

色付きジョーンズ多項式の tail と q -級数

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Friday Tea Time Zoom Seminar, (November 13, 2020)

- ① tail に関するこれまでの主な結果
- ② 結び目の量子不変量と線形スケイン理論
- ③ 線形スケイン理論による Braiding の分解公式
- ④ 結び目の tail から q -級数の恒等式を得る処方箋

色付きジョーンズ多項式

▶ 色付きジョーンズ多項式とは

| Lie 代数 $\mathfrak{sl}(2, \mathbb{C})$ の $(n+1)$ -次元の既約表現 V_{n+1} に対して定義される結び目, 絡み目の"多項式"に値を取る不变量の族 $\{J_K^{\mathfrak{sl}_2}(n)\}_n$. ($J_{K,n}^{\mathfrak{sl}_2}(q)$ と書いたりもする.)

■ Remark

- 色付きジョーンズ多項式 $\{J_K^{\mathfrak{sl}_2}(n)\}_n$ は $q^{\frac{1}{2}}\mathbb{Z}[q^{\pm 1}]$ に値を取る.
- V_2 から得られる色付きジョーンズ多項式 $J_K^{\mathfrak{sl}_2}(1)$ がジョーンズ多項式.

色付きジョーンズ多項式 $J_K^{\mathfrak{sl}_2}(n)$ に $\pm q^{\frac{1}{2}}$ をかけて q -多項式に正規化した不变量

$$\hat{J}_K^{\mathfrak{sl}_2}(n) = a_0 + a_1 q + a_2 q^2 + \cdots, \quad (a_0 > 0)$$

を考える事が出来る.

▶ 色付きジョーンズ多項式の \mathfrak{sl}_2 -tail とは

| 極限"lim _{$n \rightarrow \infty$} $\hat{J}_K^{\mathfrak{sl}_2}(n)$ " として得られる形式的幕級数 $T_K^{\mathfrak{sl}_2}(q) \in \mathbb{Z}[[q]]$.

色付きジョーンズ多項式の係数の安定性

■ **Theorem ([Armond, 2013])** L を adequate link とする。このとき、形式的 q -級数 $\mathcal{T}_L^{\mathfrak{sl}_2}(q) \in \mathbb{Z}[[q]]$ が存在して、 $\mathcal{T}_L^{\mathfrak{sl}_2}(q) - \hat{J}_{L,n}^{\mathfrak{sl}_2}(q) \in q^{n+1}\mathbb{Z}[[q]]$ 。

■ **Theorem ([Garoufalidis-Lê, 2015])** L を交代絡み目とする。このとき、任意の非負整数 k に対して $\{\hat{J}_{K,n}^{\mathfrak{sl}_2}(q)\}_n$ は k -stable. (0 -stable \iff tail の存在)

■ **Example 8** の字結び目の色付きジョーンズ多項式の係数の表

$n = 2$	1	-1	-1	0	2	0	-2	0	3	0	-3	0	3	0	-3	0
$n = 3$	1	-1	-1	0	0	3	-1	-1	-1	-1	5	-1	-2	-2	-1	6
$n = 4$	1	-1	-1	0	0	1	2	0	-2	-1	-1	1	3	1	-2	-3
$n = \infty$	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0	-1
引き算 & 左に nシフト																
$n = 2$	0	0	2	-1	-2	-1	3	0	-3	0	4	0	-2	1		
$n = 3$	0	0	2	-1	-2	-1	-1	5	-1	-1	-2	-1	7			
$n = 4$	0	0	2	-1	-2	-1	-1	1	4	1	-2	-2				
$n = \infty$	0	0	2	-1	-2	-1	-1	1

▶ 色付き \mathfrak{g} ジョーンズ多項式 (結び目の量子 $(\mathfrak{g}, V_\lambda)$ 不変量)

単純 Lie 代数 \mathfrak{g} の最高ウェイト λ を持つ既約表現 V_λ に対して定義される結び目 (絡み目) の不变量の族 $\{J_K^{\mathfrak{g}}(\lambda)\}_{\lambda \in \Lambda}$.

■ **Example** $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ とすると, $(k, l) \in \Lambda = \mathbb{N} \times \mathbb{N}$ に対して $J_K^{\mathfrak{sl}_3}(k, l)$ が定義される. 特に, $\{J_K^{\mathfrak{sl}_3}(k, 0)\}$ を一行色付き \mathfrak{sl}_3 ジョーンズ多項式と呼ぶ.

