

Skein and cluster algebras of unpunctured surfaces
for \mathbb{RP}_4 (arXiv: 2207.01540) with Tsukasa Ishibash (Tohoku Univ.)

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plan

- §1 Main results
- §2 clasped \mathbb{RP}_4 -skein alg. \mathcal{S} and \mathbb{Z}_{\pm} -form of \mathcal{S}
- §3 construction of \mathcal{A} in $\text{Frac } \mathcal{S}$
- §4 inclusion $\mathcal{S}[\partial^*]$ into \mathcal{A}
- §5 characterization of cluster variables

$\Sigma = (\Sigma, \mathbb{M})$: an unpunctured marked surface

§ Main results

Conjecture

the "clasped" \mathbb{G} -skein algebra

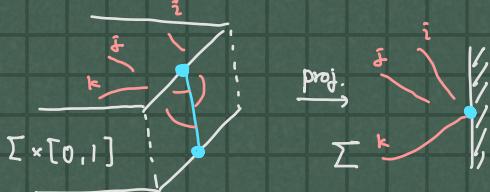
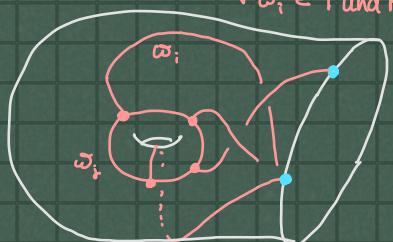
$$\mathcal{S}_{\mathbb{G}, \Sigma}[\partial^*]$$



1-3-valent graph
+ skein relation

\mathbb{G} -web

$\forall \omega_i \in \text{Fund Rep}_{\mathbb{G}}$



the quantum cluster algebra
associated with $S_{\mathbb{G}}(\Sigma)$

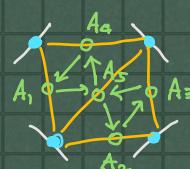
$$\mathcal{A}_{\mathbb{G}, \Sigma}^{\mathbb{Z}_{\pm}} \hookrightarrow \mathcal{U}_{\mathbb{G}, \Sigma}^{\mathbb{Z}_{\pm}}$$



=

in $\text{Frac } \mathcal{S}_{\mathbb{G}, \Sigma}$

cluster variables
+ exchange relation.



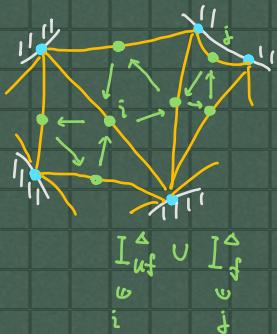
" The skein algebra gives a quantization
of the moduli space $\mathcal{A}_{\mathbb{G}, \Sigma}$ of
decorated twisted \mathbb{G} -local systems on Σ
 $(\mathcal{O}(\mathcal{A}_{\mathbb{G}, \Sigma}^*) = \mathcal{U}_{\mathbb{G}, \Sigma})$ "

• Müller ('16) $\mathfrak{g} = \mathfrak{sl}_2$ \mathfrak{sl}_2 -web = tangles on Σ
 (no trivalent vertices)

$$\left\{ \begin{array}{l} \mathcal{A}_{\mathfrak{sl}_2, \Sigma}^{\mathfrak{g}} \subset \mathcal{S}_{\mathfrak{sl}_2, \Sigma} [\delta'] \subset \mathcal{U}_{\mathfrak{sl}_2, \Sigma} \text{ (in } \text{Frac } \mathcal{S}_{\mathfrak{sl}_2, \Sigma}) \\ \mathcal{A}_{\mathfrak{sl}_2, \Sigma}^{\mathfrak{g}} = \mathcal{U}_{\mathfrak{sl}_2, \Sigma}^{\mathfrak{g}} \end{array} \right.$$

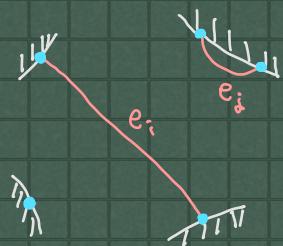
$$\rightsquigarrow \mathcal{A}_{\mathfrak{sl}_2, \Sigma}^{\mathfrak{g}} = \mathcal{S}_{\mathfrak{sl}_2, \Sigma} [\delta'] = \mathcal{U}_{\mathfrak{sl}_2, \Sigma}^{\mathfrak{g}}$$

ideal triangulation Δ



cluster variables
 $\{A_i \mid i \in I^\Delta\}$

\longleftrightarrow



\mathfrak{g} -exchange rel = skein rel

$$e_i^- e_i^+ = \mathfrak{g} \quad e_i^+ + \mathfrak{g}^{-1} e_i^-$$

we know all cluster variables in $\mathcal{S}_{\mathfrak{sl}_2, \Sigma}$

$$\Rightarrow \mathcal{A}_{\mathfrak{sl}_2, \Sigma}^{\mathfrak{g}} \subset \mathcal{S}_{\mathfrak{sl}_2, \Sigma}^{\mathfrak{g}} [\delta']$$

the quantum exchange relation

= the skein relation

\mathfrak{g} : higher rank

(Hard) Realizing all cluster variables as \mathfrak{g} -webs in $\mathcal{S}_{\mathfrak{g}, \Sigma}$

\hookrightarrow clusters do not come from
 "decorated ideal triangulations".

