Skein algebras and Quantum cluster algebras

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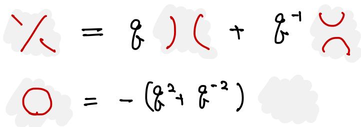
§1 Introduction

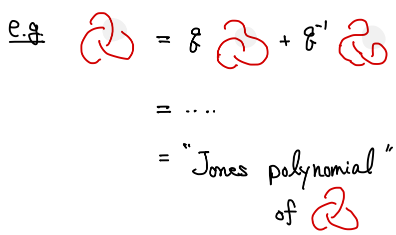
● My research interest "quantum topology"

- quantum invariants : $J_{\mathfrak{g},V}$: knots $\longrightarrow \mathbb{Z}_q := \mathbb{Z}[q^{\pm 1}]$
  ↑ rep of $\mathfrak{g}$

e.g. $\mathfrak{g} = \mathfrak{sl}_2$, $V = V_1 = \mathbb{C}_q^2$: 2-dim irrep. of $U_q(\mathfrak{sl}_2)$ $\rightsquigarrow$ Jones polynomial
(V_n : $(n+1)$-dim. irrep) (colored Jones polynomial)

the Kauffman bracket skein relation





① tail of K : $\lim_{n\to\infty} J_{\mathfrak{g},K}(n)$ = " q- series " $\in \mathbb{Z}[[q]]$
$\parallel$
$J_{\mathfrak{g},V_n}(K)$

e.g. $K = T(2,m)$ $\rightsquigarrow$ $\lim_{n\to\infty} J_{\mathfrak{g},T(2,m)}(n)$ = "(false) theta series"

$\rightsquigarrow$ higher rank $\mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sp}_4, \mathfrak{g}_2$ (rank 2)

$\lim_{n\to\infty} J_{\mathfrak{g},K}(V_{n\lambda})$ = " higher version of (false) theta series "

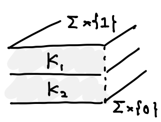
quantum modular form

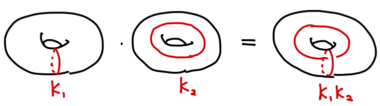
character of VOA

② Skein algebra (Σ : a surface)

$\mathbb{Z}_q\{\text{knots in } \Sigma \times [0,1]\} \twoheadrightarrow \mathscr{S}^q_{\mathfrak{g},\Sigma}$
↑ skein relation

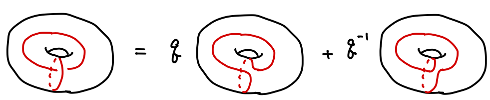
$K_1 \cdot K_2 =$



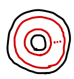


$\mathfrak{g} = \mathfrak{sl}_2$

$K_1K_2 =$



Σ : annulus

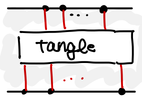
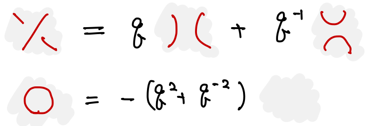
$\mathscr{S}^q_{\mathfrak{sl}_2,\Sigma} = \mathbb{Z}_q\{$  $\} \cong \mathbb{Z}_q[X]$

$\rightsquigarrow$ Σ : a marked surface : $\mathscr{S}^q_{\mathfrak{g},\Sigma} \longleftrightarrow \mathscr{A}^q_{\mathfrak{g},\Sigma}$: quantum cluster algebra

§2 skein relations

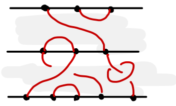
Web category

$\boxed{\mathfrak{g} = \mathfrak{sl}_2}$: the Kauffman bracket skein relation

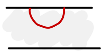
$\mathrm{Web}_{\mathfrak{sl}_2}$ $\begin{cases} \text{Obj : finite points on the interval} \\ \text{Mor : } \mathbb{C}(q)\text{-span of } \end{cases}$  / 

- web category
- Temperley-Lieb category
- spider category

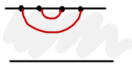
$\mathfrak{sl}_2$-web

Composition 

$\mathrm{Web}_{\mathfrak{sl}_2}(\emptyset, n\text{-points}) \cong \mathrm{Inv}_{U_q(\mathfrak{sl}_2)}(\underbrace{V_1 \otimes \cdots \otimes V_1}_{n}) \cong \mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(\mathbb{C}_q, V_1 \otimes \cdots \otimes V_1)$

