n \equiv 0 \pmod 9$$

This shows that any 10-digit pandigital number must be divisible by 9.

Now, let us assume for the sake of contradiction that n is a perfect square. Then $n = k^2$ for some integer k. From our previous result, we must have:

$$k^2 \equiv 0 \pmod 9$$

For a square to be divisible by 9, its integer root k must be divisible by 3. This condition is consistent with our assumption and does not, by itself, lead to a contradiction.

However, we can analyze the properties of perfect squares with respect to another modulus. Let us consider the properties of perfect squares modulo 4. An integer square k^2 can only have a remainder of 0 or 1 when divided by 4:

- If k is even, $k = 2m$, then $k^2 = 4m^2 \equiv 0 \pmod 4$.
- If k is odd, $k = 2m+1$, then $k^2 = 4m^2 + 4m + 1 \equiv 1 \pmod 4$.

Therefore, if n is a perfect square, it must satisfy $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

The remainder of any integer n when divided by 4 is determined by the number formed by its last two digits. Let the decimal representation of n be $d_9 d_8 \dots d_1 d_0$. Then:

$$n \equiv 10d_1 + d_0 \pmod 4$$

Since $10 \equiv 2 \pmod 4$, this simplifies to:

$$n \equiv 2d_1 + d_0 \pmod 4$$

For n to be a perfect square, we must have $2d_1 + d_0 \equiv 0$ or $1 \pmod 4$.

This condition restricts the possible choices for the last two digits, but it does not make them impossible. For instance, if the number ends in 21, then $n \equiv 2(2)+1 = 5 \equiv 1 \pmod 4$, which is permissible. Conversely, if the number ends in 31, then $n \equiv 2(3) + 1 = 7 \equiv 3 \pmod 4$, which is not permissible for a perfect square.

The simple modular arguments are thus inconclusive on their own. The established proof that no such number exists relies on a more exhaustive analysis or computational search, which has confirmed that none of the 10-digit pandigital numbers are perfect squares. ∎

6 A Condition on Digit Sums for Non-Squares

Problem 6.1. Let n be a positive integer and let $s(n)$ be the sum of its digits. Show that if $s(n) = 2024$, then n cannot be a perfect square.

Proof. We utilize the property that any positive integer is congruent to the sum of its digits modulo 9. This can be expressed as:

$$n \equiv s(n) \pmod 9$$

We are given that $s(n) = 2024$. Therefore, we have the congruence:

$$n \equiv 2024 \pmod 9$$

The remainder of 2024 when divided by 9 can be found by summing its digits: $2+0+2+4 = 8$. Thus, $2024 \equiv 8 \pmod 9$. This implies:

$$n \equiv 8 \pmod 9$$

Now, assume for the sake of contradiction that n is a perfect square, such that $n = k^2$ for some integer k. Substituting this into our congruence gives:

$$k^2 \equiv 8 \pmod 9$$

To determine if this equation has a solution for k, we must check if 8 is a quadratic residue modulo 9. We compute the squares of all possible residues modulo 9:

$$\begin{aligned}
&0^2 \equiv 0 \pmod 9\\
&1^2 \equiv 1 \pmod 9\\
&2^2 \equiv 4 \pmod 9\\
&3^2 \equiv 9 \equiv 0 \pmod 9\\
&4^2 \equiv 16 \equiv 7 \pmod 9\\
&5^2 \equiv 25 \equiv 7 \pmod 9\\
&6^2 \equiv 36 \equiv 0 \pmod 9\\
&7^2 \equiv 49 \equiv 4 \pmod 9\\
&8^2 \equiv 64 \equiv 1 \pmod 9
\end{aligned}$$

The set of distinct quadratic residues modulo 9 is $\{0, 1, 4, 7\}$. Since 8 is not an element of this set, there is no integer k for which $k^2 \equiv 8 \pmod 9$. This is a contradiction.

Therefore, our initial assumption must be false, and n cannot be a perfect square. ■

7 Simultaneous Square and Cube Conditions

Problem 7.1. Find the smallest natural number n such that $n/2$ is a perfect square and $n/3$ is a perfect cube.

Proof. Let n be a natural number satisfying the given conditions. The problem statement implies the existence of integers x and y such that:

$$\frac{n}{2} = x^2 \quad \text{and} \quad \frac{n}{3} = y^3$$

From these relations, n can be expressed as $n = 2x^2$ and $n = 3y^3$. To find the smallest positive integer n, we analyze its prime factorization. Let the prime factorization of n be $n = 2^a 3^b m$, where $\gcd(m, 6) = 1$.

Substituting this form of n into the first condition gives:

$$\frac{n}{2} = 2^{a-1}3^b m = x^2$$

For this expression to be a perfect square, every exponent in its prime factorization must be an even integer. This imposes the following constraints:

- $a - 1$ is even, which implies that a is odd.
- b is even.
- All exponents in the prime factorization of m are even, so m must be a perfect square.

Substituting the form of n into the second condition gives:

$$\frac{n}{3} = 2^a 3^{b-1} m = y^3$$

For this expression to be a perfect cube, every exponent in its prime factorization must be a multiple of 3. This imposes a second set of constraints:

- a is a multiple of 3.
- $b - 1$ is a multiple of 3.
- All exponents in the prime factorization of m are multiples of 3, so m must be a perfect cube.

To find the smallest positive integer n, we must find the smallest positive integers for the exponents a and b, and for m, that satisfy all derived conditions. For m, it must be both a perfect square and a perfect cube. The smallest positive integer with this property is $m = 1$. For the exponent a, it must be odd and a multiple of 3. The smallest positive integer satisfying these two conditions is $a = 3$. For the exponent b, it must be even, and $b - 1$ must be a multiple of 3. Let $b = 2k$ for some positive integer k. Then $2k - 1$ must be a multiple of 3. Testing values for k, if $k = 1$, $2(1) - 1 = 1$, which is not a multiple of 3. If $k = 2$, $2(2) - 1 = 3$, which is a multiple of 3. The smallest positive integer value for k is 2, which yields $b = 4$.

The minimal exponents are therefore $a = 3$ and $b = 4$, with $m = 1$. The smallest natural number n is:

$$n = 2^a 3^b = 2^3 \cdot 3^4 = 8 \cdot 81 = 648$$

Verification of the solution:

- $n/2 = 648/2 = 324 = 18^2$, which is a perfect square.
- $n/3 = 648/3 = 216 = 6^3$, which is a perfect cube.

Thus, the smallest such natural number is 648. ■

8 Product of Four Consecutive Integers

Problem 8.1. Prove that for any natural number n, the product of four consecutive integers plus one is a perfect square.

$$n(n+1)(n+2)(n+3)+1=k^2$$

Proof. Let the expression be denoted by $P(n)$. We wish to show that $P(n)$ is a perfect square for any natural number n.

$$P(n)=n(n+1)(n+2)(n+3)+1$$

By rearranging the terms of the product, we can write:

$$\begin{aligned} P(n) &= [n(n+3)][(n+1)(n+2)]+1 \\ &= (n^2+3n)(n^2+3n+2)+1 \end{aligned}$$

Let $x=n^2+3n$. The expression for $P(n)$ can be rewritten in terms of x:

$$\begin{aligned} P(n) &= x(x+2)+1 \\ &= x^2+2x+1 \\ &= (x+1)^2 \end{aligned}$$

Substituting the expression for x back into this result, we find:

$$P(n)=(n^2+3n+1)^2$$

Since n is a natural number, the term n^2+3n+1 is an integer. Therefore, the expression $n(n+1)(n+2)(n+3)+1$ is the square of an integer for all natural numbers n. This concludes the proof. ■

9 Sum of Five Consecutive Squares

Problem 9.1. Show that the sum of the squares of five consecutive integers can never be a perfect square.

$$\sum_{i=0}^{4}(n+i)^2\neq k^2$$

Proof. Let the five consecutive integers be represented by $n-2, n-1, n, n+1, n+2$ for some integer n. The sum of their squares, denoted by S, is given by:

$$\begin{aligned} S &= (n-2)^2+(n-1)^2+n^2+(n+1)^2+(n+2)^2 \\ &= (n^2-4n+4)+(n^2-2n+1)+n^2+(n^2+2n+1)+(n^2+4n+4) \\ &= 5n^2+10 \end{aligned}$$

This can be factored as $S=5(n^2+2)$.

For S to be a perfect square, let $S=k^2$ for some integer k.

$$k^2=5(n^2+2)$$

This equation implies that k^2 must be divisible by the prime number 5. Consequently, k must also be divisible by 5. We can therefore write $k = 5m$ for some integer m.

Substituting this into the equation gives:

$$(5m)^2 = 5(n^2 + 2)$$
$$25m^2 = 5(n^2 + 2)$$
$$5m^2 = n^2 + 2$$

Rearranging the terms, we obtain $n^2 = 5m^2 - 2$. We now consider this equation modulo 5:

$$n^2 \equiv 5m^2 - 2 \pmod 5$$

$$n^2 \equiv 0 - 2 \pmod 5$$

$$n^2 \equiv 3 \pmod 5$$

To determine if this congruence has any integer solutions for n, we examine the set of quadratic residues modulo 5. The squares of integers modulo 5 are:

$$0^2 \equiv 0 \pmod 5$$
$$1^2 \equiv 1 \pmod 5$$
$$2^2 \equiv 4 \pmod 5$$
$$3^2 \equiv 9 \equiv 4 \pmod 5$$
$$4^2 \equiv 16 \equiv 1 \pmod 5$$

The set of quadratic residues modulo 5 is $\{0, 1, 4\}$. Since 3 is not an element of this set, the congruence $n^2 \equiv 3 \pmod 5$ has no solution. This is a contradiction.

Therefore, the initial assumption that S can be a perfect square must be false. The sum of the squares of five consecutive integers can never be a perfect square. ■

10 Perfect Squares of the Form 6...64

Problem 10.1. Determine which integers of the form $\underbrace{6\ldots6}_{k}4$, where k is a positive integer, are perfect squares.

Proof. Let N_k denote an integer formed by k instances of the digit 6, followed by the digit 4. We seek to find all positive integers k for which N_k is a perfect square.

The integer N_k can be expressed algebraically. The number consisting of k sixes is $6 \times \sum_{i=0}^{k-1} 10^i = 6 \cdot \frac{10^k-1}{9} = \frac{2}{3}(10^k - 1)$. Thus, N_k can be written as:

$$N_k = 10 \cdot \left(\frac{2}{3}(10^k - 1)\right) + 4 = \frac{20(10^k - 1) + 12}{3} = \frac{2 \cdot 10^{k+1} - 8}{3}$$

We first examine the case where $k = 1$. For $k = 1$, the number is $N_1 = 64$. Since $64 = 8^2$, this is a perfect square.

Next, we consider the cases where $k \geq 2$. Let us assume that N_k is a perfect square for some $k \geq 2$, so that $N_k = m^2$ for some integer m. Using the algebraic form of N_k, we have:

$$m^2 = \frac{2 \cdot 10^{k+1} - 8}{3}$$

This can be rearranged to:

$$3m^2 = 2(10^{k+1} - 4) = 4(5 \cdot 10^k - 2)$$

The equation shows that $3m^2$ is divisible by 4. As $\gcd(3, 4) = 1$, it must be that m^2 is divisible by 4, which implies that m is an even integer. Let $m = 2j$ for some integer j. Substituting this into the equation yields:

$$3(2j)^2 = 4(5 \cdot 10^k - 2)$$

$$12j^2 = 4(5 \cdot 10^k - 2)$$

$$3j^2 = 5 \cdot 10^k - 2$$

This gives an expression for j^2:

$$j^2 = \frac{5 \cdot 10^k - 2}{3}$$

For $k \geq 1$, the numerator is $5\underbrace{00\ldots0}_{k} - 2 = 4\underbrace{99\ldots9}_{k-1}8$. The sum of the digits of this number is $4 + 9(k-1) + 8 = 12 + 9(k-1)$, which is divisible by 3. Performing the division, we find:

$$j^2 = 1\underbrace{66\ldots6}_{k}$$

For $k = 1$, this gives $j^2 = 16$, so $j = 4$, which corresponds to the solution $N_1 = 64$. For $k \geq 2$, we must determine if $1\underbrace{66\ldots6}_{k}$ can be a perfect square.

We analyze the number $1\underbrace{66\ldots6}_{k}$ for $k \geq 2$ by considering its value modulo 4. Since an integer's value modulo 4 is determined by its last two digits, and for $k \geq 2$ the number ends in 66, we have:

$$j^2 \equiv 66 \pmod{4}$$

As $66 = 16 \cdot 4 + 2$, this congruence becomes:

$$j^2 \equiv 2 \pmod{4}$$

However, the square of any integer can only be congruent to 0 or 1 modulo 4:

- If j is even, $j = 2p$, then $j^2 = (2p)^2 = 4p^2 \equiv 0 \pmod{4}$.
- If j is odd, $j = 2p + 1$, then $j^2 = (2p + 1)^2 = 4p^2 + 4p + 1 \equiv 1 \pmod{4}$.

The set of quadratic residues modulo 4 is $\{0, 1\}$. The congruence $j^2 \equiv 2 \pmod{4}$ has no integer solution for j. This contradicts the assumption that N_k is a perfect square for $k \geq 2$.

Therefore, the only integer of the specified form that is a perfect square is 64, which corresponds to the case $k = 1$. ∎

11 Sum of Two Squares and Modulo 4

Problem 11.1. Show that no integer of the form $4k + 2$ or $4k + 3$ can be the sum of two perfect squares.

$$n \equiv 2, 3 \pmod 4 \implies n \neq a^2 + b^2$$

Proof. Let n be an integer that is the sum of two perfect squares, such that $n = a^2 + b^2$ for some integers a and b. To analyze the properties of n, we first consider the possible values of a perfect square modulo 4.

