

Stated and Marked Skein Algebra

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# braiding for $sl_3$

$$i \in \Pi = \{1, 2, 3\}$$

$$\begin{array}{c} \text{ } \\ \text{ } \end{array} \begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-2} \begin{array}{c} \text{ } \\ \text{ } \end{array} + A \begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-2} \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

$$s < t$$

$$\begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-2} \begin{array}{c} \text{ } \\ \text{ } \end{array} + A(A^{-2}) \begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-2} (A^3 \begin{array}{c} \text{ } \\ \text{ } \end{array} + A^{-1} \begin{array}{c} \text{ } \\ \text{ } \end{array}) - A^{-\frac{5}{2}} \begin{array}{c} \text{ } \\ \text{ } \end{array} = A \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

$$\begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-2} \begin{array}{c} \text{ } \\ \text{ } \end{array} + A(A^{-1}) \begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-2} (A^{-3} \begin{array}{c} \text{ } \\ \text{ } \end{array} - A^{-2} \begin{array}{c} \text{ } \\ \text{ } \end{array}) + A^{\frac{1}{2}} \begin{array}{c} \text{ } \\ \text{ } \end{array} = A^5 \begin{array}{c} \text{ } \\ \text{ } \end{array} + A^{-\frac{5}{2}} (A^3 - A^{-3}) \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

define  $\varepsilon_{ii} = -2$   
 $\varepsilon_{st} = 1 \quad (s < t)$

$$\begin{array}{c} \text{ } \\ \text{ } \end{array} = \gamma_{uv}^{-1} \begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{\varepsilon_{1u} + \varepsilon_{1v}} \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

$$\begin{array}{c} \text{ } \\ \text{ } \end{array} = \gamma_{23}^{-1} \begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{\varepsilon_{22} + \varepsilon_{23}} \begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-1} \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

$$\begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-1} \begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-1} \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

$$\begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-1} \begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-1} \begin{array}{c} \text{ } \\ \text{ } \end{array} + A^{-1} \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

$$\begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-1} \begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-1} \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

$$\begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-2} \begin{array}{c} \text{ } \\ \text{ } \end{array} + A^{-1}(A^3 - A^{-3}) \begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-3} \begin{array}{c} \text{ } \\ \text{ } \end{array} + A^{-2}(A^3 - A^{-3}) \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

$$\begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-2} \begin{array}{c} \text{ } \\ \text{ } \end{array} + A^{-1}(A^3 - A^{-3}) \begin{array}{c} \text{ } \\ \text{ } \end{array} = A^{-1} \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

$u+v-2$	$(u, v)$
1	(1, 2)
2	(1, 3)
3	(2, 3)

braiding for  $\pi_4$

$$\begin{aligned} \text{Diagram} &= \frac{v^2}{[2]} \text{Diagram} + v^{-1} \text{Diagram} + \text{Diagram} \\ &= v^{-1} \text{Diagram} \end{aligned}$$

$$\begin{aligned} \text{Diagram} &= v \text{Diagram} + \frac{v^{-2}}{[2]} \text{Diagram} + \text{Diagram} \\ &= \frac{v^2}{[2]} \text{Diagram} + v^{-1} \text{Diagram} + \text{Diagram} \end{aligned}$$

$$\text{Diagram} - \text{Diagram} = (v - v^{-1}) (\text{Diagram} - \text{Diagram})$$

$s+t \neq 5, s < t$

$$\begin{aligned} \text{Diagram} &= \frac{v^2}{[2]} \text{Diagram} + v^{-1} \text{Diagram} + \text{Diagram} \\ &= v^{-1} \text{Diagram} + \text{Diagram} \\ &= v^{-1} \left( v \text{Diagram} + \text{Diagram} \right) - v^{-1} \text{Diagram} = \text{Diagram} \end{aligned}$$

$s+t-2$	$(s,t)$
1	(1,2)
2	(1,3)
3	(1,4) (2,3)
4	(2,4)
5	(3,4)

$$\begin{aligned} \text{Diagram} &= \frac{v^2}{[2]} \text{Diagram} + v^{-1} \text{Diagram} + \text{Diagram} \\ &= \frac{v^2}{[2]} (-v^{\frac{7}{2}}) \text{Diagram} + v^{-1} \left( v^2 \text{Diagram} + v^{\frac{1}{2}} [2]^{\frac{1}{2}} \text{Diagram} + v^{-\frac{3}{2}} [2]^{-1} \text{Diagram} \right) + (-v^{\frac{1}{2}} [2]^{-\frac{1}{2}}) \text{Diagram} \\ &= v \text{Diagram} \end{aligned}$$

$$\begin{aligned} \text{Diagram} &= \frac{v^2}{[2]} \text{Diagram} + v^{-1} \text{Diagram} + \text{Diagram} \\ &= \frac{v^2}{[2]} v^{-\frac{7}{2}} \text{Diagram} + v^{-1} \left( v^2 \text{Diagram} + (v - v^3) \text{Diagram} + v^{\frac{1}{2}} [2]^{\frac{1}{2}} \text{Diagram} - v^{-\frac{1}{2}} [2]^{-1} \text{Diagram} \right) \\ &\quad - v^{-\frac{3}{2}} [2]^{-\frac{1}{2}} \text{Diagram} \\ &= v \text{Diagram} + (1 - v^2) \text{Diagram} \end{aligned}$$

$$\text{Diagram} = \text{Diagram} - (v - v^{-1}) \text{Diagram} = \text{Diagram} - (v - v^{-1}) \text{Diagram}$$

$$\text{Diagram} = \text{Diagram} - (v - v^{-1}) \text{Diagram} = \text{Diagram} - (v - v^{-1}) \text{Diagram}$$

$$\text{Diagram} = \text{Diagram} + (v - v^{-1}) \left( \text{Diagram} - \text{Diagram} \right) = v^{-1} \text{Diagram} - (v - v^{-1}) \text{Diagram} + v^{\frac{1}{2}} (v - v^{-1}) \text{Diagram}$$

$$\begin{aligned} \text{Diagram} &= \text{Diagram} + (v - v^{-1}) \left( \text{Diagram} - \text{Diagram} \right) \\ &= v^{-1} \text{Diagram} - (v - v^{-1}) \text{Diagram} + (v - v^{-1}) \text{Diagram} - v^{-\frac{3}{2}} (v - v^{-1}) \text{Diagram} \end{aligned}$$

$$\text{Diagram} = \text{Diagram} - (v - v^{-1}) \text{Diagram}$$

$$\text{Diagram} = \text{Diagram} - (v - v^{-1}) \text{Diagram}$$

From the above,

$$\begin{aligned} \text{Diagram 1} &= v^{\varepsilon_k} \text{Diagram 2} \\ \text{Diagram 3} &= v^{\varepsilon_{5-k}} \text{Diagram 4} \end{aligned} \quad (\varepsilon_k)_{k \in \Pi} = (-1, 0, 0, 1)$$

$$\text{Diagram 1} = y_{st}^{-1} \text{Diagram 2} = y_{st}^{-1} v^{\varepsilon_{s-t} + \varepsilon_{s-s}} \text{Diagram 3} = v^{\varepsilon_{s-t} + \varepsilon_{s-s}} \text{Diagram 4}$$

$$\text{Diagram 1} = y_{st}^{-1} \text{Diagram 2} = y_{st}^{-1} v^{\varepsilon_s + \varepsilon_t} \text{Diagram 3} = v^{\varepsilon_s + \varepsilon_t} \text{Diagram 4}$$

$s+t-2$	$(s, t)$
1	(1, 2)
2	(1, 3)
3	(1, 4)
4	(2, 3)
5	(2, 4)

$$(\eta_t) = (-1, -1, 0, 1, 1)$$

$$\text{Diagram 1} = y_{21}^{-1} \text{Diagram 2} = y_{21}^{-1} v^{\varepsilon_s} \text{Diagram 3}$$

$$s=1 \quad y_{21}^{-1} v^{\varepsilon_1} \text{Diagram 1} = y_{21}^{-1} v^{\varepsilon_1} \left( \text{Diagram 2} - (v - v^{-1}) \text{Diagram 3} \right) = v^{\varepsilon_1} \text{Diagram 4}$$

$$s=2 \quad y_{21}^{-1} v^{\varepsilon_2} \text{Diagram 1} = y_{21}^{-1} v^{\varepsilon_2} v^{-1} \text{Diagram 2} = v^{\varepsilon_2 - 1} \text{Diagram 3}$$

$$s=3 \quad y_{21}^{-1} v^{\varepsilon_3} \text{Diagram 1} = v^{\varepsilon_3} \left( v \text{Diagram 2} + (1 - v^2) \text{Diagram 3}^0 \right) = v^{\varepsilon_3 + 1} \text{Diagram 4}$$

$$s=4 \quad y_{21}^{-1} v^{\varepsilon_4} \text{Diagram 1} = y_{21}^{-1} v^{\varepsilon_4} \text{Diagram 2} = v^{\varepsilon_4} \text{Diagram 3}$$

$$\varepsilon_s'' = (-1, -1, 1, 1)$$

$$\text{Diagram 1} = y_{43}^{-1} \text{Diagram 2} = v^{\varepsilon_{s-s}} \text{Diagram 3}$$

$$s=1 \quad v^{\varepsilon_4} \text{Diagram 1} = v^{\varepsilon_4 + \varepsilon_3} \text{Diagram 2} = v^{\varepsilon_4 + \varepsilon_3} \text{Diagram 3}$$

$$s=2 \quad v^{\varepsilon_3} \text{Diagram 1} = v^{\varepsilon_3} \left( v \text{Diagram 2} + (1 - v^2) \text{Diagram 3}^0 \right) = v^{\varepsilon_3 + 1} \text{Diagram 4}$$

$$s=3 \quad v^{\varepsilon_2} \text{Diagram 1} = v^{\varepsilon_2 - 1} \text{Diagram 2} = v^{\varepsilon_2 - 1} \text{Diagram 3}$$

$$s=4 \quad v^{\varepsilon_1} \text{Diagram 1} = v^{\varepsilon_1} \left( \text{Diagram 2} - (v - v^{-1}) \text{Diagram 3}^0 \right) = v^{\varepsilon_1} \text{Diagram 4}$$

$$\varepsilon_s''' = (1, 1, -1, -1)$$

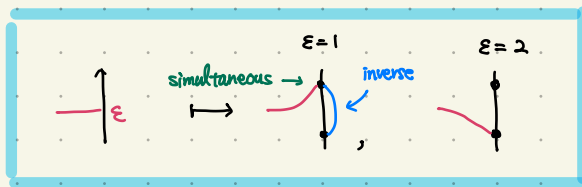
$$\text{Diagram 1} = y_{st}^{-1} \text{Diagram 2} = y_{st}^{-1} v^{\varepsilon_s^* + \varepsilon_t^*} \text{Diagram 3} = v^{\varepsilon_s^* + \varepsilon_t^*} \text{Diagram 4}$$

$$\rightsquigarrow (-2, 0, 0, 0, 2)$$

$$\text{Diagram 1} = y_{st}^{-1} \text{Diagram 2} = y_{st}^{-1} v^{\varepsilon_s^* + \varepsilon_t^*} \text{Diagram 3} = v^{\varepsilon_s^* + \varepsilon_t^*} \text{Diagram 4}$$

$$\rightsquigarrow (2, 0, 0, 0, -2)$$

⑩ well-definedness .  $\mathcal{S}_{sl_2}^{st} \longrightarrow \mathcal{S}_{sl_2}^{mk}$




