

State - Clasp correspondence for skein algebras

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§0. introduction

§1. the stated skein algebra

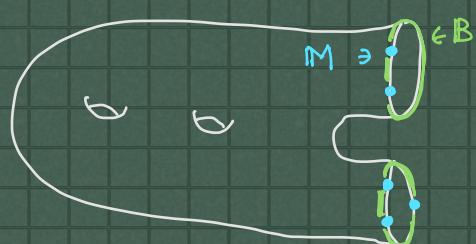
§2. the clasped skein algebra

§3. the state - clasp correspondence

§0 Introduction

(Σ, M, B) : an unpunctured marked surface
with marked points M
and boundary intervals B

||



"geometry"

the moduli space $A_{G,\Sigma}$ of decorated
twisted G -local systems on Σ

"algebra"

function ring $\mathcal{O}(A_{G,\Sigma})$

trace functions along γ

"skein algebra" $\mathcal{S}_{g,\Sigma}$

c.f. $\mathcal{I} = \text{sl}_2$,

$$\times = A) (+A^1 \times)$$

$$\circ = (A^2 + A^{-2}) \emptyset$$

§ 1 Stated skein algebras

- Bonahon-Wong introduced a skein with states to define the quantum trace map



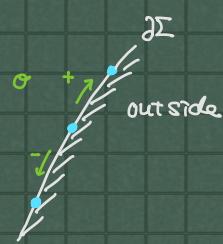
- Lê defined the stated skein algebra $\mathcal{S}_{\text{sl}, \Sigma}(B, \sigma)$

$M \subset \partial\Sigma$: a set of marked points

B : a set of connected components of $\partial\Sigma \setminus M$ (boundary intervals)

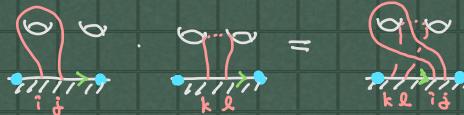
$\sigma \in \{+,-\}^B$: a choice of orientation of B

$\Lambda = \{1, 2\}$: the set of states for sl_2



Definition

$$\mathcal{S}_{\text{sl}, \Sigma}(B, \sigma) := \mathbb{Z}_A \{ \text{tangles with states } \in \Lambda \} / \text{skein relation}$$



$$\times = A \circ + A^{-1} \circ$$

$$\circ = -A^2 - A^{-2}$$

- Stated skein relation

$$\circ = U_{ij} \circ \quad \text{where } (U_{ij}) = \begin{pmatrix} 0 & -A^{-\frac{5}{2}} \\ A^{-\frac{1}{2}} & 0 \end{pmatrix},$$

$$\circ = A^{\frac{1}{2}} \circ - A^{\frac{5}{2}} \circ$$

"the sticking trick"

For simplicity, we fix an orientation of B as positive $\sigma^+ : B \rightarrow \{+\}$

$$\mathcal{S}_{\text{sl}, \Sigma}(B) := \mathcal{S}_{\text{sl}, \Sigma}(B, \sigma^+)$$

Let us define a \mathbb{Z}_A -submodule $I_{\text{bad}} := \mathbb{Z}_A \{ \text{stated tangles containing bad arcs} \}$



Lemma [Costantino-Hé '19]

I_{bad} is equal to the left and right ideal generated by bad arcs

$$\textcircled{(1)} \quad \begin{array}{c} | \\ \dots \\ | \\ \hline i_1 \dots i_k \end{array} \cdot \begin{array}{c} \text{arc} \\ \hline i_1 \dots i_k \end{array} = \begin{array}{c} | \\ \dots \\ | \\ \hline i_1 \dots i_k \end{array} \in I_{\text{bad}} \\ ? \text{ non-trivial} \end{array}$$

$$= A^{\otimes} \begin{array}{c} | \\ \dots \\ | \\ \hline i_1 \dots i_k \end{array} \in I_{\text{bad}}$$

$$\text{by } \begin{array}{c} \text{arc} \\ \hline i_1 \dots i_k \end{array} = A \begin{array}{c} | \\ \dots \\ | \\ \hline i_1 \dots i_k \end{array} \quad \begin{array}{c} \text{arc} \\ \hline i_1 \dots i_k \end{array} = A^{-1} \begin{array}{c} | \\ \dots \\ | \\ \hline i_1 \dots i_k \end{array}$$