④ $\mathfrak{G} = \mathfrak{sl}_3$ Ishibashi - Y. (2021)

$$\begin{array}{c} \textcircled{1} \quad \mathcal{A}_{\mathfrak{sl}_3, \Sigma} [\delta'] \underset{\text{(I)}}{\subset} \mathcal{A}_{\mathfrak{sl}_3, \Sigma}^{\mathbb{Z}} \subset \mathcal{U}_{\mathfrak{sl}_3, \Sigma}^{\mathbb{Z}} \subset \text{Frac } \mathcal{A}_{\mathfrak{sl}_3, \Sigma} \\ \Downarrow \qquad \qquad \qquad \Downarrow \\ \textcircled{2} \quad \begin{matrix} \text{"elevation-preserving"} \\ \mathfrak{sl}_3\text{-webs} \\ \text{w.r.t } \Delta \end{matrix} \xrightarrow{\text{(II)}} \begin{matrix} \text{Laurent polynomial in } \mathcal{L}_{\Delta} \\ \text{with coefficients in } \mathbb{Z}_+[\delta^{\pm \frac{1}{2}}] \end{matrix} \end{array}$$

(I) the sticking trick

(II) the cutting trick

④ $\mathfrak{G} = \mathfrak{sp}_4$ Ishibashi - Y. (2022)

Theorem ① & ② in a similar way (I) (II)

difference from the \mathfrak{sl}_3 case :

$$\begin{cases} \mathcal{A}_{\mathfrak{sl}_3, \Sigma} : \text{a } \mathbb{Z}[\delta^{\pm \frac{1}{2}}] \text{-algebra} \\ \mathcal{A}_{\mathfrak{sp}_4, \Sigma} : \text{a } \mathbb{Z}[\delta^{\pm \frac{1}{2}}, \frac{1}{[2]}] \text{-algebra} \end{cases}$$

- We define "the \mathbb{Z}_q -form $\mathcal{A}_{\mathfrak{sp}_4, \Sigma}^{\mathbb{Z}_q}$ " of $\mathcal{A}_{\mathfrak{sp}_4, \Sigma}$ and show $\mathcal{A}_{\mathfrak{sp}_4, \Sigma}^{\mathbb{Z}_q} [\delta'] \subset \mathcal{A}_{\mathfrak{sp}_4, \Sigma}^{\mathbb{Z}}$
- The Laurent positivity is shown in $\mathbb{Z}_+[\delta^{\pm \frac{1}{2}}, \frac{1}{[2]}]$

① Combine with a result in Ishibashi - Oya - Shen (2022) ($\mathcal{A} = \mathcal{U}$)

Corollary $\mathcal{A}_{\mathfrak{sp}_4, \Sigma}^{\mathbb{Z}} [\delta'] = \mathcal{A}_{\mathfrak{sp}_4, \Sigma} = \mathcal{U}_{\mathfrak{sp}_4, \Sigma} = \mathcal{O}(\mathcal{A}_{\mathfrak{sp}_4}^*)$

§ 2. the clasped Sp_4 -skein algebra

$$\mathcal{R} := \mathbb{Z}[v^{\pm\frac{1}{2}}, \frac{1}{[2]}] \quad \left([n] = \frac{v^n - v^{-n}}{v - v^{-1}} \right)$$

$\mathcal{S}_{Sp_4, \Sigma} := \mathcal{R} \{ Sp_4\text{-graphs on } \Sigma \} / Sp_4\text{-skein relations}$

type 1 edge $\text{---} : \omega_1$,
type 2 edge $\text{= = : } \omega_2$



$\text{---} : \omega_1$



superposition of



and

• Sp_4 -skein relations

- Kuperberg's "internal" skein relations

$$\begin{aligned} \textcircled{O} &= -\frac{[2][6]}{[3]} \textcircled{O}, \\ \textcircled{O} &= \frac{[5][6]}{[2][3]} \textcircled{O}, \\ \textcircled{O} &= 0, \\ \textcircled{O} &= -[2] \textcircled{||}, \\ \textcircled{Y} &= 0, \\ \textcircled{X} - [2] \textcircled{H} &= \textcircled{O} \textcircled{O} - [2] \textcircled{H}, \\ \textcircled{X} &= \frac{v^2}{[2]} \textcircled{O} \textcircled{O} + v^{-1} \textcircled{O} \textcircled{O} + \textcircled{H}, \\ &= v \textcircled{O} \textcircled{O} + \frac{v^{-2}}{[2]} \textcircled{O} \textcircled{O} + \textcircled{H}, \\ \textcircled{X} &= v \textcircled{H} + v^{-1} \textcircled{H}, \\ \textcircled{X} &= v \textcircled{H} + v^{-1} \textcircled{H}, \\ \textcircled{X} &= v^2 \textcircled{O} \textcircled{O} + v^{-2} \textcircled{O} \textcircled{O} + \textcircled{H}. \end{aligned}$$