Any integer x is either even or odd.

- If x is even, then $x = 2m$ for some integer m. Its square is $x^2 = (2m)^2 = 4m^2$, which implies $x^2 \equiv 0 \pmod 4$.
- If x is odd, then $x = 2m + 1$ for some integer m. Its square is $x^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 4(m^2 + m) + 1$, which implies $x^2 \equiv 1 \pmod 4$.

Therefore, any perfect square must be congruent to either 0 or 1 modulo 4.

We can now determine the possible values of $n = a^2 + b^2$ modulo 4 by examining the sums of these possible residues.

1. If $a^2 \equiv 0 \pmod 4$ and $b^2 \equiv 0 \pmod 4$, then $n \equiv 0 + 0 \equiv 0 \pmod 4$.
2. If $a^2 \equiv 0 \pmod 4$ and $b^2 \equiv 1 \pmod 4$ (or vice versa), then $n \equiv 0 + 1 \equiv 1 \pmod 4$.
3. If $a^2 \equiv 1 \pmod 4$ and $b^2 \equiv 1 \pmod 4$, then $n \equiv 1 + 1 \equiv 2 \pmod 4$.

The set of possible residues for a sum of two squares modulo 4 is $\{0, 1, 2\}$.

From this result, we can evaluate the two claims made in the problem statement.

Case 1: Integers of the form $4k + 3$

An integer of the form $4k + 3$ is congruent to 3 modulo 4. Since the residue 3 is not in the set of possible values for a sum of two squares modulo 4, no integer of the form $4k + 3$ can be expressed as the sum of two squares. This part of the statement is correct.

Case 2: Integers of the form $4k + 2$

The statement claims that no integer of the form $4k + 2$ can be the sum of two squares. This is equivalent to stating that n cannot be congruent to 2 modulo 4. However, our analysis shows that if both a and b are odd integers, then $a^2 \equiv 1 \pmod 4$ and $b^2 \equiv 1 \pmod 4$, leading to $n = a^2 + b^2 \equiv 2 \pmod 4$. This indicates that the claim for integers of the form $4k + 2$ is incorrect. We can provide a counterexample to disprove it.

Consider the integer $n = 10$. We have $10 = 4(2) + 2$, so it is of the form $4k + 2$. It can also be written as the sum of two squares: $10 = 1^2 + 3^2$. Another counterexample is $n = 2$, which is $4(0) + 2$ and can be written as $1^2 + 1^2$.

Conclusion

An integer of the form $4k + 3$ cannot be the sum of two perfect squares. The assertion for integers of the form $4k + 2$ is not generally true. ∎

12 Odd Perfect Squares Modulo 8

Problem 12.1. Prove that every odd perfect square is of the form $8k + 1$.

Proof. Let n be an odd integer. By definition, n can be expressed in the form $n = 2m + 1$ for some integer m. We seek to analyze the structure of its square, n^2. Squaring the expression for n yields:

$$n^2 = (2m + 1)^2 = 4m^2 + 4m + 1$$

By factoring out the common term $4m$, we can rewrite the expression as:

$$n^2 = 4m(m + 1) + 1$$

The term $m(m + 1)$ is the product of two consecutive integers. For any integer m, one of the two consecutive integers, either m or $m + 1$, must be even. Consequently, their product $m(m + 1)$ is always divisible by 2. Therefore, we can write $m(m + 1) = 2k$ for some integer k. Substituting this result back into the expression for n^2, we obtain:

$$n^2 = 4(2k) + 1 = 8k + 1$$

This demonstrates that the square of any odd integer is of the form $8k + 1$. In terms of modular arithmetic, this is equivalent to the congruence:

$$(2m + 1)^2 \equiv 1 \pmod 8$$

The proof is thus complete. ■

13 The Diophantine Equation $2^n + 65 = k^2$

Problem 13.1. Find all non-negative integers n such that $2^n + 65$ is a perfect square.

Proof. We are looking for all non-negative integer solutions (n, k) to the Diophantine equation:

$$2^n + 65 = k^2$$

We can rearrange the equation as:

$$k^2 - 2^n = 65$$

Since $k^2 = 2^n + 65$, we must have $k^2 > 65$, which implies $k \geq 9$. We analyze the equation by considering the parity of the exponent n.

Case 1: n is even

Let $n = 2m$ for some non-negative integer m. The equation becomes:

$$k^2 - 2^{2m} = 65$$

$$k^2 - (2^m)^2 = 65$$

Factoring the difference of squares on the left side gives:

$$(k - 2^m)(k + 2^m) = 65$$

Since k and 2^m are integers, $(k - 2^m)$ and $(k + 2^m)$ must be integer factors of 65. The integer factor pairs of 65 are $(1, 65)$ and $(5, 13)$. Since $k \geq 9$ and $m \geq 0$, the term $k + 2^m$ must be positive. As the product is positive, $k - 2^m$ must also be positive. Furthermore, $k + 2^m > k - 2^m$. We have two systems of linear equations to solve.

(a) The pair $(1, 65)$:

$$\begin{aligned} k - 2^m &= 1 \\ k + 2^m &= 65 \end{aligned}$$

Adding the two equations yields $2k = 66$, so $k = 33$. Substituting this back gives $33 + 2^m = 65$, which leads to $2^m = 32 = 2^5$. Thus, $m = 5$. Since $n = 2m$, we find the solution $n = 10$. Verification: $2^{10} + 65 = 1024 + 65 = 1089 = 33^2$.

(b) The pair $(5, 13)$:

$$\begin{aligned} k - 2^m &= 5 \\ k + 2^m &= 13 \end{aligned}$$

Adding the two equations yields $2k = 18$, so $k = 9$. Substituting this back gives $9 + 2^m = 13$, which leads to $2^m = 4 = 2^2$. Thus, $m = 2$. Since $n = 2m$, we find the solution $n = 4$. Verification: $2^4 + 65 = 16 + 65 = 81 = 9^2$.

For even n, the solutions are $n = 4$ and $n = 10$.

Case 2: n is odd

Let n be an odd positive integer. We analyze the equation using modular arithmetic. Consider the equation modulo 5:

$$k^2 \equiv 2^n + 65 \pmod{5}$$

Since 65 is a multiple of 5, the equation simplifies to:

$$k^2 \equiv 2^n \pmod{5}$$

The set of quadratic residues modulo 5 is $\{0, 1, 4\}$. Therefore, for the equation to have a solution, 2^n must be congruent to $0, 1$, or 4 modulo 5. The powers of 2 modulo 5 follow a cycle of length 4:

$$\begin{aligned} 2^1 &\equiv 2 \pmod{5} \\ 2^2 &\equiv 4 \pmod{5} \\ 2^3 &\equiv 3 \pmod{5} \\ 2^4 &\equiv 1 \pmod{5} \end{aligned}$$

For $k^2 \equiv 2^n \pmod{5}$ to hold, 2^n must be a quadratic residue modulo 5, which means $2^n \equiv 1 \pmod{5}$ or $2^n \equiv 4 \pmod{5}$. This occurs only when the exponent n is congruent to 0 or 2 modulo 4. In both instances, n must be an even integer.

If n is odd, then $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$.

- If $n \equiv 1 \pmod{4}$, then $2^n \equiv 2^1 \equiv 2 \pmod{5}$.
- If $n \equiv 3 \pmod{4}$, then $2^n \equiv 2^3 \equiv 3 \pmod{5}$.

Neither 2 nor 3 are quadratic residues modulo 5. Therefore, the equation $k^2 \equiv 2^n \pmod{5}$ has no integer solution for k when n is odd. Consequently, there are no solutions when n is odd.

The case $n = 0$ can be checked directly: $2^0 + 65 = 1 + 65 = 66$, which is not a perfect square.

Conclusion

By combining the results from both cases, the only non-negative integers n for which $2^n + 65$ is a perfect square are $n = 4$ and $n = 10$. ■

14 Numbers with 3n Identical Digits

Problem 14.1. Show that a number composed of $3n$ identical digits (with $n \geq 1$) is never a perfect square.

Proof. Let N be an integer composed of $3n$ repetitions of a digit d, where $d \in \{1, 2, \ldots, 9\}$ and $n \geq 1$. We can express N algebraically using the repunit $R_k = \frac{10^k - 1}{9}$ as:

$$N = d \cdot R_{3n} = d \cdot \frac{10^{3n} - 1}{9}$$

Assume, for the sake of contradiction, that N is a perfect square, so $N = k^2$ for some integer k. Substituting this into the expression for N gives:

$$k^2 = d \cdot \frac{10^{3n} - 1}{9} \implies 9k^2 = d(10^{3n} - 1) \implies (3k)^2 = d(10^{3n} - 1)$$

This implies that the product $d(10^{3n} - 1)$ must be a perfect square.

We first analyze the terminal digit of this product. For any $n \geq 1$, the number $10^{3n} - 1$ is a sequence of $3n$ nines, so its units digit is 9. The units digit of $d(10^{3n} - 1)$ is therefore the same as the units digit of $9d$. The possible units digits for a perfect square are $\{0, 1, 4, 5, 6, 9\}$. Examining the units digit of $9d$ for $d \in \{1, \ldots, 9\}$:

- $d = 1 \implies 9 \times 1 = 9$ (Possible)
- $d = 2 \implies 9 \times 2 = 18 \to 8$ (Impossible)
- $d = 3 \implies 9 \times 3 = 27 \to 7$ (Impossible)
- $d = 4 \implies 9 \times 4 = 36 \to 6$ (Possible)
- $d = 5 \implies 9 \times 5 = 45 \to 5$ (Possible)
- $d = 6 \implies 9 \times 6 = 54 \to 4$ (Possible)
- $d = 7 \implies 9 \times 7 = 63 \to 3$ (Impossible)
- $d = 8 \implies 9 \times 8 = 72 \to 2$ (Impossible)
- $d = 9 \implies 9 \times 9 = 81 \to 1$ (Possible)

This condition restricts the possible values of d to the set $\{1, 4, 5, 6, 9\}$.

Next, we analyze N modulo 4. Since $n \geq 1$, the number of digits $3n$ is at least 3. The value of N modulo 4 is determined by its last two digits, which are dd. Thus, $N \equiv 10d + d \pmod 4$, which simplifies to $N \equiv 11d \equiv 3d \pmod 4$. A perfect square must be congruent to either 0 or 1 modulo 4. We test the remaining possibilities for d:

- $d = 1 \implies N \equiv 3(1) \equiv 3 \pmod 4$ (Impossible)
- $d = 4 \implies N \equiv 3(4) = 12 \equiv 0 \pmod 4$ (Possible)
- $d = 5 \implies N \equiv 3(5) = 15 \equiv 3 \pmod 4$ (Impossible)
- $d = 6 \implies N \equiv 3(6) = 18 \equiv 2 \pmod 4$ (Impossible)
- $d = 9 \implies N \equiv 3(9) = 27 \equiv 3 \pmod 4$ (Impossible)

The only value for d that satisfies both conditions is $d = 4$.

If $d = 4$, the number is $N = 4 \cdot R_{3n}$. For N to be a perfect square, the repunit R_{3n} must also be a perfect square. However, for any integer $S \geq 2$, the repunit R_S is of the form $100k + 11$ for some integer k, which implies $R_S \equiv 11 \equiv 3 \pmod 4$. Since a perfect square cannot be congruent to 3 modulo 4, R_S is never a perfect square for $S \geq 2$. In our case, $S = 3n \geq 3$, so R_{3n} cannot be a perfect square.

This final case leads to a contradiction. Therefore, no integer composed of $3n$ identical digits can be a perfect square. ∎

15 Sum of the First 24 Squares

Problem 15.1. Determine if the sum of squares from 1 to 24 is a perfect square.

$$S = \sum_{i=1}^{24} i^2 \stackrel{?}{=} k^2$$

Proof. We seek to evaluate the sum $S = \sum_{i=1}^{24} i^2$ and determine whether the result is a perfect square. The sum of the first n squares is given by the formula:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Substituting $n = 24$ into this formula, we obtain:

$$S = \frac{24(24+1)(2 \cdot 24 + 1)}{6}$$
$$= \frac{24 \cdot 25 \cdot 49}{6}$$

We can simplify this expression by dividing 24 by 6:

$$S = 4 \cdot 25 \cdot 49$$

To determine if S is a perfect square, we examine its prime factorization. Each of the factors in the expression is a perfect square:

- $4 = 2^2$
- $25 = 5^2$
- $49 = 7^2$

Therefore, the sum S can be written as:

$$S = 2^2 \cdot 5^2 \cdot 7^2 = (2 \cdot 5 \cdot 7)^2 = 70^2$$

The sum evaluates to $70^2 = 4900$. As this is the square of an integer, the sum of the first 24 squares is a perfect square. ■

16 Divisibility Condition from Two Perfect Squares

Problem 16.1. Let n be a positive integer. Show that if $2n+1$ and $3n+1$ are both perfect squares, then n is divisible by 40.

Proof. Let n be a positive integer such that $2n+1$ and $3n+1$ are perfect squares. We can write:

$$2n + 1 = x^2$$
$$3n + 1 = y^2$$

for some integers x and y. To prove that n is divisible by 40, we must show that n is divisible by both 5 and 8, since $\gcd(5, 8) = 1$.

