Convex Reformulation of LMI-Based Distributed Controller Design with a Class of Non-Block-Diagonal Lyapunov Functions

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***Abstract*— This study addresses a distributed state feedback controller design problem for continuous-time linear time-invariant systems by means of linear matrix inequalities (LMIs). As structural constraints on a control gain result in non-convexity in general, the block-diagonal relaxation of Lyapunov functions has been prevalent despite its conservatism. In this work, we target a class of non-block-diagonal Lyapunov functions with the same sparsity pattern as distributed controllers. By leveraging a block-diagonal factorization of sparse matrices and Finsler's lemma, we first present a nonlinear matrix inequality for stabilizing distributed controllers with such Lyapunov functions, which boils down to a necessary and sufficient condition for such controllers if the sparsity pattern is chordal. As its relaxation, we derive novel LMIs, one of which strictly covers the conventional relaxation, and then provide analogous results for H_∞ control. Lastly, numerical examples underscore the efficacy of our results.**

I. Introduction

For the past decades, distributed control has been a pivotal topic in control as a response to the significant advancement of information and communication technologies [1]–[3]. Its practical applications include power networks [3], [4], formation control [2], [5], process control [6], and traffic systems [7]. By structuring controllers implemented only with local information, distributed controllers can substantially enhance the scalability and expandability of systems.

Nevertheless, such controller structure makes their design much more challenging even for linear systems [4]. While linear matrix inequalities (LMIs) [8] enable efficient controller design in a convex manner for centralized controllers, it is a different story in distributed ones; the structure constraint for the distributedness results in non-convexity. As a way to circumvent this issue, the most prevalent convex restriction is the block-diagonal relaxation of Lyapunov functions [9] as $\sum_i x_i^\top P_i x_i$. However, this relaxation leads to conservatism in general despite the simplicity. While several works have been proposed to resolve the issue of conservatism [5], [10]–[13], whether an equivalent convex reformulation of the structure constraint for gain matrices exists is still an open research question, except for specific classes, such as positive systems [14], [15]. Although the extended LMI approach [10], [11] gives a dense Lyapunov matrix through the introduction of a slack variable, this approach still requires a block-diagonal relaxation of the slack variable.

For the LMI-based distributed controller design problem, this study focuses on continuous-time systems and, importantly, a class of non-block-diagonal Lyapunov functions that has the same sparsity pattern as distributed controllers. For this class of Lyapunov functions, we propose a new convex relaxation not only for stabilization but also for H_∞ control [16]. Note that such Lyapunov functions play a fundamental role in gradient-flow systems [17], [18], and this approach can be easily extended to other fundamental problems, such as H_2 control. First, by leveraging a block-diagonal factorization for sparse matrices and Finsler's lemma [19], we show a new nonlinear matrix inequality for distributed controllers with such Lyapunov functions, which becomes necessary and sufficient over chordal sparsity [20], [21]. We then present new LMIs by relaxing the matrix inequality and demonstrate that the H_∞ control versions can be derived similarly. We also discuss our results in light of a state transformation called *inclusion principle* [1], [22], providing an intuitive interpretation. Finally, numerical examples illustrate the effectiveness of the proposed method. In the examples, we also show the results of another LMI obtained by slightly modifying the proposed LMI, which has no theoretical guarantee of stabilization but interestingly exhibits excellent performance.

Our contributions can be summarized as follows: a) This work presents novel LMIs for the distributed controller design problem. They can generate Lyapunov functions with a more complex sparsity pattern than block-diagonal matrices. One of the proposed methods completely covers the block-diagonal relaxation. In our numerical experiments, our proposed methods outperform not only the block-diagonal relaxation and sparsity invariance (SI) approaches [13] but also extended LMI [10], [11]; b) The computation of inverse matrices in the proposed method can be decomposed into smaller ones corresponding to subsystems formed by cliques of the communication graph, which alleviates computational burdens and enhances the scalability. Note that [12], [13] do not permit such a decomposed computation of inverse matrices; c) We show that the derived conditions provide a necessary and sufficient condition for distributed (H_∞)

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controllers with the class of non-block-diagonal Lyapunov functions under chordal sparsity. Since our convex relaxations follow from the condition, one can easily evaluate the conservatism. While the authors of [5] utilized a similar block-diagonal factorization, such a discussion cannot be found in any prior work, to our knowledge.

Note that our proposed approach can be effectively combined with many existing works, including the extended LMI approach [10], [11]. By inheriting the advantages of the two methods, our combined method enjoys both preferable theoretical and numerical performance. For the details, see our supplemental note and the discrete-time counterpart [23].

The rest of the paper is organized as follows. Section II provides the target system and problem. Section III prepares several notions and lemmas for the main results. In Section IV, we present our main results as a solution to the formulated problem, extending it to H_∞ control and presenting a control-theoretic interpretation. Section V showcases numerical examples. Finally, Section VI concludes this paper.

Notations: Let $|\cdot|$ be the number of elements in a countable finite set. Let $I_n \in \mathbb{R}^{n\times n}$ denote the $n\times n$ identity matrix. Let $O_{n_1\times n_2}$ be the $n_1 \times n_2$ zero matrix. We omit the subscript when it is obvious. Let $\mathrm{Im}(E)$ be the image space of the matrix E. Let $\mathrm{diag}(a)$ with $a = [a_1, \ldots, a_n]^\top$ denote the diagonal matrix whose ith diagonal entry is $a_i \in \mathbb{R}$. Similarly, $\mathrm{blkdiag}(\ldots, R_i, \ldots)$ represents the block diagonal matrix consisting of R_i. For $M \in \mathbb{R}^{m\times n}$ with $\mathrm{rank}(M) < m$, $M^\perp$ represents a basis of the null-space if M, i.e., $M^\perp$ satisfies $\{x : Mx = 0\} = \mathrm{Im}(M^\perp)$. For a real square matrix $A \in \mathbb{R}^{n\times n}$. the function $\mathrm{He}(\cdot)$ represents $\mathrm{He}(A) = A + A^\top$.

II. Problem Statement

A. Target system

Consider a large-scale system consisting of N subsystems. Suppose that their communication network is modeled by a time-invariant undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with $\mathcal{N} = \{1, \ldots, N\}$ and an edge set $\mathcal{E}$. Namely, the set $\mathcal{E}$ consists of pairs (i, j) of different nodes $i, j \in \mathcal{N}$, and we assume $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$.

Now, consider the following linear time-invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x = [x_1^\top, \ldots, x_N^\top]^\top \in \mathbb{R}^n$ with $x_i \in \mathbb{R}^{n_i}$, $\sum_{i=1}^N n_i = n$, $u = [u_1^\top, \ldots, u_N^\top]^\top \in \mathbb{R}^m$ with $u_i \in \mathbb{R}^{m_i}$, $\sum_{i=1}^N m_i = m$, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$. Here, x_i and u_i represent the state and control input of agent i. Besides, A and B are supposed to be partitioned with respect to the partitions $\{n_1, \ldots, n_N\}$ of $n = \sum_{i=1}^N n_i$ and $\{m_1, \ldots, m_N\}$ of $m = \sum_{i=1}^N m_i$ corresponding to the dimension of each subsystem's state and input.

Here, we consider a static state feedback controller:

$$u(t) = Kx(t). \tag{2}$$

In the following, $K_{ij} \in \mathbb{R}^{m_i\times n_j}$ represents the (i, j) block of K associated with the partitions $\{n_1, \ldots, n_N\}$ and $\{m_1, \ldots, m_N\}$.

To achieve the stabilization in a distributed fashion, we impose the following assumption on K.

Assumption 1: $K \in \mathcal{S}$, where

$$\mathcal{S} = \{K \in \mathbb{R}^{m\times n} : K_{ij} = O_{m_i\times n_j} \text{ if } (i, j) \notin \mathcal{E},\ i \neq j\}. \tag{3}$$

For $K \in \mathcal{S}$, we obtain $u_i = \sum_{j=1}^N K_{ij}x_j = \sum_{j\in\mathcal{N}_i} K_{ij}x_j$ with $\mathcal{N}_i = \{j \in \mathcal{N} : (i, j) \in \mathcal{E}\} \cup \{i\}$, which can be implemented in a distributed fashion.

For simplicity of notations in the following, we assume $n_i = m_i$ for all $i \in \mathcal{N}$ without loss of generality; otherwise, we just need to add the zero vectors to (or remove columns from) B_i to make it square, which is valid because B_i is not necessary to be column full rank in all the results below.

B. Target problem

For this system, the problem of finding a stabilizing controller satisfying Assumption 1 is equivalent to

$$\begin{aligned} &\text{Find } P \succ O,\ K \in \mathcal{S} \\ &\text{s.t. } (A + BK)^\top P + P(A + BK) \prec O. \end{aligned} \tag{4}$$

By implementing (2) with K in (4), one can guarantee the stability with the Lyapunov function $x^\top Px$.