- "clasped" skein relation (NEW relations)

$$\begin{array}{ccc} \textcircled{X} & = v & \textcircled{X}, \\ \textcircled{X} & = v & \textcircled{X}, \\ \textcircled{X} & = v & \textcircled{X}, \\ \textcircled{X} & = \frac{1}{[2]} & \textcircled{X}, \\ \textcircled{X} & = 0, & \textcircled{X} = 0, \\ \textcircled{X} & = 0, & \textcircled{X} = 0, \\ \textcircled{X} & = 0, & \textcircled{X} = 0. \end{array}$$

* We use the simultaneous crossings defined by:

$$\begin{array}{ccc} \textcircled{X} & := v^{-\frac{1}{2}} \textcircled{X}, \\ \textcircled{X} & := v^{-\frac{1}{2}} \textcircled{X}, \\ \textcircled{X} & := v^{-1} \textcircled{X}. \end{array}$$

These skein relations realize the Reidemeister moves (framed ver.)



for any colors ω_1, ω_2

④ Crossroads, rungs and legs



a rung (an internal type 2 edges)



a leg (a type 2 edge between a vertex & marked pt)

We define a new 4-valent vertex as



$$\text{a crossroad} := \left(\text{rung} - \frac{1}{[2]} \right) = \left(\text{leg} - \frac{1}{[2]} \right) \quad \mapsto \boxed{\begin{array}{l} \text{X} = u \\ \text{+ } w^{-1} \text{ X} \\ + \text{ X} \end{array}}$$

Definition

A crossroad web is an $s_{\mathbb{P}_4}$ -web represented by a 1-3-4-valent graph with no rungs.

Definition

- A basis web is a flat crossroad web with no elliptic faces.
 { no internal crossings
 only simultaneous crossings e.g. }

$$\bullet \text{BWeb}_{\mathbb{P}_4, \Sigma} := \{ \text{basis webs on } \Sigma \} \subset \mathcal{S}_{\mathbb{P}_4, \Sigma}$$

$$\left(\begin{array}{c} \pm \frac{\pi}{2} \\ \mp \frac{3}{4}\pi \end{array} \right)$$

Theorem (IT)

$\text{BWeb}_{\mathbb{P}_4, \Sigma}$ is an R -basis of $\mathcal{S}_{\mathbb{P}_4, \Sigma}$

proof By Sikora - Westbury's confluence theory

(the diamond lemma for the skein theory)

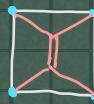
The reduction rules for spa_α -web

$$\begin{aligned}
 S_1: & \text{Diagram} \sim -\frac{[2][6]}{[3]} \text{Diagram}, & S_2: & \text{Diagram} \sim \frac{[5][6]}{[2][3]} \text{Diagram}, & S_3: & \text{Diagram} \sim 0, \\
 S_4: & \text{Diagram} \sim -[2] \text{Diagram}, & S_5: & \text{Diagram} \sim -\frac{[5]}{[2]} \text{Diagram}, & S_6: & \text{Diagram} \sim 0, \\
 S_7: & \text{Diagram} \sim -\frac{[3]}{[2]} \text{Diagram}, & S_8: & \text{Diagram} \sim \frac{1}{[2]} \text{Diagram} + \text{Diagram}, & & \\
 S_9: & \text{Diagram} \sim \frac{[6]}{[3]} \text{Diagram}, & S_{10}: & \text{Diagram} \sim -[2] \text{Diagram}, & S_{11}: & \text{Diagram} \sim \text{Diagram}, \\
 S_{12}: & \text{Diagram} \sim -[2] \text{Diagram} + \frac{[6]}{[3]} \text{Diagram}, & S_{13}: & \text{Diagram} \sim 0, & & \\
 S_{14}: & \text{Diagram} \sim \frac{[4]}{[2]} \text{Diagram} - [2] \text{Diagram}, & S_{15}: & \text{Diagram} \sim \text{Diagram} + \text{Diagram}, & & \\
 S_{16}: & \text{Diagram} \sim \text{Diagram} - \frac{[6]}{[2][3]} \text{Diagram}, & & & & \\
 S_{17}: & \text{Diagram} \sim [2] \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram}, & & & &
 \end{aligned}$$