$$\begin{array}{c} \uparrow \\ \text{red loop} \end{array} \mapsto \begin{array}{c} \text{red loop} \\ \text{blue loop} \end{array} = 0$$


$$\begin{array}{c} \uparrow \\ \text{red loop} \end{array} \mapsto \begin{array}{c} \text{red loop} \\ \text{blue loop} \end{array} = v^{\frac{1}{2}} \begin{array}{c} \uparrow \\ \text{blue loop} \end{array}$$

$$\begin{array}{c} \uparrow \\ \text{red loop} \end{array} \mapsto \begin{array}{c} \text{red loop} \\ \text{blue loop} \end{array} = \left( \begin{array}{c} \text{red loop} \\ \text{blue loop} \end{array} \right)^{\dagger} = \left( -A^3 \begin{array}{c} \text{red loop} \\ \text{blue loop} \end{array} \right)^{\dagger} \longleftarrow -\bar{A}^3 \left( \begin{array}{c} \uparrow \\ \text{red loop} \end{array} \right)^{\dagger} \quad \therefore x_{\downarrow i} = -A^{-3} x_{i \downarrow}^{\dagger}$$


$$\begin{array}{c} \uparrow \\ \text{red loop} \end{array} \mapsto \begin{array}{c} \text{red loop} \\ \text{blue loop} \end{array} = A^{\frac{1}{2}} \left( A \begin{array}{c} \text{red loop} \\ \text{blue loop} \end{array} + A^{-1} \begin{array}{c} \text{red loop} \end{array} \right) \\ = A^2 \begin{array}{c} \text{red loop} \\ \text{blue loop} \end{array} + A^{-\frac{1}{2}} \begin{array}{c} \text{red loop} \end{array} \longleftrightarrow A^2 \begin{array}{c} \uparrow \\ \text{red loop} \end{array} + A^{-\frac{1}{2}} \begin{array}{c} \text{red loop} \end{array}$$

① well-definedness :  $\mathcal{S}_{sl_3}^{st} \rightarrow \mathcal{S}_{sl_3}^{mk}$