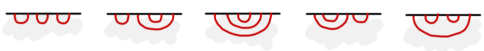
e.g.  : $\mathbb{C}(q) \longrightarrow V_1 = \mathbb{C}v_1 \oplus \mathbb{C}v_{-1}$

$1 \longmapsto v_1 \otimes v_{-1} - q^{-1} v_{-1} \otimes v_1$

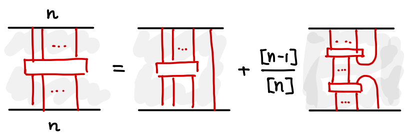


$\exists$ diagrammatic basis $\leftrightarrow$ {non-crossing matchings} $=: \mathrm{BWeb}_{\mathfrak{sl}_2}(n)$

e.g. $n = 6$

$\mathrm{BWeb}_{\mathfrak{sl}_2}(6) = \{$  $\}$

- Jones-Wenzl projectors ("internal clasp")

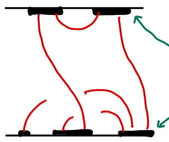


$: \quad V_1^{\otimes n} \overset{\pi_n}{\twoheadrightarrow} V_n \overset{\iota_n}{\hookrightarrow} V_1^{\otimes n}$

q-symm. power in $V^{\otimes n}$

clasped web category ($\mathfrak{g} = \mathfrak{sl}_2$)

$\widetilde{\mathrm{Web}}_{\mathfrak{sl}_2}$ $\begin{cases} \text{Obj : } (n_1, n_2, \dots, n_k) \quad (\mu \vDash n) \\ \text{Mor : morphisms of } \mathrm{Web}_{\mathfrak{sl}_2} + \text{skein relation at "external clasp"} \end{cases}$



external clasp
$V_1^{\otimes n} \overset{}{\underset{\pi_n}{\twoheadrightarrow}} V_n$

 $= 0$

[Kuperberg '96] $\quad \widetilde{\mathrm{Web}}(\emptyset, (n_1, \dots, n_k)) \cong \mathrm{Inv}_{U_q(\mathfrak{sl}_2)}(V_{n_1} \otimes V_{n_2} \otimes \cdots \otimes V_{n_k}) = \mathrm{Hom}(\mathbb{C}_q, V_{n_1} \otimes \cdots \otimes V_{n_k})$

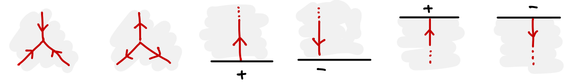
$\cup$

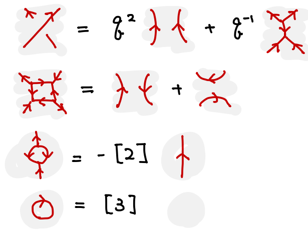
$\exists$ basis : $\widetilde{\mathrm{BWeb}}(n_1, \dots, n_k) = \{\text{non-crossing matchings}\} / \{$  $\}$

rank 2 case [Kuperberg '96]

$\mathfrak{sl}_3$

- $\mathrm{Web}_{\mathfrak{sl}_3}$
  - Objects : sequence of $\{+,-\}$ (+ upward, − downward)
  - $\mathfrak{sl}_3$-web = oriented uni-trivalent graphs with crossings

    s.t. 

  - skein relation : 

    $= q^2 \cdots + q^{-1} \cdots$, $= \cdots + \cdots$, $= -[2]$, $= [3]$

[Kuperberg '96] $\mathrm{Web}_{\mathfrak{sl}_3}(\emptyset, \varepsilon_1\varepsilon_2\cdots\varepsilon_n) \cong \mathrm{Inv}_{\mathfrak{sl}_3}(V_{\varepsilon_1}\otimes\cdots\otimes V_{\varepsilon_n})$ $\quad (V_+ = V_{\varpi_1},\ V_- = V_{\varpi_2} \cong V_{\varpi_1}^*)$