$$\begin{aligned}
 B_1: & \text{Diagram} \sim 0, & B_2: & \text{Diagram} \sim 0, & B_3: & \text{Diagram} \sim 0, \\
 B_4: & \text{Diagram} \sim 0, & B_5: & \text{Diagram} \sim 0, & B_6: & \text{Diagram} \sim 0, \\
 B_7: & \text{Diagram} \sim 0, & B_8: & \text{Diagram} \sim \frac{1}{[2]} \text{Diagram}, & & \\
 B_9: & \text{Diagram} \sim \text{Diagram}, & B_{10}: & \text{Diagram} \sim \text{Diagram}, & & \\
 B_{11}: & \text{Diagram} \sim \text{Diagram}, & B_{12}: & \text{Diagram} \sim \text{Diagram}, & & \\
 B_{13}: & \text{Diagram} \sim \text{Diagram}, & B_{14}: & \text{Diagram} \sim \text{Diagram} \quad \text{for } i < j,
 \end{aligned}$$

$$\begin{aligned}
 C_1: & \text{Diagram} \sim v \text{Diagram} + v^{-1} \text{Diagram} + \text{Diagram}, \\
 C_2: & \text{Diagram} \sim v \text{Diagram} + v^{-1} \text{Diagram}, \\
 C_3: & \text{Diagram} \sim v \text{Diagram} + v^{-1} \text{Diagram}, \\
 C_4: & \text{Diagram} \sim v^2 \text{Diagram} + v^{-2} \text{Diagram} + \text{Diagram}.
 \end{aligned}$$

X : an spa_α -graph $\xrightarrow{\text{reduction rules}} X = \sum \text{unique } a_i W_i$ \in basis web

e.g.  is not a basis web.

$$\text{Diagram} = \text{Diagram} + \frac{1}{[2]} \text{Diagram}$$

\nwarrow basis webs'

 is a basis web.

$$\text{Diagram} = \text{Diagram} + \frac{1}{[2]} \text{Diagram}$$

≈ 0

Definition The \mathbb{Z}_v -form of $\mathcal{L}_{\text{spa}, \Sigma}$ is defined by

$$\mathcal{L}_{\text{spa}, \Sigma}^{\mathbb{Z}_v} := \mathbb{Z}_v \text{BWeb}_{\text{spa}, \Sigma} \quad (\mathbb{Z}_v := \mathbb{Z}[v^{\pm \frac{1}{2}}])$$

Theorem (IT) $\mathcal{L}_{\text{spa}, \Sigma}^{\mathbb{Z}_v}$ is a \mathbb{Z}_v -algebra.

proof Show $\forall G, \forall G_1 \in \text{BWeb}_{\text{spa}, \Sigma}$, $G, G_1 \in \mathbb{Z}_v \text{BWeb}_{\text{spa}, \Sigma}$

$$\begin{aligned}
 \text{e.g. } & \text{Diagram} = v \text{Diagram} + v^{-1} \text{Diagram} \\
 & = v \text{Diagram} + v^{-1} \text{Diagram} \\
 & = v \text{Diagram} + \frac{v}{[2]} \text{Diagram} + v^{-1} \text{Diagram} + \frac{v^{-1}}{[2]} \text{Diagram} = v \text{Diagram} + v^{-1} \text{Diagram} + \text{Diagram}
 \end{aligned}$$

§ 3 Construction $\mathcal{A}_{\text{sp}_4, \Sigma}^{\mathfrak{s}}$ in $\text{Frac } \mathcal{S}_{\text{sp}_4, \Sigma}^{\mathfrak{s} = v}$

STEP 1 $T = \text{triangle}$ 

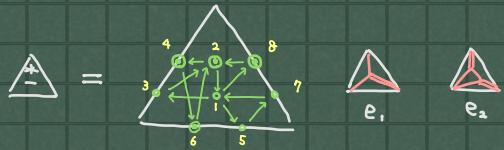
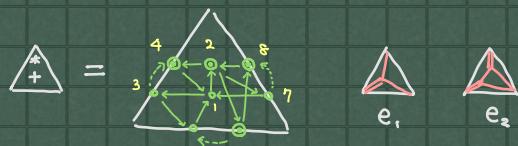
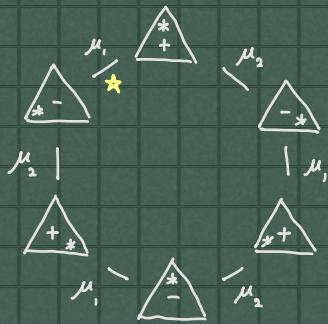
Lemma $\mathcal{S}_{\text{sp}_4, T}^{\mathfrak{s}}$ is generated by



boundary web
and  ...

