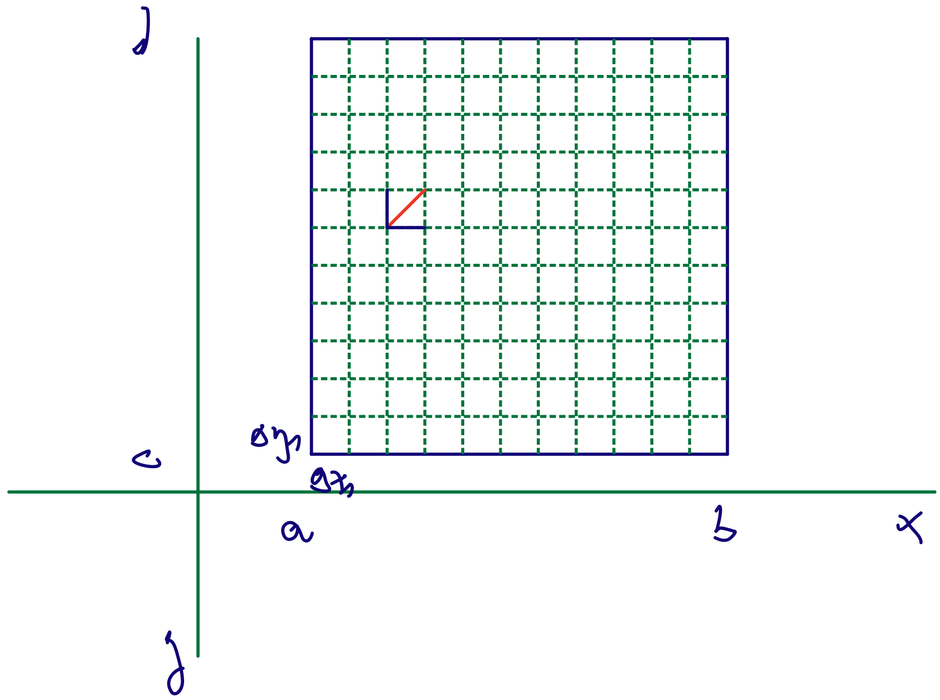
Računanje dvojnega integrala

Naj bo najprej naše integracijsko območje D pravokotnik v $\mathbb{R}^2$. Brez škode za splošnost lahko predpostavimo, da so stranice D vzporedne z osema koord. sistema na $\mathbb{R}^2$.



Naj bo torej

$$D = [a,b] \times [c,d] = \{(x,y) \,;\, x \in [a,b],\ y \in [c,d]\}$$

Naj bosta

$$a = x_0 < x_1 < x_2 < \ldots < x_m = b$$

$$c = y_0 < y_1 < y_2 < \ldots < y_m = d$$

delitvi intervalov $[a,b]$ oziroma $[c,d]$.

Označili bomo

$$x_k - x_{k-1} = \Delta x_k \quad \text{in} \quad y_k - y_{k-1} = \Delta y_k$$

Diagonala $D_{kl} = [x_{k-1}, x_k] \times [y_{l-1}, y_l]$ je vedno podana z

$$\delta_{kl}^2 = \Delta x_k^2 + \Delta y_l^2$$

Naj bo sedaj

$$f : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

funkcija dveh spremenljivk in naj območje Ω vsebuje pravokotnik D. Osnova računanja dvojnega (in večkratnega) integrala je naslednji izrek.

Izrek. Naj bo funkcija $f(x,y)$ na pravokotniku D integrabilna in naj obstaja za vsak $x \in [a,b]$ (enojni) integral

$$F(x) = \int_c^d f(x,y)\,dy .$$

Potem velja:

$$\iint_{\mathcal{D}} f(x,y)\,dx\,dy = \int_a^b F(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx \,,$$

Dokaz:

Označimo:

na $\mathcal{D}_{kl}$

$$M_{kl} = \max \{ f(x,y) \,;\, x \in [x_{k-1}, x_k] \;\&\; y \in [y_{l-1}, y_l] \}$$

$$m_{kl} = \min \{ f(x,y) \,;\, x \in [x_{k-1}, x_k] \;\&\; y \in [y_{l-1}, y_l] \}$$

Naj bo $\xi_k \in [x_{k-1}, x_k]$ poljubna točka $k = 1, \ldots, n$

Potem imamo za vsak k: (in vsako izbiro $\xi_k < \Delta x_k$)

$$\sum_{l=1}^{m} m_{kl}\,\Delta y_l \leq F(\xi_k) = \sum_{l=1}^{m} \int_{y_{l-1}}^{y_l} f(\xi_k, y)\,dy \leq \sum_{l=1}^{m} M_{kl}\,\Delta y_l$$

Zgornji neenakosti pomnožimo z Δx_k in seštejemo po k.

$$s = \sum_{k=1}^{n} \sum_{l=1}^{m} m_{kl}\,\Delta x_k \Delta y_l \leq \sum_{k=1}^{n} F(\xi_k)\,\Delta x_k \leq \sum_{k=1}^{n} \sum_{l=1}^{m} M_{kl}\,\Delta x_k \Delta y_l = S$$

Tu sta s in S spodnja in zgornja integralska vsota funkcije $f(x,y)$ za delitev $\{D_{kl} = \Delta x_k \Delta y_l\}$ pravokotnika D.