  $\epsilon \mapsto$ 

$\epsilon=1$   






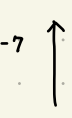
 $,$ 

$\epsilon=2$   


 $,$ 

$\epsilon=3$   


 $\quad \& \text{ Dynkin involution}$








 $\mapsto$ 

 $= A^{-8}$ 

 $= A^{-7}$ 

 $\longleftarrow A^{-7}$ 


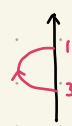







 $\mapsto$ 



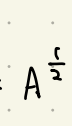





 $= 0$ 

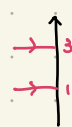

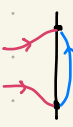




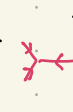
 $\mapsto 0$ 







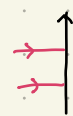

 $\mapsto 0$


 $\mapsto$ 

 $= 0$ 

 $\mapsto$ 

 $= -A^{-4}$ 

 $= -A^{-4}$ 


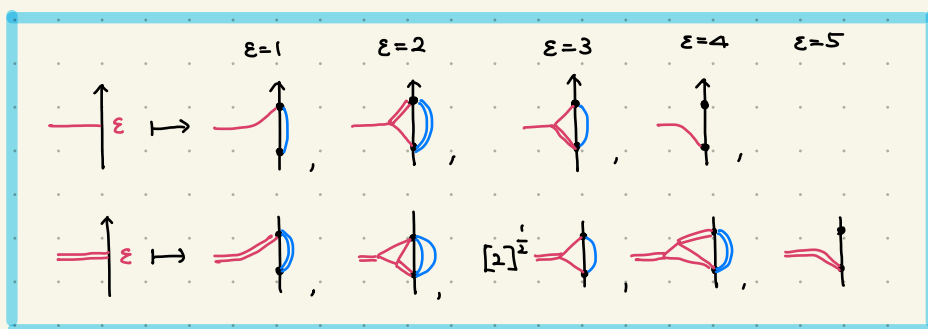

 $\mapsto$ 

 $= A^{-1}$ 


 $\mapsto$ 

 $= 0$ 

 $\mapsto$ 

 $= 0$


 $\mapsto$ 

 $= A^{\frac{1}{2}} \left( A^2 \text{  + A^{-1} \text{  \right)$   
 $= A^{\frac{5}{2}} A^{\frac{1}{2}} \text{  + A^{-\frac{1}{2}} \text{  } \longleftarrow A^3 \text{  + A^{-\frac{1}{2}} \text{ $


 $\mapsto$ 

 $= A^{\frac{1}{2}} \left( A^2 \text{  + A^{-1} \text{  \right)$   
 $= A^{\frac{5}{2}} A^{\frac{1}{2}} \text{  + A^{-\frac{1}{2}} \text{  } \longleftarrow A^3 \text{  + A^{-\frac{1}{2}} \text{ $


 $\mapsto$ 

 $= A^{\frac{1}{2}} \left( A^2 \text{  + A^{-1} \text{  \right)$   
 $= A^{\frac{5}{2}} A^{\frac{1}{2}} \text{  + A^{-\frac{1}{2}} \text{  } \longleftarrow A^3 \text{  + A^{-\frac{1}{2}} \text{ $

well-definedness :  $\mathcal{L}_{sp_4}^{\text{st}} \rightarrow \mathcal{L}_{sp_4}^{\text{mk}}$



$$\begin{array}{c} \uparrow_1 = 0 \quad \uparrow_4 = 0 \quad \uparrow_2 = \text{diagram} = 0 \quad \uparrow_3 = 0 \quad (\text{similar to } \uparrow_2) \end{array}$$

$$\begin{array}{c} \uparrow_2 = \text{diagram} = 0 \quad \uparrow_3 = \text{diagram} = 0 \quad \uparrow_4 = \text{diagram} = v^{-\frac{1}{2}} \uparrow \end{array}$$

$$\begin{array}{c} \uparrow_3 = \text{diagram} = -v^{-1} \text{diagram} = -v^{-\frac{3}{2}} \uparrow \quad \uparrow_4 = \text{diagram} = 0 \end{array}$$

$$\uparrow_4 = \text{diagram} = 0$$

$$\uparrow_i \mapsto \text{diagram} = \left( \text{diagram} \right)^\dagger = \left( -v^5 \text{diagram} \right)^\dagger \leftarrow -v^5 \left( \text{diagram} \right)^\dagger \quad \therefore x_{ji} = -v^5 x_{ij}^\dagger$$

$$\text{diagram} \mapsto \text{diagram} = 0$$

$$\text{diagram} \mapsto \text{diagram} = \text{diagram} = \text{diagram} \leftarrow \text{diagram}$$

$$\text{diagram} \mapsto \text{diagram} = \text{diagram} \leftarrow \text{diagram}$$

$$\text{diagram} = \text{diagram} = v^{-\frac{1}{2}} \text{diagram} = \frac{v^{-\frac{1}{2}}}{[2]^{\frac{1}{2}}} \text{diagram}$$

$$\text{diagram} = \text{diagram} = v^{-1} \text{diagram} = \text{diagram} - \frac{1}{[2]} \text{diagram} + \frac{1}{[2]} \text{diagram} = 0$$

$$\text{diagram} = \text{diagram} = -\frac{1}{[2]} \text{diagram} = -\frac{v^{\frac{1}{2}}}{[2]} \text{diagram} = v^{\frac{1}{2}} \text{diagram} = \frac{v^{\frac{1}{2}}}{[2]^{\frac{1}{2}}} \text{diagram}$$

$$\text{diagram} = \text{diagram} = \text{diagram} = \text{diagram}$$