Theorem $\mathcal{S}_{\text{sp}_4, T}^{\mathfrak{s}}$ = $\mathcal{A}_{\text{sp}_4, T}^{\mathfrak{s}}$ $\leftarrow \{ \text{clusters} \} = \{ \text{dec. ideal triangulations} \}$

weighted quiver



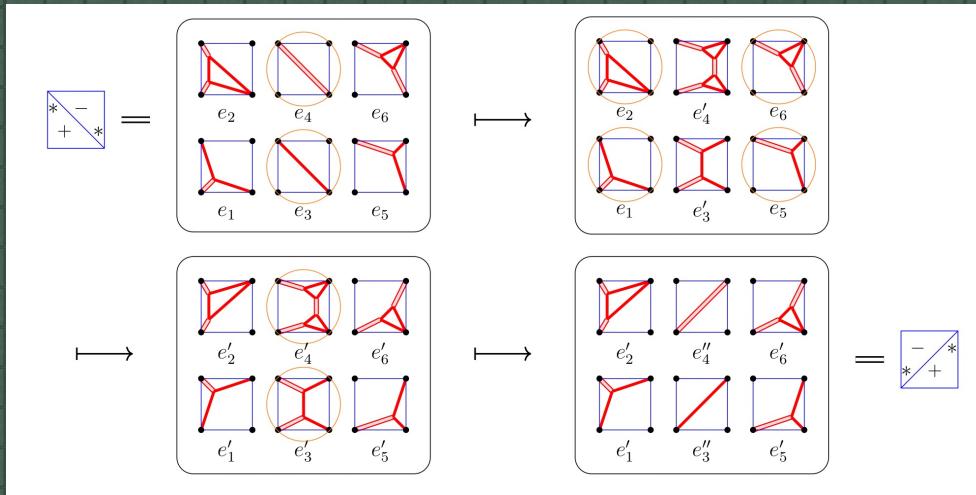
- Let us see $\star \mu_1 : e_i e'_i = g^{-\frac{1}{2}} [e_2 e_3] + g^{\frac{1}{2}} [e_4 e_5 e_7]$ $e_i = \triangle$ $e'_i = \triangle$

$$e_i e'_i = \triangle = g^{-\frac{1}{2}} \triangle$$

$$= g^{-\frac{1}{2}} \left(g \triangle + \frac{g^{-1}}{[2]} \triangle = 0 + \triangle \right)$$

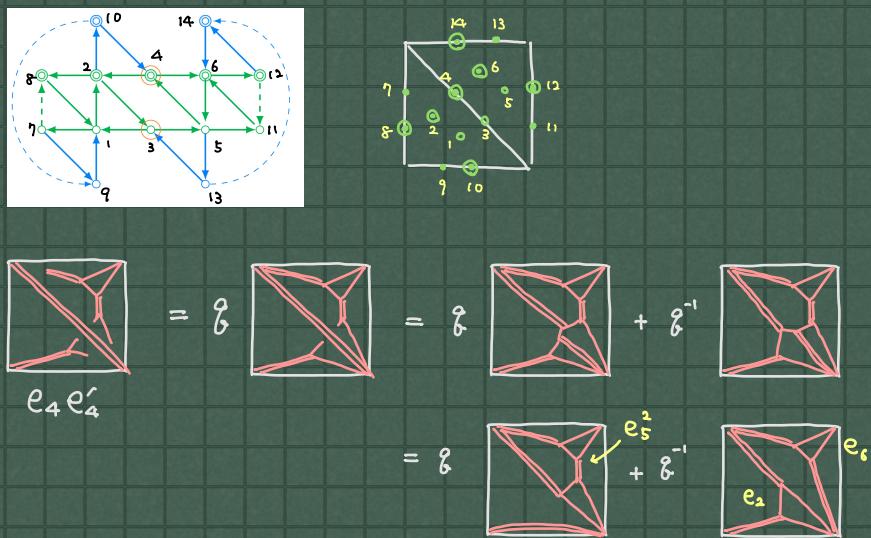
$$= g^{\frac{1}{2}} e_4 e_7 + g^{-\frac{1}{2}} e_3 e_2$$

[STEP 2] Check flips between decorated ideal triangulations.



① We can confirm that the mutation sequence is realized by skein relations

e.g.