Naj bo $\delta = \max\{\delta_{kl} = (\Delta x_k^2 + \Delta y_l^2)^{\frac{1}{2}}\}$. Tedaj velja:

$$\delta \to 0 \iff \max\{\Delta x_k\} \longrightarrow 0 \ \& \ \max\{\Delta y_l\} \longrightarrow 0 .$$

Če je $f(x,y)$ na D integrabilna, velja

$$\lim_{\delta \to 0} s(f,\delta) = \lim_{\delta \to 0} S(f,\delta) = \iint_D f(x,y)\,dx\,dy$$

V limiti $\delta \to 0$ torej dobimo,

$$\iint_D f(x,y)\,dx\,dy = \int_a^b F(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx .$$

$\square$

Seveda tudi na analogen način dokažemo:

$$\iint_D f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy .$$

Primer 1: $\quad D = [a,b] \times [c,d]$

$$\iint_D (x+y)\,dx\,dy =$$

$$= \int_a^b \left[\int_c^d (x+y)\,dy \right] dx = \int_a^b \left[xy + \frac{y^2}{2} \right]_c^d =$$

$$= \int_a^b \left(xd + \frac{d^2}{2} - xc - \frac{c^2}{2} \right) dx =$$

$$= \left(\frac{x^2 d}{2} + \frac{x d^2}{2} - \frac{x^2 c}{2} - \frac{x c^2}{2} \right)\Big|_a^b =$$

$$= \frac{b^2 d}{2} + \frac{b d^2}{2} - \frac{b^2 c}{2} - \frac{b c^2}{2} -$$

$$- \frac{a^2 d}{2} - \frac{a d^2}{2} + \frac{a^2 c}{2} + \frac{a c^2}{2} =$$

$$= \frac{1}{2} \left[(d-c)(b^2-a^2) + (d^2-c^2)(b-a) \right]$$

<u>Primer 2:</u> Naj bo $f(x,y) = f(x)\,g(y)$

$$\iint_D f(x)\,g(y) = \int_a^b \left[\int_c^d f(x)\,g(y)\,dy \right] dx =$$

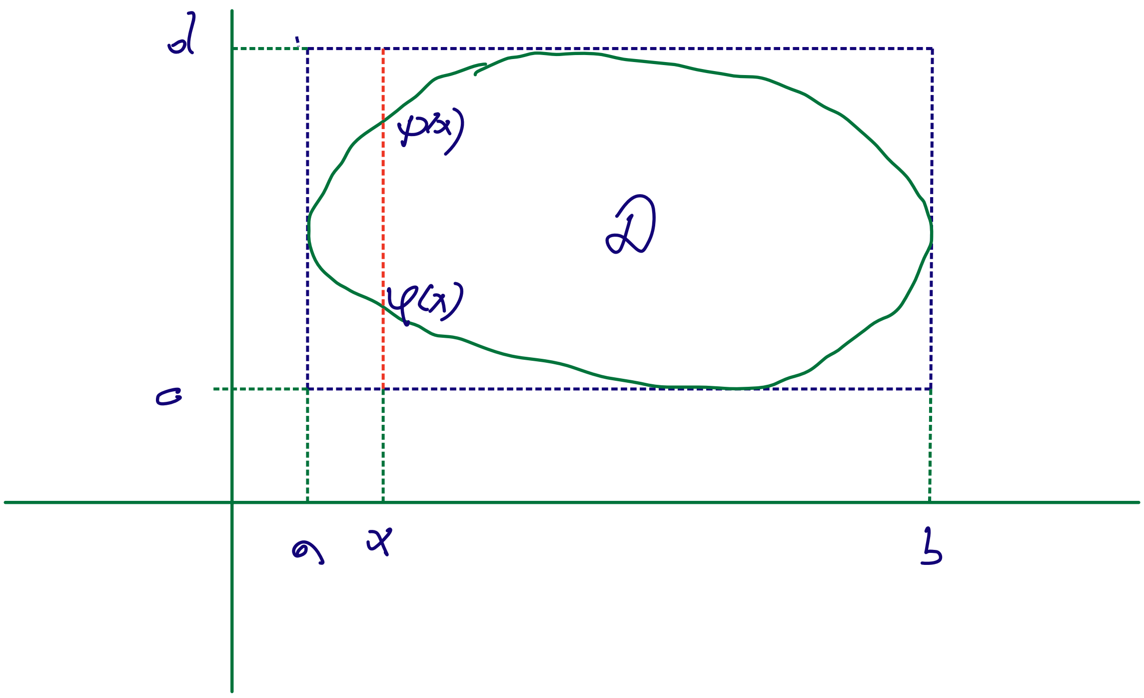
$$= \int_a^b f(x)\left(\int_c^d f(y)\,dy\right)dx =$$

