Congruent and Similar Triangles

Complete Worksheet with Solutions

Part 01

1 Congruent and Similar Triangles

1.1 Definitions

1.1.1 Congruent Triangles

Two triangles are **congruent** if they have the same size and shape. This means corresponding sides and angles are equal.

$$\triangle ABC \cong \triangle DEF \iff \begin{cases} AB = DE, \\ BC = EF, \\ CA = FD, \\ \angle A = \angle D, \\ \angle B = \angle E, \\ \angle C = \angle F \end{cases}$$

Criteria for Congruence:

- **SSS:** Side–Side–Side
- **SAS:** Side–Angle–Side
- **ASA:** Angle–Side–Angle
- **AAS:** Angle–Angle–Side
- **RHS:** Right angle–Hypotenuse–Side

1.1.2 Similar Triangles

Two triangles are **similar** if they have the same shape but not necessarily the same size.

$$\triangle ABC \sim \triangle DEF \iff \begin{cases} \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}, \\ \angle A = \angle D, \\ \angle B = \angle E, \\ \angle C = \angle F \end{cases}$$

Criteria for Similarity:

- **AA:** Angle–Angle
- **SSS:** Side–Side–Side (proportional)
- **SAS:** Side–Angle–Side (proportional)

1.2 Questions and Solutions - Part 01

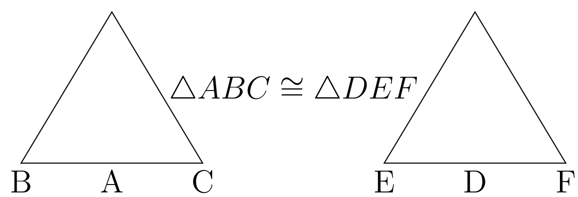
1. Define **congruent triangles**. Illustrate with a labeled diagram.

   Solution:

   Two triangles are congruent if they have exactly the same size and shape. All corresponding sides are equal in length and all corresponding angles are equal in measure.

   $\triangle ABC \cong \triangle DEF$ means:

   - $AB = DE$, $BC = EF$, $CA = FD$
   - $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$

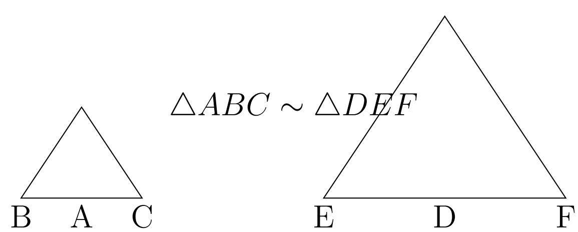
2. Define **similar triangles**. Illustrate with a labeled diagram.

   Solution:

   Two triangles are similar if they have the same shape but may differ in size. All corresponding angles are equal and all corresponding sides are proportional.

   $\triangle ABC \sim \triangle DEF$ means:

   - $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$
   - $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = k$ (scale factor)

3. State all five criteria for triangle congruence (SSS, SAS, ASA, AAS, RHS) with an example diagram.

   Solution:

   1. SSS (Side-Side-Side): If three sides of one triangle are equal to three sides of another triangle, then the triangles are congruent.

2. SAS (Side-Angle-Side): If two sides and the included angle of one triangle are equal to corresponding parts of another triangle, then they are congruent.

3. ASA (Angle-Side-Angle): If two angles and the included side of one triangle are equal to corresponding parts of another triangle, then they are congruent.

4. AAS (Angle-Angle-Side): If two angles and a non-included side of one triangle are equal to corresponding parts of another triangle, then they are congruent.

5. RHS (Right-Hypotenuse-Side): In right triangles, if the hypotenuse and one side are equal, then the triangles are congruent.

4. State the three criteria for similarity of triangles (AA, SSS, SAS). Give an example for each.

   Solution:

   1. AA (Angle-Angle): If two angles of one triangle are equal to two angles of another triangle, then the triangles are similar.

   Example: If $\angle A = \angle D$ and $\angle B = \angle E$, then $\triangle ABC \sim \triangle DEF$.

   2. SSS (Side-Side-Side): If the ratios of corresponding sides are equal, then the triangles are similar.

   Example: If $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$, then $\triangle ABC \sim \triangle DEF$.

   3. SAS (Side-Angle-Side): If two sides are proportional and the included angles are equal, then the triangles are similar.