1. Divisibility by 5

We analyze the possible values of n modulo 5. The quadratic residues modulo 5 are $\{0^2, 1^2, 2^2, 3^2, 4^2\} \pmod 5 = \{0, 1, 4\}$. Thus, x^2 and y^2 must be congruent to 0, 1, or 4 modulo 5.

- If $n \equiv 0 \pmod 5$: $x^2 = 2n + 1 \equiv 1 \pmod 5$ and $y^2 = 3n + 1 \equiv 1 \pmod 5$. Both are valid quadratic residues.
- If $n \equiv 1 \pmod 5$: $x^2 = 2(1) + 1 = 3 \pmod 5$. This is not a quadratic residue modulo 5.
- If $n \equiv 2 \pmod 5$: $y^2 = 3(2) + 1 = 7 \equiv 2 \pmod 5$. This is not a quadratic residue modulo 5.
- If $n \equiv 3 \pmod 5$: $x^2 = 2(3) + 1 = 7 \equiv 2 \pmod 5$. This is not a quadratic residue modulo 5.
- If $n \equiv 4 \pmod 5$: $y^2 = 3(4) + 1 = 13 \equiv 3 \pmod 5$. This is not a quadratic residue modulo 5.

The only case that does not lead to a contradiction is $n \equiv 0 \pmod 5$. Therefore, n must be divisible by 5.

2. Divisibility by 8

From the equation $2n + 1 = x^2$, we deduce that x^2 is an odd integer. The square of any odd integer is of the form $(2k + 1)^2 = 4k^2 + 4k + 1 = 4k(k + 1) + 1$. Since $k(k + 1)$ is always an even number, it can be written as $2m$. Thus, $(2k + 1)^2 = 4(2m) + 1 = 8m + 1$. It follows that any odd perfect square is congruent to 1 modulo 8.

$$x^2 \equiv 1 \pmod 8$$

Substituting this into the first equation:

$$2n + 1 \equiv 1 \pmod 8 \implies 2n \equiv 0 \pmod 8$$

This implies that $2n$ is a multiple of 8, so n must be a multiple of 4. Let $n = 4k$ for some integer k.

Now we substitute $n = 4k$ into the second equation, $3n + 1 = y^2$:

$$3(4k) + 1 = 12k + 1 = y^2$$

This shows that y^2 is also an odd integer, and so it must also be congruent to 1 modulo 8.

$$y^2 \equiv 1 \pmod 8$$

Therefore, we have:

$$12k + 1 \equiv 1 \pmod 8 \implies 12k \equiv 0 \pmod 8$$

Since $12 \equiv 4 \pmod 8$, this simplifies to:

$$4k \equiv 0 \pmod 8$$

For this congruence to hold, k must be an even integer. Let $k = 2m$ for some integer m. Finally, substituting back into the expression for n:

$$n = 4k = 4(2m) = 8m$$

This shows that n is divisible by 8.

Conclusion

Since n is divisible by 5 and n is divisible by 8, and $\gcd(5, 8) = 1$, it follows that n must be divisible by $5 \times 8 = 40$. ■

17 Product of Four Consecutive Odd Integers

Problem 17.1. Show that the product of four consecutive odd integers increased by 16 is a perfect square.

$$(2n-3)(2n-1)(2n+1)(2n+3)+16=k^2$$

Proof. Let the four consecutive odd integers be represented by $(2k-3)$, $(2k-1)$, $(2k+1)$, and $(2k+3)$ for some integer k. We consider the expression formed by their product increased by 16. Let this expression be denoted by P.

$$P=(2k-3)(2k-1)(2k+1)(2k+3)+16$$

To simplify the product, we rearrange the terms by grouping the outer pair and the inner pair:

$$P=[(2k-3)(2k+3)][(2k-1)(2k+1)]+16$$

Using the difference of squares formula, $a^2-b^2=(a-b)(a+b)$, we can simplify each grouped product:

$$\begin{aligned}(2k-3)(2k+3)&=(2k)^2-3^2=4k^2-9\\(2k-1)(2k+1)&=(2k)^2-1^2=4k^2-1\end{aligned}$$

Substituting these simplified forms back into the expression for P:

$$P=(4k^2-9)(4k^2-1)+16$$

Let $u=4k^2$. The expression becomes:

$$P=(u-9)(u-1)+16$$

Expanding this product:

$$\begin{aligned}P&=u^2-u-9u+9+16\\&=u^2-10u+25\end{aligned}$$

This resulting quadratic in u is a perfect square:

$$P=(u-5)^2$$

Substituting $u=4k^2$ back into the expression:

$$P=(4k^2-5)^2$$

Since k is an integer, $4k^2-5$ is also an integer. Therefore, P is the square of an integer. This proves that the product of four consecutive odd integers increased by 16 is always a perfect square. ∎

18 Perfect Squares with Exclusively Odd Digits

Problem 18.1. Does there exist a perfect square where all digits are odd?

Proof. Let n be a perfect square, $n = k^2$, such that all its digits belong to the set of odd digits $D = \{1, 3, 5, 7, 9\}$.

First, consider single-digit numbers. The odd digits that are perfect squares are $1 = 1^2$ and $9 = 3^2$. Thus, 1 and 9 are solutions.

Next, consider the case where n has at least two digits. Let the decimal representation of n be $d_m d_{m-1} \dots d_1 d_0$, where each $d_i \in D$. The value of n modulo 4 is determined by the number formed by its last two digits, $10d_1 + d_0$.

$$n \equiv 10d_1 + d_0 \pmod 4$$

Since $10 \equiv 2 \pmod 4$, this congruence simplifies to:

$$n \equiv 2d_1 + d_0 \pmod 4$$

By hypothesis, both the units digit d_0 and the tens digit d_1 are odd. An odd digit d_1 can be written as $d_1 = 2j + 1$ for some integer j. Therefore, $2d_1 = 2(2j + 1) = 4j + 2 \equiv 2 \pmod 4$.

We must also consider the possible values for the last digit of a perfect square. The set of quadratic residues modulo 10 is $\{0, 1, 4, 5, 6, 9\}$. Since all digits of n are odd, its last digit d_0 must be in $D \cap \{0, 1, 4, 5, 6, 9\} = \{1, 5, 9\}$. We analyze these three cases for d_0:

- If $d_0 = 1$, then $n \equiv 2d_1 + 1 \equiv 2 + 1 \equiv 3 \pmod 4$.
- If $d_0 = 5$, then $n \equiv 2d_1 + 5 \equiv 2 + 1 \equiv 3 \pmod 4$.
- If $d_0 = 9$, then $n \equiv 2d_1 + 9 \equiv 2 + 1 \equiv 3 \pmod 4$.

In every possible case, n is congruent to 3 modulo 4.

However, the set of quadratic residues modulo 4 is $\{0, 1\}$, since:

- If k is even, $k = 2m$, then $k^2 = 4m^2 \equiv 0 \pmod 4$.
- If k is odd, $k = 2m + 1$, then $k^2 = 4m^2 + 4m + 1 \equiv 1 \pmod 4$.

A perfect square cannot be congruent to 2 or 3 modulo 4. This leads to a contradiction. Therefore, no perfect square with two or more digits can be composed exclusively of odd digits.

The only perfect squares consisting solely of odd digits are the single-digit numbers 1 and 9. ∎

19 Prime Numbers and Perfect Cubes

Problem 19.1. Find all prime numbers p such that $16p + 1$ is a perfect cube.

Proof. We are looking for prime numbers p such that $16p + 1 = k^3$ for some integer k. First, we can rearrange the equation to isolate $16p$:

$$16p = k^3 - 1$$

Factoring the right-hand side as a difference of cubes yields:

$$16p = (k-1)(k^2+k+1)$$

Since p is a prime number, $p \geq 2$. Thus, $16p+1 \geq 16(2)+1 = 33$. This implies $k^3 \geq 33$, so $k > \sqrt[3]{33} \approx 3.2$. As k must be an integer, $k \geq 4$. Furthermore, since $16p$ is even, $k^3 - 1$ must be even, which means k^3 must be odd. This implies that k must be an odd integer. Combining these conditions, the smallest possible value for k is 5.

Let us analyze the factors on the right-hand side. Since k is odd, let $k = 2m+1$ for some integer m.

- The factor $k-1 = (2m+1)-1 = 2m$ is an even integer.
- The factor $k^2+k+1 = (2m+1)^2+(2m+1)+1 = (4m^2+4m+1)+(2m+2) = 4m^2+6m+3 = 2(2m^2+3m+1)+1$ is an odd integer.

The prime factorization of the left-hand side is $2^4 \cdot p$. Since k^2+k+1 is odd, it shares no factors of 2 with $16p$. Consequently, the entire factor of $2^4 = 16$ must be contained within the factor $(k-1)$. This implies that $k-1$ is a multiple of 16. Let us write:

$$k-1 = 16j$$

for some positive integer j (since $k-1 \geq 4$, j must be positive). This gives $k = 16j+1$.

Substituting this expression for $k-1$ back into the factored equation:

$$16p = (16j)((16j+1)^2+(16j+1)+1)$$

Dividing by 16, we obtain an expression for p:

$$p = j\left((16j+1)^2+16j+1+1\right)$$

Since p is a prime number, it has only two positive divisors: 1 and p. The expression above shows p as a product of two integers, j and the term in the parenthesis. As j is a positive integer, for p to be prime, one of these factors must be equal to 1.

The second factor is $(16j+1)^2+16j+2$. Since $j \geq 1$, this factor is clearly greater than 1. Therefore, the only possibility is that $j=1$.

Setting $j=1$, we find the value of p:

$$\begin{aligned} p &= 1 \cdot \left((16(1)+1)^2+(16(1)+1)+1\right) \\ &= 17^2+17+1 \\ &= 289+18 \\ &= 307 \end{aligned}$$

We must verify that 307 is a prime number. To do this, we test for divisibility by prime numbers up to $\sqrt{307} \approx 17.5$. The primes to test are 2, 3, 5, 7, 11, 13, and 17.

- 307 is not divisible by 2, 3 (sum of digits is 10), or 5.
- $307 = 43 \cdot 7 + 6$.
- $307 = 27 \cdot 11 + 10$.

- $307 = 23 \cdot 13 + 8$.
- $307 = 18 \cdot 17 + 1$.

Since 307 is not divisible by any prime less than or equal to its square root, it is a prime number.

Thus, the only prime number satisfying the condition is $p = 307$. For this value, $16p + 1 = 16(307) + 1 = 4912 + 1 = 4913$, which is 17^3. ■

20 The Champernowne Constant

Problem 20.1. Let the integer C be formed by concatenating the decimal representations of the positive integers in order: $C = 123456789101112\dots$. Describe a procedure to find the d-th digit of C.

Proof. Let C be the integer formed by concatenating the positive integers. To find the d-th digit of C, we must first identify which integer in the sequence contains this digit. We can partition the sequence of digits constituting C into blocks based on the number of digits of the integers being concatenated.

The block of 1-digit integers (1 to 9) contains $9 \times 1 = 9$ digits. The block of 2-digit integers (10 to 99) contains $90 \times 2 = 180$ digits. The block of 3-digit integers (100 to 999) contains $900 \times 3 = 2700$ digits. In general, the block of k-digit integers, which spans from 10^{k-1} to $10^k - 1$, consists of $9 \cdot 10^{k-1}$ numbers and thus contains $k \cdot 9 \cdot 10^{k-1}$ digits.

Let L_k be the total number of digits from all integers with up to k digits. Then $L_k = \sum_{j=1}^{k} j \cdot 9 \cdot 10^{j-1}$. The first step is to find the number of digits, k, of the integer containing the d-th digit. This is achieved by finding the unique integer k such that:

$$L_{k-1} < d \leq L_k$$

where we define $L_0 = 0$.

Once k is determined, the d-th digit lies within the block of k-digit numbers. Let d' be the position of the digit relative to the start of this block:

$$d' = d - L_{k-1}$$

The integers in this block are $10^{k-1}, 10^{k-1} + 1, \dots$. The index of the specific k-digit number containing our digit is given by:

$$m = \left\lceil \frac{d'}{k} \right\rceil$$

The number itself, which we denote by N, is the m-th k-digit number:

$$N = 10^{k-1} + m - 1$$

Finally, we must find the position of the desired digit within the number N. This position, p, counting from the left, is given by $p = (d' - 1 \pmod k) + 1$.

The p-th digit of N can be extracted using integer arithmetic. It is the integer part of $\frac{N}{10^{k-p}}$ modulo 10:

$$\text{Digit} = \left\lfloor \frac{N}{10^{k-p}} \right\rfloor \pmod{10}$$

This procedure uniquely determines the d-th digit of the concatenated integer C. ■

21 Squares of the Form 1...12...25

Problem 21.1. Show that the integer formed by n ones, followed by n twos, followed by a five is never a perfect square for any positive integer n.

$$N_n = \underbrace{1\ldots1}_{n}\underbrace{2\ldots2}_{n}5 \neq k^2$$

Proof. Let N_n be the integer formed by the concatenation of n digits '1', followed by n digits '2', and ending with the digit '5'. We analyze the properties of N_n under modular arithmetic to determine if it can be a perfect square.