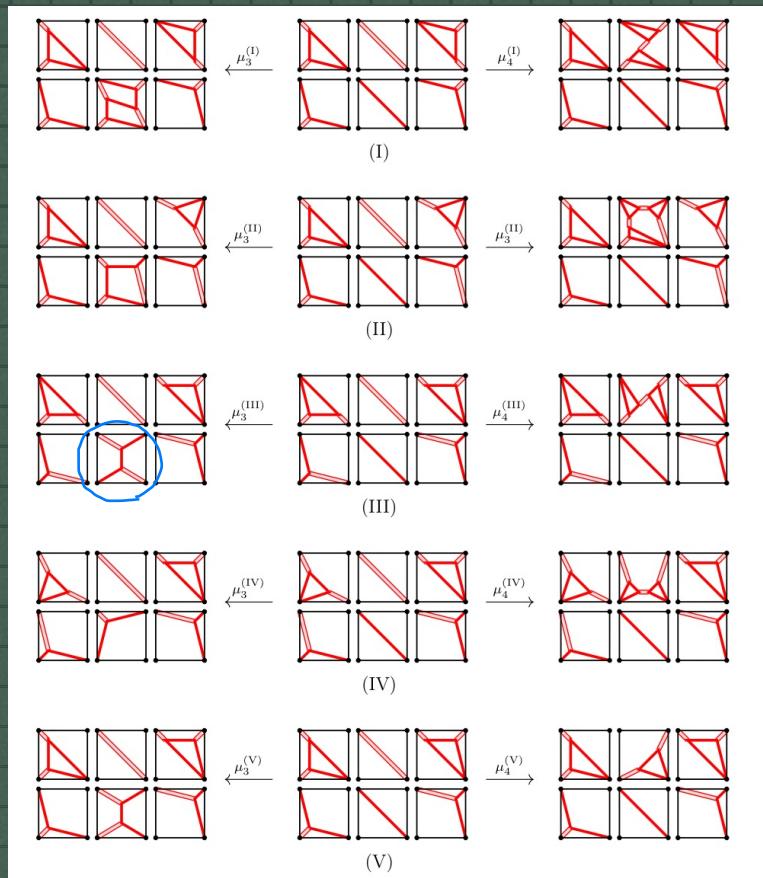
Definition

$\mathcal{A}_{\text{sp}_4, \Sigma}^g = \mathcal{A}_{S_g(\text{sp}_4, \Sigma)}$ is the quantum cluster algebra associated with the canonical mutation class $S_g(\text{sp}_4, \Sigma)$ containing $\{Q^\Delta\}$

all decorated ideal triangulation Δ .

Remark Δ, Δ' : decorated ideal triangulation $\Delta \xleftarrow[\text{mutation sequence}]{} \Delta'$

② Other Sp_4 -webs in $A_{\text{Sp}_4, \Sigma}$



$$(I) = \begin{array}{|c|c|} \hline * & + \\ + & * \\ \hline \end{array}$$

$$(II) = \begin{array}{|c|c|} \hline * & + \\ * & + \\ + & + \\ \hline \end{array}$$

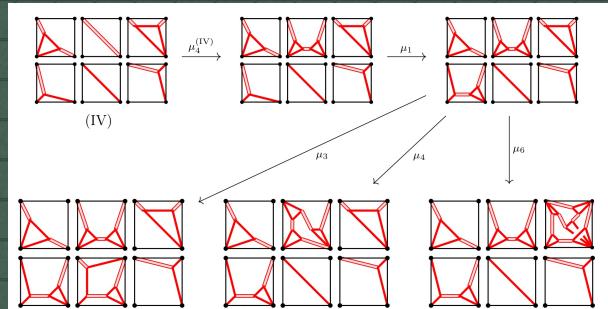
$$(III) = \begin{array}{|c|c|} \hline * & * \\ + & + \\ \hline \end{array}$$

$$(IV) = \begin{array}{|c|c|} \hline * & * \\ + & + \\ \hline \end{array}$$

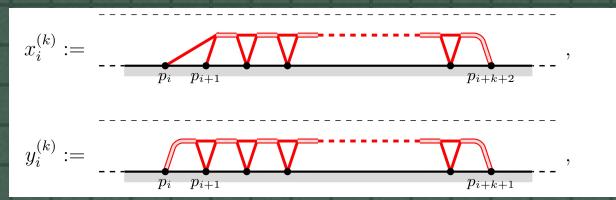
$$(V) = \begin{array}{|c|c|} \hline * & * \\ + & + \\ + & * \\ \hline \end{array}$$

③ All matrix elements of a simple Wilson line appear in these sequences. [c.f. Ishibashi - Oya - Shen '22]

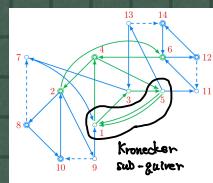
④ More examples



⑤ An infinite sequence



$$\chi_0^{(k+4)} \chi_0^{(k)} = v^\circ y_{k+2}^{(s)} \otimes \square + v^\wedge (x_2^{(k+2)})^2$$



$$\S 4. \quad \mathcal{L}_{\text{spa}, \Sigma}^{\mathbb{Z}_2} [\delta^{-}] \subset \mathcal{A}_{\text{spa}, \Sigma}^{\mathbb{Z}_2}$$

① Consequence of " $\S 3. \quad \mathcal{A}_{\text{spa}, \Sigma}^{\mathbb{Z}_2} \subset \text{Frac } \mathcal{L}_{\text{spa}, \Sigma}$ "

$$(1) \quad \mathcal{A}_{\text{spa}, \Sigma}^{\mathbb{Z}_2} \subset \text{Frac } \mathcal{L}_{\text{spa}, \Sigma}$$

$$(2) \quad \text{SimpWil}_{\text{spa}, \Sigma}^{\omega} := \left\{ \begin{array}{l} \text{spa-graph s.t.} \\ \text{"simple Wilson lines"} \\ \text{colored by } \omega, \end{array} \right. \left. \begin{array}{l} \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} \subset \Sigma, \quad \text{---} = \text{---}, \text{---}, \text{---}, \text{---}$$

■ Show $\mathcal{L}_{\text{spa}, \Sigma}^{\mathbb{Z}_2} [\delta^{-}]$ is generated by $\text{SimpWil}_{\text{spa}, \Sigma}^{\omega}$
 (and $\mathcal{L}_{\text{spa}, \Sigma}$ is an Ore domain) $\rightarrow \mathcal{L}_{\text{spa}, \Sigma}^{\mathbb{Z}_2} [\delta^{-}] \subset \mathcal{A}_{\text{spa}, \Sigma}^{\mathbb{Z}_2}$

Theorem (IT.) $\mathcal{L}_{\text{spa}, \Sigma} [\delta^{-}]$ is an Ore domain.