   Example: If $\frac{AB}{DE} = \frac{AC}{DF}$ and $\angle A = \angle D$, then $\triangle ABC \sim \triangle DEF$.

5. In $\triangle ABC$ and $\triangle DEF$, the following is given: $AB = DE$, $BC = EF$, $AC = DF$. Are the triangles congruent? Justify your answer using the appropriate test.

   Solution:

   Yes, the triangles are congruent.

   Justification: We are given that all three sides of $\triangle ABC$ are equal to the corresponding three sides of $\triangle DEF$:

   - $AB = DE$
   - $BC = EF$
   - $AC = DF$

   By the **SSS (Side-Side-Side)** criterion, $\triangle ABC \cong \triangle DEF$.

6. In $\triangle PQR$ and $\triangle XYZ$, $\angle P = \angle X$, $\angle Q = \angle Y$. Prove that the two triangles are similar.

   Solution:

   Given: $\angle P = \angle X$ and $\angle Q = \angle Y$

   To prove: $\triangle PQR \sim \triangle XYZ$

   Proof: Since the sum of angles in any triangle is 180ř:

$$\angle P + \angle Q + \angle R = 180\text{ř} \tag{1}$$
$$\angle X + \angle Y + \angle Z = 180\text{ř} \tag{2}$$

   Given that $\angle P = \angle X$ and $\angle Q = \angle Y$, we can substitute:

$$\angle X + \angle Y + \angle R = \angle X + \angle Y + \angle Z$$

   Therefore: $\angle R = \angle Z$

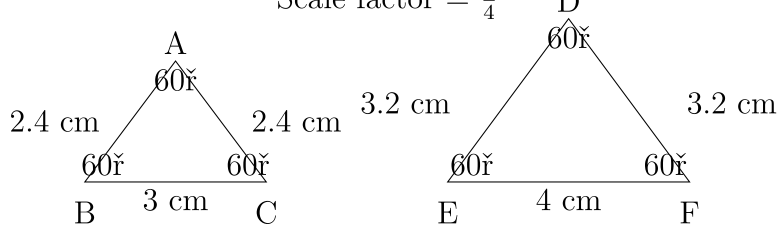
   Since all three corresponding angles are equal ($\angle P = \angle X$, $\angle Q = \angle Y$, $\angle R = \angle Z$), by the **AA criterion**, $\triangle PQR \sim \triangle XYZ$.

7. Draw two triangles such that they are similar but not congruent. Label all corresponding sides and angles.

   Solution:

$$\triangle ABC \sim \triangle DEF \text{ but } \triangle ABC \ncong \triangle DEF$$

Scale factor $= \frac{3}{4}$



   Both triangles are equilateral with all angles 60ř, but $\triangle DEF$ is larger than $\triangle ABC$.

8. The sides of two similar triangles are in the ratio 2 : 3. If the smaller triangle has area $24\,\text{cm}^2$, find the area of the larger triangle.

   Solution:

   Given:

   - Ratio of sides = 2 : 3
   - Area of smaller triangle = $24\,\text{cm}^2$

For similar triangles, the ratio of areas equals the square of the ratio of corresponding sides.

$$\frac{\text{Area of larger triangle}}{\text{Area of smaller triangle}} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

Therefore:

$$\text{Area of larger triangle} = 24 \times \frac{9}{4} = 6 \times 9 = 54\,\text{cm}^2$$

Answer: The area of the larger triangle is $54\,\text{cm}^2$.

9. In $\triangle ABC$, $\angle A = 90ř$. $AB = 6$ cm, $AC = 8$ cm. In $\triangle DEF$, $\angle D = 90ř$, $DE = 6$ cm, $DF = 8$ cm. Are the triangles congruent? If yes, state the criterion.

Solution:

Given:

- $\triangle ABC$: $\angle A = 90ř$, $AB = 6$ cm, $AC = 8$ cm
- $\triangle DEF$: $\angle D = 90ř$, $DE = 6$ cm, $DF = 8$ cm

First, let's find the hypotenuses using Pythagoras theorem:

For $\triangle ABC$:

$$BC^2 = AB^2 + AC^2 = 6^2 + 8^2 = 36 + 64 = 100$$

$$BC = 10 \text{ cm}$$

For $\triangle DEF$:

$$EF^2 = DE^2 + DF^2 = 6^2 + 8^2 = 36 + 64 = 100$$

$$EF = 10 \text{ cm}$$

Comparison:

- $\angle A = \angle D = 90ř$ (right angles)
- $AB = DE = 6$ cm
- $AC = DF = 8$ cm
- $BC = EF = 10$ cm (hypotenuses)

Yes, the triangles are congruent by the RHS (Right-Hypotenuse-Side) criterion, as they have equal right angles, equal hypotenuses, and one pair of equal sides.

