Gama in beta

1.) Eulerjeva funkcija gama je definirana s predpisom

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad (*)$$

Lahko je videti, da je za $x \geq 0$ dobro definirana (definicija za negativne x je zahtevnejša, prav tako za komplekse)

Integral $(*)$ konvergira pri $t = \infty$ za vsak $x > 0$, prav tako pri $t = 0$. Konvergira celo enakomerno glede na x. (Na intervalu $[a, b]$ je konvergenca v ∞ najpočasnejša pri $x = b$.)

Takoj vidimo:

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\, dt = \int_0^\infty e^{-t}\, dt =$$

$$= -e^{-t}\Big|_0^\infty = -(e^{-\infty} - e^{0}) = -(-1) = 1$$

Torej $\Gamma(1) = 1$

<u>Trditev</u> Za vsak $x > 0$ velja

$$\Gamma(x+1) = x\,\Gamma(x)$$

<u>Dokaz:</u>

Imamo:

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt \overset{\text{per partes}}{=}$$

$u = t^x \qquad dv = e^{-t}$

$du = x\,t^{x-1}\,dx \qquad v = -e^{-t}\,dt$

$$= \left(-t^x e^{-t}\right)\Big|_0^\infty + \int_0^\infty x\,t^{x-1} e^{-t}\,dt =$$

$$= -\lim_{t\to\infty} t^x e^{-t} + 0 + x\int_0^\infty t^{x-1} e^{-t}\,dt$$

Za vsak $x \geq 0$ velja

$$\lim_{t\to\infty} t^x e^{-t} = \lim_{t\to\infty} t^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots\right) =$$

$$= \lim_{t \to 0} \left(t^{x} - t^{x+1} + \frac{t^{2+x}}{2!} - \frac{t^{3+x}}{3!} + \dots \right)$$

$$= 0.$$

Torej res:

$$\Gamma(x+1) = x \int_0^\infty t^{x-1} e^{-t}\, dt = x\, \Gamma(x)$$

$\square$

<u>Posledica:</u> Za $n \in \mathbb{N}$ velja:

$$\Gamma(n+1) = n!$$

<u>Dokaz:</u>

Rekurzivni razmislek

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1) = n(n-1)(n-2)\,\Gamma(n-2) =$$

$$= n(n-1)(n-2) \cdots\cdots 2\,\Gamma(2) =$$

$$= n(n-1)(n-2) \cdots 2 \cdot 1 \cdot \underbrace{\Gamma(1)}_{1} =$$

$$= n!$$

$\square$

Pomembna vrednost je tudi $\Gamma\left(\frac{1}{2}\right)$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt$$

Vpeljimo novo spremenljivko:

$$t = u^2 \qquad u = t^{\frac{1}{2}}$$

$$dt = 2u\, du$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{1}{u}\, 2u\, e^{-u^2}\, du = 2\int_0^\infty e^{-u^2}\, du$$

Kasneje bomo izračunali:

$$\int_0^\infty e^{-u^2}\, du = \frac{\sqrt{\pi}}{2} \qquad \text{(Gaussov integral)}$$

Opomba:

$$\int_0^\infty e^{-r^2}\, dA = \int_0^\infty e^{-r^2}\, \tfrac{1}{2}\, dr\, r\, d\varphi =$$

$$= \frac{1}{2}\pi \int_0^\infty r\, e^{-r^2}\, dr \qquad r^2 = u$$

$$dr = \frac{1}{r}\, \frac{1}{2}\, du$$

$$= \frac{1}{4}\pi \int_0^\infty e^{-u^2}\, du = \frac{\pi}{4}$$

Po drugi strani:

$$\int_0^\infty e^{-r^2} \, dA = \int_0^\infty e^{-(x^2+y^2)} \, dx\,dy =$$

$$= \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \left(\int_0^\infty e^{-x^2} dx \right)^2 = I^2$$

$$I^2 = \frac{\pi}{4}$$

$$I = \frac{\sqrt{\pi}}{2}$$

$\square$

S pomočjo rezultata $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ dobimo z relacijo $\Gamma(x+1) = x\Gamma(x)$ še naslednjo pomembno formulo:

<u>Trditev</u>: $$\Gamma(n+\tfrac{1}{2}) = \frac{(2n)!}{2^n \, n!}\sqrt{\pi}$$

Res:

$$\Gamma(n+\tfrac{1}{2}) = (n-\tfrac{1}{2})\Gamma(n-\tfrac{1}{2}) = (n-\tfrac{1}{2})(n-\tfrac{3}{2})\cdots\tfrac{3}{2}\cdot\tfrac{1}{2}\cdot\Gamma(\tfrac{1}{2})$$

$$= \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdots \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} = \frac{(2n)!}{2^n \, n!} \sqrt{\pi} ,$$

2.) Eulerjeva funkcija Beta.

Definicija

Eulerjeva funkcija B je funkcija dveh spremenljivk $B(p,q)$, podana s predpisom

$$B(p,q) = \int_0^1 t^{p-1} (1-t)^{q-1} \, dt , \qquad p,q > 0$$

Trditev

$$B(p,q) = B(q,p)$$

Dokaz:

Uvedemo novo spremenljivko

$$u = 1-t$$

Związek

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

Dowód: [illegible]: