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Mathematical Foundation of Quantum Measurement Theory

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Abstract In this paper, we attempt to establish quantum measurement theory. First, we introduce the concept of an instrument. Then, we define a measuring process which is another formulation of a quantum measurement. We show a one-to-one correspondence between them. Next, we define a concept of a system of measurement correlations. We then show a one-to-one correspondence between systems of measurement correlations and measuring processes up to statistical equivalence.

1 Introduction

In this paper, we mathematically investigate measuring processes. Quantum measurement theory was established by J. von Neumann in [8]. Von Neumann formulated quantum measurements mathematically. When a discrete physical quantity $A = \sum_j a_j E^A(\{a_j\})$ lies in $\Delta \in \mathscr{B}(\mathbb{R})$, a change of a state ρ is discribed as follows under the repeatability hypothesis in the condition that a physical quantity A is non-degenerate,

$$\rho \mapsto \frac{\sum_{a\in\Delta} E^M(a)\rho E^A)}{\mathrm{Tr}[\rho E^A(\Delta)]} = \frac{\sum_{a\in\Delta} E^M(a)\rho E^A)}{\mathrm{Tr}[\rho E^A(\Delta)]}. \tag{1.1}$$

E.B. Davies and J.T. Lewis [7] introduced the concept of an instrument which enabled a formalization of quantum measurements without the repeatability hypothesis. M. Ozawa [1] introduced concepts of a CP (completely positive) instrument and a measurement process which characterized quantum measurements completely. K. Okamura [3] established systems of measurement correlations which are the exact conterparts of instruments in the (generalized) Heisenberg picture.

This paper is based on papers [1] and [3]. Let us describe how this paper is organized. In Section 2, we define CP instruments and measuring processes and show equivalence of them. We discuss about weak repetability of CP instruments in Section 3. In Section 4, we define systems of measurement correlations and prove a one-to-one correspondence between systems of measurement correlations and classes of measuring processes in the situation that an algebra of quantities is $\boldsymbol{B}(\mathcal{H})$.

2 Measuring Processes

Definition 2.1. Let $(S, \mathcal{F})$ be a measurable space. A positive operator-valued measure $X : \mathcal{F} \to \boldsymbol{B}(\mathcal{H})$ is called a semiobservable on $\mathcal{H}$ with value space $(S, \mathcal{F})$ if it satisfies $X(S) = 1$.

An observable X is a semiobservable which is projection-valued.

Definition 2.2. Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. The map $\mathcal{I}$ is called a CP (completely positive) instrument on $\mathcal{M}$ with $(S, \mathcal{F})$, if $\mathcal{I}$ is a subtransition map-valued measure satisfying the following two conditions for any mutually disjoint and countable family $\{\Delta_i\} \subset \mathcal{F}$ and $\rho \in \mathcal{M}_*$, $M \in \mathcal{M}$ where $\mathcal{M}_*$ is a predual of $\mathcal{M}$.

$$\langle \rho, \sum_i \mathcal{I}(\Delta_i M) \rangle = \sum_i \langle \rho, \mathcal{I}(\Delta_i M) \rangle, \tag{2.1}$$

$$\mathcal{I}(S)1 = 1. \tag{2.2}$$

Before entering main discussion in this section, we introduce some conditional expectations on Hilbert spaces.

Definition 2.3. The map T on $\boldsymbol{B}(\mathcal{H})$ onto $\mathcal{M}$ is called a conditional expectation if T is normal and completely positive and satisfies $T(AMB) = AT(M)B$ for all $A, B \in \boldsymbol{B}(\mathcal{H})$, $M \in \mathcal{M}$.

Let $\mathcal{K}$ be another Hilbert space and σ be a state on $\mathcal{K}$. An equation

$$\mathrm{Tr}[\rho E_\sigma(x)] = \mathrm{Tr}[(\rho \otimes \sigma)x], x \in \boldsymbol{B}(\mathcal{H} \otimes \mathcal{K}), \rho \in \boldsymbol{T}(\mathcal{H}) \tag{2.3}$$

defines a CP map $E_\sigma : \boldsymbol{B}(\mathcal{H} \otimes \mathcal{K}) \to \boldsymbol{B}(\mathcal{H})$. As E_σ is the adjoint of the map $\rho \mapsto \rho \otimes \sigma$, the map $x \mapsto E_\sigma(x) \otimes 1$ is a conditional expectation.

Similarly, let $E_\mathcal{K} : \boldsymbol{T}(\mathcal{H} \otimes \mathcal{K}) \to \boldsymbol{T}(\mathcal{H})$ be a CP map defined by

$$\mathrm{Tr}[E_\mathcal{K}(\phi)x] = \mathrm{Tr}[\phi(x \otimes 1)], \phi \in \boldsymbol{T}(\mathcal{H} \otimes \mathcal{K}), x \in \boldsymbol{B}(\mathcal{H}). \tag{2.4}$$

Its adjoint $x \mapsto x \otimes 1$ is also a conditinal expectation.

Definition 2.4. Let $\mathcal{H}$ be a Hilbert space and X be a semiobservable on $\mathcal{H}$ with value space $(S, \mathcal{F})$. A measuring process $\mathbb{M}$ of X is a 4-tuple $\mathbb{M} = (\mathscr{K}, \sigma, \tilde{X}, U)$ consisting of a Hilbert space $\mathscr{K}$, an observable $\tilde{X}$ on $\mathscr{K}$ with value space $(S, \mathcal{F})$, a state σ on $\mathscr{K}$, and a unitary operator U on $\mathscr{K} \otimes \mathscr{K}$ satisfying the relation

$$X(\Delta) = E_\sigma[U^*(1 \otimes \tilde{X}(\Delta))U] \tag{2.5}$$

for any B in $\mathcal{F}$.

Suppose that a measuring process $\mathbb{M} = (\mathscr{K}, \sigma, \tilde{X}, U)$ of X is carried out in the initial state ρ of $\mathcal{H}$. Let ρ^Δ denote the state at the instant after the measurement. Suppose that the observer were to measure simultaneously measurable observables M and $\tilde{X}$. Then we have the joint probability distribution of their values:

$$\mathrm{Prob}(M \in d\lambda), \tilde{X} \in d\Delta = \mathrm{Tr}[U(\rho \otimes \sigma)U^*(M(d\lambda) \otimes \tilde{X}(d\Delta))]. \tag{2.6}$$

Thus, if $\mathrm{Prob}(\tilde{X} \in \Delta) \neq 0$, we have also the conditional probability distribution of M conditioned by the value of $\tilde{X}$ lying in Δ,

$$\begin{aligned} \mathrm{Prob}(M \in d\lambda | \tilde{X} \in \Delta) &= \mathrm{Prob}(M \in d\lambda, \tilde{X} \in \Delta)/\mathrm{Prob}(\tilde{X} \in \Delta) \\ &= \mathrm{Tr}[U(\rho \otimes \sigma)U^*(M(d\lambda) \otimes \tilde{X}\Delta)]/\mathrm{Tr}[\rho X(\Delta)]. \end{aligned} \tag{2.7}$$