$$\begin{array}{c} \text{arc} \\ \hline i_1 \dots i_k \end{array} \cdot \begin{array}{c} | \\ \dots \\ | \\ \hline i_1 \dots i_k \end{array} = \begin{array}{c} | \\ \dots \\ | \\ \hline i_1 \dots i_k \end{array} = A^{\bullet} \begin{array}{c} | \\ \dots \\ | \\ \hline i_1 \dots i_k \end{array} \in I_{\text{bad}}$$

Definition

$\mathcal{S}_{sl_2, \Sigma}(B)_{\text{rd}} = \mathcal{S}_{sl_2, \Sigma}(B) / I_{\text{bad}}$: the reduced stated skein algebra.

Similarly, we define $\mathcal{S}_{g, \Sigma}(B)$ and $\mathcal{S}_{g, \Sigma}(B)_{\text{rd}}$ for $g = sl_3, sp_4$

\oplus sl_3
(Higgins '20)

sl_3 -web : tangled trivalent graph on Σ , with 

$$\begin{aligned} \text{Diagram 1} &= A^2 \text{Diagram 2} + A^{-1} \text{Diagram 3}, \\ \text{Diagram 4} &= A^{-2} \text{Diagram 2} + A \text{Diagram 5}, \\ \text{Diagram 6} &= \text{Diagram 2} + \text{Diagram 7}, \\ \text{Diagram 8} &= -[2] \text{Diagram 9}, \\ \text{Diagram 10} &= [3] \text{Diagram 11} = \text{Diagram 12}. \end{aligned}$$

$$\begin{aligned} \text{Diagram 13} &= U_{ij} \text{Diagram 14} \quad \text{where } (U_{ij}) = \begin{pmatrix} 0 & 0 & A^{-7} \\ 0 & -A^{-4} & 0 \\ A^{-1} & 0 & 0 \end{pmatrix}, \\ \text{Diagram 15} &= V_{ij} \text{Diagram 16} \quad \text{where } (V_{ij}) = \begin{pmatrix} 0 & -A^{-\frac{7}{2}} & -A^{-\frac{7}{2}} \\ A^{-\frac{1}{2}} & 0 & -A^{-\frac{7}{2}} \\ A^{-\frac{1}{2}} & A^{-\frac{1}{2}} & 0 \end{pmatrix}, \\ \text{Diagram 17} &= A^3 \text{Diagram 18} + A^{-\frac{1}{2}} \text{Diagram 19} \quad \text{for } i < j. \end{aligned}$$

the stated skein relations
with states $\Lambda = \{1, 2, 3\}$

Kuperberg's internal skein relations

$\leadsto \mathcal{S}_{sl_3, \Sigma}(\mathbb{B})$

Lemma (the sticking trick)

$$\begin{aligned} \text{Diagram 20} &= A^7 \text{Diagram 21} - A^4 \text{Diagram 22} + A \text{Diagram 23} \\ \text{Diagram 24} &= A^7 \text{Diagram 25} - A^4 \text{Diagram 26} + A \text{Diagram 27} \end{aligned}$$

Lemma (Ishibashi - Y.)

$$I_{\text{bad}} := \mathbb{Z}_A \left\{ \begin{array}{c} \text{sl}_3\text{-webs containing} \\ \text{bad arcs} \quad \text{Diagram 28} \quad \mid \\ \text{bad arcs} \quad \text{Diagram 29} \quad \mid \\ i < j \\ i, j \in \Lambda = \{1, 2, 3\} \end{array} \right\}$$

coincides with the left and right ideal generated by bad arcs.

$\leadsto \mathcal{S}_{sl_3, \Sigma}(\mathbb{B})_{\text{rd}}$

$\otimes \text{RP}_4$
(Ishibashi - Y.)