Sketch of proof $\mathcal{L}_{\text{spa}, \Sigma} [\delta^{-}]$

$$\downarrow \cong \quad \mathcal{L}_{\text{spa}, \Sigma} \xrightarrow{\text{stated, red}} \bigotimes_{T \in \Delta} \mathcal{L}_{\text{spa}, T}^{\text{stated, red}}$$

[IT, in preparation]

$\downarrow \cong$: the state-clasp (c.f. sl₂: Lê-Yu)

correspondence

\hookrightarrow : the splitting homomorphism

(c.f. sl₂: T.T.Q. Lê, sl₃: Higgins, sl_n: Lê-Sikora)

$$\bigotimes_{T \in \Delta} \mathcal{L}_{\text{spa}, T} [\delta^{-}] = \bigotimes_{T \in \Delta} \mathcal{A}_{\text{spa}, T}^{\mathbb{Z}_2} (\mathcal{R})$$

Corollary - $\mathcal{L}_{\text{spa}, \Sigma}^{\mathbb{Z}_2} [\delta^{-}] \subset \text{Frac } \mathcal{L}_{\text{spa}, \Sigma}$

⑩ The Cutting trick & the sticking trick.

Lemma (the cutting trick)

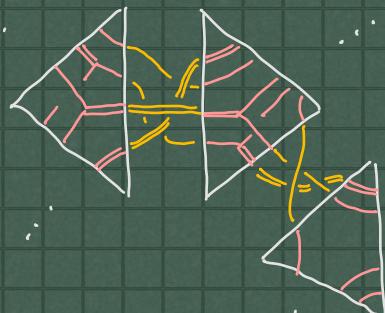
$$\begin{array}{c} \text{Diagram} \\ = v^2 \text{ Diagram} + v \text{ Diagram} + v^{-1} \text{ Diagram} + v^{-2} \text{ Diagram} \end{array} \quad (3.1)$$

$$\begin{array}{c} \text{Diagram} \\ = v^4 \text{ Diagram} + v^2 \text{ Diagram} \\ + [2] \text{ Diagram} + v^{-2} \text{ Diagram} + v^{-4} \text{ Diagram} \end{array} \quad (3.2)$$

Remark • The coefficients are positive

$$\bullet \mathcal{S}_{\mathrm{sp}_4, \Sigma}^{\mathbb{Z}_8} \subset \mathcal{U}_{\mathrm{sp}_4, \Sigma}$$

→ Laurent positivity for
“elevation preserving webs”.



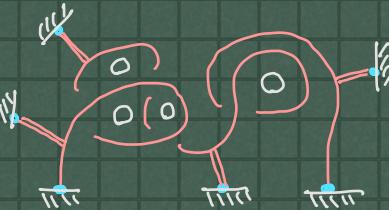
$$\in \mathcal{A}_{\mathrm{sp}_4, \Sigma}^{\mathbb{Z}_8} \otimes R \\ \mathbb{Z}_8[[\frac{1}{[2]}]]$$

Lemma (the sticking trick)

$$\begin{array}{c} \text{Diagram} \\ = v \text{ Diagram} - v^2 \text{ Diagram} + v^3 \text{ Diagram} - v^4 \text{ Diagram}, \\ \text{Diagram} \\ = v^2 \text{ Diagram} - v^4 \text{ Diagram} \\ + v^4[2] \text{ Diagram} - v^4 \text{ Diagram} + v^7 \text{ Diagram}. \end{array}$$

Proposition $\mathcal{S}_{\text{sp}_4, \Sigma}^{\mathbb{Z}_2}$ is generated by

- "descending loops & arcs with/without legs" of type 1
- simple loops/arcs of type 2



proof. Use a filtration by the "number" of crossings and crossroads

Theorem $\mathcal{S}_{\text{sp}_4, \Sigma}^{\mathbb{Z}_2} [\delta]$ localization by boundary webs is generated by $\text{SimpWil}_{\text{sp}_4, \Sigma}^{\omega}$ for Σ with $\#\{\text{marked points}\} \geq 2$.

proof

① Arcs of type 2

$$\text{Diagram of an arc with two endpoints and two cusps.} = \text{Diagram of an arc with three cusps and a crossing.} - v^{-1} \text{Diagram of an arc with four cusps and a crossing.}$$