Then,

$$\begin{aligned} \mathrm{Ex}(M|\tilde{X} \in \Delta) &= \int_\mathbb{R} \lambda \mathrm{Prob}(M \in d\lambda | \tilde{X} \in \Delta) \\ &= \mathrm{Tr}[U(\rho \otimes \sigma)U * (m \otimes \tilde{X}(\Delta))]/\mathrm{Tr}[\rho X(\Delta)], \end{aligned} \tag{2.8}$$

where $m = \int_\mathbb{R} \lambda m(d\lambda)$. On the other hand, by the probabilistic interpretation of the state ρ^Δ, it must satisfy the relation

$$\mathrm{Prob}(M \in d\lambda | \tilde{X} \in \Delta) = \mathrm{Tr}[\rho^\Delta M(d\lambda)], \tag{2.9}$$

or, equivalently,

$$\mathrm{Ex}(M|\tilde{X} \in \Delta) = \mathrm{Tr}[\rho^{\Delta} M]. \tag{2.10}$$

By equations (2.8), (2.10) and easy calculations, we have

$$\rho^{\Delta} = E_{\mathcal{K}}[U(\rho \otimes \sigma)U^*(1 \otimes \tilde{X}(\Delta))]/\mathrm{Tr}[\rho X(\Delta)]. \tag{2.11}$$

Therefore, we have determined the change of the states $\rho \mapsto \rho^{\Delta}$ caused by the measuring process $\mathbb{M} = (\mathscr{K}, \sigma, \tilde{X}, U)$ of the semiobservable X on $\mathcal{H}$ with value space $(S, \mathcal{F})$.

For any M in $\boldsymbol{B}(\mathcal{H})$, $\mathrm{Ex}^{\mathbb{M}}(a|\Delta; \rho)$ will denote the conditional expectation of the outcome of a measurement of M at that instant after the measuring process $\mathbb{M}$ under the condition that the measuring process $\mathbb{M}$ of X has been carried out in the initial state ρ on $\mathcal{H}$ and its outcome lies in $\Delta \in \mathcal{F}$. Then from the above discussions, we have

$$\begin{aligned} \mathrm{Ex}^{\mathbb{M}}(M|\Delta; \rho) &= \mathrm{Tr}[\rho^{\Delta} M] \\ &= \mathrm{Tr}[U(\rho \otimes \sigma)U^*(M \otimes \tilde{X}(B))]. \end{aligned} \tag{2.12}$$

We define a state ρ^{Δ} and an expectation $\mathrm{Ex}^{\mathbb{M}}(M|\Delta; \rho)$ formally.

Definition 2.5. Let $\mathbb{M} = (\mathscr{K}, \sigma, \tilde{X}, U)$ be a measuring process of X. For each $\Delta \in \mathcal{F}$, we define a state $\rho^{\Delta} \in \boldsymbol{T}(\mathcal{H})$ after the measurement as the state holding (2.11) and the conditional expectation of outcome of measurement of M by (2.12).

Proposition 2.6. Let $\mathcal{I}$ be a CP instrument of $\mathcal{M}$. There is a Hilbert space $\mathcal{H}_0$, a spectral measure $E : \mathcal{F} \to \boldsymbol{B}(\mathcal{H}_0)$, a non-degenerate normal representation $\pi : \mathcal{M} \to \boldsymbol{B}(\mathcal{H}_0)$ and a linear isometry $V : \mathcal{H} \to \mathcal{H}_0$ such that

$$\mathcal{I}(\Delta, M) = V^* E(\Delta)\pi(M)V \tag{2.13}$$

for all $\Delta \in \mathcal{F}, M \in \mathcal{M}$.

Proof. We difine a sesquilinear form on $\mathcal{L}^{\infty}(S) \otimes \mathcal{M} \otimes \mathcal{H}$ as follows:

$$\langle \xi, \eta \rangle = \sum_i \int_{\mathcal{F}} g_j(\Delta)^* f_i(\Delta)(\mathcal{I}(d\Delta, N_j^* M_i)\xi_i, \eta_j), \tag{2.14}$$

where

$$\xi = \sum_i f_i \otimes M_i \otimes \xi_i,$$

$$\eta = \sum_i g_i \otimes N_i \otimes \eta_i.$$

Define an action of $\mathcal{M}$ and $\mathcal{F}$ on $\mathcal{L}^\infty(S) \otimes \mathcal{M} \otimes \mathcal{H}$:

$$\pi(x)\xi = \sum_i f_i \otimes xM_i \otimes \xi_i, \tag{2.15}$$

$$E(\Delta)\xi = \sum_i \chi_\Delta f_i \otimes M_i \otimes \xi_i. \tag{2.16}$$

The maps π and E are well-defined on the completion of $\mathcal{H}_0$ which is the completion of $\mathcal{L}^\infty(S) \otimes \mathcal{M} \otimes \mathcal{H}$. The isometry V is defined as $V\phi = [1 \otimes 1 \otimes \phi], \phi \in \mathcal{H}$. □

Definition 2.7. Let $\mathcal{I}$ be a CP instrument, X be a semiobservable. The map X is an associate map of $\mathcal{I}$ and $\mathcal{I}$ is X-*compatible* if $X(\Delta) = \mathcal{I}(\Delta, 1)$ for all $\Delta \in S$. A transition map T is an associate map of $\mathcal{I}$, if $T(M) = \mathcal{I}(S, M)$. The map T is said to be X-*compatible* if $T(\mathcal{M})$ is included in $X(S)'$. The CP instrument $\mathcal{I}$ is called decomposable, if it is of the form $\mathcal{I}(\Delta, M) = X(\Delta)T(M)$ where X, T are associates of $\mathcal{I}$.

Proposition 2.8. A CP instrument $\mathcal{I}$ is decomposable if (1) an associate observable X is projection-valued or (2) an associate transition map T is holomorphic, i.e., $T(M)^*T(M) = T(M^*M)$.

Proof. Suppose (2) and $\mathcal{I}(\Delta, M) = V^*E(\Delta)\pi(M)V$ as in Proposition 2.6. We have $T(M) = V^*\pi(M)V$, $V^*V = \mathcal{I}(S, 1) = 1$. Then,

$$(VT(M) - \pi(M)V)^*(VT(M) - \pi(M)V) = T(M^*M) - T(M)^*T(M) = 0.$$

We obtain $VT(M) = \pi(M)V$. Thus, $\mathcal{I}(\Delta, M) = V^*E(\Delta)VT(M) = X(\Delta)T(M)$. Case(1) is proved similarly. □

Proposition 2.9. Let X be an observable. There is one-to-one correspondence between X-*compatible* CP instruments on $\mathcal{M}$ and X-*compatible* transition maps on $\mathcal{M}$.