However, the inequality in (4) involves non-convexity, which complicates distributed controller design. Indeed, the standard change of variables, i.e., $Q = P^{-1}$ and $Z = KQ$, gives the following equivalent problem to (4):

$$\begin{aligned} &\text{Find } Q \succ O,\ Z \\ &\text{s.t. } QA^\top + AQ + Z^\top B^\top + BZ \prec O,\ ZQ^{-1} \in \mathcal{S}. \end{aligned} \tag{5}$$

Importantly, $K = ZQ^{-1} \in \mathcal{S}$ is a non-convex constraint, and the exact convexification is still an open question.

A popular convex relaxation is the following LMI that restricts Q to a block-diagonal matrix, i.e., assumes $x^\top Q^{-1}x = \sum_{i=1}^N x_i^\top Q_i^{-1}x$ as a Lyapunov function, which guarantees $ZQ^{-1} \in \mathcal{S} \Leftrightarrow Z \in \mathcal{S}$. Then, (5) is reduced to the following relaxed LMI:

$$\begin{aligned} &\text{Find } Q = \mathrm{blkdiag}(Q_1, \ldots, Q_N) \succ O,\ Z \in \mathcal{S} \\ &\text{s.t. } QA^\top + AQ + Z^\top B^\top + BZ \prec O, \end{aligned} \tag{6}$$

where $Q_i \in \mathbb{R}^{n_i\times n_i}$. Let $\mathcal{K}_{\mathcal{S},\mathrm{diag}}$ be the set of all K given by (6) as follows:

$$\begin{aligned} \mathcal{K}_{\mathcal{S},\mathrm{diag}} = \{&K = ZQ^{-1} : \exists Q = \mathrm{blkdiag}(Q_1, \ldots, Q_N) \succ O, \\ &Z \in \mathcal{S} \text{ s.t. } QA^\top + AQ + Z^\top B^\top + BZ \prec O\}. \end{aligned}$$

Since the matrix Q is non-block-diagonal in general, this relaxation causes conservatism.

To mitigate the conservatism, our goal is to find a new subclass of K satisfying (4) that generalizes $\mathcal{K}_{\mathcal{S},\mathrm{diag}}$ and can be obtained in a convex manner, e.g., via LMIs. For this purpose, we here focus on the following problem and give a solution in the form of LMI.

Problem 1: Consider linear time-invariant system (1) with a static state feedback controller in (2) over an undirected graph $\mathcal{G}$ and the set $\mathcal{S}$ in (3). Then, solve the following:

$$\begin{aligned} &\text{Find} \quad P \succ O,\ P \in \mathcal{S},\ K \in \mathcal{S} \\ &\text{s.t.} \quad (A + BK)^\top P + P(A + BK) \prec O. \end{aligned} \tag{7}$$

Now, we let $\mathcal{K}_\mathcal{S}$ be the set of all K satisfying (7), i.e.,

$$\mathcal{K}_\mathcal{S} = \{K \in \mathcal{S} : \exists P \in \mathcal{S},\ P \succ O \text{ s.t. (7)}\}. \tag{8}$$

Although (7) still involves a relaxation owing to $P \in \mathcal{S}$ compared with the original problem (5), this can generalize the conventional block-diagonal relaxation in (6), i.e., $\mathcal{K}_{\mathcal{S},\text{diag}} \subset \mathcal{K}_{\mathcal{S}}$. Note that for gradient-flow systems, this class of Lyapunov functions characterizes all quadratic potential functions that generate a distributed gradient [17]. In addition, even if $n_i \neq m_i$, we can consider the same sparsity pattern for K by setting an appropriate dimension.

To take the robustness and exogenous disturbances into account, we also tackle H_∞ control in Subsection IV-B.

III. Preliminaries

Before establishing our main results in Section IV, let us introduce several notions and supporting lemmas.

a) Graph theory: First, we prepare graph theoretic concepts. Consider an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. A *clique* $\mathcal{C}$ is a subset of $\mathcal{N}$ satisfying $(i,j) \in \mathcal{E}$ for all $i, j \in \mathcal{C}$, $i \neq j$. Namely, a clique is a node set of a complete subgraph of $\mathcal{G}$. We assign indices to the cliques of $\mathcal{G}$ as $\mathcal{C}_1, \mathcal{C}_2, \ldots$ and let $\mathcal{Q}_{\mathcal{G}}^{\text{all}}$ denote the set of all cliques' indices. If $\mathcal{C}$ is not included in any other cliques, $\mathcal{C}$ is said to be *maximal*. Let $\mathcal{Q}_{\mathcal{G}}^{\max} (\subset \mathcal{Q}_{\mathcal{G}}^{\text{all}})$ represent the index set of all maximal cliques of $\mathcal{G}$. In the following, we use $\mathcal{Q}_{\mathcal{G}}$ as a subset of $\mathcal{Q}_{\mathcal{G}}^{\text{all}}$ (not necessarily as $\mathcal{Q}_{\mathcal{G}}^{\max}$) and $\mathcal{Q}_{\mathcal{G}}^i$ as the subset of cliques in $\mathcal{Q}_{\mathcal{G}}$ by which node i is contained, i.e., $\mathcal{Q}_{\mathcal{G}}^i := \{k \in \mathcal{Q}_{\mathcal{G}} : i \in \mathcal{C}_k\}$. A graph $\mathcal{G}$ is said to be *chordal* if every cycle of length at least four has a chord [20], [21]. Note that this class is not restrictive, and readers can find examples of chordal graphs in [20], [21].

b) Matrix E: Consider a subset $\mathcal{Q}_{\mathcal{G}}$ of $\mathcal{Q}_{\mathcal{G}}^{\text{all}}$. For a clique $\mathcal{C}_k$, $k \in \mathcal{Q}_{\mathcal{G}}$, we define $E_{\mathcal{C}_k} \in \mathbb{R}^{n_{\mathcal{C}_k} \times n}$ with $n_{\mathcal{C}_k} = \sum_{j \in \mathcal{C}_k} n_j$ as

$$E_{\mathcal{C}_k} = [\ldots, \underset{j \in \mathcal{C}_k}{E_j^\top}, \ldots]^\top, \tag{9}$$

where $E_j = [O_{n_j \times n_1}, \ldots, I_{n_j}, \ldots, O_{n_j \times n_N}] \in \mathbb{R}^{n_j \times n}$. This matrix generates the clique-wise copy of $x = [x_1^\top, \ldots, x_N^\top]^\top$ as $E_{\mathcal{C}_k} x = [\ldots, \underset{j \in \mathcal{C}_k}{x_j^\top}, \ldots]^\top$.

Interestingly, for a subset $\mathcal{Q}_{\mathcal{G}}$ of $\mathcal{Q}_{\mathcal{G}}^{\text{all}}$, the matrices $E_{\mathcal{C}_k}$, $k \in \mathcal{Q}_{\mathcal{G}}$ have the following beneficial properties for distributed controller design. The proof can be found in [24, Lemma 2].

Proposition 1: Consider the following matrix E consisting of $E_{\mathcal{C}_k}$ in (9):

$$E = [\ldots, \underset{k \in \mathcal{Q}_{\mathcal{G}}}{E_{\mathcal{C}_k}^\top}, \ldots]^\top \in \mathbb{R}^{(\sum_{k \in \mathcal{Q}_{\mathcal{G}}} n_{\mathcal{C}_k}) \times n}.$$

Assume that $\mathcal{Q}_{\mathcal{G}}^i \neq \emptyset$ for all $i \in \mathcal{N}$. Then, the matrix E satisfies the following properties.

(a) E is column full rank.
(b) $E^\top E = \text{blkdiag}(|\mathcal{Q}_{\mathcal{G}}^1| I_{n_1}, \ldots, |\mathcal{Q}_{\mathcal{G}}^N| I_{n_N}) \succ O$.
(c) For $\tilde{x} = [\ldots, \tilde{x}_k^\top, \ldots]^\top \in \mathbb{R}^{\sum_{k \in \mathcal{Q}_{\mathcal{G}}} n_{\mathcal{C}_k}}$ with $\tilde{x}_k \in \mathbb{R}^{n_{\mathcal{C}_k}}$,

$$E^\top \tilde{x} = \begin{bmatrix} \sum_{k \in \mathcal{Q}_{\mathcal{G}}^1} E_{k,1} \tilde{x}_k \\ \vdots \\ \sum_{k \in \mathcal{Q}_{\mathcal{G}}^N} E_{k,N} \tilde{x}_k \end{bmatrix} \in \mathbb{R}^n, \tag{10}$$

where $E_{k,i} \in \mathbb{R}^{n_i \times n_{\mathcal{C}_k}}$ for $k \in \mathcal{Q}_{\mathcal{G}}$ and $i \in \mathcal{C}_k$ satisfies $E_{k,i} x_{\mathcal{C}_k} = x_i$ with $x_{\mathcal{C}_k} = [\ldots, \underset{j \in \mathcal{C}_k}{x_j^\top}, \ldots]^\top \in \mathbb{R}^{n_{\mathcal{C}_k}}$ for any $x = [x_1^\top, \ldots, x_N^\top]^\top \in \mathbb{R}^n$.