$\text{RP}_4\text{-webs} : \text{tangled triv. graphs on } \Sigma \text{ with } \mathbb{X}$

$$\begin{aligned} \text{Diagram 1:} & \quad = -\frac{[2][6]}{[3]} \text{Diagram 2}, \quad \text{Diagram 3:} \quad = \frac{[5][6]}{[2][3]} \text{Diagram 4}, \quad \text{Diagram 5:} \quad = 0 \\ \text{Diagram 6:} & \quad = -[2] \text{Diagram 7}, \quad \text{Diagram 8:} \quad = 0, \\ \text{Diagram 9:} & \quad - [2] \text{Diagram 10} = \text{Diagram 11} - [2] \text{Diagram 12}, \\ \text{Diagram 13:} & \quad = \frac{v^2}{[2]} \text{Diagram 14} + v^{-1} \text{Diagram 15} + \text{Diagram 16}, \\ & \quad = v \text{Diagram 14} + \frac{v^{-2}}{[2]} \text{Diagram 15} + \text{Diagram 16}, \\ \text{Diagram 17:} & \quad = v \text{Diagram 18} + v^{-1} \text{Diagram 19}, \\ \text{Diagram 20:} & \quad = v \text{Diagram 21} + v^{-1} \text{Diagram 22}, \\ \text{Diagram 23:} & \quad = v^2 \text{Diagram 24} + v^{-2} \text{Diagram 25} + \text{Diagram 26}. \end{aligned}$$

$$\begin{aligned} \text{Diagram 27:} & \quad = U_{ij} \text{Diagram 28} \quad \text{where } (U_{ij}) = \begin{pmatrix} 0 & 0 & v^{-\frac{3}{2}} & -v^{-\frac{9}{2}} \\ 0 & 0 & v^{-\frac{3}{2}} & 0 \\ 0 & -v^{-\frac{3}{2}} & 0 & 0 \\ v^{-\frac{1}{2}} & 0 & 0 & 0 \end{pmatrix}, \\ \text{Diagram 29:} & \quad = V_{ij} \text{Diagram 30} \quad \text{where } (V_{ij}) = \begin{pmatrix} 0 & -v^{-1} & -v^{-1} & -v^{-\frac{1}{2}}[2]^{-\frac{1}{2}} \\ 1 & 0 & -v^{-\frac{3}{2}}[2]^{-\frac{1}{2}} & -v^{-1} \\ 1 & v^{\frac{1}{2}}[2]^{-\frac{1}{2}} & 0 & -v^{-1} \\ v^{-\frac{1}{2}}[2]^{-\frac{1}{2}} & 1 & 1 & 0 \end{pmatrix}, \\ \text{Diagram 31:} & \quad = v \text{Diagram 32} + \text{Diagram 33} \quad (i < j, i+j \neq 5) \\ \text{Diagram 34:} & \quad = v^2 \text{Diagram 35} + v^{\frac{1}{2}}[2]^{-\frac{1}{2}} \text{Diagram 36} + v^{-\frac{3}{2}}[2]^{-1} \text{Diagram 37} \end{aligned}$$

the stated skein relation

with states $\Lambda_1 = \{1, 2, 3, 4\}$ for |

$\Lambda_2 = \{1, 2, 3, 4, 5\}$ for ||

Kuperberg's internal skein relations

$\leadsto \mathcal{S}_{\text{RP}_4, \Sigma}(\mathbb{B}) \quad (\mathcal{R}_v = \mathbb{Z}[v^{\pm\frac{1}{2}}, \frac{1}{[2]}] - \text{algebra})$

Lemma (the sticking trick)

$$\begin{aligned} \text{Diagram 38:} & \quad = v^{\frac{1}{2}} \text{Diagram 39} - v^{\frac{3}{2}} \text{Diagram 40} + v^{\frac{7}{2}} \text{Diagram 41} - v^{\frac{9}{2}} \text{Diagram 42} \\ \text{Diagram 43:} & \quad = v \text{Diagram 44} - v^3 \text{Diagram 45} + v^4 \text{Diagram 46} - v^5 \text{Diagram 47} + v^7 \text{Diagram 48} \end{aligned}$$

Lemma $I_{\text{bad}} = \mathcal{R}_v \left\{ \begin{array}{l} \text{RP}_4\text{-webs containing} \\ \text{bad arcs} \end{array} \right. \left| \begin{array}{l} \bar{i}_1 < \bar{i}_2 \quad (\bar{i}_1, \bar{i}_2 \in \Lambda_1) \\ \bar{j}_1 < \bar{j}_2 \quad (\bar{j}_1, \bar{j}_2 \in \Lambda_2) \end{array} \right. \right\}$

coincides with the left and right ideal generated by bad arcs.