Proof. If $\mathcal{I}$ is X-*compatible*, $\mathcal{I}$ is decomposable. There exist an associate transition map T and $\mathcal{I} = X \cdot T$. For any $0 \leq M \in \mathcal{M}$, $X(\Delta)T(M) = \mathcal{I}(\Delta, M) = \mathcal{I}(\Delta, M)^* = T(M)X(\Delta)$. Then $T(\mathcal{M}) \subset X(\Delta)'$. $\square$

Definition 2.10. Measuring processes $\mathbb{M}_1, \mathbb{M}_1$ of X are said to be statistically equivalent if

$$\mathrm{Ex}^{\mathbb{M}_1}(M|\Delta;\rho) = \mathrm{Ex}^{\mathbb{M}_2}(M|\Delta;\rho) \tag{2.17}$$

for any $a \in \mathcal{M}, \Delta \in \mathcal{F}, \rho \in \mathcal{M}_*$.

Proposition 2.11. Let $\mathbb{M} = (\mathscr{K}, \sigma, \tilde{X}, U)$ be a measuring process $\mathcal{M}$ of X. A map $\mathcal{I}$ defined as following is a CP intrument.

$$\mathcal{I}(\Delta)M = E_\sigma[U^*(M \otimes \tilde{X}(\Delta))U]. \tag{2.18}$$

Then, $\mathcal{I}$ is X-compatible and the following identity holds.

$$\rho\mathcal{I}(\Delta) = E_{\mathcal{K}}[U(\rho \otimes \sigma)U^*(1 \otimes \tilde{X}(\Delta))U]. \tag{2.19}$$

Proof. We have that

$$\begin{aligned}
\mathrm{Tr}[\rho\mathcal{I}(\Delta)M] &= \mathrm{Tr}[\rho E_\sigma[U^*(M \otimes \tilde{X}(\Delta))U]] \\
&= \mathrm{Tr}[(\rho \otimes \sigma)U^*(M \otimes \tilde{X}(\Delta))U] \\
&= \mathrm{Tr}[U(\rho \otimes \sigma)U^*(M \otimes \tilde{X}(\Delta))] \\
&= \mathrm{Tr}[E_{\mathcal{K}}[(U(\rho \otimes \sigma)U^*)(1 \otimes \tilde{X}(\Delta))]M] \qquad (2.20)
\end{aligned}$$

for any state $\rho \in \boldsymbol{T}(\mathcal{H})$ and $M \in \mathcal{M}$. $\square$

By the equation (2.11), the state ρ^Δ holds

$$\rho^\Delta = (1/\mathrm{Tr}[\rho\mathcal{I}(\Delta)])\rho\mathcal{I}(\Delta). \tag{2.21}$$

Theorem 2.12. Let X be an semiobservable. There is a one-to-one correspondence between X-*compatible* CP instruments and statistically equivalent classes of measuring processes of X which is given by

$$\mathrm{Tr}[\rho\mathcal{I}(\Delta)]\mathrm{Ex}^{\mathbb{M}}(M|\Delta;\rho) = \mathrm{Tr}[\rho\mathcal{I}(\Delta)M]. \tag{2.22}$$

Proof. For each measuring process $\mathbb{M}$, a CP instrument $\mathcal{I}$ on $\boldsymbol{B}(\mathcal{H})$ defined by equation (2.18) is a unique CP instrument satisfying (2.22).

Let $\mathcal{I}$ be a X-*compatible* CP instrument. We construct a measuring process giving $\mathcal{I}$ by (2.18). Let $\mathcal{H}_0, \pi, V$ and E be such as obtaind in theorem 2.6. We can assume that $\pi \cdot E = E \cdot \pi$. Since a non-degenerate $*$-representation of $\boldsymbol{B}(\mathcal{H})$ is a multiple of the identity representation (See [6], Lemma 9.2.2), there exist a Hilbert space $\mathcal{H}_1$ and a projection-valued measure $E_1 : \mathcal{F} \to \boldsymbol{B}(\mathcal{H}_1)$ such that

$$\mathcal{H}_0 = \mathcal{H} \otimes \mathcal{H}_1, \pi(a) = a \otimes 1, \tag{2.23}$$

$$E = 1 \otimes E_1. \tag{2.24}$$

We have

$$\mathcal{I}(\Delta, a) = V^*(a \otimes E_1(\Delta))V. \tag{2.25}$$

Let η_0 be a unit vector of $\mathcal{H}_0$ and η_1 be a unit vector of $\mathcal{H}_1$. Define an isometry V_0 from $\mathcal{H} \otimes \mathbb{C}\eta_1 \otimes \mathbb{C}\eta_0$ into $\mathcal{H} \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$ as

$$V_0(\xi \otimes \eta \otimes \eta_0) = V\xi \otimes \eta_0.$$

The operator V_0 can be extended to a unitary operator U on $\mathcal{H} \otimes \mathcal{H}_1 \otimes \mathcal{H}_0$.

We define a Hilbert space $\mathcal{K}$, a state σ on $\mathcal{K}$ and and an observable $\tilde{X}$ on $\mathcal{K}$ as follows.

$$\mathcal{K} = \mathcal{H}_1 \otimes \mathcal{H}_0, \tag{2.26}$$

$$\sigma = |\eta_1 \otimes \eta_0\rangle\langle \eta_0 \otimes \eta_1|, \tag{2.27}$$

$$\tilde{X}(\Delta) = E_1(\Delta) \otimes 1. \tag{2.28}$$

A 4-tuple $\mathbb{M} =: (\mathscr{K}, \sigma, \tilde{X}, U)$ is a measuring process. For any $\xi \in \mathcal{H}$,we have

$$\begin{aligned}
\langle \mathcal{I}(\Delta)M\xi, \xi\rangle &= \langle V^*(M \otimes E_1(\Delta))V\xi, \xi\rangle \\
&= \langle M \otimes E_1(\Delta)V\xi, V\xi\rangle \\
&= \langle M \otimes E_1(\Delta)V\xi \otimes \eta_0, V\xi \otimes \eta_0\rangle \\
&= \langle (M \otimes E_1(\Delta) \otimes 1)U(\xi \otimes \eta_1 \otimes \eta_0), U(\xi \otimes \eta_1 \otimes \eta_0)\rangle \\
&= \langle U^*(M \otimes \tilde{X}(\Delta))U(\xi \otimes \eta_1 \otimes \eta_0), \xi \otimes \eta_1 \otimes \eta_0\rangle \\
&= \langle E_\sigma[U^*(M \otimes \tilde{X}(\Delta))U]\xi, \xi\rangle
\end{aligned}$$

Thus, (2.18) holds for any $\Delta \in \mathcal{F}$ and $M \in \mathcal{M}$. $\square$

3 Repeatability

Consider von Neumann's repeatability hypothesis ([8], pp. 214, 335):

(M) If the physical quantity is measured twice in succession in a system, then we get the same value each time.

Let $\mathbb{M} = (\mathscr{K}, \sigma, \tilde{X}, U)$ be a measuring process of a semiobservable X. We say that a positive operator-valued measure X is discrete if there is a countable set $\Delta_0 \in \mathcal{F}$ such that $X(\mathcal{F} \backslash \Delta_0) = 0$. If X is discrete,

(M') $\mathrm{Ex}^{\mathbb{M}}(X(\{\lambda\})|\mu; \rho) = \delta_{\lambda,\mu'}$

for all state on $\mathcal{H}$ and all λ, μ in $\mathcal{F}$, whenever $\mathrm{Tr}[\rho X(\{\mu\})] \neq 0$. We expand this observation to general observables.