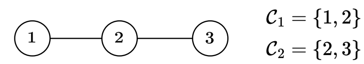

Fig. 1: An example of systems with $N = 3$. This network is chordal with maximal cliques $\mathcal{C}_1 = \{1, 2\}$ and $\mathcal{C}_2 = \{2, 3\}$.

This proposition together with $\bigcup_{k \in \mathcal{Q}_{\mathcal{G}}^i} \mathcal{C}_k \subset \mathcal{N}_i$ [18] implies that one can compute the least squares solution $(E^\top E)^{-1} E^\top \tilde{x}$ of $\tilde{x} = Ex$ and the projection $E(E^\top E)^{-1} E^\top \tilde{x}$ onto $\text{Im}(E)$ in a distributed fashion. Additionally, the assumption $\mathcal{Q}_{\mathcal{G}}^i \neq \emptyset$, $i \in \mathcal{N}$, which means that each node belongs to at least one clique, is always satisfied for $\mathcal{Q}_{\mathcal{G}} = \mathcal{Q}_{\mathcal{G}}^{\max}$, $\mathcal{Q}_{\mathcal{G}}^{\text{all}}$, regardless of the connectivity of $\mathcal{G}$.

Example 1: Consider a system with $N = 3$ over the chordal graph in Fig. 1 with maximal cliques $\mathcal{Q}_{\mathcal{G}} = \mathcal{Q}_{\mathcal{G}}^{\max} = \{1, 2\}$ with $\mathcal{C}_1 = \{1, 2\}$ and $\mathcal{C}_2 = \{2, 3\}$. Then, $\mathcal{Q}_{\mathcal{G}}^1 = 1$, $\mathcal{Q}_{\mathcal{G}}^2 = 2$, and $\mathcal{Q}_{\mathcal{G}}^3 = 1$. When $n_1 = n_2 = n_3 = 1$, the E matrix is given by $E = [E_{\mathcal{C}_1}^\top, E_{\mathcal{C}_2}^\top]^\top \in \mathbb{R}^{4 \times 3}$ with

$$E_{\mathcal{C}_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_{\mathcal{C}_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This E matrix satisfies $E_{\mathcal{C}_1} x = [x_1, x_2]^\top$, $E_{\mathcal{C}_2} x = [x_2, x_3]^\top$ for $x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3$, and $E^\top E = \text{diag}(1, 2, 1) = \text{diag}(|\mathcal{Q}_{\mathcal{G}}^1|, |\mathcal{Q}_{\mathcal{G}}^2|, |\mathcal{Q}_{\mathcal{G}}^3|)$.

Remark 1: The matrix E is called the *clique-wise duplication (CD) matrix* in our previous work [24] in the context of distributed optimization, which provides more detailed properties of this matrix. Moreover, there is more flexibility in the dimension of each block in E; see [23, Appendix A].

c) Block-diagonal factorization of sparse matrices: Leveraging the matrix E, we first present the positive definite version of Agler's theorem [20], [21], [25]. This theorem ensures the existence of a block-diagonal factorization of sparse positive definite matrices over chordal graphs.

Lemma 1 (positive definite version of Agler's theorem): Consider undirected graph $\mathcal{G}$ with cliques $\mathcal{Q}_{\mathcal{G}} = \{1, \ldots, q\}$. Then, given a partition $\{n_1, \ldots, n_N\}$ of n, for $\tilde{P} = \text{blkdiag}(\tilde{P}_1, \ldots, \tilde{P}_q) \succ O$ with $\tilde{P}_k \in \mathbb{R}^{n_{\mathcal{C}_k} \times n_{\mathcal{C}_k}}$, the matrix $P = E^\top \tilde{P} E = \sum_{k=1}^q E_{\mathcal{C}_k}^\top \tilde{P}_k E_{\mathcal{C}_k}$ is positive definite and belongs to $\mathcal{S}$. Moreover, if $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is chordal with maximal cliques $\mathcal{Q}_{\mathcal{G}} = \mathcal{Q}_{\mathcal{G}}^{\max} = \{1, \ldots, q\}$, the following equivalence holds:

$$\begin{aligned} & P \in \mathcal{S} \text{ and } P \succ O \\ \Leftrightarrow & \exists \tilde{P} = \text{blkdiag}(\tilde{P}_1, \ldots, \tilde{P}_q) \succ O \text{ with } \tilde{P}_k \in \mathbb{R}^{n_{\mathcal{C}_k} \times n_{\mathcal{C}_k}} \\ & \text{s.t. } P = E^\top \tilde{P} E = \sum_{k=1}^q E_{\mathcal{C}_k}^\top \tilde{P}_k E_{\mathcal{C}_k}. \end{aligned} \tag{11}$$

Next, for all undirected graphs (not necessarily chordal), we show that every matrix K in $\mathcal{S}$ has the following block-diagonal factorization. This lemma implies that $K \in \mathcal{S}$ can be parameterized by a clique-wise block-diagonal matrix $\tilde{K}$.

Lemma 2: Consider a partition $\{n_1, \ldots, n_N\}$ of n. Let $\mathcal{Q}_{\mathcal{G}}^i$, $i \in \mathcal{N}$ satisfy $\mathcal{Q}_{\mathcal{G}}^i \neq \emptyset$. Suppose $\mathcal{Q}_{\mathcal{G}}^i \cap \mathcal{Q}_{\mathcal{G}}^j \neq \emptyset \Leftrightarrow$

$(i,j) \in \mathcal{E}$. Then, it holds that

$$\mathcal{S} = \{E^\top \tilde{K} E : \tilde{K} = \text{blkdiag}(\cdots, \underset{k\in\mathcal{Q}_\mathcal{G}}{\tilde{K}_k}, \cdots),\ \tilde{K}_k \in \mathbb{R}^{n_{\mathcal{C}_k}\times n_{\mathcal{C}_k}}\}.$$

Proof: See Appendix A in the supplemental note. ∎

Example 2: For the graph in Fig. 1, Lemma 2 can be verified as follows. Here $*_{2\times 2}$ is an arbitrary 2×2 matrix.

$$E^\top \begin{bmatrix} *_{2\times 2} & O_{2\times 2} \\ O_{2\times 2} & *_{2\times 2} \end{bmatrix} E = \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix} \in \mathcal{S}.$$

IV. Main results

This section presents our main results for Problem 1. The notion of chordal graphs and Lemmas 1 and 2 prepared in Section III allow us to derive a (nonlinear) matrix inequality to solve Problem 1, which yields novel LMIs. Over chordal graphs, this nonlinear matrix inequality boils down to a necessary and sufficient condition for $K \in \mathcal{K}_\mathcal{S}$. Additionally, we apply the same strategy to the H_∞ control problem, presenting analogous LMIs. Note that our proposed approach can be extended to other important scenarios, such as polytopic uncertainties and output feedback control. For the details, see Appendix E in our supplemental note.

In what follows, let $\mathcal{Q}_\mathcal{G} = \{1,\ldots,q\}$ without loss of generality, and for notational simplicity, define the following sets of block-diagonal matrices corresponding to $\mathcal{Q}_\mathcal{G}$:

$$\mathbb{D}_{\mathcal{Q}_\mathcal{G}} = \{\text{blkdiag}(\tilde{P}_1,\ldots,\tilde{P}_q) : \tilde{P}_k \in \mathbb{R}^{n_{\mathcal{C}_k}\times n_{\mathcal{C}_k}},\ \forall k \in \mathcal{Q}_\mathcal{G}\},$$
$$\mathbb{D}^{++}_{\mathcal{Q}_\mathcal{G}} = \{\text{blkdiag}(\tilde{P}_1,\ldots,\tilde{P}_q) \in \mathbb{D}_{\mathcal{Q}_\mathcal{G}} : \tilde{P}_k \succ 0,\ \forall k \in \mathcal{Q}_\mathcal{G}\}.$$

A. Solution to Problem 1

A solution to Problem 1 is presented in this subsection. First, we impose the following assumption which is not strict as it is always satisfied for $\mathcal{Q}_\mathcal{G} = \mathcal{Q}^{\text{all}}_\mathcal{G}$ and $\mathcal{Q}^{\max}_\mathcal{G}$, regardless of the connectivity of $\mathcal{G}$.

Assumption 2: For $\mathcal{G}$, the index set $\mathcal{Q}_\mathcal{G} = \{1,\ldots,q\}$ of its cliques satisfies the following:

- $\mathcal{Q}^i_\mathcal{G} \neq \emptyset$ for all $i \in \mathcal{N}$.
- $\mathcal{Q}^i_\mathcal{G} \cap \mathcal{Q}^j_\mathcal{G} \neq \emptyset \Leftrightarrow (i,j) \in \mathcal{E}$.

Next, the following lemma, known as *Finsler's lemma*, plays a key role together with the lemmas in Section III.