$\cup$

$\mathrm{BWeb}_{\mathfrak{sl}_3}(\varepsilon_1\cdots\varepsilon_n) = \{$ non-elliptic $\mathfrak{sl}_3$-webs $\}$

elliptic faces : 

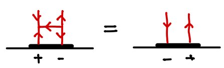
- $\widetilde{\mathrm{Web}}_{\mathfrak{sl}_3}$
  - Obj : $(s_1, s_2, \ldots, s_k)$ $\quad s_i$: a sequence of $\{+,-\}$
  - Mor : $\mathfrak{sl}_3$-web + skein relation at "external clasps"

    $=0 \quad =0 \quad =0 \quad =0$

[Frohman-Sikora '21, Ishibashi-Y. '22]

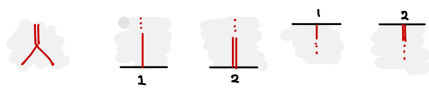
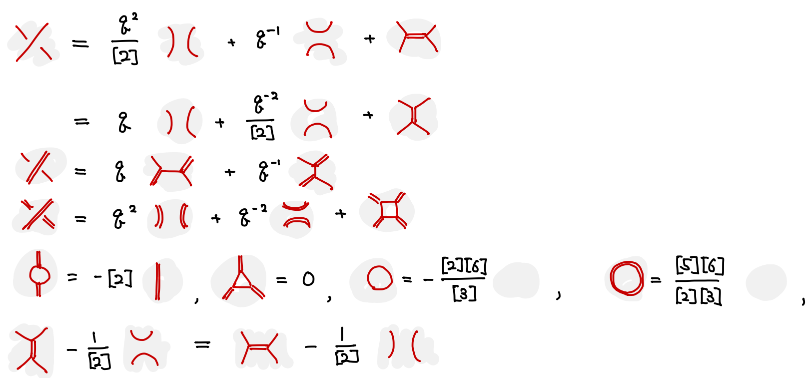
s_i : a multiset on $\{+,-\}$



$\widetilde{\mathrm{Web}}_{\mathfrak{sl}_3}(\emptyset, (s_1, s_2, \ldots, s_k)) \cong \mathrm{Inv}_{U_q(\mathfrak{sl}_3)}(V_{s_1}\otimes\cdots\otimes V_{s_k})$ $\quad V_s := V_{a\varpi_1 + b\varpi_2}$

$\begin{pmatrix} a := \#\text{ of } + \text{ in } s \\ b := \#\text{ of } - \text{ in } s \end{pmatrix}$

$\mathfrak{sp}_4$

$\mathrm{Web}_{\mathfrak{sp}_4}$
- Obj : a sequence of $\{1,2\}$
- $\mathfrak{sp}_4$-web : 
- skein relation : 

  $= \frac{q^2}{[2]} \cdots + q^{-1} \cdots + \cdots$

  $= q \cdots + \frac{q^{-2}}{[2]} \cdots + \cdots$

  $= q \cdots + q^{-1} \cdots$

  $= q^2 \cdots + q^{-2} \cdots + \cdots$

  $= -[2] \cdots, \quad = 0, \quad = -\frac{[2][6]}{[3]}, \quad = \frac{[5][6]}{[2][3]},$

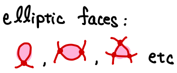
  $- \frac{1}{[2]} \cdots = \cdots - \frac{1}{[2]} \cdots$

[Kuperberg '96] $\mathrm{Web}_{\mathfrak{sp}_4}(\emptyset, \varepsilon_1\varepsilon_2\cdots\varepsilon_n) = \mathrm{Inv}_{U_q(\mathfrak{sp}_4)}(V_{\varepsilon_1}\otimes\cdots\otimes V_{\varepsilon_n})$ $\quad \begin{pmatrix} V_1 = V_{\varpi_1} : \text{4-dim irrep} \\ V_2 = V_{\varpi_2} : \text{5-dim irrep} \end{pmatrix}$