Definition 3.1. A measuring process $\mathbb{M}$ is said to be weakly repeatable, if it satisfies

$$\mathrm{Ex}^{M}[\tilde{X}(\Delta_1|\tilde{X} \in \Delta_2 : \rho)] = \mathrm{Tr}[\rho X(\Delta_1 \cap \Delta_2)]. \tag{3.1}$$

for all states ρ on $\mathscr{H}$ and $\Delta_1, \Delta_2 \in \mathcal{F}$.
A CP instrument $\mathscr{I}$ is repeatable if it satisfies

$$\mathcal{I}(\Delta_1)X(\Delta_2) = X(\Delta_1 \cap \Delta_2). \tag{3.2}$$

Definitions (3.1) and (3.2) are equivalent under the one-to-one correspondence between measuring processes and CP instruments.

Let $\mathcal{I}$ be a weakly repeatable CP instrument on a von Neumann algebra $\mathcal{M}$. The map X is an associate semiobservable and T is an associate map of $\mathcal{I}$.

Lemma 3.2. For any $\Delta_1, \Delta_2 \in \mathcal{F}$ and $a \in \mathcal{M}$, it satisfies the following.
(1) $T(X(\Delta_1)^2) = X(\Delta_1)$,
(2) $\mathcal{I}(\Delta_1 \cap \Delta_2, a) = \mathcal{I}(\Delta_2, aX(\Delta_1)) = \mathcal{I}(\Delta_2, X(\Delta_1)a)$,
(3) $\mathcal{I}(\Delta_1, a) = T(aX(\Delta_1)) = T(X(\Delta_1)a)$.

Proof. We can assume a triplet (E, π, V) is of the form $\mathcal{I}(\Delta, a) = V^*E(\Delta)\pi(a)$ as in Theorem 2.6.

(1) By weak repeatability of $\mathcal{I}$, we have $\mathcal{I}(\Delta, X(\Delta)) = X(\Delta)$.

$$(\pi(X(\Delta))V - E(\Delta)V)^*(\pi(X(\Delta))V - E(\Delta)V) = T(X(\Delta)^2) - X(\Delta) \geq 0. \tag{3.3}$$

Since $0 \leq X(\Delta) \leq 1$, $T(X(\Delta)) = \mathcal{I}(\mathcal{F}, X(\Delta)) = X(\Delta)$. Thus, $X(\Delta) = T(X(\Delta)) \geq T((X\Delta)^2) \geq 0$. It follows that $T(X(\Delta)^2) = X(\Delta)$.

(2) The left-hand side of (3.3) is 0, so we have $\pi(X(\Delta))V = E(\Delta)V$. Then, $V^*\pi(X(\Delta)) = V^*E(\Delta)$. Hence,

$$\begin{aligned} V^*E(\Delta_1)E(\Delta_2)\pi(a)V &= V^*\pi(X\Delta_1)E(\Delta_2)\pi(a)V \\ &= V^*E(\Delta_2)\pi(X(\Delta_1)a)V. \end{aligned}$$

It holds that $\mathcal{I}(\Delta_1 \cap \Delta_2, a) = \mathcal{I}(\Delta_2, X(\Delta_1)a)$.

(3) This relation is obtained by putting $\Delta_2 = \mathcal{F}$ in (2) $\square$

Lemma 3.3. Let p be the least projection in $X(\mathcal{F})''$ such that $T(p)$=1. For any x in $\mathcal{M}$,

$$T(x) = T(xp) = T(px) = T(pxp).$$

Proof. We prove only $T(x) = T(px)$. The others are proved similarly. Let ξ, η be in $\mathcal{H}$. We have,

$$\begin{aligned} |\langle T(x - px)\xi, \eta\rangle| &= |\langle V^*\pi(1-p)\pi(x)V\xi, \eta\rangle| \\ &= |\langle \pi(x)V\xi, \pi(1-p)V\eta\rangle| \\ &\leq ||\pi(x)V\xi|| \cdot ||\pi(1-p)V\eta|| \\ &= |\pi(x)V\xi|| \cdot ||\langle V^*\pi(1-p)V\eta, \eta\rangle||^{\frac{1}{2}} \\ &= |\pi(x)V\xi|| \cdot ||\langle T(1-p)\eta, \eta\rangle||^{\frac{1}{2}} = 0. \end{aligned}$$

It follows that $T(x) = T(px)$. $\square$

Lemma 3.4. If $x \geq 0$ in $X(\mathcal{F})''$ satisfies $T(x) = 0$, then $pxp = 0$.

Proof. We fix $0 \leq x \in X(\mathcal{F})''$. Set e be a projection map to the range of x. There exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ of polynomials of x which converges to e in the strong operator topology. We can take each p_n not containing a constant term. Then, $T(e) = \lim T(p_n(x)) = 0$. Thus, $1 - e \geq p$ so that $pe = ep = 0$. It follows that $pxp = 0$. $\square$

We construct a map $P : \mathcal{F} \to X(\mathcal{F})''$ as $P(\Delta) = pX(\Delta)p, \Delta \in \mathcal{F}$.

Lemma 3.5. Let P be a projection-valued measure such that $P(\Delta) = pX(\Delta) = X(\Delta)p$ for all $\Delta \in \mathcal{F}$.

Proof. By Lemma 3.3, $T(P(\Delta)) = T(pX(\Delta)p) = T(X(\Delta))$. Then,

$$\begin{aligned} T(P(\Delta)^2) &= T(X(\Delta)pX(\Delta)) = \mathcal{I}(\Delta, pX(\Delta)) \\ &= I(\Delta \cap \Delta, p) = T(pX(\Delta)) = T(X(\Delta)). \end{aligned}$$

□

The following theorem is easily seen by Lemma 3.2, 3.4 and 3.5.

Theorem 3.6. Let $\mathcal{I}$ be a weakly repeatable CP instrument on $\mathcal{M}$. There exists a projection-valued measure $P : \mathcal{F} \to X(\mathcal{F})''$ such that

$$\begin{aligned} &\mathcal{I}(\Delta, a) = T(aP(\Delta)) = T(P(\Delta)a), \\ &P(\Delta) = P(\mathcal{F}X(\Delta)) = X(\Delta)P(\mathcal{F}). \end{aligned}$$

Theorem 3.7. Let $(S, \mathcal{F})$ be a standard Borel space and $\mathcal{H}$ be a separable Hilbert space. We say that a CP instrument is discrete if the associate semiobservable is discrete. Then every weakly repeatable CP instrument $\mathcal{I}$ on $\boldsymbol{B}(\mathcal{H})$ with value space $(S, \mathcal{F})$ is discrete.

