Kitaev's quantum double model for abelian groups

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Def (Quantum Spin Systems)

Γ : set (e.g. $\mathbb{Z}^\nu$: lattice)

$x \in \Gamma$, $\mathcal{H}_x$: Hilb sp., $\dim \sup_{x \in \Gamma} \mathcal{H}_x < \infty$

we define local algebra as,

$$\Lambda \subset\subset \Gamma, \quad \mathcal{A}_\Lambda := \bigotimes_{x \in \Lambda} B(\mathcal{H}_x)$$

quasi-local algebra as,

$$\mathcal{A} := \overline{\bigcup_{\Lambda \subset\subset \Gamma} \mathcal{A}_\Gamma}^{\,\text{nor}} . \qquad \mathcal{A}_{loc} := \bigcup_{\Lambda \subset\subset \Gamma} \mathcal{A}_\Gamma$$

Def (Quantum Spin Systems)

interaction Φ is a map

$$\Phi : \mathcal{P}(\Gamma) \ni \Lambda \longmapsto \Phi(\Lambda) \in \mathbb{A}$$

s.t. $\Phi(\Lambda) \in \mathbb{A}_\Lambda$, self-adjoint

$$H_\Phi(\Lambda) := \sum_{O \subset\subset \Lambda} \Phi(O), \quad \Lambda \subset\subset \Gamma$$: local Hamiltonian

local automorphisms on $\mathbb{A}_\Lambda$: $\tau_t^\Lambda := \mathrm{Ad}\exp itH_\Lambda$

Def (Quantum Spin Systems)

if interaction Φ is finite range

(i.e. $\exists d > 0$, $\operatorname{diam} \Lambda \geq d \Rightarrow \Phi(\Lambda) = 0$)

$\Rightarrow \exists \tau_t : \mathbb{R} \curvearrowright \mathbb{A}$: $*$-automorphism

s.t. $\forall A \in \mathbb{A}_{loc}$, $\lim_{\Lambda \nearrow \Gamma} \| \tau_t(A) - \tau_t^{\Lambda}(A) \| = 0$

$\rightsquigarrow (\mathbb{A}, \tau)$: C^*-dynamical system

generator of τ_t : $\delta(A) = i \sum_{\substack{\Lambda \subset\subset \Gamma \\ \Lambda \cap O \neq \emptyset}} [\Phi(\Lambda), A]$, $A \in \mathbb{A}_O$, $O \subset\subset \Gamma$

Ground State

<u>Def</u>

$(\mathbb{A}, \tau)$: C^*-dynamical system,

δ : generator of τ

state ω on $\mathbb{A}$ is ground state

$\Longleftrightarrow {}^\forall A \in D(\delta)$: $-i\,\omega(A^*\delta(A)) \geq 0$

$\Longleftrightarrow {}^\forall A, B \in \mathbb{A}$, ${}^\exists F_{A,B}$ s.t. analytic on $\mathrm{Im}\, z \geq 0$

$$F_{A,B}(t) = \omega(A\,\tau_t(B)),\quad t \in \mathbb{R}$$

Ground State

Def

$\beta \in \mathbb{R} \setminus \{0\}$: inverse temperature

state ω on $\mathbb{A}$ is (τ, β)-KMS state

$\Longleftrightarrow {}^{\forall} A, B \in \mathbb{A}$, B : analytic, $\omega(A \tau_{i\beta}(B)) = \omega(BA)$

$\Longleftrightarrow {}^{\forall} A, B \in \mathbb{A}$, ${}^{\exists} F_{A,B}$ s.t. analytic on $D_\beta = \{z \in \mathbb{C},\ 0 < \mathrm{Im}\, z < \beta\}$

$$F_{A,B}(t) = \omega(A \tau_t(B)).$$

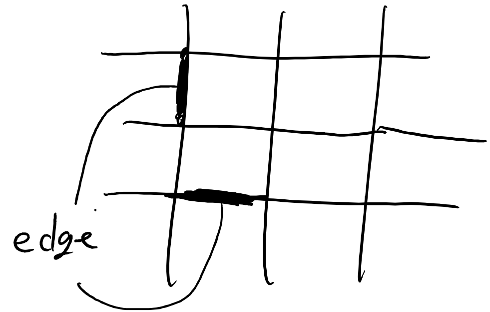
$$F_{A,B}(t + i\beta) = \omega(\tau_t(B) A), \quad t \in \mathbb{R}$$

$\rightsquigarrow$ ground state is KMS-state at value $+\infty$

Kitaev's Quantum Double Model

G : fin grp

$\mathbb{Z}^2$ lattice, Γ := set of all edge



${}^\forall e \in \Gamma$, $\mathcal{H}_e := \overline{\mathrm{span}}\{|g\rangle ; g \in G\}$

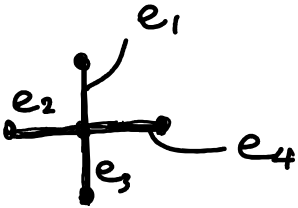
$T_\tau^g |h\rangle := \delta_{g^{-1},h} |h\rangle$, $L_{\tau'}^g := |k g^{-1}\rangle$

$F_\tau^{h,g} := T_\tau^g$, $F_{\tau'}^{h,g} := \delta_{g,e} L_{\tau'}^h$, $F_\varepsilon^{h,g} = \delta_{g,e} 1$