$\cup$

$\mathrm{BWeb}_{\mathfrak{sp}_4}(\varepsilon_1\varepsilon_2\cdots\varepsilon_n) = \{$ non-elliptic "crossroad" webs $\}$

replace all with

elliptic faces :  etc

$:= \cdots - \frac{1}{[2]} \cdots = \cdots - \frac{1}{[2]} \cdots$

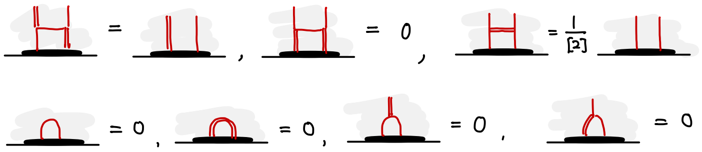
crossroad

(c.f. [Bodish '22] Light Ladder basis

$\mathrm{Web}_{\mathfrak{sp}_4}^{\mathbb{Z}_q[1/[2]]} \longrightarrow U_{\mathbb{Z}_q}(\mathfrak{sp}_4)\text{-mod}$ $\quad$ (c.f. Elias $\mathfrak{gl}_n$)

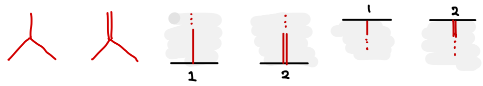
monoidal equivalence)

- skein relation at "external clasps" [Ishibashi - Y. '22+]

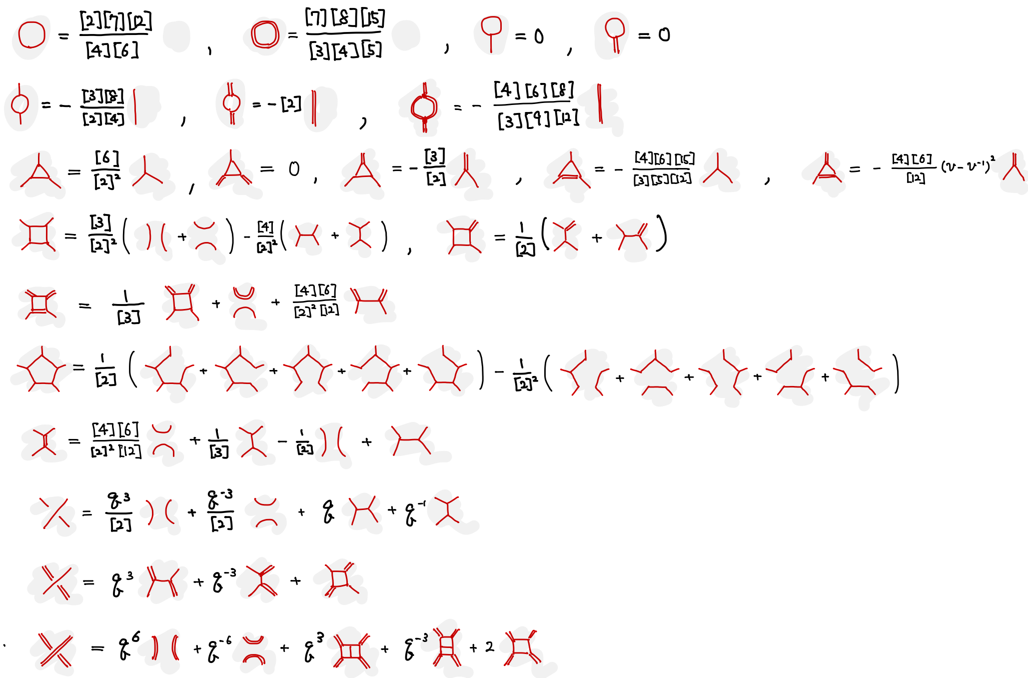


(g₂) $\mathrm{Web}_{\mathfrak{g}_2}$ { Obj : a sequence of $\{1, 2\}$

$\mathfrak{g}_2$- web



skein rel.
[Sakamoto
-Yonezawa '17]



[Kuperberg '96 (?)] $\mathrm{Web}_{\mathfrak{g}_2}(\emptyset, \varepsilon_1\varepsilon_2\cdots\varepsilon_n) = \mathrm{Inv}_{U_q(\mathfrak{g}_2)}(V_{\varepsilon_1}\otimes\cdots\otimes V_{\varepsilon_n})$ $\left(\begin{array}{l} V_1 = V_{\varpi_1} : 7\text{-dim irrep} \\ V_2 = V_{\varpi_2} : 14\text{-dim irrep.} \end{array}\right)$