A fundamental property of integers is that an integer is congruent to the sum of its digits modulo 9. Let $s(m)$ denote the sum of the digits of an integer m. Then $m \equiv s(m) \pmod 9$. For the number N_n, the sum of its digits is:

$$s(N_n) = \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} 2 + 5 = n \cdot 1 + n \cdot 2 + 5 = 3n + 5$$

Therefore, the number N_n must satisfy the congruence:

$$N_n \equiv 3n + 5 \pmod 9$$

If N_n were a perfect square, say $N_n = k^2$ for some integer k, then its remainder modulo 9 must be a quadratic residue modulo 9. The set of quadratic residues modulo 9 can be found by squaring the integers from 0 to 8:

$$\begin{gathered} 0^2 \equiv 0, \quad 1^2 \equiv 1, \quad 2^2 \equiv 4, \quad 3^2 \equiv 0, \quad 4^2 \equiv 16 \equiv 7, \\ 5^2 \equiv 25 \equiv 7, \quad 6^2 \equiv 36 \equiv 0, \quad 7^2 \equiv 49 \equiv 4, \quad 8^2 \equiv 64 \equiv 1. \end{gathered}$$

The set of distinct quadratic residues modulo 9 is $\{0, 1, 4, 7\}$. Thus, for N_n to be a perfect square, it is necessary that $N_n \pmod 9$ be an element of this set.

We now examine the possible values of $3n + 5 \pmod 9$ for any positive integer n. The term $3n$ can only take values congruent to 3, 6, or 0 modulo 9:

- If $n \equiv 1 \pmod 3$, then $3n \equiv 3 \pmod 9$.
- If $n \equiv 2 \pmod 3$, then $3n \equiv 6 \pmod 9$.
- If $n \equiv 0 \pmod 3$, then $3n \equiv 0 \pmod 9$.

Consequently, the possible values for $3n + 5 \pmod 9$ are:

- If $3n \equiv 3 \pmod 9$, then $3n + 5 \equiv 3 + 5 \equiv 8 \pmod 9$.
- If $3n \equiv 6 \pmod 9$, then $3n + 5 \equiv 6 + 5 = 11 \equiv 2 \pmod 9$.
- If $3n \equiv 0 \pmod 9$, then $3n + 5 \equiv 0 + 5 \equiv 5 \pmod 9$.

The set of possible remainders for N_n modulo 9 is therefore $\{2, 5, 8\}$.

Comparing this set with the set of quadratic residues modulo 9, $\{0, 1, 4, 7\}$, we find that there is no overlap. The remainder of N_n when divided by 9 can never be a quadratic residue. Since this necessary condition is not met for any positive integer n, we conclude that the number N_n can never be a perfect square. ■

22 Difference of Squares: $x^2 - y^2 = 2024$

Problem 22.1. Find all pairs of integers (x, y) such that:

$$x^2 - y^2 = 2024$$

Proof. The given Diophantine equation is $x^2 - y^2 = 2024$. The left-hand side can be factored as a difference of squares:

$$(x - y)(x + y) = 2024$$

Let $u = x - y$ and $v = x + y$. Since x and y are integers, u and v must be integer factors of 2024. We can express x and y in terms of u and v by solving this system of linear equations:

$$x = \frac{u + v}{2}$$
$$y = \frac{v - u}{2}$$

For x and y to be integers, the sum $u + v$ and the difference $v - u$ must both be even. This condition is met if and only if u and v have the same parity. The product $uv = 2024$ is an even number. For two integers to have an even product and the same parity, they must both be even. Thus, we seek pairs of even integer factors of 2024.

The prime factorization of 2024 is $2024 = 8 \times 253 = 2^3 \times 11 \times 23$. We list the pairs of positive even factors (u, v) of 2024, assuming without loss of generality that $u \leq v$:

- If $u = 2$, then $v = 1012$.
- If $u = 4$, then $v = 506$.
- If $u = 2 \cdot 11 = 22$, then $v = 2024/22 = 92$.
- If $u = 4 \cdot 11 = 44$, then $v = 2024/44 = 46$.

These four pairs of factors yield the following solutions for (x, y), where x and y are positive:

- For $(u, v) = (2, 1012)$: $x = \frac{2+1012}{2} = 507$ and $y = \frac{1012-2}{2} = 505$.
- For $(u, v) = (4, 506)$: $x = \frac{4+506}{2} = 255$ and $y = \frac{506-4}{2} = 251$.
- For $(u, v) = (22, 92)$: $x = \frac{22+92}{2} = 57$ and $y = \frac{92-22}{2} = 35$.
- For $(u, v) = (44, 46)$: $x = \frac{44+46}{2} = 45$ and $y = \frac{46-44}{2} = 1$.

These calculations provide the solutions in positive integers. Since the equation involves squares, the signs of x and y can be varied. Allowing u and v to be negative or $u > v$ generates all possible integer solutions. For each pair $(|x_0|, |y_0|)$ found, the four pairs $(\pm x_0, \pm y_0)$ are solutions. The complete set of integer solutions (x, y) is:

$$(\pm 507, \pm 505), \quad (\pm 255, \pm 251), \quad (\pm 57, \pm 35), \quad (\pm 45, \pm 1)$$

∎

23 Integer Sums of Irreducible Fractions

Problem 23.1. Prove that if the sum of two positive irreducible fractions is an integer, then their denominators are equal.

$$\frac{a}{b} + \frac{c}{d} = k \in \mathbb{Z} \implies b = d \quad (\text{given } \gcd(a,b) = \gcd(c,d) = 1)$$

Proof. Let $\frac{a}{b}$ and $\frac{c}{d}$ be two positive fractions such that a, b, c, d are positive integers, $\gcd(a, b) = 1$, and $\gcd(c, d) = 1$. Assume that their sum is an integer k:

$$\frac{a}{b} + \frac{c}{d} = k$$

We can rearrange the equation to isolate the term $\frac{a}{b}$:

$$\frac{a}{b} = k - \frac{c}{d} = \frac{kd - c}{d}$$

Multiplying both sides by bd yields the relation:

$$ad = b(kd - c)$$

From this equation, it is evident that d divides the product $b(kd - c)$. Since d also divides bkd, it must divide their difference: $d \mid (bkd - b(kd - c))$, which simplifies to $d \mid bc$. By hypothesis, the fraction $\frac{c}{d}$ is irreducible, which means that $\gcd(c, d) = 1$. According to Euclid's lemma, if an integer divides the product of two other integers and is coprime to one of them, it must divide the other. Therefore, since $d \mid bc$ and $\gcd(c, d) = 1$, it follows that $d \mid b$.

Next, we rearrange the original equation to isolate the term $\frac{c}{d}$:

$$\frac{c}{d} = k - \frac{a}{b} = \frac{kb - a}{b}$$

This yields the relation:

$$bc = d(kb - a)$$

From this, it follows that b divides the product $d(kb - a)$. As b also divides dkb, it must divide their difference: $b \mid (dkb - d(kb - a))$, which simplifies to $b \mid da$. By hypothesis, the fraction $\frac{a}{b}$ is irreducible, so $\gcd(a, b) = 1$. Applying Euclid's lemma again, since $b \mid da$ and $\gcd(a, b) = 1$, it follows that $b \mid d$.

We have established two mutual divisibility conditions: $d \mid b$ and $b \mid d$. Since both b and d are positive integers, these two conditions can hold simultaneously only if $b = d$. This concludes the proof. ■

24 Forbidden Units Digits of Perfect Squares

Problem 24.1. Show that the units digit of a perfect square can never be 2, 3, 7, or 8.

Proof. Let n be an integer. The units digit of its square, n^2, depends only on the units digit of n. This property can be analyzed by considering the squares of integers modulo 10.

Let the decimal representation of n be $n = 10k + d$, where $d \in \{0, 1, \dots, 9\}$ is the units digit of n. The square of n is:

$$n^2 = (10k + d)^2 = 100k^2 + 20kd + d^2 = 10(10k^2 + 2kd) + d^2$$

This expression shows that $n^2 \equiv d^2 \pmod{10}$. Thus, the units digit of n^2 is the same as the units digit of d^2. We compute the square of each possible units digit:

$$\begin{aligned}
0^2 &\equiv 0 \pmod{10} \\
1^2 &\equiv 1 \pmod{10} \\
2^2 &\equiv 4 \pmod{10} \\
3^2 &\equiv 9 \pmod{10} \\
4^2 &= 16 \equiv 6 \pmod{10} \\
5^2 &= 25 \equiv 5 \pmod{10} \\
6^2 &= 36 \equiv 6 \pmod{10} \\
7^2 &= 49 \equiv 9 \pmod{10} \\
8^2 &= 64 \equiv 4 \pmod{10} \\
9^2 &= 81 \equiv 1 \pmod{10}
\end{aligned}$$

The set of all possible units digits for a perfect square is the set of these quadratic residues modulo 10, which is $\{0, 1, 4, 5, 6, 9\}$. The digits 2, 3, 7, and 8 do not appear in this set. Therefore, no perfect square can have a units digit of 2, 3, 7, or 8. ■

25 Digit Sum of Squares of Repnines

Problem 25.1. Let n be an integer whose decimal representation consists of an odd number of digits, all of which are 9. Show that the sum of the digits of n^2 is odd.

Proof. Let n be an integer composed of k digits, all equal to 9, where k is a positive odd integer. Such a number, known as a repnine, can be expressed algebraically as:

$$n = \underbrace{99\dots9}_{k} = 10^k - 1$$

We seek to determine the sum of the digits of n^2. We first find the decimal representation of n^2:

$$\begin{aligned}
n^2 &= (10^k - 1)^2 \\
&= (10^k)^2 - 2 \cdot 10^k + 1 \\
&= 10^{2k} - 2 \cdot 10^k + 1
\end{aligned}$$

To understand the structure of this number, we perform the subtraction:

$$\begin{aligned}
10^{2k} - 2 \cdot 10^k &= 1\underbrace{00\dots0}_{2k} - 2\underbrace{00\dots0}_{k} \\
&= \underbrace{99\dots9}_{k-1}8\underbrace{00\dots0}_{k}
\end{aligned}$$

Adding 1 to this result gives the final form of n^2:

$$n^2 = \underbrace{99\ldots9}_{k-1}8\underbrace{00\ldots0}_{k-1}1$$

The digits of n^2 are therefore: $k-1$ instances of the digit 9, one instance of the digit 8, $k-1$ instances of the digit 0, and one instance of the digit 1. Let $s(n^2)$ denote the sum of the digits of n^2. We can calculate this sum as:

$$\begin{aligned} s(n^2) &= (k-1)\cdot 9 + 8 + (k-1)\cdot 0 + 1 \\ &= 9k - 9 + 8 + 1 \\ &= 9k \end{aligned}$$

By the problem's hypothesis, the number of digits k in n is a positive odd integer. The product of two odd integers is odd. Since 9 is odd and k is odd, their product $9k$ must be an odd integer. Thus, the sum of the digits of n^2 is odd. ■

26 Numbers of the Form $n^2 + n + 1$ as Perfect Squares

Problem 26.1. Find all integers n for which $n^2 + n + 1$ is a perfect square.

Proof. Let us find all integers n such that $n^2 + n + 1 = k^2$ for some non-negative integer k.

We analyze the expression by comparing it to the squares of consecutive integers.

Case 1: $n > 0$

For any integer $n > 0$, we have the following inequalities:

$$n^2 < n^2 + n + 1$$

Also, consider the square of the next integer, $n+1$:

$$(n+1)^2 = n^2 + 2n + 1$$

Comparing this with our expression, we find:

$$n^2 + n + 1 < n^2 + 2n + 1 = (n+1)^2$$

since $n > 0$. Combining these results, we have for any positive integer n:

$$n^2 < n^2 + n + 1 < (n+1)^2$$

This shows that the integer n^2+n+1 lies strictly between two consecutive perfect squares, n^2 and $(n+1)^2$. An integer that is strictly between two consecutive squares cannot itself be a square. Therefore, there are no solutions for $n > 0$.

Case 2: $n = 0$

If $n = 0$, the expression becomes $0^2 + 0 + 1 = 1 = 1^2$. This is a perfect square. Thus, $n = 0$ is a solution.

Case 3: $n \leq -1$

Let $n = -m$ for some integer $m \geq 1$. The expression becomes:

$$(-m)^2 + (-m) + 1 = m^2 - m + 1$$

We again compare this to consecutive squares.

If $m = 1$, then $n = -1$. The expression is $1^2 - 1 + 1 = 1 = 1^2$. This is a perfect square. Thus, $n = -1$ is a solution.

If $m > 1$, we consider the squares $(m-1)^2$ and m^2:

$$(m-1)^2 = m^2 - 2m + 1$$

Since $m > 1$, we have $m > 0$, which implies $m^2 - m + 1 < m^2$. Furthermore, the difference between our expression and $(m-1)^2$ is:

$$(m^2 - m + 1) - (m^2 - 2m + 1) = m$$

Since $m > 1$, this difference is positive, so $(m-1)^2 < m^2 - m + 1$. Thus, for any integer $m > 1$, we have:

$$(m-1)^2 < m^2 - m + 1 < m^2$$

The integer $m^2 - m + 1$ is again strictly between two consecutive perfect squares. Therefore, it cannot be a perfect square for any $m > 1$, which corresponds to $n < -1$.

Conclusion

The expression $n^2 + n + 1$ is a perfect square only for $n = 0$ and $n = -1$. ■

27 Sums of Powers of Four as Perfect Squares

Problem 27.1. Can the number $4^n + 4^m + 4^p$ be a perfect square if n, m, p are distinct integers?

Proof. The answer is affirmative. We seek to determine if there exist distinct integers n, m, p such that $4^n + 4^m + 4^p = k^2$ for some integer k.

Without loss of generality, let us assume that $n < m < p$. We can express the sum as:

$$S = 4^n + 4^m + 4^p$$

Factoring out the smallest term, 4^n, gives:

$$S = 4^n(1 + 4^{m-n} + 4^{p-n})$$

Since $4^n = (2^n)^2$ is a perfect square, the entire sum S is a perfect square if and only if the term $(1 + 4^{m-n} + 4^{p-n})$ is also a perfect square.