Lemma 3 (Finsler's lemma [19]): Let $x \in \mathbb{R}^n$, $Q = Q^\top \in \mathbb{R}^{n\times n}$, and $M \in \mathbb{R}^{r\times n}$ such that $\text{rank}(M) < n$. The following statements are equivalent:

1) $x^\top Q x < 0$, $\forall Mx = 0,\ x \neq 0$.
2) $M^{\perp\top} Q M^\perp \prec O$.
3) $\exists \rho \in \mathbb{R}$ s.t. $Q + \rho M M^\top \prec O$.
4) $\exists X \in \mathbb{R}^{n\times r}$ s.t. $Q + M^\top X^\top + XM \prec O$.

In our main result, we use $I - E(E^\top E)^{-1}E^\top$ as the matrix M in Finsler's lemma based on the following proposition, which follows from Proposition 1.

Proposition 2: Consider undirected graph $\mathcal{G} = (\mathcal{N},\mathcal{E})$ with a set of cliques $\mathcal{Q}_\mathcal{G} \subset \mathcal{Q}^{\text{all}}_\mathcal{G}$. Assume that $\mathcal{G}$ is not complete. Suppose $\mathcal{Q}^i_\mathcal{G} \neq \emptyset$ for all $i \in \mathcal{N}$. Consider

$$M = I - E(E^\top E)^{-1}E^\top. \tag{12}$$

Then, $\text{rank}(M) < \sum_{k\in\mathcal{Q}_\mathcal{G}} n_{\mathcal{C}_k}$ and $M^\perp = E$ hold.

Proof: See Appendix B in the supplemental note. ∎

As a preliminary for controller design, let us present the following lemma for the internal stability, which characterizes $P \in \mathcal{S}$ in Problem 1 with $K = O$ if $\mathcal{G}$ is chordal. For non-chordal graphs, this lemma still gives a sufficient condition for such a $P \in \mathcal{S}$. Note that for the system matrices A, B in (1), the matrices $\tilde{A}$, $\tilde{B}$ below represent

$$\tilde{A} = EA(E^\top E)^{-1}E^\top,\ \tilde{B} = EB(E^\top E)^{-1}E^\top, \tag{13}$$

whose interpretation is presented in Subsection IV-C.

Lemma 4: Suppose Assumption 2. Let

$$\begin{aligned} \mathcal{P}_\mathcal{S} &= \{P \in \mathcal{S} : P \succ O,\ A^\top P + PA \prec O\} \\ \hat{\mathcal{P}}_\mathcal{S} &= \{P = E^\top \tilde{P} E : \exists \tilde{P} \in \mathbb{D}^{++}_{\mathcal{Q}_\mathcal{G}},\ \rho \in \mathbb{R} \\ &\qquad \text{s.t. } \tilde{A}^\top \tilde{P} + \tilde{P}\tilde{A} + \rho M \prec O\} \end{aligned}$$

with M in (12). Then, $\hat{\mathcal{P}}_\mathcal{S} \subset \mathcal{P}_\mathcal{S}$ holds. Moreover, $\hat{\mathcal{P}}_\mathcal{S} = \mathcal{P}_\mathcal{S}$ holds if $\mathcal{G}$ is chordal and $\mathcal{Q}_\mathcal{G} = \mathcal{Q}^{\max}_\mathcal{G}$.

Proof: When $\mathcal{G}$ is complete, then E is reduced to the identity matrix and thus we have $M = O$. Therefore $\mathcal{P}_\mathcal{S}$ and $\hat{\mathcal{P}}_\mathcal{S}$ are equivalent.

It is assumed below that $\mathcal{G}$ is not complete. First, $\hat{\mathcal{P}}_\mathcal{S} \subset \mathcal{P}_\mathcal{S}$ is shown as follows. Consider $E^\top \tilde{P} E \in \hat{\mathcal{P}}_\mathcal{S}$. Then, we obtain $E^\top(\tilde{A}^\top \tilde{P} + \tilde{P}\tilde{A} + \rho(I - E(E^\top E)^{-1}E^\top))E = A^\top(E^\top \tilde{P} E) + (E^\top \tilde{P} E)A \prec O$. Since Lemma 2 guarantees $E^\top \tilde{P} E \in \mathcal{S}$, we have $\hat{\mathcal{P}}_\mathcal{S} \subset \mathcal{P}_\mathcal{S}$.

To prove $\hat{\mathcal{P}}_\mathcal{S} = \mathcal{P}_\mathcal{S}$ for chordal graph $\mathcal{G}$ with $\mathcal{Q}_\mathcal{G} = \mathcal{Q}^{\max}_\mathcal{G}$, we show $\mathcal{P}_\mathcal{S} \subset \hat{\mathcal{P}}_\mathcal{S}$. Let $P \in \mathcal{P}_\mathcal{S}$. Then, by Lemma 1, P always admits the factorization $P = \sum_{k=1}^q E^\top_{\mathcal{C}_k} \tilde{P}_k E_{\mathcal{C}_k} = E^\top \tilde{P} E$ as $\mathcal{G}$ is chordal, where $\tilde{P} \in \mathbb{D}^{++}_{\mathcal{Q}_\mathcal{G}}$. Then, by leveraging this, we obtain

$$\begin{aligned} &A^\top P + PA = A^\top(E^\top \tilde{P} E) + (E^\top \tilde{P} E)A \\ &= E^\top(E(E^\top E)^{-1}A^\top E^\top \tilde{P} + \tilde{P} EA(E^\top E)^{-1}E^\top)E \\ &= E^\top(\tilde{A}^\top \tilde{P} + \tilde{P}\tilde{A})E \prec O. \end{aligned}$$

By Finsler's lemma, the above inequality is equivalent to $\tilde{A}^\top \tilde{P} + \tilde{P}\tilde{A} + \rho M \prec O$ for some $\rho \in \mathbb{R}$. Therefore $P \in \hat{\mathcal{P}}_\mathcal{S}$ holds. ∎

We now proceed to distributed controller design. As the first main result, we present the following theorem as a sufficient condition for distributed controllers $K \in \mathcal{K}_\mathcal{S}$. Furthermore, this condition satisfies the necessity over chordal graphs, meaning that this provides a characterization of the solutions to Problem 1. Recall that $E^\top E$ is diagonal from Proposition 1, which guarantees $(E^\top E)^{-1}F \in \mathcal{S}$ for $F \in \mathcal{S}$.

Theorem 1: Suppose Assumption 2. Let

$$\begin{aligned} &\hat{\mathcal{K}}_\mathcal{S} = \{K = (E^\top E)^{-1}E^\top(\tilde{Z}\tilde{Q}^{-1})E : \\ &\exists \tilde{Z} \in \mathbb{D}_{\mathcal{Q}_\mathcal{G}},\ \tilde{Q} \in \mathbb{D}^{++}_{\mathcal{Q}_\mathcal{G}},\ \rho \in \mathbb{R} \text{ s.t. } \Phi(\tilde{Q},\tilde{Z}) + \rho\tilde{Q}M\tilde{Q} \prec O\} \end{aligned}$$

with M in (12) and

$$\Phi(\tilde{Q},\tilde{Z}) = \tilde{Q}\tilde{A}^\top + \tilde{A}\tilde{Q} + \tilde{Z}^\top \tilde{B}^\top + \tilde{B}\tilde{Z}. \tag{14}$$

Then, $\hat{\mathcal{K}}_\mathcal{S} \subset \mathcal{K}_\mathcal{S}$ holds for $\mathcal{K}_\mathcal{S}$ in (8). Moreover, $\hat{\mathcal{K}}_\mathcal{S} = \mathcal{K}_\mathcal{S}$ holds if $\mathcal{G}$ is chordal and $\mathcal{Q}_\mathcal{G} = \mathcal{Q}^{\max}_\mathcal{G}$.

Proof: If $\mathcal{G}$ is complete, both $\hat{\mathcal{K}}_\mathcal{S}$ and $\mathcal{K}_\mathcal{S}$ coincide with the set of all the stabilizing controllers because $E = I$ and $M = O$. Therefore, the statement is true.

In what follows, we assume that $\mathcal{G}$ is not complete. First, we prove $\hat{\mathcal{K}}_\mathcal{S} \subset \mathcal{K}_\mathcal{S}$. For $(E^\top E)^{-1}E^\top \tilde{K} E \in \hat{\mathcal{K}}_\mathcal{S}$ with $\tilde{K} =$

$\tilde{Z}\tilde{Q}^{-1}$, it can be seen that

$$\begin{aligned}O \succ&\tilde{Q}^{-1}(\tilde{Q}\tilde{A}^\top+\tilde{A}\tilde{Q}+\tilde{Z}^\top\tilde{B}^\top+\tilde{B}\tilde{Q}+\rho\tilde{Q}M\tilde{Q})\tilde{Q}^{-1}\\ =&(\tilde{A}+\tilde{B}\tilde{K})^\top\tilde{Q}^{-1}+\tilde{Q}^{-1}(\tilde{A}+\tilde{B}\tilde{K})+\rho M.\end{aligned}$$

Recalling Finsler's lemma and defining $K = (E^\top E)^{-1}E^\top\tilde{K}E$ and $P = E^\top\tilde{Q}^{-1}E$ yield the following equivalent inequality:

$$\begin{aligned}O \succ&E^\top((\tilde{A}+\tilde{B}\tilde{K})^\top\tilde{Q}^{-1}+\tilde{Q}^{-1}(\tilde{A}+\tilde{B}\tilde{K}))E\\ =&(A+BK)^\top P+P(A+BK).\end{aligned}$$

Since Lemma 2 ensures $K \in \mathcal{S}$ and $P \in \mathcal{S}$, we obtain $\hat{\mathcal{K}}_{\mathcal{S}} \subset \mathcal{K}_{\mathcal{S}}$.