Proof. By the relation $X(\Delta) = \mathcal{I}(\Delta, 1) = T(\Delta)$, we only have to show P is discrete. By Ref [6], Lemma 4.4.1, there is a countable set $\Delta_0 \in \mathcal{F}$ such that $\Delta \mapsto P(\Delta \cap \Delta_0)$ is a discrete projection-valued measure on $\boldsymbol{B}(P(\Delta_0)\mathcal{H})$ and $\Delta \mapsto P(\Delta \backslash \Delta_0)$ dtinuous projection-valued measure on $\boldsymbol{B}(P(S \backslash \Delta_0)\mathcal{H})$. It suffices to show $Q \equiv P(S \backslash \Delta_0) = 0$.

We set $T_0(a) = QT(a)Q$ where T is an asssociate map of $\mathcal{I}$. Then,

$$\begin{aligned} T_0(Q) &= QT(Q)Q \\ &= QT(P(S \backslash \Delta_0))Q \\ &= Q\mathcal{I}(S \backslash \Delta_0, 1)Q \\ &= QX(S \backslash \Delta_0)Q \\ &= pX(S \backslash \Delta_0)pX(S \backslash \Delta_0)pX(S \backslash \Delta_0)p \\ &= pX(S \backslash \Delta_0)p = Q. \end{aligned}$$

The third equation holds by Theorem 3.6. It follows that T_0 is a transition map on $\boldsymbol{B}(Q\mathcal{H})$. There is a linear map $S : \mathscr{T}(Q\mathcal{H}) \to \mathscr{T}(Q\mathcal{H})$ such that

$S^* = T_0$. Let $a \in \boldsymbol{B}(Q\mathcal{H}), \Delta \in \mathcal{F}, \rho \in T(Q\mathcal{H})$. We thus have

$$\begin{aligned}
\mathrm{Tr}[aP(\Delta \backslash \Delta_0)S(\rho)] &= \mathrm{Tr}[T_0(aP(\Delta \backslash \Delta_0))\rho] \\
&= \mathrm{Tr}[QT(aP(\Delta \backslash \Delta_0))Q\rho] \\
&= \mathrm{Tr}[QT(P(\Delta \backslash \Delta_0)a)Q\rho] \\
&= \mathrm{Tr}[P(\Delta \backslash \Delta_0)aS(\rho)].
\end{aligned}$$

We have $P(\Delta \backslash \Delta_0)S(\rho) = P(\Delta \backslash \Delta_0)S(\rho)$ for any $\Delta \in \mathcal{F}, \rho \in \mathscr{T}(Q\mathcal{H})$. By Ref. [6], Theorem 4.3.3, $S = 0$. As Q is the identity of $\boldsymbol{B}(Q\mathcal{H})$, we can conclude that $Q = T_0(Q) = 0$. $\square$

4 Systems of Measurement correlations

Before going to discussion in this section, we introduce some notation.
Notation Let $\mathscr{T}_{(1)}$ be a set. We define a set $\mathscr{T}$ by

$$\mathscr{T} = \bigcup_{j=1}^{\infty} (\mathscr{T}^{(1)})^j. \tag{4.1}$$

(1) For each T in $\mathscr{T}$, $|T|$ denotes natural number which satisfies

$$T \in (\mathscr{T}^{(1)})^{|T|}. \tag{4.2}$$

(2) For each $T = (t_1, ..., t_n) \in \mathscr{T}$, we set

$$T^{\#} = (t_n, ..., t_1). \tag{4.3}$$

(3) For any $T_1 = (t_{1,1}, ..., t_{1,n}), T_2 = (t_{2,1}, ..., t_{2,m}) \in \mathscr{T}$, we set

$$T_1 \times T_2 = (t_{1,1}, ..., t_{1,m}, t_{2,1}, ..., t_{2,m}) \in \mathscr{T}. \tag{4.4}$$

(4) Let $\mathcal{M}$ be a von Neumann algebra. For $\vec{M} = (M_1, ..., M_n) \in \mathcal{M}^n$, we set

$$\vec{M}^{\#} = (M_n^*, ..., M_1^*). \tag{4.5}$$

(5) For $\vec{M}_1 = (M_{1,1}, ..., M_{1,n}) \in \mathcal{M}^n, \vec{M}_2 = (M_{2,1}, ..., M_{2,m}) \in \mathcal{M}^m$, we set

$$\vec{M}_1 \times \vec{M}_2 = (M_{1,1}, ..., M_{1,n}, M_{2,1}, ..., M_{2,m}) \in \mathcal{M}^{n+m}. \tag{4.6}$$

(6) For a family of representations $\{\Pi_t\}_{t\in\mathscr{T}}$ of $\mathcal{M}, T = (t_1, ..., t_n) \in \mathcal{T}$ and $\vec{M} = (M_1, ..., M_2) \in \mathcal{M}^n$, we set

$$\Pi_T(\vec{M}) = \Pi_{t_1}(M_1)\Pi_{t_2}(M_2)\cdots\Pi_{t_n}(M_n). \tag{4.7}$$

(7) Let $(S, \mathcal{F})$ be a measurable space.

$$\mathscr{T} = \bigcup_{j=1}^{\infty} (\mathscr{T}_S^{(1)})^j \tag{4.8}$$

$$\mathscr{T}^{(1)} = \{\text{in}\} \cup \mathscr{F} \tag{4.9}$$

where in is a symbol.

Definition 4.1. Let $\mathcal{M}$ be a von Neumann algebra, $M \in \mathcal{M}$ and $\vec{M} \in \mathcal{M}^n$ for some $n \in \mathbb{N}$. A family of maps $\{W_T : \mathscr{M}^{|T|} \to \mathscr{M}\}_{T\in\mathscr{T}}$ is called a system of measurement correlations for $(\mathscr{M}, S)$ if it satisfies $\mathscr{T}^{(1)} = \mathscr{T}_S^{(1)}$ and the following six conditions.
(MC1) For each $T \in \mathscr{T}$, $W(M_1, ..., M_n)$ is multilinear and σ-weakly continuous for each variable.

(MC2) For any $(T_1, \vec{M}_1), ..., (T_n, \vec{M}_n) \in \bigcup_{T\in\mathscr{T}}(\{\mathscr{T}\} \times \mathcal{M}^{|\mathcal{T}|})$ and $\xi_1, ...\xi_n \in \mathcal{H}$,

$$\sum_{i,j}^{n} \langle \xi_i, W_{T_i^{\#}\times T_j}(\vec{M}_i^{\#} \times M_j)\xi_j\rangle \geq 0, \tag{4.10}$$

(MC3)

$$MW_T(\vec{M}) = W_{\{\text{in}\}\times \mathrm{T}}((M) \times \vec{M}), \tag{4.11}$$

$$W_T(\vec{M})M = W_{T\times\{\text{in}\}}(\vec{M} \times M). \tag{4.12}$$

(MC4) Let $T = (t_1, ..., t_n) \in \mathscr{T}$. For some $1 \leq k \leq |T| - 1$,

$$W_T(\vec{M}) = W_{(t_1,,,,,t_k\cap t_{k+1},...,t_n}(M_1, ..., \underset{\hat{k}}{M_k M_{k+1}}, ...M_n) \tag{4.13}$$

where

$$t_k \cap t_{k+1} = \begin{cases} \text{in}, & (\text{if} \quad t_k = t_k + 1 = \text{in}) \\ t_k \cap t_{k+1}, & (\text{if} \quad t_k, t_{k+1} \in \mathcal{F}) \end{cases}$$