10. (Challenging) A vertical stick of height 1.5 m casts a shadow 2 m long. At the same time, a tree casts a shadow 12 m long. Using similarity of triangles, find the height of the tree.

Solution:

Let the height of the tree be h meters.

Since both the stick and tree are measured at the same time, the sun's rays create similar triangles.

For the stick: Height = 1.5 m, Shadow = 2 m For the tree: Height = h m, Shadow = 12 m

Using similarity of triangles:

$$\frac{\text{Height of stick}}{\text{Shadow of stick}} = \frac{\text{Height of tree}}{\text{Shadow of tree}}$$

$$\frac{1.5}{2} = \frac{h}{12}$$

Cross-multiplying:

$$1.5 \times 12 = 2 \times h$$

$$18 = 2h$$

$$h = 9 \text{ m}$$

Answer: The height of the tree is 9 meters.

Part 02

2 Extended Content and Solutions

2.1 Important Properties

2.1.1 Isosceles Triangles:

If two sides of a triangle are equal, the angles opposite to them are also equal.

$$AB = AC \implies \angle B = \angle C$$

2.1.2 Right-Angle Triangles:

For right-angled triangles, the **RHS (Right–Hypotenuse–Side)** test is often used.

2.1.3 Pythagoras Theorem:

In a right triangle, the square of the hypotenuse equals the sum of the squares of the other two sides:

$$c^2 = a^2 + b^2$$

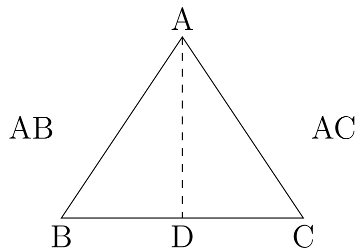
2.2 Questions and Solutions - Part 02

1. In an isosceles triangle, prove that the angles opposite the equal sides are equal. Illustrate with a diagram.

   Solution:

   Given: $\triangle ABC$ with $AB = AC$

   To prove: $\angle B = \angle C$

   Proof: Draw AD perpendicular to BC, where D is the midpoint of BC.

   In $\triangle ABD$ and $\triangle ACD$:

   - $AB = AC$ (given)
   - $AD = AD$ (common side)
   - $\angle ADB = \angle ADC = 90$ř (construction)

   By RHS criterion, $\triangle ABD \cong \triangle ACD$

   Therefore, $\angle B = \angle C$ (corresponding angles of congruent triangles)

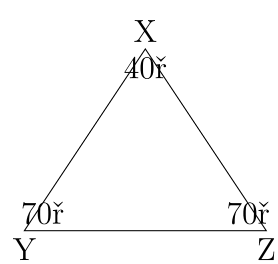
2. In $\triangle XYZ$, $XY = XZ$. If $\angle Y = 70$ř, find $\angle Z$. Draw a diagram.

   Solution:

   Given: $XY = XZ$ and $\angle Y = 70$ř

   Since $\triangle XYZ$ is isosceles with $XY = XZ$, the angles opposite to equal sides are equal.

   Therefore: $\angle Y = \angle Z = 70$ř

Using angle sum property: $\angle X + \angle Y + \angle Z = 180$ř $\angle X + 70$ř $+ 70$ř $= 180$ř $\angle X = 40$ř

Answer: $\angle Z = 70$ř

3. Two right triangles are given: $\triangle PQR$ with $\angle Q = 90$ř, and $\triangle LMN$ with $\angle M = 90$ř. If $PQ = LM$ and $PR = LN$, prove that the triangles are congruent (RHS criterion).

   Solution:

   Given:

   - $\triangle PQR$ with $\angle Q = 90$ř
   - $\triangle LMN$ with $\angle M = 90$ř
   - $PQ = LM$ (one side)
   - $PR = LN$ (hypotenuse)

   To prove: $\triangle PQR \cong \triangle LMN$

   Proof: In $\triangle PQR$ and $\triangle LMN$:

   - $\angle Q = \angle M = 90$ř (given - right angles)
   - $PR = LN$ (given - hypotenuses)
   - $PQ = LM$ (given - one side)

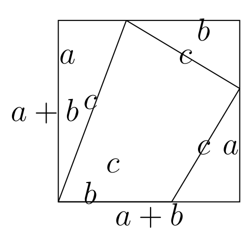
   By the RHS (Right-Hypotenuse-Side) criterion, $\triangle PQR \cong \triangle LMN$.