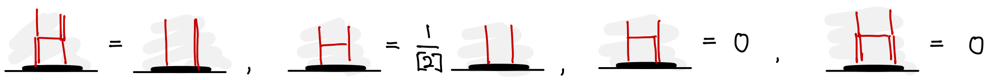
$\cup$

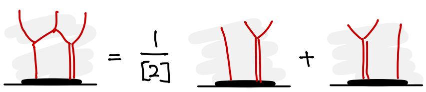
$\mathrm{BWeb}_{sp_4}(\varepsilon_1\varepsilon_2\cdots\varepsilon_n) = \{$ non-elliptic no internal $\|$ webs $\}$

$\uparrow$

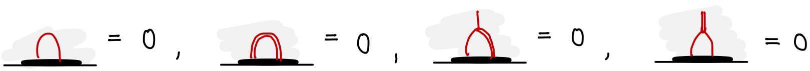
[Sikora-Westbury '07]

- skein relation at "external clasps"

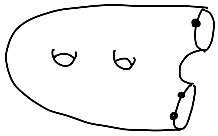




$\leftarrow$ $\dim(\mathrm{Hom}(V_{\varpi_1}^{\otimes 3}, V_{\varpi_1+\varpi_2})) = 2$

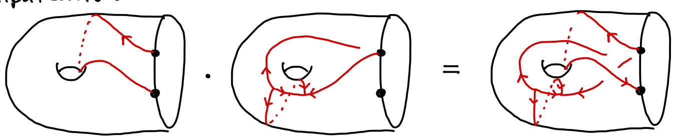


§ Skein & Cluster algebra

- skein algebra of marked surface Σ = 

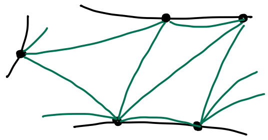
$$\mathcal{S}^{q}_{\mathfrak{g},\Sigma} = R\{\mathfrak{g}\text{-webs on } \Sigma\} \Big/ \begin{array}{l}\text{skein relation in } \Sigma\setminus\partial\Sigma \\ \text{“skein relation at marked points”}\end{array}$$

- multiplication

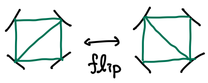


- “marked points” $\longleftrightarrow$ “external clasps” $\rightsquigarrow$ • external clasps define skein relation at marked points

- (quantum) cluster algebra of Σ

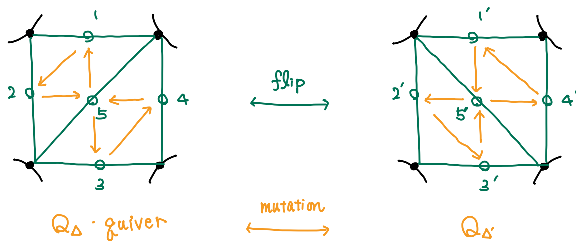
 $\in \mathrm{Tri}(\Sigma)$

$\mathrm{Tri}(\Sigma)$: the set of ideal triangulation of Σ

Remark $\forall \Delta, \Delta' \in \mathrm{Tri}(\Sigma)$ Δ is related to Δ' by flips 

quantum torus

$$\mathcal{A}^{q}_{\mathfrak{sl}_2,\Sigma} := \bigsqcup_{\Delta\in\mathrm{Tri}(\Sigma)} \mathbb{T}_\Delta \Big/ \text{“exchange relation”}$$: the quantum cluster algebra of Σ

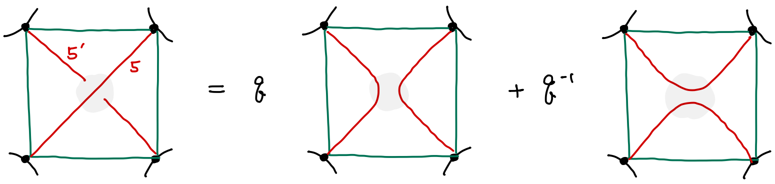


$i \in$ boundary arc $\quad A_i$: invertible (frozen variable)