$$= \int_a^b f(x)\,dx \cdot \int_c^d g(y)\,dy .$$

Naj bo sedaj D neko konveksno območje v $\Omega \subset \mathbb{R}^2$ in

$$f : \Omega \longrightarrow \mathbb{R}^2$$

integrabilna funkcija.



Projekcija D na x je $[a,b]$

Projekcija D na y je $[c,d]$

Označimo pravokotnik $[a,b]\times[c,d]$ z $\tilde{D}$

in domeno $\tilde{D}\setminus D$ z $\tilde{\tilde{D}}$

Definirajmo novo funkcijo

$$\varphi(x,y): \tilde{D} \longrightarrow \mathbb{R}$$

s predpisom

$$\varphi(x,y) = \begin{cases} f(x,y) & ; \quad (x,y) \in D \\ 0 & ; \quad (x,y) \in \tilde{\tilde{D}} . \end{cases}$$

Ker je $f(x,y)$ integrabilna, je integrabilna tudi funkcija $\varphi(x,y)$. Zaradi aditivnosti integrala velja

$$\iint_{\tilde{D}} \varphi(x,y)\,dx\,dy = \iint_{D} \varphi(x,y)\,dx\,dy + \iint_{\tilde{\tilde{D}}} \varphi(x,y)\,dx\,dy$$

$$= \iint_{D} f(x,y)\,dx\,dy + \iint_{\tilde{\tilde{D}}} \varphi(x,y)\,dx\,dy$$

Ker je $\varphi(x,y)$ na $\tilde{\tilde{D}}$ enaka 0, velja

$$\iint_{\tilde D} \varphi(x,y)\,dx\,dy = \iint_{D} f(x,y)\,dx\,dy.$$

Vsaka premica $x = \text{const}$ ~~se~~ z D seka v dveh točkah

$\varphi(x)$ in $\psi(x)$

pri čemer je $\varphi(x) < \psi(x)$.

Pri $x = a$ in $x = b$ se $x = a$ in $x = b$ sekata z D v dotikališčih $\varphi(a) = \psi(a)$ in $\varphi(b) = \psi(b)$.

Pravzaprav $\tilde D$ smo konstruirali tako, da se dotika $x = a$ in $x = b$ dotikata roba $\partial \tilde D$.

Torej se vsaka seka z $\tilde D$ samo v eni točki.

Sedaj lahko uporabimo prejšnji izrek in računamo:

$$\iint_{\tilde D} \varphi(x,y)\,dx\,dy = \int_a^b \left[\int_c^d \varphi(x,y)\,dy \right] dx =$$

$$= \int_a^b \left[\underbrace{\cancel{\int_c^{\varphi(x)}}}_{0} + \int_{\varphi(x)}^{\psi(x)} + \underbrace{\cancel{\int_{\psi(x)}^{d}}}_{0} \right] dx =$$

$$= \int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy \right] dx.$$

Dolezali zmo

Izrek Naj za funkcijo $f(x,y)$ velja:

1.) $f(x,y)$ je integrabilna na D

2.) Območje D je konveksno in ima odsekoma gladek rob.

3.) Naj bo $\alpha \in (a,b)$. Premica $x = \alpha$ seka ∂D v dveh točkah; v točkah $\varphi(\alpha)$ in v $\psi(\alpha)$