Next, let us move on to the proof of $\hat{\mathcal{K}}_{\mathcal{S}} = \mathcal{K}_{\mathcal{S}}$ under chordal graphs with $\mathcal{Q}_{\mathcal{G}} = \mathcal{Q}_{\mathcal{G}}^{\max}$. We shall show $\mathcal{K}_{\mathcal{S}} \subset \hat{\mathcal{K}}_{\mathcal{S}}$. Consider $K \in \mathcal{K}_{\mathcal{S}}$. By Lemma 2, K can be represented by $K = (E^\top E)^{-1}E^\top\tilde{K}E$ with $\tilde{K} = \mathrm{blkdiag}(\tilde{K}_1,\dots,\tilde{K}_q) \in \mathbb{D}_{\mathcal{Q}_{\mathcal{G}}}$, where $(E^\top E)^{-1}$ is diagonal. Moreover, the existence of a positive definite Lyapunov matrix $P \in \mathcal{S}$ is guaranteed. Then, leveraging Lemma 1 for $P \in \mathcal{S}$, we obtain $P = E^\top\tilde{P}E$ with some $\tilde{P} = \mathrm{blkdiag}(\tilde{P}_1,\dots,\tilde{P}_q) \succ O$. Consequently, Lemma 4 gives the following inequality with some $\rho \in \mathbb{R}$:

$$(\tilde{A}+\tilde{B}\tilde{K})^\top\tilde{P}+\tilde{P}(\tilde{A}+\tilde{B}\tilde{K})+\rho M \prec O.$$

Then, letting $\tilde{Q} = \tilde{P}^{-1}$ and $\tilde{Z} = \tilde{K}\tilde{Q}$ gives the equivalent inequality below:

$$\tilde{Q}\tilde{A}^\top+\tilde{A}\tilde{Q}+\tilde{Z}^\top\tilde{B}^\top+\tilde{B}\tilde{Z}+\rho\tilde{Q}M\tilde{Q} \prec O.$$

Hence, we obtain $K = (E^\top E)^{-1}E^\top\tilde{K}E \in \hat{\mathcal{K}}_{\mathcal{S}}$. Therefore we arrive at $\hat{\mathcal{K}}_{\mathcal{S}} = \mathcal{K}_{\mathcal{S}}$. ■

Theorem 1 gives the following novel convex relaxation that strictly contains the block-diagonal relaxation $\mathcal{K}_{\mathcal{S},\mathrm{diag}}$. A solution to the LMI in $\hat{\mathcal{K}}_{\mathcal{S}}$ admits $xE^\top\tilde{Q}^{-1}Ex$ as a Lyapunov function, where $E^\top\tilde{Q}^{-1}E \in \mathcal{S}$ is satisfied from Lemma 2.

Theorem 2 (Proposed method 1): Suppose Assumption 2. Let

$$\begin{aligned}\hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx1}} = \{&(E^\top E)^{-1}E^\top(\tilde{Z}\tilde{Q}^{-1})E : \exists\tilde{Z} \in \mathbb{D}_{\mathcal{Q}_{\mathcal{G}}},\, \tilde{Q} \in \mathbb{D}_{\mathcal{Q}_{\mathcal{G}}}^{++},\\ &\rho \in \mathbb{R},\, \eta > 0 \text{ s.t. } \Phi(\tilde{Q},\tilde{Z})+\rho M \prec O,\, \tilde{Q}M+M\tilde{Q} \succeq \eta M\}\end{aligned}$$

with $\Phi(\tilde{Q},\tilde{Z})$ in (14). Then, the following inclusion holds:

$$\mathcal{K}_{\mathcal{S},\mathrm{diag}} \subset \hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx1}} \subset \hat{\mathcal{K}}_{\mathcal{S}} \subset \mathcal{K}_{\mathcal{S}}.$$

Moreover, $\mathcal{K}_{\mathcal{S},\mathrm{diag}} \subset \hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx1}} = \hat{\mathcal{K}}_{\mathcal{S}} = \mathcal{K}_{\mathcal{S}}$ if $\mathcal{G}$ is complete.

Proof: First, let us show $\hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx1}} \subset \hat{\mathcal{K}}_{\mathcal{S}}$. Consider $K = (E^\top E)^{-1}E^\top(\tilde{Z}\tilde{Q}^{-1})E \in \hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx1}}$. Lemma 2 guarantees $K \in \mathcal{S}$. Then, combining the two inequalities in $\hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx1}}$ gives $\Phi(\tilde{Q},\tilde{Z}) \prec \frac{|\rho|}{\eta}(\tilde{Q}M+M\tilde{Q})$. Thus, pre- and post-multiplying this by $E^\top\tilde{Q}^{-1}$ and $\tilde{Q}^{-1}E$ respectively and utilizing Finsler's lemma yield $E^\top\tilde{Q}^{-1}\Phi(\tilde{Q},\tilde{Z})\tilde{Q}^{-1}E \prec O \Leftrightarrow \exists\beta \in \mathbb{R}$ s.t. $\tilde{Q}^{-1}\Phi(\tilde{Q},\tilde{Z})\tilde{Q}^{-1}+\beta M \prec O \Leftrightarrow \exists\beta \in \mathbb{R}$ s.t. $\Phi(\tilde{Q},\tilde{Z})+\beta\tilde{Q}M\tilde{Q} \prec O$. Therefore, $\hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx1}} \subset \hat{\mathcal{K}}_{\mathcal{S}}$.

Next, we prove $\mathcal{K}_{\mathcal{S},\mathrm{diag}} \subset \hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx1}}$. Consider $K \in \mathcal{K}_{\mathcal{S},\mathrm{diag}}$. Then, we have $K = ZQ^{-1}$ with some $Z \in \mathcal{S}$ and block-diagonal matrix $Q = \{Q_1,\dots,Q_N\} \succ O$. Additionally, Lemma 2 guarantees that there exists a block-diagonal matrix $\tilde{K}$ such that $K = (E^\top E)^{-1}E^\top\tilde{K}E$. Now, by setting $\tilde{Q} = \mathrm{blkdiag}(\cdots,\tilde{Q}_k,\cdots) \succ O$ with $\tilde{Q}_k = \mathrm{blkdiag}_{j\in\mathcal{C}_k}(\cdots,|\mathcal{Q}_{\mathcal{G}}^j|Q_j,\cdots) \succ O$ and $\tilde{Z} = \tilde{K}\tilde{Q}$, we have $(E^\top E)^{-1}E^\top\tilde{Z}\tilde{Q}^{-1}E = (E^\top E)^{-1}E^\top\tilde{Z}E(E^\top E)^{-1}Q^{-1} = ZQ^{-1} = K$ since $E(E^\top E)^{-1}Q^{-1} = \tilde{Q}^{-1}E$ from the proof of Proposition 3 in the supplemental note. Note that Z can be represented as $Z = (E^\top E)^{-1}E^\top\tilde{Z}E(E^\top E)^{-1} \in \mathcal{S}$ and it holds for $\tilde{Q}$ and M that

$$\tilde{Q}M = M\tilde{Q} = \tilde{Q}-E^\top QE \tag{15}$$

from Proposition 3. Hence, we obtain $O \succ (A+BK)^\top Q^{-1}+Q^{-1}(A+BK) = (A+BK)^\top(E^\top\tilde{Q}^{-1}E)+(E^\top\tilde{Q}^{-1}E)(A+BK) = E^\top((\tilde{A}+\tilde{B}\tilde{K})^\top\tilde{Q}^{-1}+\tilde{Q}^{-1}(\tilde{A}+\tilde{B}\tilde{K}))E$. Thus, by using Finsler's lemma and following the same procedure as the proof $\mathcal{K}_{\mathcal{S}} \subset \hat{\mathcal{K}}_{\mathcal{S}}$ over chordal graphs above, one can obtain $K \in \hat{\mathcal{K}}_{\mathcal{S}}$, which gives $\mathcal{K}_{\mathcal{S},\mathrm{diag}} \subset \hat{\mathcal{K}}_{\mathcal{S}}$. Then, utilizing Theorem 1 and (15), we get $O \succ \Phi(\tilde{Q},\tilde{Z})+\nu\tilde{Q}M\tilde{Q} = \Phi(\tilde{Q},\tilde{Z})+\nu M\tilde{Q}^2M \succeq \Phi(\tilde{Q},\tilde{Z})+\rho M$ for some $\nu \in \mathbb{R}$ and $\rho = \nu\lambda^2 \in \mathbb{R}$, where λ represents either the maximal or minimum eigenvalue of $\tilde{Q}$. Moreover, (15) also gives $M\tilde{Q}+\tilde{Q}M = M^2\tilde{Q}+\tilde{Q}M^2 = 2M\tilde{Q}M \succeq \eta M$, where $\eta = 2\lambda_{\min}(\tilde{Q}) > 0$ and $\lambda_{\min}(\tilde{Q})$ is the minimum eigenvauel of $\tilde{Q}$. Therefore, we obtain $K \in \hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx1}}$, and thus $\mathcal{K}_{\mathcal{S},\mathrm{diag}} \subset \hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx1}}$. Hence, combining the above discussion with Theorem 1, we obtain $\mathcal{K}_{\mathcal{S},\mathrm{diag}} \subset \hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx1}} \subset \hat{\mathcal{K}}_{\mathcal{S}} \subset \mathcal{K}_{\mathcal{S}}$.