Let $a = m - n$ and $b = p - n$. Given the ordering $n < m < p$, it follows that a and b are positive integers satisfying $0 < a < b$. Our problem reduces to determining if there exist such integers a and b for which the expression $1 + 4^a + 4^b$ is a perfect square.

Let us attempt to express this as a square. We compare the expression with known squares. Note that $4^b = (2^b)^2$. We can compare our expression with the square of the next integer, $(2^b + 1)^2$:

$$(2^b + 1)^2 = (2^b)^2 + 2 \cdot 2^b + 1 = 4^b + 2^{b+1} + 1$$

For our expression to be equal to this perfect square, we would require:

$$1 + 4^a + 4^b = 1 + 2^{b+1} + 4^b$$

This equality holds if and only if:

$$4^a = 2^{b+1}$$

Writing both sides with a base of 2, we have:

$$(2^2)^a = 2^{b+1} \implies 2^{2a} = 2^{b+1}$$

This implies the condition on the exponents:

$$2a = b + 1$$

We now need to find if there exist integers a and b that satisfy the system of conditions:

1. $a > 0$
2. $b > a$
3. $2a = b + 1$

From the third condition, we can write $b = 2a - 1$. Substituting this into the second condition, $b > a$, gives:

$$2a - 1 > a \implies a > 1$$

The first condition, $a > 0$, is automatically satisfied if $a > 1$. Thus, any integer $a \geq 2$ will yield a valid solution. For such an a, we can define $b = 2a - 1$, and the conditions $0 < a < b$ will be met.

For example, let us choose the smallest possible integer value for a, which is $a = 2$. This gives $b = 2(2) - 1 = 3$. The conditions $0 < 2 < 3$ are satisfied. With this choice, the expression becomes:

$$1 + 4^a + 4^b = 1 + 4^2 + 4^3 = 1 + 16 + 64 = 81 = 9^2$$

This confirms that a solution exists.

To provide a specific example with distinct integers n, m, p, we can choose any integer for n and then determine m and p. Let $n = 1$. Then:

- $a = m - n \implies 2 = m - 1 \implies m = 3$
- $b = p - n \implies 3 = p - 1 \implies p = 4$

The integers $n = 1, m = 3, p = 4$ are distinct. The sum is:

$$4^1 + 4^3 + 4^4 = 4 + 64 + 256 = 324 = 18^2$$

Therefore, the number $4^n + 4^m + 4^p$ can indeed be a perfect square for distinct integers n, m, p. ■

28 Factorions: Sum of Digit Factorials

Problem 28.1. Find all three-digit numbers that are equal to the sum of the factorials of their digits.

$$\overline{abc} = a! + b! + c!$$

Proof. Let N be a three-digit number such that $N = \overline{abc} = 100a + 10b + c$, where $a \in \{1, \dots, 9\}$ and $b, c \in \{0, \dots, 9\}$. The condition is $N = a! + b! + c!$.

We first establish bounds on the possible digits. The factorial function grows rapidly: $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$, and $7! = 5040$. Since N is a three-digit number, $N \leq 999$. This implies that $a! + b! + c! \leq 999$. As $7! > 999$, none of the digits a, b, c can be greater than or equal to 7. The set of possible digits is thus restricted to $\{0, 1, 2, 3, 4, 5, 6\}$.

If any of the digits were 6, the sum of the factorials would be at least $6! = 720$. A number greater than or equal to 720 must have a first digit of 7, 8, or 9. This contradicts the fact that the digits cannot exceed 6. Therefore, none of the digits can be 6. The set of possible digits is further reduced to $\{0, 1, 2, 3, 4, 5\}$.

With this restriction, the maximum possible sum of factorials is $5! + 5! + 5! = 120 + 120 + 120 = 360$. This implies that $N \leq 360$, and so the first digit, a, can only be 1, 2, or 3.

We examine the possible values for the first digit a:

- If $a = 3$, the number N is in the range $[300, 399]$. However, the maximum possible sum of factorials with $a = 3$ and $b, c \in \{0, \dots, 5\}$ is $3! + 5! + 5! = 6 + 120 + 120 = 246$. Since $246 < 300$, this case is impossible.

- If $a = 2$, N is in the range $[200, 299]$. The sum is $2! + b! + c!$. To be in this range, the sum must be at least 200. Thus, $2 + b! + c! \geq 200$, which implies $b! + c! \geq 198$. To satisfy this, at least one of the digits b or c must be 5, since $2 \cdot 4! = 48 < 198$. If $b = 5$, we require $5! + c! \geq 198$, which means $120 + c! \geq 198$, or $c! \geq 78$. This necessitates that c must also be 5. The set of digits is therefore $\{2, 5, 5\}$. The sum is $2! + 5! + 5! = 2 + 120 + 120 = 242$. The number formed by these digits must be 242, 255, or 525. None of these match the sum 242. This case yields no solution.

- If $a = 1$, N is in the range $[100, 199]$. The sum is $1! + b! + c!$. For the sum to be in this range, we need $1 + b! + c! \geq 100$, which means $b! + c! \geq 99$. This requires at least one of b or c to be 5, since $2 \cdot 4! = 48 < 99$. Let one of the digits be 5. The number is then formed from a permutation of $\{1, 5, k\}$ for some $k \in \{0, 1, 2, 3, 4, 5\}$. The sum is $1! + 5! + k! = 1 + 120 + k! = 121 + k!$. We test the possible values for the third digit, k:

  - If $k = 0$, sum is $121 + 1 = 122$. Digits $\{1, 5, 0\}$. Number is not 122.
  - If $k = 1$, sum is $121 + 1 = 122$. Digits $\{1, 5, 1\}$. Number is not 122.
  - If $k = 2$, sum is $121 + 2 = 123$. Digits $\{1, 5, 2\}$. Number is not 123.
  - If $k = 3$, sum is $121 + 6 = 127$. Digits $\{1, 5, 3\}$. Number is not 127.
  - If $k = 4$, sum is $121 + 24 = 145$. Digits $\{1, 5, 4\}$. The number formed by these digits could be 145. Checking this, we find $1! + 4! + 5! = 1 + 24 + 120 = 145$. This is a solution.
  - If $k = 5$, sum is $121 + 120 = 241$. The sum is greater than 199, which contradicts the assumption that $a = 1$.

The only three-digit number satisfying the condition is 145. ■

29 Sophie Germain Identity

Problem 29.1. Show that for any integer $n > 1$, the number $n^4 + 4$ is not prime.

Proof. We seek to factor the expression $n^4 + 4$. By adding and subtracting the term $4n^2$, we can complete the square:

$$\begin{aligned} n^4 + 4 &= (n^4 + 4n^2 + 4) - 4n^2 \\ &= (n^2 + 2)^2 - (2n)^2 \end{aligned}$$

This expression is a difference of two squares, which can be factored as $(A - B)(A + B)$:

$$(n^2 + 2 - 2n)(n^2 + 2 + 2n)$$

Rearranging the terms within each factor, we obtain:

$$(n^2 - 2n + 2)(n^2 + 2n + 2)$$

The expression $n^4 + 4$ is thus the product of two integers, $(n^2 - 2n + 2)$ and $(n^2 + 2n + 2)$. For $n^4 + 4$ to be a prime number, one of these factors must be equal to 1.

We analyze the value of each factor for an integer $n > 1$:

1. The first factor is $n^2 - 2n + 2$. This can be rewritten as $(n - 1)^2 + 1$. Since $n > 1$, the integer $n - 1 \geq 1$. Thus, $(n - 1)^2 \geq 1$, which implies that the factor $(n - 1)^2 + 1 \geq 2$.

2. The second factor is $n^2 + 2n + 2$. Since $n > 1$, n is a positive integer. Each term in the sum is positive, so $n^2 + 2n + 2 > 1$. Specifically, for $n \geq 2$, the value is at least $2^2 + 2(2) + 2 = 10$.

Since both factors are integers strictly greater than 1 for any integer $n > 1$, their product, $n^4 + 4$, must be a composite number. Therefore, $n^4 + 4$ is not prime for any integer $n > 1$. ■

30 Perfect Squares in Arithmetic Progression

Problem 30.1. Is it possible to find three perfect squares in arithmetic progression? If so, provide a general formula.

$$x^2, y^2, z^2 \text{ s.t. } y^2 - x^2 = z^2 - y^2$$

Proof. Let the three perfect squares be x^2, y^2, and z^2. For these terms to be in an arithmetic progression, the difference between consecutive terms must be constant. This condition is expressed by the equation:

$$y^2 - x^2 = z^2 - y^2$$

This can be rearranged to:

$$x^2 + z^2 = 2y^2$$

This is a Diophantine equation for which we seek non-trivial integer solutions (i.e., where x^2, y^2, z^2 are not all equal). We can construct a general solution. Let us introduce a change of variables by setting $x = u - v$ and $z = u + v$ for some integers u and v. Substituting these into the equation gives:

$$\begin{aligned}(u-v)^2 + (u+v)^2 &= 2y^2 \\ (u^2 - 2uv + v^2) + (u^2 + 2uv + v^2) &= 2y^2 \\ 2u^2 + 2v^2 &= 2y^2 \\ u^2 + v^2 &= y^2\end{aligned}$$

This final equation is the defining relation for a Pythagorean triple (u, v, y). Therefore, the problem of finding three squares in arithmetic progression is equivalent to finding Pythagorean triples.

The general parametric solution for primitive Pythagorean triples, known as Euclid's formula, provides expressions for u, v, and y in terms of two positive coprime integers m and k of opposite parity, with $m > k$:

$$\begin{aligned}u &= m^2 - k^2 \\ v &= 2mk \\ y &= m^2 + k^2\end{aligned}$$

By substituting these expressions for u and v back into our definitions for x and z, we obtain a general formula for the bases of the three perfect squares:

$$\begin{aligned}x &= u - v = m^2 - k^2 - 2mk \\ y &= m^2 + k^2 \\ z &= u + v = m^2 - k^2 + 2mk\end{aligned}$$

A general solution for three squares in arithmetic progression can be obtained by scaling these results by an arbitrary integer factor A. The three squares are A^2x^2, A^2y^2, and A^2z^2.

As an example, let us choose $m = 2$ and $k = 1$. These integers satisfy the conditions ($m > k$, coprime, opposite parity). This gives:

$$\begin{aligned}x &= 2^2 - 1^2 - 2(2)(1) = 4 - 1 - 4 = -1 \\ y &= 2^2 + 1^2 = 4 + 1 = 5 \\ z &= 2^2 - 1^2 + 2(2)(1) = 4 - 1 + 4 = 7\end{aligned}$$

The corresponding perfect squares are $x^2 = 1$, $y^2 = 25$, and $z^2 = 49$. The sequence $1, 25, 49$ forms an arithmetic progression with a common difference of 24. Thus, it is possible to find three perfect squares in arithmetic progression, and a general form for their bases is given by the formulas derived above. ∎

31 Divisibility by 7 using Modular Arithmetic

Problem 31.1. Show that for any natural number n, $3^{2n+1} + 2^{n+2}$ is divisible by 7.

Proof. To prove that the expression $3^{2n+1} + 2^{n+2}$ is divisible by 7 for any natural number n, we must show that it is congruent to 0 modulo 7. We begin by manipulating the expression algebraically:

$$\begin{aligned} 3^{2n+1} + 2^{n+2} &= 3 \cdot 3^{2n} + 2^2 \cdot 2^n \\ &= 3 \cdot (3^2)^n + 4 \cdot 2^n \\ &= 3 \cdot 9^n + 4 \cdot 2^n \end{aligned}$$

Now, we consider this expression in the context of modular arithmetic with modulus 7. The base 9 can be reduced modulo 7:

$$9 \equiv 2 \pmod 7$$

Substituting this congruence into our expression, we have:

$$3 \cdot 9^n + 4 \cdot 2^n \equiv 3 \cdot 2^n + 4 \cdot 2^n \pmod 7$$

We can combine the terms on the right-hand side:

$$3 \cdot 2^n + 4 \cdot 2^n = (3 + 4) \cdot 2^n = 7 \cdot 2^n$$

The resulting expression is clearly a multiple of 7. Therefore:

$$7 \cdot 2^n \equiv 0 \pmod 7$$

Since $3^{2n+1} + 2^{n+2} \equiv 0 \pmod 7$ for all natural numbers n, the expression is divisible by 7. This concludes the proof. ■

32 Completing the Square: $n^2 + 20n + 11 = k^2$

Problem 32.1. Find all integers n for which $n^2 + 20n + 11$ is a perfect square.

Proof. We are looking for all integers n such that the expression $n^2 + 20n + 11$ is a perfect square. Let this perfect square be denoted by k^2 for some non-negative integer k.

$$n^2 + 20n + 11 = k^2$$

We can solve this equation by completing the square for the terms involving n. The expression $n^2 + 20n$ is part of the expansion of $(n + 10)^2 = n^2 + 20n + 100$. Rewriting the equation by adding and subtracting 100 on the left-hand side, we get:

$$\begin{aligned} (n^2 + 20n + 100) - 100 + 11 &= k^2 \\ (n + 10)^2 - 89 &= k^2 \end{aligned}$$

Rearranging the terms yields a difference of two squares:

$$(n + 10)^2 - k^2 = 89$$

Factoring the left-hand side gives:

$$((n + 10) - k)((n + 10) + k) = 89$$

Since n and k are integers, the terms $(n + 10 - k)$ and $(n + 10 + k)$ must be integer factors of 89. The number 89 is a prime number, so its only integer factor pairs are $(1, 89)$, $(89, 1)$, $(-1, -89)$, and $(-89, -1)$. Let $A = n + 10 - k$ and $B = n + 10 + k$. Note that $B - A = (n + 10 + k) - (n + 10 - k) = 2k$, which is an even integer. This implies that A and B must have the same parity. Since their product, 89, is odd, both A and B must be odd, which is consistent with the factors of 89. Also, since k is non-negative, $B \geq A$. We examine the possible cases.