4. Prove Pythagoras theorem using two congruent right triangles arranged in a square.

   Solution:

   Consider a right triangle with sides a, b, and hypotenuse c.

   Arrange four such congruent triangles to form a square of side $(a + b)$.



   Area of large square $= (a + b)^2 = a^2 + 2ab + b^2$

   Area of inner square $= c^2$

   Area of four triangles $= 4 \times \frac{1}{2}ab = 2ab$

Since: Area of large square = Area of inner square + Area of four triangles

$$a^2 + 2ab + b^2 = c^2 + 2ab$$

$$a^2 + b^2 = c^2$$

This proves the Pythagoras theorem.

5. In $\triangle ABC$, $\angle A = \angle B = 45$ř. If $AB = 5\,\text{cm}$, prove that $\triangle ABC$ is an isosceles right triangle. Draw and label all sides.

Solution:

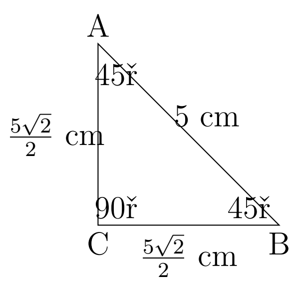
Given: $\angle A = \angle B = 45$ř and $AB = 5$ cm

Since $\angle A = \angle B$, by the property of isosceles triangles, the sides opposite to equal angles are equal. Therefore: $BC = AC$

Using angle sum property: $\angle A + \angle B + \angle C = 180$ř 45ř $+ 45$ř $+ \angle C = 180$ř $\angle C = 90$ř

Since $\angle C = 90$ř and $BC = AC$, $\triangle ABC$ is an isosceles right triangle.

Using Pythagoras theorem in the right triangle: $AB^2 = BC^2 + AC^2$ $5^2 = BC^2 + BC^2$ (since $BC = AC$) $25 = 2BC^2$ $BC^2 = 12.5$ $BC = AC = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$ cm



6. Two similar triangles have corresponding sides 4 cm, 5 cm, 6 cm and 8 cm, 10 cm, 12 cm. Verify similarity using the SSS test.

Solution:

Given:

- Triangle 1: sides = 4 cm, 5 cm, 6 cm
- Triangle 2: sides = 8 cm, 10 cm, 12 cm

For SSS similarity test, we need to check if the ratios of corresponding sides are equal:

$$\frac{8}{4} = 2, \quad \frac{10}{5} = 2, \quad \frac{12}{6} = 2$$

Since all ratios are equal $(= 2)$, the triangles satisfy the SSS similarity criterion.

Conclusion: The triangles are similar with a scale factor of 2:1.

7. In the figure, $\triangle ABC \sim \triangle DEF$. If $AB = 3\,\text{cm}$, $DE = 6\,\text{cm}$, and $AC = 4\,\text{cm}$, find DF.

   Solution:

   Given: $\triangle ABC \sim \triangle DEF$, $AB = 3$ cm, $DE = 6$ cm, $AC = 4$ cm

   Since the triangles are similar, the ratios of corresponding sides are equal:

$$\frac{AB}{DE} = \frac{AC}{DF}$$

   Substituting the known values:

$$\frac{3}{6} = \frac{4}{DF}$$

   Cross-multiplying:

$$3 \times DF = 6 \times 4$$
$$3 \times DF = 24$$
$$DF = 8 \text{ cm}$$

   Answer: $DF = 8$ cm

8. The ratio of areas of two similar triangles is $25 : 36$. If one side of the smaller triangle is $10\,\text{cm}$, find the corresponding side of the larger triangle.

   Solution:

   Given: Ratio of areas = 25:36, side of smaller triangle = 10 cm

   For similar triangles, if the ratio of areas is $m : n$, then the ratio of corresponding sides is $\sqrt{m} : \sqrt{n}$.

$$\frac{\text{Area of smaller triangle}}{\text{Area of larger triangle}} = \frac{25}{36}$$

   Therefore:

$$\frac{\text{Side of smaller triangle}}{\text{Side of larger triangle}} = \sqrt{\frac{25}{36}} = \frac{5}{6}$$

   Let the corresponding side of the larger triangle be x cm:

$$\frac{10}{x} = \frac{5}{6}$$

   Cross-multiplying:

$$5x = 60$$

$$x = 12 \text{ cm}$$

Answer: The corresponding side of the larger triangle is 12 cm.