$\leadsto \mathcal{S}_{\text{RP}_4, \Sigma}(\mathbb{B})_{\text{rd}}$

§ 3. the clasped skein algebra $\mathcal{S}_{g,\Sigma}(M)$

① the clasped skein algebra $\mathcal{S}_{sl_2,\Sigma}(M)$ is introduced by Muller ('16)

and he showed $\mathcal{S}_{sl_2,\Sigma}(M) = \underline{\mathcal{A}_{sl_2,\Sigma}^{\mathbb{C}}}$ the quantum cluster algebra

- endpoints have elevation :



(quantization
of $\mathcal{O}(\mathcal{A}_{g,\Sigma})$)

② the clasped skein relations

- sl_2
(Muller '16)

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = A \quad \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}, \quad \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} = 0.$$

- sl_3
(Frohman-Sikora '20)

	$= A^2$		$= A^2$
	$= A$		$= A$
	$=$		$=$
	$= 0$		$= 0$
	$= 0$		$= 0$

- sl_4
(Ishibashi-Y '22)

	$= v$		$= v^2$
	$= v$		$= v$
	$=$		$=$
	$= \frac{1}{[2]}$		$= 0,$
	$= 0,$		$= 0,$
	$= 0,$		$= 0,$
	$= 0,$		$= 0,$

We consider the boundary-localized clasped skein algebra $\mathcal{S}_{g,\Sigma}(M)[\partial']$
which has inverses of boundary web $\text{Diagram} := (\text{Diagram})^{-1}$

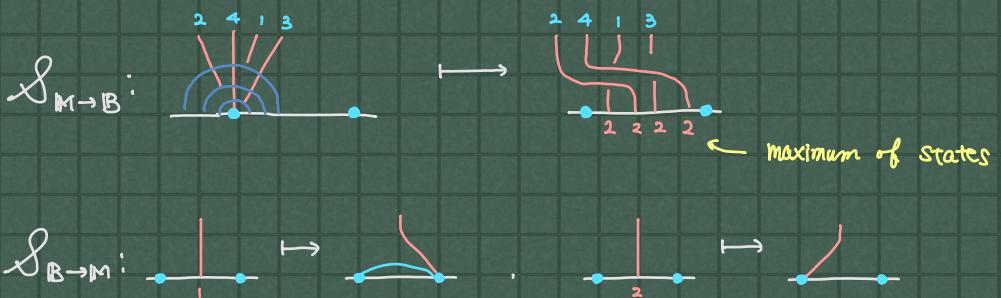
§4. the state - clasp correspondence

We define algebra homomorphisms

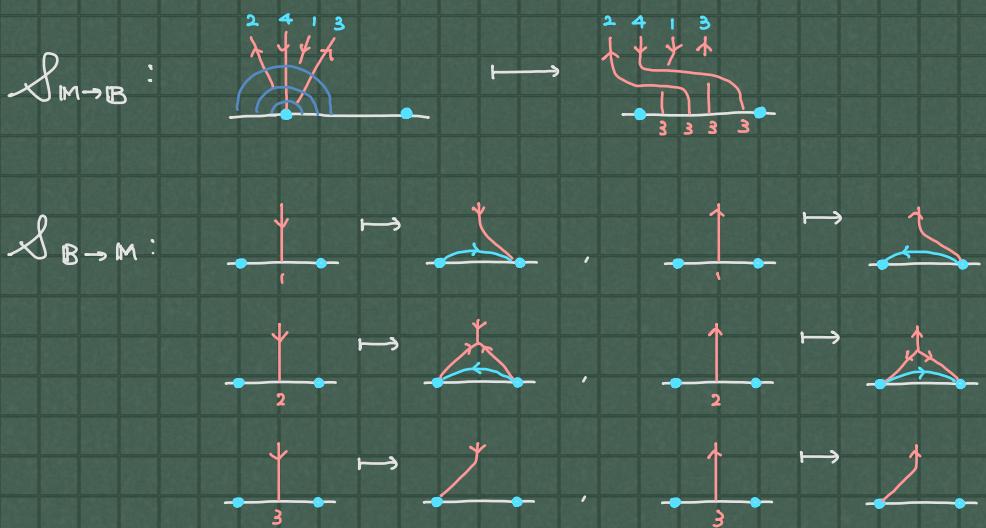
$$\begin{cases} \mathcal{S}_{M \rightarrow B} : \mathcal{S}_{g, \Sigma}(M)[\partial^{-1}] \rightarrow \mathcal{S}_{g, \Sigma}(B)_{rd} \\ \mathcal{S}_{B \rightarrow M} : \mathcal{S}_{g, \Sigma}(B)_{rd} \rightarrow \mathcal{S}_{g, \Sigma}(M)[\partial^{-1}] \end{cases}$$