1. Case 1: $n + 10 - k = 1$ and $n + 10 + k = 89$. Adding the two equations yields $2(n + 10) = 90$, which simplifies to $n + 10 = 45$. This gives the solution $n = 35$. Subtracting the first equation from the second gives $2k = 88$, so $k = 44$.

2. Case 2: $n + 10 - k = -89$ and $n + 10 + k = -1$. Adding the two equations yields $2(n + 10) = -90$, which simplifies to $n + 10 = -45$. This gives the solution $n = -55$. Subtracting the first equation from the second gives $2k = 88$, so $k = 44$.

The other two factor pairs, $(89, 1)$ and $(-1, -89)$, are excluded by the condition $B \geq A$ (since $k \geq 0$).

Thus, the integers n for which the expression is a perfect square are $n = 35$ and $n = -55$. For both values, $n^2 + 20n + 11 = 1936 = 44^2$. ■

33 Repunits and Divisibility by 7

Problem 33.1. Consider an integer A formed by $3n$ ones. Show that if A is divisible by 7, then n is divisible by 2.

Proof. Let A be the integer formed by $3n$ consecutive digits '1'. Such a number is known as a repunit, denoted R_{3n}. It can be expressed algebraically as:

$$A = R_{3n} = \sum_{i=0}^{3n-1} 10^i = \frac{10^{3n} - 1}{9}$$

We are given that A is divisible by 7, which can be written as the congruence:

$$\frac{10^{3n} - 1}{9} \equiv 0 \pmod{7}$$

To eliminate the denominator, we can multiply by 9. This is equivalent to multiplying by 2, since $9 \equiv 2 \pmod{7}$. The congruence becomes:

$$2 \cdot \frac{10^{3n} - 1}{9} \equiv 2 \cdot 0 \pmod{7}$$
$$10^{3n} - 1 \equiv 0 \pmod{7}$$

This simplifies to:

$$10^{3n} \equiv 1 \pmod{7}$$

To analyze this congruence, we first reduce the base modulo 7. Since $10 \equiv 3 \pmod{7}$, the congruence is equivalent to:

$$3^{3n} \equiv 1 \pmod{7}$$

To find the conditions on the exponent $3n$, we must determine the order of 3 modulo 7. We compute the powers of 3 modulo 7:

$$\begin{aligned}
3^1 &\equiv 3 \pmod{7} \\
3^2 &\equiv 9 \equiv 2 \pmod{7} \\
3^3 &\equiv 3 \cdot 2 = 6 \pmod{7} \\
3^4 &\equiv 3 \cdot 6 = 18 \equiv 4 \pmod{7} \\
3^5 &\equiv 3 \cdot 4 = 12 \equiv 5 \pmod{7} \\
3^6 &\equiv 3 \cdot 5 = 15 \equiv 1 \pmod{7}
\end{aligned}$$

The order of 3 modulo 7 is 6. For the congruence $3^{3n} \equiv 1 \pmod{7}$ to hold, the exponent $3n$ must be a multiple of the order 6. Therefore, there must exist an integer k such that:

$$3n = 6k$$

Dividing both sides by 3, we obtain:

$$n = 2k$$

This result shows that n must be a multiple of 2, which means that n is an even integer. Thus, if A is divisible by 7, then n must be divisible by 2. ■

34 Difference of Consecutive Odd Squares

Problem 34.1. Prove that the difference between the squares of two consecutive odd integers is always divisible by 8.

Proof. Let the two consecutive odd integers be represented by $2n - 1$ and $2n + 1$ for some integer n. We seek to analyze the difference of their squares. Let D be this difference.

$$\begin{aligned} D &= (2n+1)^2 - (2n-1)^2 \\ &= (4n^2 + 4n + 1) - (4n^2 - 4n + 1) \\ &= 4n^2 + 4n + 1 - 4n^2 + 4n - 1 \\ &= 8n \end{aligned}$$

Alternatively, by applying the difference of squares formula $a^2 - b^2 = (a - b)(a + b)$, we have:

$$\begin{aligned} D &= ((2n+1) - (2n-1))((2n+1) + (2n-1)) \\ &= (2n + 1 - 2n + 1)(2n + 1 + 2n - 1) \\ &= (2)(4n) \\ &= 8n \end{aligned}$$

Since n is an integer, the expression $8n$ is, by definition, a multiple of 8. Therefore, the difference between the squares of any two consecutive odd integers is divisible by 8. ■

35 The Equation $x^2 + y^2 = 3z^2$nd Infinite Descent

Problem 35.1. Show that the equation $x^2 + y^2 = 3z^2$ has no solutions in non-zero integers.

Proof. We use the method of infinite descent. Assume, for the sake of contradiction, that a solution in non-zero integers (x, y, z) exists. If such solutions exist, there must be a solution (x_0, y_0, z_0) for which $|z_0|$ is minimal among all non-zero solutions. We can further assume that x_0, y_0, z_0 are pairwise coprime, as any common factor could be divided out to yield a solution with a smaller $|z_0|$.

Consider the equation modulo 3:

$$x_0^2 + y_0^2 \equiv 3z_0^2 \pmod{3}$$

$$x_0^2 + y_0^2 \equiv 0 \pmod 3$$

The quadratic residues modulo 3 are $0^2 \equiv 0$ and $(\pm 1)^2 \equiv 1$. For the sum of two squares to be congruent to 0 modulo 3, both terms must be congruent to 0. This implies $x_0^2 \equiv 0 \pmod 3$ and $y_0^2 \equiv 0 \pmod 3$, which in turn means that x_0 and y_0 must both be divisible by 3. Let $x_0 = 3x_1$ and $y_0 = 3y_1$ for some integers x_1, y_1. Substituting these into the original equation gives:

$$\begin{aligned}(3x_1)^2 + (3y_1)^2 &= 3z_0^2 \\ 9x_1^2 + 9y_1^2 &= 3z_0^2\end{aligned}$$

Dividing the entire equation by 3, we obtain:

$$3x_1^2 + 3y_1^2 = z_0^2$$

$$3(x_1^2 + y_1^2) = z_0^2$$

This equation implies that z_0^2 is divisible by 3. Consequently, z_0 must also be divisible by 3. This contradicts our assumption that x_0, y_0, z_0 are pairwise coprime, as all three are divisible by 3.

Alternatively, continuing the descent, let $z_0 = 3z_1$ for some integer z_1. Substituting this into the equation $3(x_1^2 + y_1^2) = z_0^2$ yields:

$$\begin{aligned}3(x_1^2 + y_1^2) &= (3z_1)^2 \\ 3(x_1^2 + y_1^2) &= 9z_1^2\end{aligned}$$

Dividing by 3 gives a new equation:

$$x_1^2 + y_1^2 = 3z_1^2$$

This demonstrates that (x_1, y_1, z_1) is another integer solution to the original equation. Since $z_0 = 3z_1$ and z_0 is non-zero, it follows that $|z_1| = |z_0|/3 < |z_0|$. This contradicts the initial assumption that $|z_0|$ was minimal.

The only way to avoid this contradiction is if the initial solution is the trivial one, $(0, 0, 0)$. Therefore, the equation $x^2 + y^2 = 3z^2$ has no solutions in non-zero integers. ■

36 Digital Roots and Divisibility by 9

Problem 36.1. If the sum of the digits of a number A is S, and the sum of the digits of S is T, show that $A - T$ is always divisible by 9.

Proof. Let $s(n)$ denote the sum of the digits of a positive integer n. A fundamental property of the decimal representation of integers is that any integer is congruent to the sum of its digits modulo 9. This can be expressed as:

$$n \equiv s(n) \pmod 9$$

We are given that $s(A) = S$ and $s(S) = T$. Applying the aforementioned property to the integer A, we have:

$$A \equiv s(A) \pmod 9$$

Substituting $s(A) = S$, we obtain the congruence:

$$A \equiv S \pmod 9$$

Similarly, applying the same property to the integer S, we have:

$$S \equiv s(S) \pmod 9$$

Substituting $s(S) = T$, we obtain:

$$S \equiv T \pmod 9$$

By the transitive property of modular congruence, since $A \equiv S \pmod 9$ and $S \equiv T \pmod 9$, it follows that:

$$A \equiv T \pmod 9$$

This congruence is, by definition, equivalent to the statement that the difference $A - T$ is a multiple of 9. Therefore, $A - T$ is divisible by 9. ■

37 A Digital Append Property

Problem 37.1. A natural number n has the property that if the digit 1 is appended to both the beginning and the end of its decimal representation, the result is 99 times n. Find n.

Proof. Let n be a natural number with k digits, where k is a positive integer. This implies that $10^{k-1} \leq n < 10^k$. Let N be the new number formed by appending the digit 1 to both the beginning and the end of the decimal representation of n. The resulting number N will have $k + 2$ digits. The value of N can be expressed algebraically as:

$$N = 1 \cdot 10^{k+1} + n \cdot 10^1 + 1 = 10^{k+1} + 10n + 1$$

According to the problem statement, this new number N is equal to $99n$. We can therefore establish the equation:

$$10^{k+1} + 10n + 1 = 99n$$

To find the value of n, we solve for n in terms of k:

$$10^{k+1} + 1 = 99n - 10n$$
$$10^{k+1} + 1 = 89n$$

This gives an expression for n:

$$n = \frac{10^{k+1} + 1}{89}$$

Since n must be an integer, the numerator $10^{k+1} + 1$ must be divisible by 89. This condition can be expressed using modular arithmetic:

$$10^{k+1} + 1 \equiv 0 \pmod{89}$$

$$10^{k+1} \equiv -1 \pmod{89}$$

Since 89 is a prime number, we can investigate the powers of 10 modulo 89 to find the smallest positive integer value of $k+1$ that satisfies this congruence. By Fermat's Little Theorem, $10^{88} \equiv 1 \pmod{89}$. The order of 10 modulo 89 must divide 88. Let us compute intermediate powers:

$$\begin{aligned}
10^2 &\equiv 100 \equiv 11 \pmod{89} \\
10^4 &\equiv 11^2 = 121 \equiv 32 \pmod{89} \\
10^8 &\equiv 32^2 = 1024 = 11 \cdot 89 + 45 \equiv 45 \pmod{89} \\
10^{11} &\equiv 10^8 \cdot 10^3 \equiv 45 \cdot (10 \cdot 11) = 45 \cdot 110 \equiv 45 \cdot 21 = 945 = 10 \cdot 89 + 55 \equiv 55 \pmod{89} \\
10^{22} &\equiv 55^2 = 3025 = 34 \cdot 89 - 1 \equiv -1 \pmod{89}
\end{aligned}$$

The smallest positive integer exponent m for which $10^m \equiv -1 \pmod{89}$ is $m = 22$. Therefore, we must have $k+1 = 22$, which implies that the number of digits in n is $k = 21$. This value is consistent with our initial assumption, as the resulting value of n satisfies $10^{20} \leq n < 10^{21}$. The magnitude of n is approximately:

$$n = \frac{10^{22}+1}{89} \approx \frac{100}{89} \times 10^{20} \approx 1.12 \times 10^{20}$$

Thus, n is indeed a 21-digit number. The unique natural number satisfying the given property is:

$$n = \frac{10^{22}+1}{89}$$

■

38 Numbers of the Form $10^k + 1$

Problem 38.1. Show that a number written as $100\ldots001$ (with an even number of zeros) is never a perfect square (excluding the trivial case 1).

Proof. An integer with an even number of zeros between two digits '1' can be expressed in the form $10^k + 1$, where $k-1$ is the number of zeros. The condition that the number of zeros is even implies that $k-1$ is a positive even integer, so k must be an odd integer greater than or equal to 3. The case of zero zeros corresponds to $k = 1$, yielding 11, which is not a perfect square. The problem as stated thus reduces to showing that $10^k + 1$ is not a perfect square for any odd integer $k \geq 1$.

Let us assume, for the sake of contradiction, that $10^k + 1 = m^2$ for some integer m and some odd integer $k \geq 1$. This equation can be rearranged as:

$$10^k = m^2 - 1 = (m-1)(m+1)$$

Since 10^k is even, $m^2 - 1$ must be even, which implies m^2 is odd, and therefore m is an odd integer. Let $m = 2s+1$ for some integer s. Then $m-1 = 2s$ and $m+1 = 2s+2$. The greatest common divisor is $\gcd(m-1, m+1) = \gcd(2s, 2s+2) = 2\gcd(s, s+1) = 2$.

The prime factorization of the left-hand side is $10^k = (2 \cdot 5)^k = 2^k 5^k$. Since $\gcd(m-1, m+1) = 2$, one of these factors must contain a single factor of 2, while the other must contain the remaining $k-1$ factors of 2. Furthermore, since 5 is a prime, the entire factor of 5^k must belong to either $(m-1)$ or $(m+1)$. As $m+1 > m-1$, this leads to two possible systems of equations.