9. A ladder 6 m long leans against a wall making a right triangle with the ground. The foot of the ladder is 3.5 m from the wall. Using Pythagoras theorem, find the height at which the ladder touches the wall.

   Solution:

   Given:

   - Length of ladder (hypotenuse) = 6 m
   - Distance from wall (base) = 3.5 m
   - Height on wall = ? (to find)

   Using Pythagoras theorem:

$$(\text{hypotenuse})^2 = (\text{base})^2 + (\text{height})^2$$

$$6^2 = 3.5^2 + h^2$$

$$36 = 12.25 + h^2$$

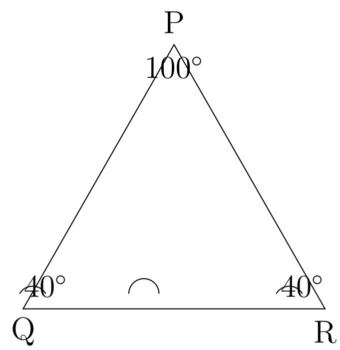
$$h^2 = 36 - 12.25 = 23.75$$

$$h = \sqrt{23.75} = 4.87 \text{ m (approximately)}$$

Part 03 - Final Questions Solutions

Question 1

Problem: In $\triangle PQR$, $PQ = PR$. If $\angle Q = 40°$, find $\angle R$ and $\angle P$. Draw a neat diagram.
Solution:
Since $PQ = PR$, triangle PQR is isosceles with P as the vertex angle.
In an isosceles triangle, base angles are equal. Therefore: $\angle Q = \angle R$
Given: $\angle Q = 40°$ So: $\angle R = 40°$
Using angle sum property of triangles: $\angle P + \angle Q + \angle R = 180°$ $\angle P + 40° + 40° = 180°$ $\angle P = 180° - 80° = 100°$
Answer: $\angle R = 40°$ and $\angle P = 100°$



Question 2

Problem: In the adjoining figure, $\triangle ABC \cong \triangle PQR$. Write all equal sides and equal angles.

Solution:

When $\triangle ABC \cong \triangle PQR$, corresponding parts are equal.

Equal sides:

$$AB = PQ \tag{3}$$
$$BC = QR \tag{4}$$
$$CA = RP \tag{5}$$

Equal angles:

$$\angle A = \angle P \tag{6}$$
$$\angle B = \angle Q \tag{7}$$
$$\angle C = \angle R \tag{8}$$

Question 3

Problem: Two right triangles have hypotenuses of equal length and one side equal. Prove they are congruent by the RHS criterion.

Solution:

Given: Two right triangles $\triangle ABC$ and $\triangle PQR$ where:

- $\angle B = \angle Q = 90°$ (right angles)
- $AC = PR$ (hypotenuses are equal)
- $AB = PQ$ (one side is equal)

To Prove: $\triangle ABC \cong \triangle PQR$
Proof:
In right triangles $\triangle ABC$ and $\triangle PQR$:

1. Hypotenuse: $AC = PR$ (given)
2. Right angle: $\angle B = \angle Q = 90^\circ$ (given)
3. One side: $AB = PQ$ (given)

By RHS (Right angle-Hypotenuse-Side) congruence criterion: $\triangle ABC \cong \triangle PQR$
Hence proved.

Question 4

Problem: A triangle has sides $7\,\text{cm}, 24\,\text{cm}, 25\,\text{cm}$. Verify whether it is a right triangle using Pythagoras theorem.
Solution:
Given sides: $a = 7$ cm, $b = 24$ cm, $c = 25$ cm
For a right triangle, Pythagoras theorem states: $a^2 + b^2 = c^2$ (where c is the longest side)
Checking:

$$7^2 + 24^2 = 49 + 576 = 625 \tag{9}$$
$$25^2 = 625 \tag{10}$$

Since $7^2 + 24^2 = 25^2$, the triangle satisfies Pythagoras theorem.
Answer: Yes, it is a right triangle.