(MC5) Let $T = (t_1, ..., t_n) \in \mathscr{T}$. If $t_k = \text{in}, M_k = 1$ for some $1 \leq k \leq n$, we have

$$\begin{aligned} &W_{(t_1,...,t_n)}(M_1, ..., M_{k-1}, 1, M_{k+1}, ..., M_n) \\ &\qquad = W_{(t_1,...,t_{k-1},t_{k+1}...,t_n)}(M_1, ..., M_{k-1}, M_{k+1}, ..., M_n) \end{aligned} \tag{4.14}$$

(MC6) Let $\{t_{k,j}\}_j \subset \mathcal{F}$ be a disjoint sequence for some $1 \leq k \leq n$ and $\rho \in \mathcal{M}_*$.

$$\langle \rho, W_{(t_1,,,,.\bigcup_k t_{k,j},...,t_n)}(M) \rangle = \sum_j \langle \rho, W_{(t_1,...,t_{k,j},...t_n)}(\vec{M}) \rangle \tag{4.15}$$

In this section, we show the equivalence of measurement correlations and measurement processes.

Definition 4.2. Let C be a set. A map $K : C \times C \to \boldsymbol{B}(\mathcal{H})$ is called a kernel of C on $\mathcal{H}$. The kernel K is said to be positive definite if it satisfies

$$\sum_{i,j}^{n} \langle \xi_i, K(c_i, c_j)\xi_j) \rangle \geq 0 \tag{4.16}$$

for all $n \in \mathbb{N}, \xi_1, ..., \xi_n \in \mathcal{H}, c_1, ..., c_n \in C$.

Proposition 4.3. If a kernel K is positeive definite, there exist a Hilbert space $\mathcal{L}$ and $\Lambda : C \to \boldsymbol{B}(\mathcal{H}, \mathcal{L})$ such that

$$K(c, c') = \Lambda^*(c)\Lambda(c'). \tag{4.17}$$

A map Λ is called a Kolmogorov decomposition. A Kolmogorov decomposition of K is said to be minimal if $\mathcal{L} = \overline{\text{span}}(\Lambda(C)\mathcal{H})$. A minimal Kolmogorov decomposition is unique up to unitary equivalence.

Theorem 4.4. Let $\{W_T\}_{T\in\mathscr{T}}$ be measurement correlations for $(\mathcal{M},S)$. There exist a Hilbert space $\mathcal{L}$, normal representations $\{\Pi_t\}_{t\in\mathscr{T}^{(1)}}$ of $\mathcal{M}$ on $\mathcal{L}$ and an isometry $V:\mathcal{H}\to\mathcal{L}$ such that

$$\Pi_{\mathrm{in}}(M)V = VM, \tag{4.18}$$
$$W_T(\vec{M}) = V^*\Pi_T(\vec{M})V \tag{4.19}$$

for all $M\in\mathcal{M}$.

Proof. Let $C=\bigcup_{T\in\mathscr{T}}(T\times M^{|T|})$ and set $K: C\times C\to \boldsymbol{B}(\mathcal{H})$ as follows.

$$K(a,b) = W_{T_1^{\#}\times T_2}(\vec{M}_1\times\vec{M}_2). \tag{4.20}$$

where $a=(T_1,\vec{M}_1), b=(T_2,\vec{M}_2)$. It is obvious K is positive definite. There exist $\Lambda: C\to\boldsymbol{B}(\mathcal{H},\mathcal{L})$ and a minimal Kolmogorov decomposition $K(a,b)=\Lambda(a)^*\Lambda(b)$ by Theorem 4.3. We remark that $\Lambda(C)\mathcal{H}$ is dence in $\mathcal{L}$. We define Π_t as follows.

$$\Pi_t(M)\Lambda(a)\xi = \Lambda((t)\times T,(M)\times\vec{M})\xi. \tag{4.21}$$

Now, we show that Π_t is a normal $*$-representation of $\mathcal{M}$. For $\xi_1,\xi_2\in\mathcal{H}, a=(T_1,\vec{M}_1), b=(T_2,\vec{M}_2)\in C$,

$$\begin{aligned}\langle\Lambda(a)\xi_1,\Pi_t(\alpha M+\beta N)\Lambda(b)\xi_2\rangle &= \langle\xi_1,\Lambda(a)^*\Pi_t(\alpha M+\beta N)\Lambda(b)\xi_2\rangle\\ &= \langle\xi_1,\Lambda(a)^*\Lambda((t)\times T_2,(\alpha M+\beta N)\times\vec{M}_2)\xi_2\rangle\\ &= \langle\xi_1,W_{T_1^{\#}\times(t)\times T_2}(\vec{M}_1^{\#}\times(\alpha M+\beta N)\times\vec{M}_2)\xi_2\rangle.\end{aligned}$$

Thus, $\Pi_t(MN)=\Pi_t(M)\Pi_t(N), \Pi_t(M^*)=\Pi_t(M)^*$ for all $t\in\mathscr{T}^{(1)}, M,N\in\mathcal{M}$.

For any $M\in\mathcal{M}_+,\xi_i\in\mathcal{H},a_i=(T_i,\vec{M}_i)$,

$$\begin{aligned}\sum_{i,j}^{n}\langle\Lambda(a_i)\xi_i,\Pi_t(M)\Lambda(a_j)\xi_j\rangle &= \sum_{i,j}^{n}\langle\xi_i,W_{T_i^{\#}\times(t)\times T_j}(\vec{M}_i\times(M)\times\vec{M}_j)\xi_j\rangle\\ &= \sum_{i,j}^{n}\langle\xi_i,W_{T_i^{\#}\times(t)\times(t)\times T_j}(\vec{M}_i\times(\sqrt{M})\times(\sqrt{M})\times\vec{M}_j)\xi_j\rangle\\ &= \sum_{i,j}^{n}\langle\Lambda((t)\times T_i,(\sqrt{M})\times T_i)\xi_i,\Lambda((t)\times T_j,(\sqrt{M})\times M_j)\xi_j\rangle\geq 0.\end{aligned}$$

Thus,

$$\sum_{i,j}^{n} \langle \Lambda(a_i)\xi_i, \Pi_t(||M||^2 - M^*M)\Lambda(a_j)\xi_j\rangle \geq 0,$$
$$||\sum_{i,j}^{n} \Pi_t(M)\Lambda(a_i)\xi_i|| \leq ||M|| || \sum_{i,j}^{n} \Lambda(a_i)\xi_i||.$$

for all $M \in \mathcal{M}$. It follows that Π_t is bounded.

We define $V : \mathcal{H} \to \mathcal{L}$ as $V = \Lambda((\mathrm{in}, 1))$.

$$\begin{aligned}\langle V\xi, V\xi\rangle &= \langle \xi, \Lambda((\mathrm{in}, 1))^*\Lambda((\mathrm{in}, 1))\xi\rangle \\ &= \langle \xi, W_{(\mathrm{in})\times(\mathrm{in})}(1, 1)\xi\rangle \qquad = \langle \xi, \xi\rangle.\end{aligned}$$

Since $V * V = 1$ is obvious from the construction of V, it holds that V is an isometry.