$\mathbb{T}_\Delta := \langle \{A_i \mid i \in Q_\Delta\} \rangle$ ← cluster

$\begin{cases} A_5 A_{5'} = A_2 A_4 + A_1 A_3 \\ A_i = A_{i'} \quad (i \neq 5) \end{cases}$: exchange relation

in $\underline{\mathcal{S}^{q}_{\mathfrak{sl}_2,\Sigma}}$



[Muller 2016] • $\mathcal{A}^{q}_{\mathfrak{sl}_2,\Sigma} \subset \mathcal{S}^{q}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] \subset \mathcal{U}^{q}_{\mathfrak{sl}_2,\Sigma} \subset \mathrm{Frac}\,\mathcal{S}^{q}_{\mathfrak{sl}_2,\Sigma}$

localization at ∂-arcs

upper cluster algebra $\mathcal{U}_{\mathfrak{sl}_2,\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{\mathfrak{sl}_2,\Sigma})$

• $\mathcal{A}^{q} = \mathcal{U}^{q}$ (acyclic exchange type)

$\rightsquigarrow \mathcal{A}^{q}_{\mathfrak{sl}_2,\Sigma} = \mathcal{S}^{q}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] = \mathcal{U}^{q}_{\mathfrak{sl}_2,\Sigma}$

Main results

[Ishibashi - Y. 2023] $\mathscr{S}^{q}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \subset \mathscr{A}^{q}_{\mathfrak{sl}_3,\Sigma} \subset \mathrm{Frac}\,\mathscr{S}^{q}_{\mathfrak{sl}_3,\Sigma}$

(c.f. [Fomin - Pylyavskyy '16])

[I - Y. 2022 +] $\mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}] \subset \mathscr{A}^{q}_{\mathfrak{sp}_4,\Sigma} \subset \mathrm{Frac}\,\mathscr{S}^{q}_{\mathfrak{sp}_4,\Sigma}$

$\cap$ $\mathbb{Z}_q$ - subalgebra

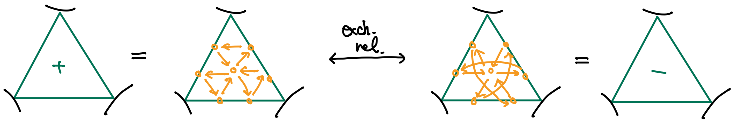
$\mathscr{S}^{q}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}]$

[I-Y. in progress] $\mathscr{A}^{q}_{\mathfrak{g}_2,\Sigma} \subset \mathrm{Frac}\,\mathscr{S}^{q}_{\mathfrak{g}_2,\Sigma}$

Difficulty for higher-rank case

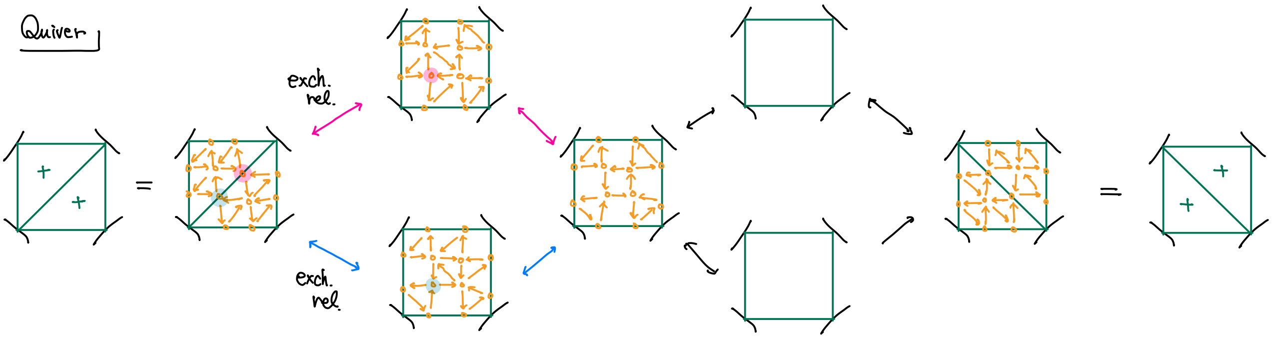
$\mathrm{Tri}_{\mathfrak{sl}_3}(\Sigma) \ni \Delta$: decorated ideal triangulation

