Uvedba novih spremenljivk

Spomnimo se formule za uvedbo nove spremenljivke v integralu funkcije ene spremenljivke:

$$\int_a^b f(x)\, dx$$

$$x(u) : [\alpha, \beta] \longrightarrow [a, b]$$

naj bo monotono naraščajoča odvedljiva funkcija.

Potem velja:

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f(x(u)) \cdot \frac{dx}{du}\, du =$$

$$= \int_\alpha^\beta f(x(u)) \cdot x'\, du$$

Naš cilj v tem razdelku je izpeljati analogno formulo za dvojni integral.

Naj bo

$$F : \Omega \longrightarrow D$$

$$(u,v) \longmapsto (x(u,v), y(u,v))$$

preslikava, ki je bijektivna in odvedljiva.

Spomnimo se, kaj je odvod (diferencial) preslikave F.

Odvod preslikave F v točki (u_0,v_0) je linearna preslikava

$$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$

za katero velja:

$$F\big((u_0,v_0)+(h,k)\big) = F(u_0,v_0) + A\begin{pmatrix} h \\ k \end{pmatrix} + o(h,k),$$

$$\lim_{\|(h,k)\| \to 0} \frac{o(h,k)}{\|(h,k)\|} = 0$$

Če taka preslikava obstaja, pravimo, da je F v (u_0,v_0) odvedljiva in odvod $D_{(u_0,v_0)}F$ je enak

$$D_{(u_0,v_0)}F = A.$$

Lahek premislek pokaže:

$$D_{(u_0,v_0)} F = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}_{(u_0,v_0)}$$

Označimo:

$$\mathrm{Jac}(F)_{(u_0,v_0)} = \det\left(D_{(u_0,v_0)} F\right)$$

Formula za uvedbo nove spremenljivke v dvojni integral se glasi:

$$\boxed{\iint_D f(x,y)\,dx\,dy = \iint_\Omega f(x(u,v), y(u,v))\,\left|\mathrm{Jac}(F)_{(u,v)}\right|\,du\,dv}$$

Zgornjo formulo moramo dokazati.

Ena pot do njenega dokaza je prek Greenove formule.

Greenova formula pravi:

Naj bo

$$(x,y) \longmapsto (P(x,y), Q(x,y))$$

preslikava iz $D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (oz. vektorsko polje)

in

$$\gamma(t) = (x(t), y(t)) : [\alpha, \beta] \longrightarrow \mathbb{R}^2$$

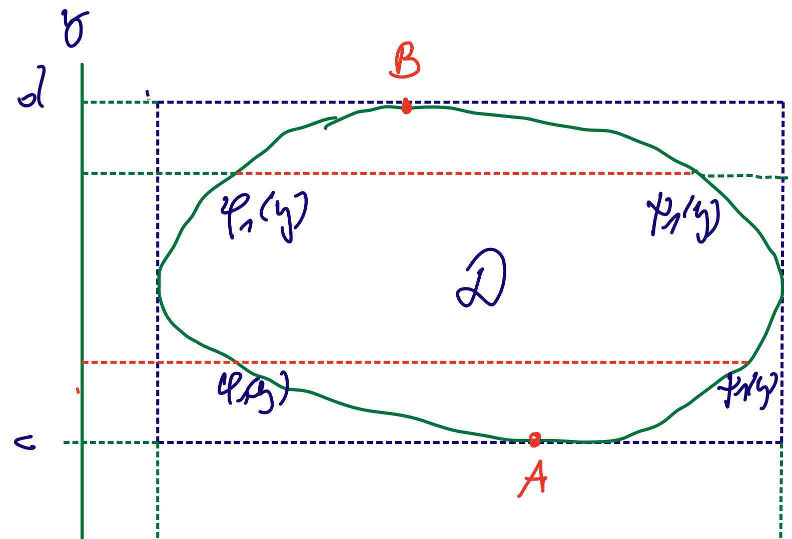
pot, ki parametrizira rob ∂D

$$\int_{\partial D} \langle P, \dot{\gamma} \rangle \, dt = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

oz.:

$$\int_\alpha^\beta (P \cdot \dot{x} + Q \dot{y}) \, dt = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

Dokaz Greenove formule



Naj bo D spet območje v ravnini (xy).

Naj bosta $P(x,y)$ in $Q(x,y)$ dve zvezni in odvedljivi funkciji, oz.:

$$(x, y) \longmapsto (P(x,y), Q(x,y))$$

odvedljiva preslikava na D.

Potem imamo:

$$\iint_D \frac{\partial Q}{\partial x}\, dx\, dy = \int_c^d \left[\int_{\varphi_1(y)}^{\varphi_2(y)} \frac{\partial Q}{\partial x}(x,y)\, dx \right] dy =$$

$$= \int_c^d \left[Q(\varphi_2(y), y) - Q(\varphi_1(y), y) \right] dy$$

$$\iint_D \frac{\partial Q}{\partial x}\, dx\, dy = \int_c^d Q(\varphi_2(y), y)\, dy - \int_c^d Q(\varphi_1(y), y)\, dy \qquad (*)$$

Naj bo spet

$$\gamma : (\alpha, \tilde{\alpha}) \longrightarrow D$$

parametrizacija roba ∂D.

$$\gamma(t) = (x(t), y(t))$$

in naj bo $\gamma(\alpha) = A$, $\gamma(\beta) = B$ in $\gamma(\tilde{\alpha}) = A$.

Če na desni strani (*) vpeljemo novo spremenljivko t,

$y = y(t)$; $dy = \dot{y}\, dt$

tu po integraciji naredil pog?

$$\iint_{D} \frac{\partial Q}{\partial x}\, dx\, dy = \int_{\alpha}^{\beta} Q \cdot \dot{y}\, dt + \int_{\beta}^{\tilde{\alpha}} Q \cdot \dot{y}\, dt$$

$$= \int_{\alpha}^{\tilde{\alpha}} Q \cdot \dot{y}\, dt \qquad (1)$$

Analogno:

$$\iint_{D} \frac{\partial P}{\partial y}\, dx\, dy = -\int_{\alpha}^{\tilde{\alpha}} P \cdot \dot{x}\, dt \qquad (2)$$

Iz (1) in (2) dobimo res

$$\int_{\alpha}^{\tilde{\alpha}} P(x(t), y(t))\, \dot{x}\, dt + Q(x(t), y(t))\, \dot{y}\, dt = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy \qquad \square$$