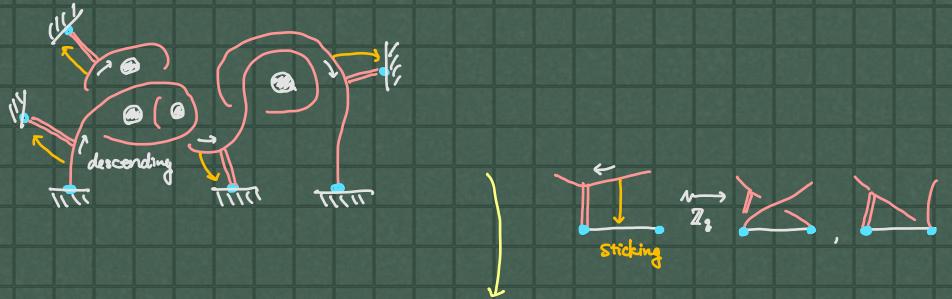
① Loops of type 2

$$\text{Diagram of a loop with one cusp.} \xrightarrow{\text{Localization}} \text{Diagram of a loop with two cusps.} , \text{Diagram of a loop with three cusps labeled } [2]^{\frac{1}{2}}, \text{Diagram of a loop with four cusps.}$$

$$\begin{aligned} \text{Diagram of a loop with two cusps.} &= \text{Diagram of a loop with three cusps.} - v^{-1} \text{Diagram of a loop with four cusps.} \\ \text{Diagram of a loop with three cusps.} &= \text{Diagram of a loop with four cusps.} - v^{-1} \text{Diagram of a loop with five cusps.} \end{aligned}$$

$$[2] \quad \text{Diagram} = [2] \quad \text{Diagram} - v^2 \quad \text{Diagram} - v^{-1} \quad \text{Diagram}$$

④ descending curves with / without legs



a \mathbb{Z}_q -polynomial in

$$\left\{ \text{Diagram} \mid \text{descending arc of with } n \leq 2 \text{ legs} \right\}$$

a \mathbb{Z}_q -polynomial in

$$\left\{ \text{Diagram} \mid \text{Diagram} = \text{Diagram}_1 + \text{Diagram}_2 + \dots + \text{Diagram}_n \right\}$$

a \mathbb{Z}_q -polynomial in

$$\text{SimpWil}_{\mathcal{M}_4, \Sigma}^{\infty}$$

use $\#\{\text{marked points}\} \geq 2$

Corollary $\mathcal{I} = \mathcal{A} = \mathcal{U} = \emptyset$ at $q=1$

§5 Characterization of cluster variables

$$\underline{\text{Conjecture}} \quad E\text{Web}_{mp_4, \Sigma} \setminus (E\text{Web}_{mp_4, \Sigma})^{\text{DT}} \xleftarrow{\text{invariant under Donaldson-Thomas transformation}} \text{Tree}_{mp_4, \Sigma} = CV_{mp_4, \Sigma}$$

Definition $G \in BWeb_{spa, \Sigma}$ is an elementary web if

- G is indecomposable and
 - $\mathcal{A}S \subset \{\text{rings of } G\}$, $G|S = 0$ in $\mathcal{S}_{\text{sys}, \mathbb{Z}_2}$

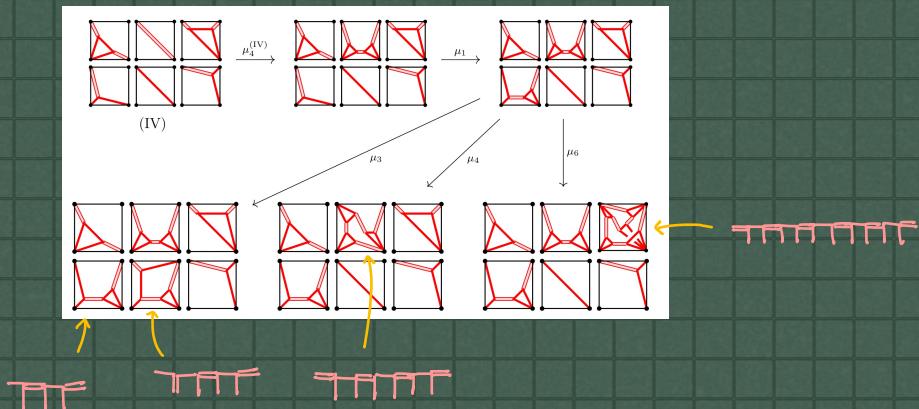
$$\text{EWeb}_{\text{mpa}, \Sigma} := \{ \text{elementary webs on } \Sigma \}$$

$$\xrightarrow{\text{e.g.}} \begin{array}{c} \text{Diagram of a surface with boundary components} \\ \sim \rightarrow \end{array} = 0$$

Definition

$G \in EWeb_{apq,\Sigma}$ is tree-type : $\Leftrightarrow \widetilde{G} : \text{tree}$ s.t. $G = g^\otimes \widetilde{G}$

e.g.



$$x_i^{(k)} := \text{---} \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad \text{---} \quad \text{---} \quad ,$$

$$y_i^{(k)} := \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \dots \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \quad ,$$

Thank you