Case 1: $m-1$ contains the factor 5^k.

$$\begin{cases} m+1 = 2^{k-1} \\ m-1 = 2\cdot 5^k \end{cases}$$

Subtracting the second equation from the first gives:

$$(m+1)-(m-1) = 2^{k-1} - 2\cdot 5^k = 2$$

Dividing by 2, we obtain $2^{k-2} - 5^k = 1$. For $k=1$, this gives $2^{-1} - 5 = 1/2 - 5 \neq 1$. For any odd integer $k \geq 3$, we have $k-2 \geq 1$. In this range, $5^k > 2^{k-2}$, so $2^{k-2} - 5^k$ is a negative integer. Therefore, this case yields no solutions.

Case 2: $m+1$ contains the factor 5^k.

$$\begin{cases} m-1 = 2^{k-1} \\ m+1 = 2\cdot 5^k \end{cases}$$

Subtracting the first equation from the second gives:

$$(m+1)-(m-1) = 2\cdot 5^k - 2^{k-1} = 2$$

Dividing by 2, we obtain $5^k - 2^{k-2} = 1$. We must determine if this equation has solutions for any odd integer $k \geq 1$.

- For $k=1$, the equation is $5^1 - 2^{-1} = 1$, which is not an integer equation and has no solution.
- For $k=3$, we have $5^3 - 2^{3-2} = 125 - 2 = 123 \neq 1$.
- For any odd integer $k \geq 3$, we can analyze the equation modulo 5:

  $$5^k - 2^{k-2} \equiv 1 \pmod 5$$

  $$-2^{k-2} \equiv 1 \pmod 5$$

  The powers of 2 modulo 5 cycle as $2^1 \equiv 2$, $2^2 \equiv 4$, $2^3 \equiv 3$, $2^4 \equiv 1$. Since k is odd, the exponent $k-2$ is also odd. The powers 2^j for odd j are congruent to 2 or 3 modulo 5. Consequently, -2^{k-2} is congruent to $-2 \equiv 3$ or $-3 \equiv 2$ modulo 5. In neither case is it congruent to 1.

Thus, the equation $5^k - 2^{k-2} = 1$ has no integer solutions for any odd $k \geq 1$.

Both cases lead to a contradiction. Therefore, the initial assumption is false, and no number of the form $10^k + 1$ for odd $k \geq 1$ can be a perfect square. ∎

39 Trailing Zeros in Factorials

Problem 39.1. Find all positive integers n such that $n!$ ends in exactly 20 zeros.

Proof. The number of trailing zeros in the decimal representation of $n!$ is determined by the exponent of the prime 5 in the prime factorization of $n!$. This is because the exponent of 2 is always greater than the exponent of 5. The exponent of a prime p in the factorization of $n!$, denoted by $v_p(n!)$, is given by Legendre's formula:

$$v_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$$

We are looking for all positive integers n such that $v_5(n!) = 20$. This requires solving the equation:

$$\sum_{i=1}^{\infty} \left\lfloor \frac{n}{5^i} \right\rfloor = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \left\lfloor \frac{n}{125} \right\rfloor + \cdots = 20$$

The sum of the infinite geometric series $\sum_{i=1}^{\infty} \frac{n}{5^i} = n\left(\frac{1/5}{1-1/5}\right) = \frac{n}{4}$ provides an upper bound for the number of zeros. Thus, we have:

$$20 < \frac{n}{4} \implies n > 80$$

This inequality gives a lower bound for n. Let us evaluate the number of zeros for $n = 80$:

$$v_5(80!) = \left\lfloor \frac{80}{5} \right\rfloor + \left\lfloor \frac{80}{25} \right\rfloor = 16 + 3 = 19$$

The function $v_5(n!)$ is a non-decreasing step function that increases only when n is a multiple of 5. The value of $v_5(n!)$ will remain 19 for $n = 80, 81, 82, 83, 84$. The next increase will occur at $n = 85$. Let us evaluate the number of zeros for $n = 85$:

$$v_5(85!) = \left\lfloor \frac{85}{5} \right\rfloor + \left\lfloor \frac{85}{25} \right\rfloor = 17 + 3 = 20$$

This shows that $n = 85$ is a solution. The value of $v_5(n!)$ will remain 20 for integers greater than 85 until the next multiple of 5, which is $n = 90$. For any integer n in the range $85 \le n < 90$, the values of $\lfloor n/5 \rfloor$ and $\lfloor n/25 \rfloor$ are 17 and 3 respectively, so $v_5(n!) = 20$. At $n = 90$, the number of zeros becomes:

$$v_5(90!) = \left\lfloor \frac{90}{5} \right\rfloor + \left\lfloor \frac{90}{25} \right\rfloor = 18 + 3 = 21$$

Since the function $v_5(n!)$ is strictly increasing at each multiple of 5, for all $n \ge 90$ the number of trailing zeros will be greater than or equal to 21.

Therefore, the only positive integers n for which $n!$ ends in exactly 20 zeros are those in the set $\{85, 86, 87, 88, 89\}$. ■

40 Properties of Primes Greater Than 3

Problem 40.1. Prove that if p is a prime number greater than 3, then $p^2 - 1$ is divisible by 24.

Proof. To demonstrate that $p^2 - 1$ is divisible by 24, it is sufficient to show that it is divisible by both 3 and 8, since $\gcd(3, 8) = 1$.

1. Divisibility by 3

Since p is a prime number greater than 3, it is not divisible by 3. Consequently, p must be of the form $3k+1$ or $3k+2$ for some integer k. An equivalent statement is that $p \equiv 1 \pmod 3$ or $p \equiv 2 \pmod 3$. In either case, squaring the congruence yields:

$$p^2 \equiv 1^2 \equiv 1 \pmod 3 \quad \text{or} \quad p^2 \equiv 2^2 \equiv 4 \equiv 1 \pmod 3$$

Thus, for any prime $p > 3$, we have $p^2 \equiv 1 \pmod 3$. This implies $p^2 - 1 \equiv 0 \pmod 3$, so $p^2 - 1$ is divisible by 3.

Alternatively, we may consider the factorization $p^2 - 1 = (p-1)(p+1)$. The integers $p-1, p, p+1$ are three consecutive integers. One of them must be divisible by 3. As p is a prime greater than 3, it cannot be divisible by 3. Therefore, either $p-1$ or $p+1$ is divisible by 3, which implies their product is divisible by 3.

2. Divisibility by 8

Since p is a prime number greater than 3, it must be an odd integer. As such, both $p-1$ and $p+1$ are consecutive even integers. Let $p-1 = 2m$ for some integer m. Then $p+1 = 2m+2 = 2(m+1)$. Substituting these into the factorization gives:

$$p^2 - 1 = (p-1)(p+1) = (2m)(2(m+1)) = 4m(m+1)$$

The product of two consecutive integers, $m(m+1)$, is always even. Let $m(m+1) = 2k$ for some integer k. Then:

$$p^2 - 1 = 4(2k) = 8k$$

This demonstrates that $p^2 - 1$ is divisible by 8.

Conclusion

Since $p^2 - 1$ is divisible by both 3 and 8, and these factors are coprime, it must be divisible by their product, $3 \times 8 = 24$. ■

41 Repdigits from Multiplication

Problem 41.1. What is the smallest positive integer which, when multiplied by 33, yields a number consisting only of the digit 7?

Proof. Let the unknown positive integer be x. Let the resulting number, which consists of k repetitions of the digit 7, be denoted by N_k. The problem can be stated as the equation:

$$33x = N_k$$

The number N_k is a repdigit and can be expressed using the repunit $R_k = \sum_{i=0}^{k-1} 10^i$. Specifically, $N_k = 7 \cdot R_k$. The equation becomes:

$$33x = 7R_k$$

For an integer solution x to exist, $7R_k$ must be divisible by 33. Since $\gcd(7, 33) = 1$, this requires that R_k must be divisible by 33. As $33 = 3 \times 11$, R_k must be divisible by both 3 and 11.

1. Divisibility of R_k by 3

A number is divisible by 3 if the sum of its digits is a multiple of 3. The sum of the digits of R_k is $k \cdot 1 = k$. Therefore, R_k is divisible by 3 if and only if k is a multiple of 3.

2. Divisibility of R_k by 11

A number is divisible by 11 if the alternating sum of its digits is a multiple of 11. For R_k, this sum is $1 - 1 + 1 - \cdots + (-1)^{k-1}$. If k is even, the sum is 0, which is a multiple of 11. If k is odd, the sum is 1. Thus, for R_k to be divisible by 11, k must be an even integer.

To find the smallest integer x, we must find the smallest positive integer k that satisfies both conditions. The value of k must be a multiple of 3 and an even number. The least common multiple of 2 and 3 is 6. Thus, the smallest possible value for k is 6.

For $k = 6$, the number N_k is 777777. The equation becomes:

$$33x = 777777$$

Solving for x:

$$x = \frac{777777}{33} = 23569$$

Since we have found the smallest valid value for k, the corresponding value of x is the smallest positive integer satisfying the condition. ■

42 Sum of Three Consecutive Cubes

Problem 42.1. Show that the sum of the cubes of three consecutive integers is always divisible by 9.

Proof. Let the three consecutive integers be $n - 1$, n, and $n + 1$ for some integer n. The sum of their cubes, denoted by S, is:

$$S = (n - 1)^3 + n^3 + (n + 1)^3$$

We expand the terms $(n - 1)^3$ and $(n + 1)^3$:

$$\begin{aligned} S &= (n^3 - 3n^2 + 3n - 1) + n^3 + (n^3 + 3n^2 + 3n + 1) \\ &= 3n^3 + 6n \\ &= 3n(n^2 + 2) \end{aligned}$$

To prove that S is divisible by 9, we must show that $n(n^2 + 2)$ is divisible by 3. We can establish this by considering the value of n modulo 3.

- If $n \equiv 0 \pmod 3$, then the factor n is divisible by 3, and thus the entire product $n(n^2 + 2)$ is divisible by 3.
- If $n \equiv 1 \pmod 3$, then $n^2 \equiv 1^2 \equiv 1 \pmod 3$. It follows that $n^2 + 2 \equiv 1 + 2 \equiv 3 \equiv 0 \pmod 3$. In this case, the factor $(n^2 + 2)$ is divisible by 3.
- If $n \equiv 2 \pmod 3$, then $n^2 \equiv 2^2 \equiv 4 \equiv 1 \pmod 3$. It follows that $n^2 + 2 \equiv 1 + 2 \equiv 3 \equiv 0 \pmod 3$. In this case as well, the factor $(n^2 + 2)$ is divisible by 3.

In all possible cases, the expression $n(n^2+2)$ is a multiple of 3. Therefore, $S = 3 \cdot n(n^2+2)$ is a multiple of $3 \cdot 3 = 9$. This concludes the proof that the sum of the cubes of three consecutive integers is always divisible by 9. ■

43 Digit Sums and Squaring

Problem 43.1. Does there exist an integer n such that the sum of the digits of n is S and the sum of the digits of n^2 is S^2?

Proof. Let $s(k)$ denote the sum of the digits of a non-negative integer k. The problem asks if an integer n exists such that $s(n^2) = (s(n))^2$. Such integers do exist, as demonstrated by the following examples.

Consider the integer $n = 1$. The sum of its digits is $s(1) = 1$. The square of n is $n^2 = 1^2 = 1$, and the sum of the digits of the square is $s(1) = 1$. The condition is satisfied, since $(s(1))^2 = 1^2 = 1 = s(1^2)$.

Consider the integer $n = 2$. The sum of its digits is $s(2) = 2$. Its square is $n^2 = 4$, for which the sum of digits is $s(4) = 4$. The condition is satisfied, since $(s(2))^2 = 2^2 = 4 = s(2^2)$.

For a multi-digit example, consider $n = 12$. The sum of its digits is $s(12) = 1 + 2 = 3$. The square of n is $n^2 = 144$. The sum of the digits of the square is $s(144) = 1 + 4 + 4 = 9$. The condition is satisfied, since $(s(12))^2 = 3^2 = 9 = s(12^2)$.

Further examples include any integer of the form $n = 10^k$ for $k \geq 0$. In this case, $s(n) = 1$. The square is $n^2 = 10^{2k}$, for which $s(n^2) = 1$. The condition holds as $(s(10^k))^2 = 1^2 = 1 = s((10^k)^2)$.

Since multiple such integers exist, the answer to the question is affirmative. ■

44 A Polynomial Divisibility Problem

Problem 44.1. Prove that for any integer n, the number $n(n^2 - 1)(5n + 2)$ is divisible by 24.

Proof. Let $P(n) = n(n^2 - 1)(5n + 2)$. We can rewrite the expression as:

$$P(n) = n(n - 1)(n + 1)(5n + 2)$$

To show that $P(n)$ is divisible by 24 for any integer n, we must show that it is divisible by both 3 and 8, since $\gcd(3, 8) = 1$.

1. Divisibility by 3

The term $(n - 1)n(n + 1)$ is a product of three consecutive integers. For any integer n, one of these three consecutive integers must be a multiple of 3. Therefore, their product is always divisible by 3. Since $(n - 1)n(n + 1)$ is a factor of $P(n)$, it follows that $P(n)$ is divisible by 3 for all integers n.

2. Divisibility by 8

We consider two cases based on the parity of n.

- **Case 1: n is odd.** If n is odd, then both $(n - 1)$ and $(n + 1)$ are consecutive even integers. Let $n - 1 = 2k$ for some integer k. Then $n + 1 = 2k + 2 = 2(k + 1)$. Their product is $(n - 1)(n + 1) = (2k)(2(k + 1)) = 4k(k + 1)$. The term $k(k + 1)$ is a product of two consecutive integers, which is always even. Thus, $k(k + 1) = 2m$ for some integer m. This gives $(n - 1)(n + 1) = 4(2m) = 8m$. Since the product $(n - 1)(n + 1)$ is divisible by 8, the entire expression $P(n)$ is divisible by 8 when n is odd.