If $\mathcal{G}$ is complete, we have $E = I$ and $M = O$, which obviously yields $\mathcal{K}_{\mathcal{S},\mathrm{diag}} \subset \hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx1}} = \hat{\mathcal{K}}_{\mathcal{S}} = \mathcal{K}_{\mathcal{S}}$. ■

Furthermore, we present another convex restriction that imposes $\rho = 0$ on $\hat{\mathcal{K}}_{\mathcal{S}}$. Note that there is no extra constraint as $\tilde{Q}M+\tilde{Q}M \succeq \eta M$ in this variant, which enables searching solutions that cannot be obtained by the proposed method 1.

Corollary 1 (Proposed method 2): Suppose Assumption 2. Let

$$\begin{aligned}\hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx2}} =&\{(E^\top E)^{-1}E^\top(\tilde{Z}\tilde{Q}^{-1})E :\\ &\exists\tilde{Z} \in \mathbb{D}_{\mathcal{Q}_{\mathcal{G}}},\, \tilde{Q} \in \mathbb{D}_{\mathcal{Q}_{\mathcal{G}}}^{++} \text{ s.t. } \Phi(\tilde{Q},\tilde{Z}) \prec O\}\end{aligned}$$

with $\Phi(\tilde{Q},\tilde{Z})$ in (14). Then, $\hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx2}} \subset \hat{\mathcal{K}}_{\mathcal{S}}$ holds. Moreover, $\mathcal{K}_{\mathcal{S},\mathrm{diag}} \subset \hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx2}} = \hat{\mathcal{K}}_{\mathcal{S}} = \mathcal{K}_{\mathcal{S}}$ if $\mathcal{G}$ is complete.

Remark 2: In the LMI in Corollary 1, we restrict ρ to be zero. A similar approach can be found in [5]. Note that in general, we cannot expect $\mathcal{K}_{\mathcal{S},\mathrm{diag}} \subset \hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx2}}$ because this approach cannot cover the case where the optimal value of ρ is negative.

Remark 3 (Proposed method 3): Removing the constraint $\tilde{Q}M+M\tilde{Q} \succeq \eta M$, $\eta > 0$ from $\hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx1}}$ in Theorem 2 gives another set of $K \in \mathcal{S}$ with an LMI:

$$\begin{aligned}\hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx3}} = \{&(E^\top E)^{-1}E^\top(\tilde{Z}\tilde{Q}^{-1})E : \exists\tilde{Z} \in \mathbb{D}_{\mathcal{Q}_{\mathcal{G}}},\, \tilde{Q} \in \mathbb{D}_{\mathcal{Q}_{\mathcal{G}}}^{++},\\ &\rho \in \mathbb{R} \text{ s.t. } \tilde{Q}\tilde{A}^\top+\tilde{A}\tilde{Q}+\tilde{Z}^\top\tilde{B}^\top+\tilde{B}\tilde{Z}+\rho M \prec O\}.\end{aligned}$$

This LMI can be obtained just by replacing the nonlinear term $\rho\tilde{Q}M\tilde{Q}$ in $\hat{\mathcal{K}}_{\mathcal{S}}$ (Theorem 1) with the linear term ρM. Since the constraint $\tilde{Q}M+M\tilde{Q} \succeq \eta M$, $\eta > 0$ has been removed, there is no guarantee that any $K \in \hat{\mathcal{K}}_{\mathcal{S},\mathrm{rlx3}}$ is stabilizing. Nevertheless, this approach usually achieves not only stabilization but also high numerical performance, as shown in Section V.

B. H_∞ control

This subsection presents the H_∞ control version of the results in Subsection IV-A. Note that the following discussion can be applied to H_2 control analogously.

Consider the following perturbed system:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t) \tag{16}$$
$$y(t) = Cx(t) + Du(t) + D_w w(t), \tag{17}$$

where w and y are an exogenous disturbance and the output, respectively. Under $u = 0$, strict upper bounds of H_∞ norm of the transfer function matrix $C(sI - A)^{-1}B_w + D_w$ are characterized by the following *bounded real lemma*.

Lemma 5 (Bounded real lemma [16]): Let $G(s) = C(sI - A)^{-1}B_w + D_w$. Then, A is Hurwitz and $\|G(s)\|_\infty < \gamma$ if and only if there exists $P \succ O$ such that

$$\begin{bmatrix} A^\top P + PA & PB_w & C^\top \\ B_w^\top P & -\gamma I & D_w^\top \\ C & D_w & -\gamma I \end{bmatrix} \prec O. \tag{18}$$

Consider static state feedback in (2) for the system above. Then, by this lemma and the notions in Section III, one can obtain an H_∞ control version of Lemma 4 as follows. For C, D, B_w in (16) and (17), we define $\tilde{C}$, $\tilde{D}$, and $\tilde{B}_w$ as

$$\tilde{C} = C(E^\top E)^{-1}E^\top,\ \tilde{D} = D(E^\top E)^{-1}E^\top,\ \tilde{B}_w = EB_w. \tag{19}$$

For an interpretation of them, see Subsection IV-C.

Lemma 6: Suppose Assumption 2. Let

$$\mathcal{P}_{\mathcal{S}}^{\infty,\gamma} = \{P \in \mathcal{S} : P \succ O \text{ s.t. (18) holds.}\},$$
$$\hat{\mathcal{P}}_{\mathcal{S}}^{\infty,\gamma} = \{P = E^\top \tilde{P} E : \exists \tilde{P} \in \mathbb{D}_{\mathcal{Q}_\mathcal{G}}^{++},\ \rho \in \mathbb{R}$$
$$\text{s.t.} \begin{bmatrix} \tilde{A}^\top \tilde{P} + \tilde{P}\tilde{A} + \rho M & \tilde{P}\tilde{B}_w & \tilde{C}^\top \\ \tilde{B}_w^\top \tilde{P} & -\gamma I & D_w^\top \\ \tilde{C} & D_w & -\gamma I \end{bmatrix} \prec O\}.$$

Then, $\hat{\mathcal{P}}_{\mathcal{S}}^{\infty,\gamma} \subset \mathcal{P}_{\mathcal{S}}^{\infty,\gamma}$ holds. Moreover, $\hat{\mathcal{P}}_{\mathcal{S}}^{\infty,\gamma} = \mathcal{P}_{\mathcal{S}}^{\infty,\gamma}$ holds if $\mathcal{G}$ is chordal and $\mathcal{Q}_\mathcal{G} = \mathcal{Q}_\mathcal{G}^{\max}$.

Proof: Let $\Theta = \text{blkdiag}(M, O_{n\times n}, O_{n\times n})$, for which we have $\Theta^\perp = \text{blkdiag}(E, I_n, I_n)$. Then, this lemma can be shown in a similar way to Lemma 2 from the following relationship for (18) with $P = E^\top \tilde{P} E$:

$$O \succ \begin{bmatrix} A^\top (E^\top \tilde{P} E) + (E^\top \tilde{P} E)A & (E^\top \tilde{P} E)B_w & C^\top \\ B_w^\top (E^\top \tilde{P} E) & -\gamma I & D_w^\top \\ C & D_w & -\gamma I \end{bmatrix}$$
$$= \Theta^{\perp\top} \begin{bmatrix} \tilde{A}^\top \tilde{P} + \tilde{P}\tilde{A} & \tilde{P}\tilde{B}_w & \tilde{C}^\top \\ \tilde{B}_w^\top \tilde{P} & -\gamma I & D_w^\top \\ \tilde{C} & D_w & -\gamma I \end{bmatrix} \Theta^\perp.$$

Therefore, by Finsler's lemma and $\Theta^\top \Theta = \text{blkdiag}(M, O_{n\times n}, O_{n\times n})$, we obtain the LMI with respect to $\tilde{P}$, ρ, and γ in $\hat{\mathcal{P}}_{\mathcal{S}}^{\infty,\gamma}$. ■

The H_∞ version of Theorem 1 can also be obtained as follows. Here, let $\mathcal{K}_{\mathcal{S}}^{\infty,\gamma}$ represent all distributed control gains $K \in \mathcal{S}$ with $P \in \mathcal{S}$ that achieve $\|G(s)\|_\infty < \gamma$.