For any $t \in \mathcal{T}$ and $\vec{M} \in \mathcal{M}^{|T|}$,

$$\begin{aligned}V^*\Pi_T(\vec{M})V &= \Lambda((\mathrm{in}, 1))^*\Pi_T(\vec{M})\Lambda((\mathrm{in}, 1)) \\ &= W_{(\mathrm{in})\times T\times(\mathrm{in})}((1 \times M \times (1)) \\ &= 1 \cdot W_T(\vec{M}) \cdot 1 = W_T(\vec{M}).\end{aligned}$$

We have

$$\begin{aligned}&(VM - \Pi_{\mathrm{in}}(M)V)^*(VM - \Pi_{\mathrm{in}}(M)V) \\ &= M^*V^*VM - M^*V^*\Pi_{\mathrm{in}}(M)V - V^*\Pi_{\mathrm{in}}(M)VM + V^*\Pi_{\mathrm{in}}(M * M)V \\ &= M^*M - M^*V_{\mathrm{in}}(M) - W_{\mathrm{in}}M + W_{\mathrm{in}}(M^*M) \\ &= 0.\end{aligned}$$

It follows that $P_{\mathrm{in}}(M)V = VM$ for all $M \in \mathcal{M}$. □

The following theorem is a main statement of this section.

Theorem 4.5. There is a one-to-one correspondence between statistically equivalent classes of measuring processes and systems of measurement correlations $\{W_T\}_{T\in\mathscr{T}}$ for $\boldsymbol{B}(\mathcal{H})$, which is given by the relation

$$W_T(\vec{M}) = W_T^{\mathbb{M}}(\vec{M}) \tag{4.22}$$

for all $T \in \mathcal{T}$ and $\vec{M} \in \mathcal{M}^{|T|}$.

Before proving the theorem, we admit the following two theorems and prove a lemma.

Theorem 4.6. ([9] Theorem 1.3.1) Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, $\mathcal{B}$ be a unital C^*-subalgebra of $\boldsymbol{B}(\mathcal{K})$ and $V \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ satisfy $\mathcal{K} = \overline{\mathrm{span}} \mathcal{B} V \mathcal{H}$.
For all $A \in (V^* \mathcal{B} V)'$, there exists unique $A_1 \in \mathcal{B}'$ such that $VA = A_1 V$. A map $A \in (V^* \mathcal{B} V)' \mapsto A_1 \in \mathcal{B} \cup \{VV^*\}'$ is σ-weakly continuous, surjective and a $*$-homomorphism.

The following theorem holds as a corollary of [10], Part I, Chapter 4 Theorem 3 and [11], Chapter IV, Theorem 5.5.

Theorem 4.7. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. If π is a normal representation of $\boldsymbol{B}(\mathcal{H}_1)$ on $\mathcal{H}_2$, there exist a Hilbert space $\mathcal{K}$ and a unitary operator U of $\mathcal{H}_1 \otimes \mathcal{K}$ onto $\mathcal{H}_2$ such that

$$\pi(X) = U(X \otimes 1_{\mathcal{K}})U^* \tag{4.23}$$

for all $X \in \boldsymbol{B}(\mathcal{H}_1)$.

Lemma 4.8. Let $\mathcal{H}_1, \mathcal{H}_2$ be a Hilbert spaces and $V : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ such that $(X \otimes 1)V = VX$ for all $X \in \boldsymbol{B}(\mathcal{H})$. Then, there is $\eta \in \mathcal{H}_2$ which satisfies $V\xi = \xi \otimes \eta$ for any $\xi \in \mathcal{H}_1$.

Proof. Let $x \in \mathcal{H}_1 \backslash \{0\}$ and P_x be the projection onto $\mathbb{C}x$. It holds that $(P_x \otimes 1)Vx = VP_x x = Vx$. Hence, there exists $\eta_x \in \mathcal{H}_2$ such that $||\eta_x|| = 1, Vx = x \otimes \eta_x$ for each $x \in \mathcal{H}_1 \backslash \{0\}$. For any $y \in \mathcal{H} \backslash \{0\}$,

$$\begin{aligned}
\langle x, x \rangle y \otimes \eta_y &= \langle x, x \rangle V y \\
&= V(y|\rangle\langle x|)x \\
&= (|y\rangle\langle x| \otimes 1)Vx \\
&= (|y\rangle\langle x| \otimes 1)(x \otimes \eta_x) \\
&= \langle x, x \rangle y \otimes \eta_x.
\end{aligned}$$

Thus, $\eta := \eta_x$ is independent of x and $Vx = x \otimes \eta$. □

Proof. (Proof of Theorem 4.5) Let $\{W_T\}_{T \in \mathscr{T}}$ be a system of measurement correlations for $\boldsymbol{B}(\mathcal{H})$. We can assume a triplet $(\mathcal{L}_0, \{\Pi_t\}_{t \in \mathscr{T}^{(1)}}, V_0)$ satisfies

$$W_T(\vec{M}) = V_0^* \Pi_T(\vec{M}) V_0$$

as in Thorem 4.4.

By Theorem 4.7, there exists a Hilbert space $\mathcal{L}_1$ and a unitary operator $U_1 : \mathcal{L}_0 \to \mathcal{H}_0 \otimes \mathcal{L}_1$ satisfies

$$\Pi_{\mathrm{in}}(M) = U_1^*(M \otimes 1)U_1$$

for any $M \in \mathcal{M}$. Similarly, there are a Hilbert space $\mathcal{L}_2$ and a unitary operator $U_2 : \mathcal{L}_0 \to \mathcal{H}_0 \otimes \mathcal{L}_1$ satisfying

$$\Pi_S(M) = U_2^*(M \otimes 1)U_2$$

for any $M \in \mathcal{M}$. For Theorem 4.6, there exists a projection-valued measure $E_0 : \mathcal{F} \to \boldsymbol{B}(\mathcal{L}_2)$ such that

$$\Pi_\Delta(1) = U_2^*(1 \otimes E(\Delta))U_2 \tag{4.24}$$

for any $\Delta \in \mathcal{F}$.

We define $V : \mathcal{H} \to \mathcal{H}_1 \otimes \mathcal{L}_1$ as $V = U_1V_0$. The unitary operator V is obviously an isometry. Then,

$$V^*(M \otimes 1)V = W_{\mathrm{in}}(M) = M. \tag{4.25}$$

Thus, $((M\otimes 1)V - VM)^*((M\otimes 1)V - VM) = 0$. We have $(M\otimes 1)V = VM$.