Question 5

Problem: $\triangle ABC \sim \triangle DEF$. If $AB = 5\,\text{cm}$, $BC = 7.5\,\text{cm}$, $DE = 10\,\text{cm}$, find EF.
Solution:
Since $\triangle ABC \sim \triangle DEF$, corresponding sides are proportional.
The ratio of similarity $= \frac{DE}{AB} = \frac{10}{5} = 2$
Therefore: $\frac{EF}{BC} = 2$
$EF = 2 \times BC = 2 \times 7.5 = 15$ cm
Answer: $EF = 15$ cm

Question 6

Problem: The sides of two triangles are in the ratio 3 : 5. If the area of the smaller triangle is $27\,\text{cm}^2$, find the area of the larger triangle.

Solution:

Given: Ratio of corresponding sides = 3 : 5

For similar triangles, the ratio of areas = (ratio of corresponding sides)^2

Ratio of areas $= \left(\frac{3}{5}\right)^2 = \frac{9}{25}$

Let the area of larger triangle $= A$

$\frac{\text{Area of smaller triangle}}{\text{Area of larger triangle}} = \frac{9}{25}$

$\frac{27}{A} = \frac{9}{25}$

$A = \frac{27\times 25}{9} = \frac{675}{9} = 75\ \text{cm}^2$

Answer: Area of larger triangle = 75 cm^2

Question 7

Problem: In $\triangle XYZ$, $XY \parallel BC$ of $\triangle ABC$. Prove that $\triangle AXY \sim \triangle ABC$. (Basic proportionality theorem).

Solution:

Given: In $\triangle ABC$, $XY \parallel BC$ where X lies on AB and Y lies on AC.

To Prove: $\triangle AXY \sim \triangle ABC$

Proof:

In $\triangle AXY$ and $\triangle ABC$:

1) $\angle A = \angle A$ (common angle)

2) $\angle AXY = \angle ABC$ (corresponding angles, since $XY \parallel BC$)

3) $\angle AYX = \angle ACB$ (corresponding angles, since $XY \parallel BC$)

By AA similarity criterion: $\triangle AXY \sim \triangle ABC$

Hence proved.

Question 8

Problem: The perimeters of two similar triangles are 36 cm and 24 cm. If one side of the smaller triangle is 8 cm, find the corresponding side of the larger triangle.

Solution:

For similar triangles, the ratio of perimeters = ratio of corresponding sides

Ratio $= \frac{\text{Perimeter of larger triangle}}{\text{Perimeter of smaller triangle}} = \frac{36}{24} = \frac{3}{2}$

Let the corresponding side of larger triangle $= x$ cm

$\frac{x}{8} = \frac{3}{2}$

$x = 8 \times \frac{3}{2} = 12$ cm

Answer: Corresponding side of larger triangle = 12 cm

Question 9

Problem: (Challenging) A 10 m tall lamp post casts a shadow of 8 m. At the same time, a nearby tower casts a shadow of 40 m. Find the height of the tower using similar triangles.

Solution:

Since the shadows are cast at the same time, the triangles formed are similar (same angle of elevation of sun).

Let height of tower = h m

For lamp post: Height = 10 m, Shadow = 8 m For tower: Height = h m, Shadow = 40 m

Using similarity: $\frac{\text{Height of tower}}{\text{Shadow of tower}} = \frac{\text{Height of lamp post}}{\text{Shadow of lamp post}}$

$\frac{h}{40} = \frac{10}{8}$

$h = \frac{10 \times 40}{8} = \frac{400}{8} = 50$ m

Answer: Height of tower = 50 m

Question 10

Problem: (Challenging) Two isosceles triangles have equal vertex angles and their areas are in the ratio 16 : 25. Show that the ratio of their corresponding sides is 4 : 5.

Solution:

Given:

- Two isosceles triangles with equal vertex angles
- Ratio of areas = 16 : 25

To Show: Ratio of corresponding sides = 4 : 5

Solution:

Since the triangles are isosceles with equal vertex angles, they are similar triangles.

For similar triangles: $\frac{\text{Area of triangle 1}}{\text{Area of triangle 2}} = \left(\frac{\text{Side of triangle 1}}{\text{Side of triangle 2}}\right)^2$

Given: $\frac{\text{Area 1}}{\text{Area 2}} = \frac{16}{25}$

Therefore: $\left(\frac{\text{Side 1}}{\text{Side 2}}\right)^2 = \frac{16}{25}$

Taking square root of both sides: $\frac{\text{Side 1}}{\text{Side 2}} = \sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5}$

Hence, the ratio of corresponding sides is 4 : 5.

Proved.