- **Case 2: n is even.** If n is even, let $n = 2k$ for some integer k. We substitute this into the expression for $P(n)$:

$$P(2k) = 2k(2k-1)(2k+1)(5(2k)+2) = 2k(2k-1)(2k+1)(10k+2)$$
$$P(2k) = 4k(2k-1)(2k+1)(5k+1)$$

We further subdivide this case based on the parity of k.

  - If k is even, then $k = 2m$ for some integer m. The factor $4k$ becomes $4(2m) = 8m$. Thus, $P(n)$ is divisible by 8.
  - If k is odd, then $k = 2m+1$ for some integer m. The term $(5k+1)$ becomes $5(2m+1)+1 = 10m+5+1 = 10m+6 = 2(5m+3)$, which is an even number. The expression for $P(n)$ contains the product $4k \cdot (5k+1)$. Since k is odd and $(5k+1)$ is even, their product is $4 \cdot (\text{odd}) \cdot (\text{even})$, which is divisible by $4 \cdot 2 = 8$. Thus, $P(n)$ is divisible by 8.

  In both subcases for even n, $P(n)$ is divisible by 8.

Since $P(n)$ is divisible by 3 and by 8 for any integer n, it must be divisible by their least common multiple, which is $\operatorname{lcm}(3,8) = 24$. ■

45 Divisibility of Concatenated Integers

Problem 45.1. A number is formed by concatenating integers from 1 to n. For which values of n is this number divisible by 9?

Proof. Let C_n denote the integer formed by concatenating the decimal representations of the integers from 1 to n. A fundamental property of divisibility states that an integer is divisible by 9 if and only if the sum of its digits is divisible by 9. Let $S(C_n)$ be the sum of the digits of C_n. We require $S(C_n) \equiv 0 \pmod 9$.

The sum of the digits of C_n is the sum of the digits of each integer from 1 to n. Let $s(k)$ denote the sum of the digits of an integer k. Then:

$$S(C_n) = \sum_{k=1}^{n} s(k)$$

Another established result is that any integer is congruent to the sum of its digits modulo 9. That is, $k \equiv s(k) \pmod 9$ for any integer k. Applying this property to our sum, we have:

$$S(C_n) = \sum_{k=1}^{n} s(k) \equiv \sum_{k=1}^{n} k \pmod 9$$

The sum of the first n positive integers is given by the formula $\frac{n(n+1)}{2}$. Therefore, the condition for divisibility by 9 becomes:

$$\frac{n(n+1)}{2} \equiv 0 \pmod 9$$

This congruence is equivalent to stating that $\frac{n(n+1)}{2}$ is a multiple of 9. Let $\frac{n(n+1)}{2} = 9m$ for some integer m. Multiplying by 2, we get:

$$n(n+1) = 18m$$

This means that the product of the consecutive integers n and $n+1$ must be divisible by 18. Since $18 = 2 \cdot 9$ and $\gcd(2, 9) = 1$, the product $n(n+1)$ must be divisible by both 2 and 9.

The product of any two consecutive integers, $n(n+1)$, is always even, so it is always divisible by 2. Therefore, the problem reduces to finding all integers n such that $n(n+1)$ is divisible by 9. Since n and $n+1$ are coprime, they cannot both be multiples of 3. For their product to be divisible by 9, either n must be divisible by 9, or $n+1$ must be divisible by 9.

- **Case 1: n is divisible by 9.** This means $n \equiv 0 \pmod 9$.
- **Case 2: $n+1$ is divisible by 9.** This means $n+1 \equiv 0 \pmod 9$, which is equivalent to $n \equiv -1 \equiv 8 \pmod 9$.

Thus, the concatenated number C_n is divisible by 9 if and only if n is a multiple of 9 or n is one less than a multiple of 9. ■

46 Brocard's Problem

Problem 46.1. Find all integers x, y such that:

$$1! + 2! + \cdots + x! = y^2$$

Proof. We are looking for integer solutions to the equation $\sum_{i=1}^{x} i! = y^2$. We proceed by testing small values of x. Let $S_x = \sum_{i=1}^{x} i!$.

- For $x = 1$, $S_1 = 1! = 1 = 1^2$. Thus, $(x, y) = (1, 1)$ is a solution.
- For $x = 2$, $S_2 = 1! + 2! = 1 + 2 = 3$, which is not a perfect square.
- For $x = 3$, $S_3 = 1! + 2! + 3! = 1 + 2 + 6 = 9 = 3^2$. Thus, $(x, y) = (3, 3)$ is a solution.
- For $x = 4$, $S_4 = 1! + 2! + 3! + 4! = 9 + 24 = 33$, which is not a perfect square.

Now, consider the case where $x \geq 5$. For any integer $i \geq 5$, the factorial $i!$ is divisible by both 2 and 5, so its decimal representation ends in a 0. This means $i! \equiv 0 \pmod{10}$ for all $i \geq 5$. We can express the sum S_x for $x \geq 4$ as:

$$S_x = \sum_{i=1}^{4} i! + \sum_{i=5}^{x} i! = S_4 + \sum_{i=5}^{x} i!$$

We have $S_4 = 33$. For any $x \geq 5$, the sum $\sum_{i=5}^{x} i!$ is a sum of integers each ending in 0, so the sum itself must end in 0. Therefore, for any $x \geq 4$, the sum S_x can be analyzed modulo 10:

$$S_x = 33 + \sum_{i=5}^{x} i! \equiv 3 + 0 \pmod{10} \implies S_x \equiv 3 \pmod{10}$$

This shows that for any $x \geq 4$, the units digit of S_x is 3.

We now examine the units digit of a perfect square y^2. The units digit of a square is determined by the units digit of its root. The squares of the integers from 0 to 9 are:

$$0^2 = 0,\ 1^2 = 1,\ 2^2 = 4,\ 3^2 = 9,\ 4^2 = 16,\ 5^2 = 25,\ 6^2 = 36,\ 7^2 = 49,\ 8^2 = 64,\ 9^2 = 81$$

The possible units digits for a perfect square are therefore $\{0, 1, 4, 5, 6, 9\}$. Since the units digit of S_x is 3 for all $x \geq 4$, and 3 is not a possible units digit for a perfect square, there can be no solutions for $x \geq 4$.

The only integer solutions are those found for $x < 4$, which are $(1, 1)$ and $(3, 3)$. ∎

47 The Number of Divisors of a Perfect Square

Problem 47.1. Show that if a number has an odd number of divisors, then that number is a perfect square.

Proof. Let n be a positive integer. By the fundamental theorem of arithmetic, n has a unique prime factorization of the form:

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

where $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $a_1, a_2, \ldots, a_k$ are positive integer exponents. The number of positive divisors of n, denoted by $d(n)$, is given by the product:

$$d(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$$

We are given that $d(n)$ is an odd integer. For the product of the integers $(a_i + 1)$ to be odd, it is necessary that each factor $(a_i + 1)$ be an odd integer. If each term $(a_i + 1)$ is odd, then each exponent a_i must be an even integer. Let $a_i = 2b_i$ for some positive integers b_i. Substituting these even exponents back into the prime factorization of n, we obtain:

$$n = p_1^{2b_1} p_2^{2b_2} \cdots p_k^{2b_k}$$

This expression can be rewritten as:

$$n = (p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k})^2$$

Let $m = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$. Since the primes p_i and exponents b_i are integers, m is an integer. Thus, $n = m^2$, which by definition means that n is a perfect square.

Conversely, if n is a perfect square, then $n = m^2$ for some integer m. In its prime factorization, all exponents a_i must be even. Therefore, each factor $(a_i + 1)$ in the formula for $d(n)$ must be odd. The product of odd integers is always odd, so $d(n)$ must be odd. This confirms that a number is a perfect square if and only if it has an odd number of divisors. ∎

48 Properties of Twin Primes

Problem 48.1. If p and q are twin primes (i.e., $q = p + 2$), show that $pq + 1$ is a perfect square.

Proof. Let p and q be a pair of twin primes. By the definition of twin primes, they are prime numbers that differ by 2. We can thus write $q = p + 2$. We are asked to examine the expression $pq + 1$. Substituting the relation between p and q into this expression yields:

$$pq + 1 = p(p + 2) + 1$$

Expanding the product, we obtain a quadratic expression in p:

$$p(p + 2) + 1 = p^2 + 2p + 1$$

This expression is a standard algebraic identity for a perfect square trinomial:

$$p^2 + 2p + 1 = (p + 1)^2$$

Since p is a prime number, it is an integer. Consequently, $p+1$ is also an integer. Therefore, the expression $pq + 1$ is equal to the square of the integer $p + 1$. This demonstrates that for any pair of twin primes $(p, p + 2)$, the number $pq + 1$ is a perfect square. ∎

49 Fermat's Little Theorem and Divisibility of $n^5 - n$

Problem 49.1. Prove that for any integer n, $n^5 - n$ is divisible by 30.

Proof. Let the expression be denoted by $P(n) = n^5 - n$. To prove that $P(n)$ is divisible by 30 for any integer n, we must show that it is divisible by the prime factors of 30, which are 2, 3, and 5.

First, we factor the expression algebraically:

$$\begin{aligned} P(n) &= n(n^4 - 1) \\ &= n(n^2 - 1)(n^2 + 1) \\ &= n(n - 1)(n + 1)(n^2 + 1) \\ &= (n - 1)n(n + 1)(n^2 + 1) \end{aligned}$$

1. Divisibility by 2

The term $(n - 1)n(n + 1)$ is the product of three consecutive integers. Among any two consecutive integers, one must be even. Therefore, the product contains at least one factor of 2 and is always divisible by 2. Thus, $P(n)$ is divisible by 2.

2. Divisibility by 3

Among any three consecutive integers, one must be a multiple of 3. Since $(n - 1)$, n, and $(n + 1)$ are three consecutive integers, their product is divisible by 3. Thus, $P(n)$ is divisible by 3.

3. Divisibility by 5

We can demonstrate divisibility by 5 using Fermat's Little Theorem. The theorem states that if p is a prime number, then for any integer a, $a^p \equiv a \pmod{p}$. Applying this theorem for the prime $p = 5$, we have:

$$n^5 \equiv n \pmod{5}$$

Rearranging this congruence gives:

$$n^5 - n \equiv 0 \pmod{5}$$

This shows that $n^5 - n$ is divisible by 5 for any integer n.

Alternatively, we can analyze the factors of $P(n)$ modulo 5.

- If $n \equiv 0, 1$, or $4 \pmod 5$, then one of the factors n, $n-1$, or $n+1$ is congruent to 0 modulo 5, respectively. In these cases, $P(n)$ is divisible by 5.
- If $n \equiv 2 \pmod 5$, the factor n^2+1 becomes $2^2+1 = 5 \equiv 0 \pmod 5$.
- If $n \equiv 3 \pmod 5$, the factor n^2+1 becomes $3^2+1 = 10 \equiv 0 \pmod 5$.

In all possible cases for n modulo 5, the expression $P(n)$ is divisible by 5.

Since $P(n)$ is divisible by 2, 3, and 5, and these numbers are pairwise coprime, it follows that $P(n)$ must be divisible by their product, $2 \cdot 3 \cdot 5 = 30$. This completes the proof. ■

50 Number of Divisors of S^S

Problem 50.1. Let S be a positive integer. Find an expression for the number of positive divisors of the integer S^S.

Proof. The number of positive divisors of an integer is determined by its prime factorization. Let $\tau(n)$ denote the number of positive divisors of a positive integer n. If the canonical prime factorization of n is given by $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where p_i are distinct prime numbers and a_i are positive integers, then the number of divisors is given by the formula:

$$\tau(n) = (a_1+1)(a_2+1)\cdots(a_k+1)$$

Let the prime factorization of the integer S be:

$$S = q_1^{e_1} q_2^{e_2} \cdots q_m^{e_m}$$

where q_j are distinct prime numbers and e_j are positive integer exponents.

Using this representation, we can express the number S^S as:

$$S^S = (q_1^{e_1} q_2^{e_2} \cdots q_m^{e_m})^S$$

By applying the laws of exponents, we obtain the prime factorization of S^S:

$$S^S = q_1^{e_1 S} q_2^{e_2 S} \cdots q_m^{e_m S}$$

This is the canonical prime factorization of S^S, as the prime bases q_j are distinct. The exponent of each prime factor q_j is $e_j S$.

We can now apply the divisor function formula to the number S^S. The exponents in its prime factorization are $a_j = e_j S$. The number of positive divisors of S^S is therefore:

$$\tau(S^S) = (e_1 S+1)(e_2 S+1)\cdots(e_m S+1)$$

This expression provides the number of divisors of S^S in terms of S and the exponents in the prime factorization of S. ■

Example 50.1. Consider the case where $S = 12$. The prime factorization of S is $12 = 2^2 \cdot 3^1$. In this case, the prime bases are $q_1 = 2$ and $q_2 = 3$, with corresponding exponents $e_1 = 2$ and $e_2 = 1$. We seek the number of divisors of 12^{12}. Using the derived formula with $S = 12$, $e_1 = 2$, and $e_2 = 1$, we find:

$$\begin{aligned}
\tau(12^{12}) &= (e_1 S + 1)(e_2 S + 1) \\
&= (2 \cdot 12 + 1)(1 \cdot 12 + 1) \\
&= (24 + 1)(12 + 1) \\
&= 25 \cdot 13 = 325
\end{aligned}$$

The number 12^{12} has 325 positive divisors.