The Problem's statement:

5. **Scheduling with types to minimize the makespan.**

Input: An integer number of (identical) machines $m \geq 2$. A set of n jobs $J = \{1, 2, \ldots, n\}$, where every job j has an integer processing time $p_j > 0$ and a positive integer type $t_j \in \{1, 2, 3, \ldots, 7\}$.

Goal: Find a partition of the jobs of to the machines, $J_1, J_2, \ldots, J_m$, such that every subset has jobs of at most three types (for any i there are three values $k_{i_1}, k_{i_2}, k_{i_3} \in \{1, 2, 3, \ldots, 7\}$, such that if $j \in J_i$, then $t_j \in \{k_{i_1}, k_{i_2}, k_{i_3}\}$).

Objective:: Minimize $\max_{1 \leq i \leq m} \sum_{j \in J_i} p_j$.

*5 types instead of 7.

Input i:

m=2 machines.

j	1	2	3	4	5	6	7	8	9
p_j	5	9	9	6	10	3	3	10	7
t_j	1	3	1	1	2	1	2	4	4

*Note: sum on all processing times is 62, so 62/2=31 is a lower bound

Bad Solution:

m_1 = {1, 2, 6}, time=5+9+3=17

m_2 = {3, 4, 5, 7, 8, 9}, time =9+6+10+3+10+17 = 45

$$Objective\ function = \max\{17, 45\} = 45$$

So-so solution: (view all jobs belonging to the same type as one super-job, and arbitrarily place types 1,2 together, and 3,4,5 on the other machine)

m_1 = {2, 8, 9}, time = 9+10+7 = 26

m_2 = {1, 3, 4, 5, 6, 7}, time = 5+9+6+10+3+3 = 36

$$Objective\ function = \max\{26, 36\} = 36$$

Optimal solution:

m_1 = {1, 9, 8, 2}, time = 5+7+10+9 = 31

m_2 = {3, 6, 4, 5, 7}, time = 9+3+6+10+3 = 31

$$Objective\ function = \max\{31, 31\} = 31$$

Invalid solution:

m_1 = {1, 4, 8, 7, 2}*, time = 5+6+10+3+9 = 33

(but this machine has **4 different types, therefore the solution is not valid!)*

m_2 = {3, 6, 9, 5}, time = 9+3+7+10 = 29

Input ii:

m=3 machines

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
p_j	4	5	10	13	3	7	8	9	13	8	16	8	7	2	4	5	8	11	9
t_j	1	5	3	3	4	2	1	4	3	5	3	2	5	1	3	4	2	1	1

*Note: sum on all processing times is 150, so 150/3=50 is a lower bound

Bad Solution:

m_1 = {1,7,14,18,19,6,12,17,3,4,9,11,15}, time = 4+8+2+11+9+7+8+8+10+13+13+16+4 = 113

m_2 = {5,8,16}, time = 3+9+5 = 17

m_3 = {2,10,13}, time = 5+8+7 = 20

$$Objective\ function = \max\{113, 17, 20\} = 45$$

So-so solution:

m_1 = {3,4,9,11,15}, time = 10+13+13+16+4 = 56

m_2 = {6,12,17,2,10,13}, time = 7+8+8+5+8+7 = 43

m_3 = {5,8,16,1,7,14,18,19}, time = 3+9+5+4+8+2+11+9 = 51

$$Objective\ function = \max\{56, 43, 51\} = 56$$

Optimal/near optimal solution:

m_1 = {3,9,11,15,6}, time = 10+13+16+4+7 = 50

m_2 = {12,17,2,10,13,4}, time = 8+8+5+8+7+13 = 49

m_3 = {5,8,16,1,7,14,18,19}, time = 3+9+5+4+8+2+11+9 = 51

$$Objective\ function = \max\{50, 49, 51\} = 51$$

Invalid solution:

m_1 = {3,9,11,15,6}, time = 10+13+16+4+7 = 50

m_2 = {12,17,2,13,4,19}*, time = 8+8+5+7+13+9 = 50

(but this machine has **4 different types, therefore the solution is not valid!)*

m_3 = {5,8,16,1,7,14,18,10}, time = 3+9+5+4+8+2+11+8 = 50