$$\varphi(\alpha) < \psi(\alpha).$$

4.) Za vsak $x_0 \in [a,b]$ je

$$\int_{\varphi(x_0)}^{\psi(x_0)} f(x_0,y)\, dy$$

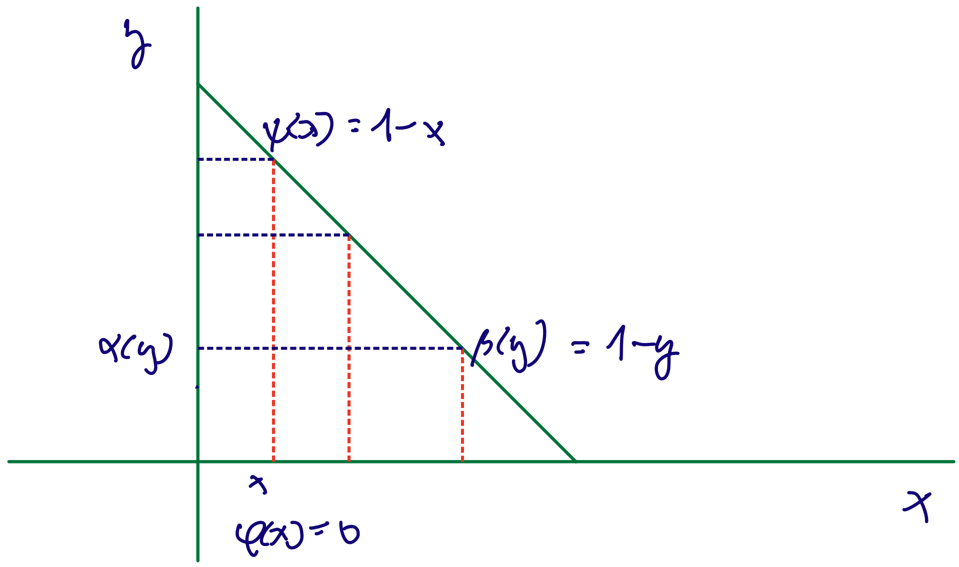
dobro definiran. Tj., $f(x_0,y)$ je integrabilna po y.

Potem velja:

$$\iint_D f(x,y) = \int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)\, dy \right] dx .$$

Primer: 1.) Naj bo $D = \{(x,y);\ x \geq 0,\ y \geq 0,\ x+y \leq 1\}$ in $f(x,y) = xy^2$.

Izračunajmo $\iint\limits_D xy^2\,dx\,dy$



Tedaj:

$$\iint\limits_D xy^2\,dx\,dy = \int_0^1 \left[\int_0^{1-x} xy^2\,dy\right] dx =$$

$$= \int_0^1 x\,\frac{y^3}{3}\Big|_0^{1-x} = \int_0^1 x\,\frac{(1-x)^3}{3}\,dx = \frac{1}{60}$$

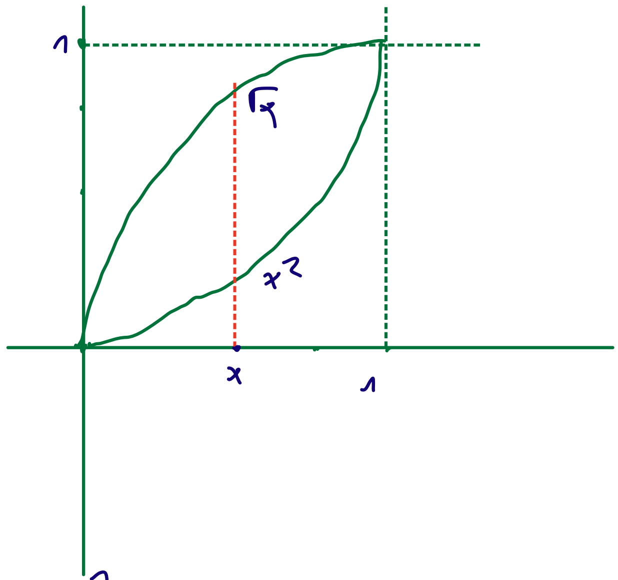
Integral lahko izračunamo tudi z obratnim vrstnim redom integriranja:

$$\iint\limits_D xy^2\,dx\,dy = \int_0^1 \left[\int_0^{1-y} xy^2\,dx\right] dy =$$

$$= \int_0^1 \frac{x^2 y^2}{2}\Big|_0^{1-y} dy = \frac{1}{2}\int_0^1 (1-y)^2 y^2 \, dy = \frac{1}{60}$$

2.) $D = x^2 \leq y \leq \sqrt{x}$

$$\iint_D (x+1)\, y$$



$$\iint_D (x+1)\, y \; dx\, dy = \int_0^1 \left[\int_{x^2}^{\sqrt{x}} (x+1)\, y \; dy \right] dx$$

$$= \frac{1}{2}\int_0^1 (x+1)\, y^2 \Big|_{x^2}^{\sqrt{x}} dx = \frac{1}{2}\int_0^1 (x+1)(x - x^4)\, dx$$

$$\frac{1}{2}\int_0^1 (x^2 + x - x^5 - x^5)\, dx = \frac{7}{20}$$