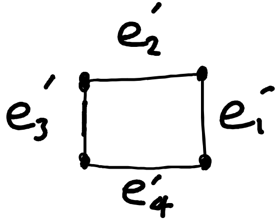
$F_\rho^{h,g} := \sum_{k \in G} F_{\rho_1}^{h,k} F_{\rho_2}^{k^{-1}hk,\, kg}$

$B_s^g := F_{\beta_s}^{e,g}$, $A_s^h := F_{\alpha_s}^{h,e}$

$B_s := B_s^e$, $A_s := \frac{1}{|G|} \sum_{h \in G} A_s^h$

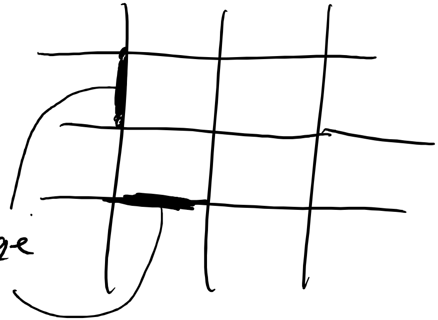


star(s)



plaquet(p)

edge



A_s : star operator on $\bigotimes_{i=1}^{4} \mathcal{H}_{e_i}$

B_p : plaquet operator on $\bigotimes_{i=1}^{4} \mathcal{H}_{e'_i}$

local Hamiltonian: $H_\Lambda := -\sum_{star(s)\subset\Lambda} A_s - \sum_{plaq(p)\subset\Lambda} B_p$, $\Lambda \subset\subset \mathbb{B}$

Thm (Naaijkens '12)

G : fin grp

There exists unique translational invariant ground state ω_0

This state is uniquely determined by,

$$\omega_0(A_s) = \omega_0(B_p) = 1 \quad , \quad \forall s, p$$

☺ ground state minimize $H_\Lambda := -\sum_{\mathrm{star}(s) \subset \Lambda} A_s - \sum_{\mathrm{plaq}(p) \subset \Lambda} B_p$

$\|A_s\| = \|B_p\| = 1$

uniqueness : technical

Kitaev's Toric Code Model

$\uparrow$ case of quantum double model $G = \mathbb{Z}/2\mathbb{Z}$

$\rightsquigarrow$ $\forall e \in \Gamma$, $\mathcal{H}_e = \overline{\mathrm{span}}\{|0\rangle, |1\rangle\} = \mathbb{C}^2$

$$A_s = \bigotimes_{j \in star(s)} \sigma_j^x \quad , \qquad B_p = \bigotimes_{j \in plaq(p)} \sigma_j^z$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Thm

ground state ω_0 has spectral gap

i.e. $^\exists M>0$, $\mathrm{sp}H_0 \subset \{0\} \cup [M, \infty)$ —— (*)

$$\left[\begin{array}{l} (\pi, H, \Omega) : \text{GNS of } \omega \\ U_t \pi(A)\Omega = \pi(\tau_t(A)) \\ ^\exists H_0 : \text{self adjoint}, \quad U_t = e^{itH_0} \end{array}\right]$$

sketch of pf

P_Ω : spectral proj of H_0 corr to $\{0\}$

$(*) \Longleftrightarrow \langle \psi, (H_0 + MP_\Omega)\psi \rangle = \langle \psi, H_0\psi \rangle + M|\langle \Omega, \psi \rangle|^2 \geq M\|\psi\|^2, \quad \psi \in D(H_0)$

$\Longleftrightarrow -i\,\omega_0(X^*\delta(X)) \geq M(\omega_0(X^*X) - |\omega_0(X)|^2)$ —— (**)

$$-i\,\omega_0(X^*\delta(X)) \geq M(\omega_0(X^*X) - |\omega_0(X)|^2) \quad \text{—— } (**)$$

$$\mathbb{A}_{xz} := C^*(A_s, B_p \;;\; s, p)$$

$$X \text{ or } Y \in \mathbb{A}_{xz} \Rightarrow -i\,\omega_0(X^*\delta(Y)) = 4(\omega_0(X^*Y) - \overline{\omega_0(X)}\,\omega_0(Y)) = 0$$

let $X = X_0 + \sum_{i \in I} \lambda_i X_i$, $X_0 \in \mathbb{A}_{xz}$, $X_i \notin \mathbb{A}_{xz}$; monomial of pauli matrix

$\forall i$, $\exists A_s$, or B_p s.t. not commute with X_i

$\{A_s, X_i\} = A_s X_i + X_i A_s = 0 \rightarrow 2\omega(X_i) = \omega(\{X_i, A_s\}) = 0$

$$(**) \Leftrightarrow -i\sum_{i,j} \omega_0(X_i^*\delta(X_j)) \geq \sum_{i,j} \omega_0(X_i^* X_j)$$

$$(**) \iff -i\sum_{i,j}\omega_0(X_i^*\delta(X_j)) \geq M\sum_{i,j}\omega_0(X_i^*X_j)$$

$$n_i := | A_s , B_p ; \text{ not commute with } X_i \}|$$

$$-i\sum_{i,j}\omega_0(X_i^*\delta(X_j)) = \sum_{i,j}\omega_0\left(X_i^*\left[\left(-\sum_s A_s - \sum_p B_p\right), X_j\right]\right)$$

$$= \sum_{i,j} 2n_i\,\omega_0(X_i^*X_j)$$

$$\overset{n_i\geq 2}{\geq} 4\sum_{i,j}\omega_0(X_i X_j)$$

$M = 4$!

Thank you!