Theorem 3: Suppose Assumption 2. Let

$$\hat{\mathcal{K}}_{\mathcal{S}}^{\infty,\gamma} = \{(E^\top E)^{-1}E^\top (\tilde{Z}\tilde{Q}^{-1})E : \exists \tilde{Z} \in \mathbb{D}_{\mathcal{Q}_\mathcal{G}},\ \tilde{Q} \in \mathbb{D}_{\mathcal{Q}_\mathcal{G}}^{++},$$
$$\rho \in \mathbb{R} \text{ s.t. } \Gamma_\gamma(\tilde{Q}, \tilde{Z}) + \text{blkdiag}(\rho \tilde{Q} M \tilde{Q}, O_{n\times n}, O_{n\times n}) \prec O\},$$

where

$$\Gamma_\gamma(\tilde{Q}, \tilde{Z}) = \begin{bmatrix} \text{He}(\tilde{A}\tilde{Q} + \tilde{B}\tilde{Z}) & \tilde{B}_w & \tilde{Q}\tilde{C}^\top + \tilde{Z}^\top \tilde{D}^\top \\ \tilde{B}_w^\top & -\gamma I & D_w^\top \\ \tilde{C}\tilde{Q} + \tilde{D}\tilde{Z} & D_w & -\gamma I \end{bmatrix}. \tag{20}$$

Then, $\hat{\mathcal{K}}_{\mathcal{S}}^{\infty,\gamma} \subset \mathcal{K}_{\mathcal{S}}^{\infty,\gamma}$ holds. Moreover, $\hat{\mathcal{K}}_{\mathcal{S}}^{\infty,\gamma} = \mathcal{K}_{\mathcal{S}}^{\infty,\gamma}$ holds if $\mathcal{G}$ is chordal and $\mathcal{Q}_\mathcal{G} = \mathcal{Q}_\mathcal{G}^{\max}$.

Proof: This theorem can be shown in the same way as the proof of Theorem 1 by pre- and post-multiplying the LMI in $\hat{\mathcal{P}}_{\mathcal{S}}^{\infty,\gamma}$ with $A + BK$, $K = (E^\top E)^{-1}E^\top \tilde{K} E$ as A by blkdiag$(\tilde{Q}, I, I)$ with $\tilde{Q} = \tilde{P}^{-1}$ and setting $\tilde{Z} = \tilde{K}\tilde{Q}$. ■

A similar convex relaxation to Theorem 2 can also be obtained as follows. This theorem guarantees that the proposed relaxation in $\hat{\mathcal{K}}_{\mathcal{S},\text{rlx1}}^{\infty,\gamma}$ strictly covers the conventional block-diagonal relaxation.

Theorem 4 (Proposed method 1 for H_∞ control): Suppose Assumption 2. Let

$$\mathcal{K}_{\mathcal{S},\text{diag}}^{\infty,\gamma} = \{K = ZQ^{-1} :$$
$$\exists Q = \text{blkdiag}(Q_1, \ldots, Q_N) \succ O,\ Z \in \mathcal{S} \text{ s.t.}$$
$$\begin{bmatrix} \text{He}(AQ + BZ) & B_w & QC^\top + Z^\top D^\top \\ B_w^\top & -\gamma I & D_w^\top \\ CQ + DZ & D_w & -\gamma I \end{bmatrix} \prec O\},$$

$$\hat{\mathcal{K}}_{\mathcal{S},\text{rlx1}}^{\infty,\gamma} = \{(E^\top E)^{-1}E^\top (\tilde{Z}\tilde{Q}^{-1})E : \exists \tilde{Z} \in \mathbb{D}_{\mathcal{Q}_\mathcal{G}},\ \tilde{Q} \in \mathbb{D}_{\mathcal{Q}_\mathcal{G}}^{++},$$
$$\rho \in \mathbb{R},\ \eta > 0 \text{ s.t. } \tilde{Q}M + M\tilde{Q} \succeq \eta M,$$
$$\Gamma_\gamma(\tilde{Q}, \tilde{Z}) + \text{blkdiag}(\rho M, O_{n\times n}, O_{n\times n}) \prec O\}$$

with $\Gamma_\gamma(\tilde{Q}, \tilde{Z})$ in (20). Then, the following inclusion holds:

$$\mathcal{K}_{\mathcal{S},\text{diag}}^{\infty,\gamma} \subset \hat{\mathcal{K}}_{\mathcal{S},\text{rlx1}}^{\infty,\gamma} \subset \hat{\mathcal{K}}_{\mathcal{S}}^{\infty,\gamma} \subset \mathcal{K}_{\mathcal{S}}^{\infty,\gamma}.$$

Moreover, $\mathcal{K}_{\mathcal{S},\text{diag}}^{\infty,\gamma} \subset \hat{\mathcal{K}}_{\mathcal{S},\text{rlx1}}^{\infty,\gamma} = \hat{\mathcal{K}}_{\mathcal{S}}^{\infty,\gamma} = \mathcal{K}_{\mathcal{S}}^{\infty,\gamma}$ if $\mathcal{G}$ is complete.

Similarly, we can present the following variant based on the same approach as Corollary 1.

Corollary 2 (Proposed method 2 for H_∞ control): Suppose Assumption 2. Let

$$\hat{\mathcal{K}}_{\mathcal{S},\text{rlx2}}^{\infty,\gamma} = \{K = (E^\top E)^{-1}E^\top (\tilde{Z}\tilde{Q}^{-1})E :$$
$$\exists \tilde{Z} \in \mathbb{D}_{\mathcal{Q}_\mathcal{G}},\ \tilde{Q} \in \mathbb{D}_{\mathcal{Q}_\mathcal{G}}^{++} \text{ s.t. } \Gamma_\gamma(\tilde{Q}, \tilde{Z}) \prec O\}$$

with $\Gamma_\gamma(\tilde{Q}, \tilde{Z})$ in (20). Then, $\hat{\mathcal{K}}_{\mathcal{S},\text{rlx2}}^{\infty,\gamma} \subset \hat{\mathcal{K}}_{\mathcal{S}}^{\infty,\gamma}$ holds. Moreover, $\mathcal{K}_{\mathcal{S},\text{diag}}^{\infty,\gamma} \subset \hat{\mathcal{K}}_{\mathcal{S},\text{rlx2}}^{\infty,\gamma} = \hat{\mathcal{K}}_{\mathcal{S}}^{\infty,\gamma} = \mathcal{K}_{\mathcal{S}}^{\infty,\gamma}$ if $\mathcal{G}$ is complete.

Remark 4 (Proposed method 3 for H_∞ control): In the same spirit as $\hat{\mathcal{K}}_{\mathcal{S},\text{rlx3}}$ in Remark 3, we present the following set with an LMI, which shows interestingly strong numerical performance despite the lack of theoretical guarantee of stabilization:

$$\hat{\mathcal{K}}_{\mathcal{S},\text{rlx3}}^{\infty,\gamma} = \{(E^\top E)^{-1}E^\top (\tilde{Z}\tilde{Q}^{-1})E : \exists \tilde{Z} \in \mathbb{D}_{\mathcal{Q}_\mathcal{G}},\ \tilde{Q} \in \mathbb{D}_{\mathcal{Q}_\mathcal{G}}^{++},$$
$$\rho \in \mathbb{R} \text{ s.t. } \Gamma_\gamma(\tilde{Q}, \tilde{Z}) + \text{blkdiag}(\rho M, O_{n\times n}, O_{n\times n}) \prec O\}.$$

C. An interpretation of the proposed methods

We here view the proposed methods and the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{B}_w$ in (13) and (19), which are naturally obtained by combining Lemma 1 with Finsler's lemma, from the perspective of a state transformation with E.

Consider a LTI system in (16) and (17) with $u = Kx$ in (2). For this system, transforming the state x as $\tilde{x} = Ex$

with $\tilde{x}(0) = Ex(0)$ and plugging in $K = (E^\top E)^{-1}E^\top \tilde{K}E$ with a $\tilde{K}$ provide the following expanded system:

$$\begin{aligned}\dot{\tilde{x}} &= E\dot{x} = E(A+BK)x + EB_w w\\ &= (EA(E^\top E)E^\top + EB(E^\top E)^{-1}E^\top\tilde{K})Ex + EB_w w\\ &= (\tilde{A}+\tilde{B}\tilde{K})\tilde{x} + \tilde{B}_w w\\ y &= (C+DK)x + D_w w\\ &= (C(E^\top E)^{-1}E^\top + D(E^\top E)^{-1}E^\top\tilde{K})Ex + D_w w\\ &= (\tilde{C}+\tilde{D}\tilde{K})\tilde{x} + D_w w.\end{aligned}$$

From Proposition 1a, one can recover x from $\tilde{x}$ by $x = (E^\top E)^{-1}E^\top\tilde{x}$.