$$\text{s.t. } \mathcal{S}_{B \rightarrow M} \circ \mathcal{S}_{M \rightarrow B} = \text{id}_{\mathcal{S}_{g, \Sigma}(M)[\partial^{-1}]}, \quad \mathcal{S}_{M \rightarrow B} \circ \mathcal{S}_{B \rightarrow M} = \text{id}_{\mathcal{S}_{g, \Sigma}(B)_{rd}}$$

④ \mathfrak{sl}_2



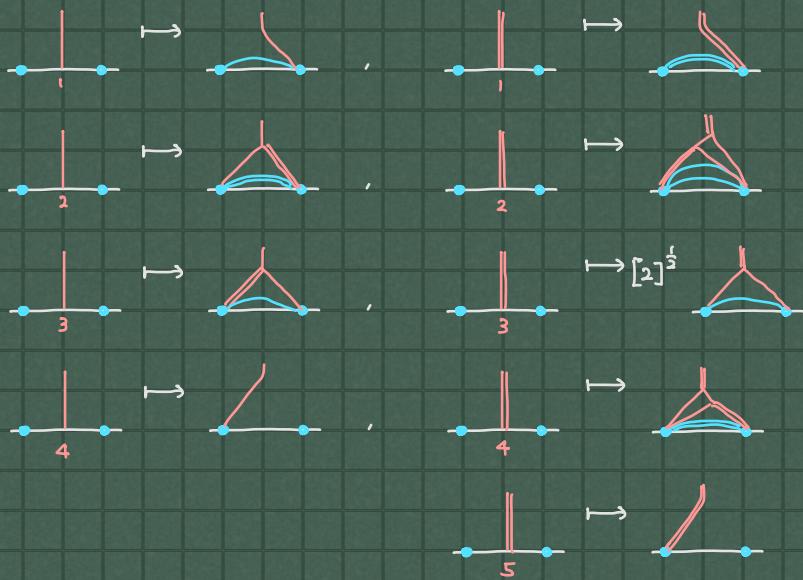
④ \mathfrak{sl}_3



② $\text{Sp}_4 \quad \mathcal{S}_{M \rightarrow B} :$



$\mathcal{S}_{B \rightarrow M} :$



By definition, $\mathcal{S}_{B \rightarrow M} \circ \mathcal{S}_{M \rightarrow B} = \text{id}$ is easy.

$\mathcal{S}_{M \rightarrow B} \circ \mathcal{S}_{B \rightarrow M} = \text{id}$: use the sticking trick

e.g. Sp_4

$$\begin{aligned}
 & \text{Initial configuration: } \text{B} \rightarrow \text{M} \\
 & \text{Result after } \mathcal{S}_{B \rightarrow M}: u^{\frac{1}{2}} \\
 & \text{Result after } \mathcal{S}_{M \rightarrow B}: u^{\frac{1}{2}} - u^{\frac{3}{2}} + u^{\frac{7}{2}} - u^{\frac{9}{2}} \\
 & \text{add webs with bad arcs: } u^{\frac{1}{2}} - u^{\frac{3}{2}} + u^{\frac{7}{2}} - u^{\frac{9}{2}} \\
 & \text{the sticking trick: } u^{\frac{1}{2}}
 \end{aligned}$$