By Lemma 4.8, there is $\eta_1 \in \mathcal{L}_1$ which satisfies $V\xi = \xi \otimes \eta_1, \xi \in \mathcal{H}$. We pick up $\eta_2 \in \mathcal{L}_2$ such that $||\eta_2|| = 1$. Let ζ be an isomorphism from $\mathcal{L}_1 \otimes \mathcal{L}_2$ to $\mathcal{L}_2 \otimes \mathcal{L}_1$ defined by $\zeta(\xi_1 \otimes \xi_2) = \xi_2 \otimes \xi_1$. We define unitary operators $U_3 : \mathcal{H} \otimes \mathcal{L}_1 \otimes \mathbb{C}\eta_2 \to \mathcal{H} \otimes \mathbb{C}\eta_1 \otimes \mathcal{L}_2$ and $U_5 : \mathbb{C}\eta_2 \to \mathbb{C}\eta_1$ as follows.

$$U_3(\xi \otimes \eta_2) = (1 \otimes \zeta)(U_2U_1^*\xi \otimes \eta_1), \tag{4.26}$$

$$U_5x = \langle \eta_2, x\rangle\eta_1, \tag{4.27}$$

for all $\xi \in \mathcal{H} \otimes \mathcal{L}_1, x \in \mathbb{C}\eta_2$. We can see $\mathcal{H} \otimes \mathcal{L}_1 \otimes \mathbb{C}\eta_2, \mathcal{H} \otimes \mathbb{C}\eta_1 \otimes \mathcal{L}_2$ as subspaces of $\mathcal{H} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2$. Since U_3 is unitary,

$$\dim(\mathcal{H} \otimes \mathcal{L}_1 \otimes \mathbb{C}\eta_2) = \dim(\mathcal{H} \otimes \mathbb{C}\eta_1 \otimes \mathcal{L}_2).$$

Thus, there is a unitary operator $U_4 : (\mathcal{H}\otimes\mathcal{L}_1\otimes\mathbb{C}\eta_2)^\perp \to (\mathcal{H}\otimes\mathbb{C}\eta_1\otimes\mathcal{L}_2)^\perp$. Let Q, R be projection operators from $\mathcal{H}\otimes\mathcal{L}_1\otimes\mathcal{L}_2$ onto $\mathcal{H}\otimes\mathcal{L}_1\otimes\mathbb{C}\eta_2, \mathcal{H}\otimes\mathbb{C}\eta_1\otimes\mathcal{L}_2$ respectively We define a unitary operator U on $\mathcal{H} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2$ as follows.

$$U = U_3Q + U_4(1 - Q). \tag{4.28}$$

It it obvious that $UQ = U_3Q = RU_3 = RU$.

We put $\mathcal{K} = \mathcal{L}_1 \otimes \mathcal{L}_2$, $\sigma(Y) = \langle \eta_1 \otimes \eta_2, Y(\eta_1 \otimes \eta_2)\rangle$ for all $Y \in \boldsymbol{B}(\mathcal{K})$. Define a spectral measure $E : \mathcal{F} \to \boldsymbol{B}(\mathcal{K})$ as

$$E(\Delta) = 1 \otimes E_0(\Delta). \tag{4.29}$$

for all $\Delta \in \mathcal{F}$. Then, $\mathbb{M} := (\mathcal{K}, \sigma, E, U)$ is a measuring process for $\boldsymbol{B}(\mathcal{H})$. We show that the measuring process $\mathcal{M}$ satisfies (4.22). Since $Q = 1 \otimes 1 \otimes P_{\eta_2}$,

$$\pi_{\mathrm{in}}(M) = M \otimes 1 \otimes 1 = U_1 \Pi_{\mathrm{in}}(M) U_1^* \otimes 1_{\boldsymbol{B}(\mathcal{L}_\in)}. \tag{4.30}$$

We have

$$\pi_{\mathrm{in}}(M)Q = Q\pi_{\mathrm{in}}(M). \tag{4.31}$$

for all $M \in \mathcal{M}$.

$$\begin{aligned}
\pi_\Delta(M)Q &= U^*(M \otimes E(\Delta))UQ \\
&= U^*(M \otimes E(\Delta))RU_3Q \\
&= U^*R(M \otimes E(\Delta))U_3Q \\
&= U_3^*R(M \otimes E(\Delta))U_3Q \\
&= U_3^*(M \otimes E(\Delta))U_3Q \\
&= ((1 \otimes \zeta)(U_2U_1^* \otimes U_5))^*(M \otimes E(\Delta))(1 \otimes \zeta)(U_2U_1^* \otimes U_5)Q \\
&= (U_2U_1^* \otimes U_5)^*(M \otimes E_0(\Delta) \otimes 1_{\boldsymbol{B}(\mathcal{L}_1)})(U_2U_1^* \otimes U_5)Q \\
&= (U_1\Pi_\Delta(M)U_1^* \otimes 1_{\boldsymbol{B}(\mathcal{L}_1)}).
\end{aligned} \tag{4.32}$$

Thus,

$$\pi_\Delta(M)Q = Q\pi_\Delta(M). \tag{4.33}$$

By (4.31), (4.32) and (4.33),

$$Q\pi_\Delta(\vec{M})Q = Q(U_1\Pi_T(\vec{M})U_1^* \otimes 1_{\boldsymbol{B}(\mathcal{L}_2)})Q. \tag{4.34}$$

For any $T \in \mathscr{T}, M \in \mathcal{M}^{|T|}$ and $\xi \in \mathcal{H}$,

$$
\begin{aligned}
\langle \xi, W_T^{\mathbb{M}}(\vec{M})\xi \rangle &= \langle \xi, (id \otimes \sigma)(\pi_T(\vec{M}))\xi \rangle \\
&= \langle \xi \otimes \eta_1 \otimes \eta_2, \pi_T(\vec{M})(\xi \otimes \eta_1 \otimes \eta_2) \rangle \\
&= \langle V\xi \otimes \eta_2, Q\pi_T(\vec{M})Q(V\xi \otimes \eta_2) \rangle \\
&= \langle V\xi \otimes \eta_2, Q(U_1 \Pi_T(\vec{M})U_1^* \otimes 1_{\boldsymbol{B}(\mathcal{L}_2)})Q(V\xi \otimes \eta_2) \rangle \\
&= \langle \xi \otimes \eta_2, (V^* U_1 \Pi_T(\vec{M})U_1^* V \otimes 1_{\boldsymbol{B}(\mathcal{L}_2)})(\xi \otimes \eta_2) \rangle \\
&= \langle \xi, V_0^* \Pi_T(\vec{M})V_0 \xi) \rangle \\
&= \langle \xi, W_T(\vec{M})\xi \rangle.
\end{aligned}
$$

The theorem is proved. $\square$

The next corollary is easily seen by Theorem 4.5.

Corollary 4.9. Let $\{W_T\}_{T \in \mathscr{T}}$ be a system of measurement correlations for $(\mathscr{M}, S)$. The following are equivalent.
(1) There exists a system of measurement correlations $\{\tilde{W}_T\}_{T \in \mathscr{T}}$ for $(\boldsymbol{B}(\mathcal{H}), S)$ such that $W_T(\vec{M}) = \tilde{W}_T(\vec{M})$.
(2) There is a measuring process $\mathbb{M} = (\mathscr{K}, E, \sigma, U)$ for $(\mathscr{M}, S)$.

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