- switch of decoration

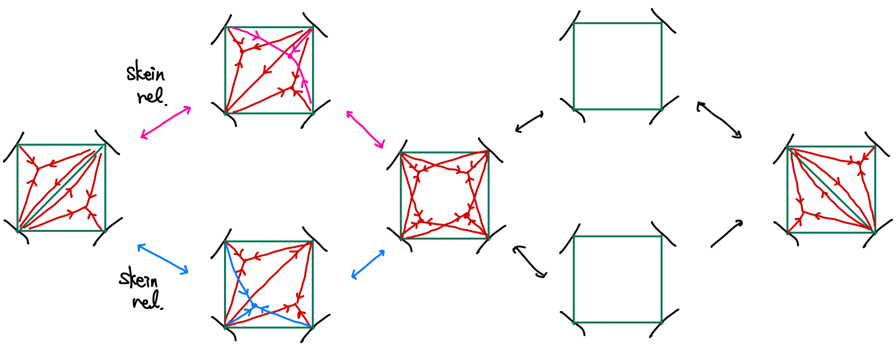


- Construction of flip as $\mathfrak{sl}_3$ - webs.

Quiver

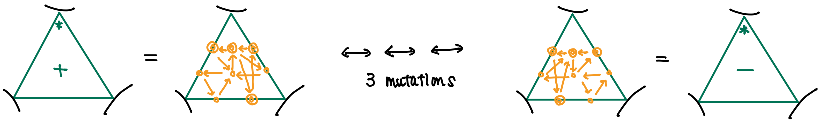


$\mathfrak{sl}_3$-web



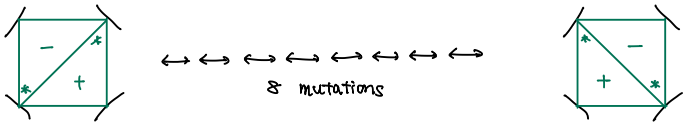
- Expand any $\mathfrak{sl}_3$ - webs as a polynomial of known cluster variables

(sp_4-case) • decoration



3 mutations

• flip.



8 mutations

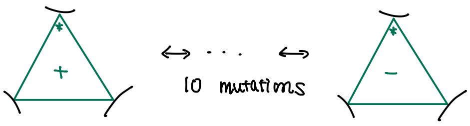
☆ $\mathscr{S}^{q}_{sp_4,\Sigma}$: $\mathbb{Z}_q[1/[2]]$-algebra

$\mathscr{A}^{q}_{sp_4,\Sigma}$: $\mathbb{Z}_q$-algebra

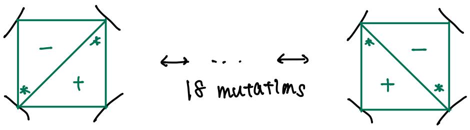
related to Lusztig's integer form?

$\rightsquigarrow$ Define $\mathbb{Z}_q$-subalgebra $\mathscr{S}^{\mathbb{Z}_q}_{sp_4,\Sigma}$ of $\mathscr{S}^{q}_{sp_4,\Sigma}$

($\mathfrak{g}_2$-case)



10 mutations



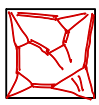
18 mutations

§ Conjectures, other works,

Conjecture. • $\mathscr{A}^{q}_{\mathfrak{g},\Sigma} = \mathscr{S}^{q}_{\mathfrak{g},\Sigma}[\partial^{-1}]$ for $\mathfrak{g} = sl_3, sp_4, \mathfrak{g}_2$

(※. $q = 1$ is OK for sl_3, sp_4
via $\mathscr{A} = \mathcal{U}$ theorem in [Ishibashi - Oya - Shen '23])

• {cluster variables} = {tree-type webs}

e.g. sp_4  is a cluster variable

Problem. • Construct positive basis (c.f. [Mandel - Qin '23+] $\mathfrak{g} = sl_2$
theta basis = bracelet basis)

Other work [Ishibashi - Kano - Y. 24+] $\mathfrak{g} = sl_2$

the skein algebra of a "walled surface"

$\cong$ the quantum cluster algebra with coefficients