The expanded system above with $\tilde{x} = Ex$ tells us that our main results and proposed methods provide a distributed controller as $\tilde{K}\tilde{x}$ with $\tilde{K} = \text{blkdiag}(\ldots, \tilde{K}_k, \ldots)$ for the expanded system of $\tilde{x}$, and the terms ρM in $\hat{\mathcal{P}}_{\mathcal{S}}$ (Lemma 4) and $\rho\tilde{Q}M\tilde{Q}$ in $\hat{\mathcal{K}}_{\mathcal{S}}$ (Theorem 1) are needed due to the constraint $\tilde{x} \in \text{Im}(E)$. Interestingly, this state transformation was studied under the framework of *inclusion principle* [1], [22] in the 1980s in the control community while Agler's theorem (Lemma 1) was initially exploited in the operations research community [20], [21], [25].

V. Numerical Examples

Here, we conduct numerical experiments for the stabilization and H_∞ control problems using the `mosek` [26], a standard commercial SDP solver. For the details of their implementation, see our codes available on GitHub.[1]

a) Stabilization problem: In the stabilization case, we set the following system and parameters. We set $N = 32$, $n_1 = m_1 = \ldots = n_{32} = m_{32} = 1$, and $B = \text{diag}(b_1, \ldots, b_{32})$ with $b_i = 0$, $i = 1, 16$ and $b_i = 1$, $i \neq 1, 16$. As A matrix, we generated 200 unstable matrices such that (A, B) are stabilizable. Each entry is generated by the normal Gaussian distribution. As undirected graph $\mathcal{G}$, we use two different graphs: ring and wheel graphs [27]. [2]

The simulation results are shown in Table I. In the simulations, we design a stabilizing control gain for the system (1) with the input in (2) by using our proposed methods 1–3, the block-diagonal relaxation, SI-based relaxation [13], and extended LMI approach [10], [11]. In the extended LMI approach, we set the additional scalar parameter as $\alpha = 1$, similarly to [10], [12]; see Proposition 4 in our supplemental note for the details. Table I indicates the number of cases where a stabilizing gain is generated out of 200 samples.

From Table I, our proposed methods outperformed the others in all the graphs, including the extended LMI approach that generates a dense Lyapunov function. Moreover, the proposed method 3 always provides a stabilizing control gain despite the absence of its guarantee. The SI-based relaxation ends in the same performance as the block-diagonal relaxation, which implies the conservativeness[3].

[1] `https://github.com/WatanabeYuto/Distributed_Control_with_Non-Block-Diagonal_Lyapunov`

[2] In the wheel graph, we assign node 1 to the central node.

[3] This is mainly due to the symmetry of Lyapunov matrices. Note that [13] has multiple notable benefits and insights, especially in its frequency domain formulation.

TABLE I: The comparison of how many cases a stabilizing control gain is obtained out of 200 different examples. We compare P1–P3 (proposed methods 1–3) with BD (the block-diagonal relaxation), SI (SI-based relaxation [13]), and Extended LMI [10], [11].

Graph	P1	P2	P3	BD	SI	Extended LMI
Ring	130	114	200	51	51	73
Wheel	149	178	200	54	54	106

TABLE II: The comparison of the optimal H_∞ norm bound γ^* for three systems (DIS1, BDT1, and DIS3) from [28]. The second column shows the optimal value γ^*_{cen} of the centralized controller, and the third to seventh columns show the value of the ratio $\gamma^*/\gamma^*_{\text{cen}}$. The cases where a stabilizing controller could not be generated are denoted by "fail".

	Centralized	P1	P2	P3	BD	SI
DIS1	289.41	9.6728	78.874	1.210	fail	fail
BDT1	55.702	1.0101	1.6027	1.0015	1.0504	1.0504
DIS3	204.886	1.1015	1.5313	fail	1.1016	1.1016

b) H_∞ controller design: We also present numerical results of H_∞ controller design for several realistic models from $COMPle_ib$ [28], a standard benchmark library for controller evaluation. We remark that the supplemental note presents additional simulation results for randomly generated unstable A matrices. Here, we test our three proposed methods for the three models DIS1, BDT1, and DIS3 from [28] in comparison with the block-diagonal relaxation (6), SI-based relaxation [13], and the centralized H_∞ controller. The systems' dimensions are given by $(n, m) = (8, 4)$, $(6, 4)$, and $(11, 3)$, respectively. As the H_∞ performance measure, we define $C = [20I_n, O_{m\times m}]^\top$, $D = [O_{n\times n}, 200I_m]^\top$, and $D_w = O$. As the sparsity pattern $\mathcal{S}$ for DIS1, we consider

$$P = \begin{bmatrix} * & * & * & * & * & * & * & * \\ * & * & * & 0 & 0 & 0 & 0 & * \\ * & * & * & * & 0 & 0 & 0 & 0 \\ * & 0 & * & * & * & 0 & 0 & 0 \\ * & 0 & 0 & * & * & * & 0 & 0 \\ * & 0 & 0 & 0 & * & * & * & 0 \\ * & 0 & 0 & 0 & 0 & * & * & * \\ * & * & 0 & 0 & 0 & 0 & * & * \end{bmatrix}.$$

As the sparsity pattern of gain K for DIS1, we take the first 4 rows of P's above. Then, we can apply our present formulation by replacing the nonsquare B matrix in $\mathbb{R}^{8\times 4}$ by $[B, O_{8\times 4}] \in \mathbb{R}^{8\times 8}$ without loss of generality. (For the treatment of non-square matrices, see [23, Appendix A].) Note that for the other examples, we define the sparsity pattern of P and K using the wheel graph in the same way.

The simulation results are presented in Table II. For three systems (DIS1, BDT1, and DIS3) from [28], the second column represents the optimal value γ^*_{cen} of the centralized controller. The third to seventh columns show the value of the ratio $\gamma^*/\gamma^*_{\text{cen}}$, where γ^* is the optimal H_∞ norm bound by each method[4]. When a stabilizing controller could not be obtained, we denote "fail". From this table, our proposed methods outperform the other relaxation methods, as expected from our theoretical results. In particular, all the proposed methods achieved minimization for DIS1 while the previous methods failed. Note that despite the failure of the

[4] Only in the proposed method 3, we compute $\gamma^* = \|G\|_\infty$ with the optimal gain K^* using the MATLAB function "`hinfnorm`", due to the lack of theoretical guarantee for recovering the original bounded real lemma.

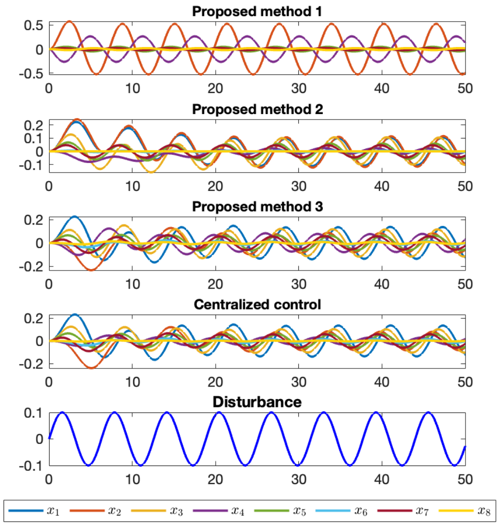

Fig. 2: The evolution of the state vector $x(t) = [x_1(t), \ldots, x_8(t)]^\top$ in the system DIS1 [28] with our proposed methods and the centralized H_∞ controller against $w(t) = 0.1\sin(t)$.

proposed method 3 for DIS3, this method achieved stabilization and the highest performance in the other examples, showing its value as a heuristic algorithm.

Furthermore, to test their disturbance attenuation performance, we also present the plots of the evolution of the states $x(t) = [x_1(t), \ldots, x_8(t)]^\top$ against $w(t) = 0.1\sin(t)$ for DIS1 in Fig. 2. We note that $B_w = [1, 0, 1, 0, 1, 0, 1, 0]^\top$ in DIS1, and we set the initial state as $x(0) = 0$. This figure reveals that all the methods successfully remain bounded against the disturbance. In particular, the magnitude of variation due to the disturbance is also maintained small, showing the efficacy of our proposed methods.

VI. Conclusion

This study considered the LMI-based distributed controller design problem with non-block-diagonal Lyapunov functions with the same sparsity pattern as controllers. Based on a block-diagonal factorization of sparse matrices and Finsler's lemma, we derived a new matrix inequality condition for distributed controllers with such Lyapunov functions and then showed that this inequality is reduced to a necessary and sufficient condition under chordal sparsity. We further derived a new LMI as the convex relaxation and also provided analogous results for H_∞ control. Finally, numerical results demonstrated the effectiveness of our proposed methods.

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