■ **Theorem (Integrality theorem [Lê, 2000])**

$$J_K^{\mathfrak{g}}(\lambda) \in q^{\frac{1}{2}} \mathbb{Z}[q^{\pm 1}]$$

■ **Remark**

- $\mathfrak{g} = \mathfrak{sl}_2$ 以外での”非自明”な結び目に対する色付きジョーンズ多項式の明示式はほとんど知られていない.
- q -多項式への正規化 $\{\hat{J}_K^{\mathfrak{g}}(\lambda)\}_{\lambda \in \Lambda}$ を考えることは出来るが, 一般にその”極限”である \mathfrak{g} -tail が存在するかは知られていない.

■ **Theorem ([Garoufalidis-Vuong, 2017])** K をトーラス結び目, \mathfrak{g} をランク 2 の単純 Lie 代数とする. このとき, 任意の基本表現 λ に対して $\{\hat{J}_K^{\mathfrak{g}}(n\lambda)\}_{n \in \mathbb{N}}$ は k -stable. ($k \in \mathbb{N}$)

tail に関して得られた結果の紹介

- **Theorem ([Y. 2018])** Anti-parallel な $(2, 2m)$ -torus link $T_{\leftarrow}(2, 2m)$ に対して, $J_K^{\mathfrak{sl}_3}(k, 0)$ に関する二通りの明示式を与え, その \mathfrak{sl}_3 -tail から以下の恒等式を得た.

$$\sum_{i=0}^{\infty} q^{-2i} q^{m(i^2+2i)} \frac{(1-q^{i+1})^3(1+q^{i+1})}{(1-q)^2(1-q^2)} \\ = \frac{(q)_{\infty}}{(1-q)(1-q^2)} \sum_{k_1 \geq k_2 \geq \dots \geq k_m \geq 0} \frac{q^{-2k_m} q^{\sum_{i=1}^m k_i^2 + 2k_i}}{(q)_{k_1-k_2} (q)_{k_2-k_3} \cdots (q)_{k_{m-1}-k_m} (q)_{k_m}^2}$$

- **Remark** 左辺は [Bringmann-Kaszian-Milas, 2019] の \mathfrak{sl}_3 false theta series の "diagonal summand" と一致する.

- **Theorem ([Y. 2020])** Parallel な $(2, 2m)$ -torus link $T_{\rightarrow}(2, 2m)$ のに対して, \mathfrak{sl}_2 -tail と \mathfrak{sl}_3 -tail は有理関数の差を除いて一致する.

- **Theorem ([Y. 2020])** "minus-adequate" な向き付き絡み目 K に対して, 一行色付きジョーンズ多項式の極限 $\mathcal{T}_K^{\mathfrak{sl}_3}(q)$ が存在する.

結び目図式による結び目の定義

▶ 絡み目とは

| \mathbb{R}^3 に埋め込まれた $\sqcup^l S^1$ の全域 isotopy 類.

▶ 絡み目図式とは

| \mathbb{R}^3 に埋め込まれた $\sqcup^l S^1$ の \mathbb{R}^2 への generic な射影に交点における上下情報を付加した図.

■ Remark l を結び目の成分数といい、特に $l=1$ のときに結び目と呼ぶ。 S^1 をアニュラス $S^1 \times [0, 1]$ に置き換えたものを枠付き結び目と呼ぶ。

▶ 絡み目図式と結び目の対応

| 絡み目図式 D_1 と D_2 が Reidemeister move の列と \mathbb{R}^2 の isotopy で移り合うなら、 D_1 を結び目図式にもつ結び目と D_2 を結び目図式にもつ結び目は一致する。



$\left\{ \begin{array}{l} \text{framed : (R1)を (R0)} \\ \text{に replace} \end{array} \right.$



枠付き結び目の量子不变量

▶ 結び目の量子不变量 (色付きジョーンズ多項式) の定義

- ① 表現圏を用いた定義,
- ② 結び目図式を用いた定義,
- ③ その他いろいろ

▶ 表現圏を用いた量子不变量の構成 (概要)

- strict なリボン圏 \mathcal{C} を固定すると, \mathcal{C} の対象で色付けしたタングル図式の圏から \mathcal{C} への函手 $F_{\mathcal{C}}$ が一意に存在する.
- $c \in \mathcal{C}$ で色付けられた絡み目 L_c は単位対象から単位対象への射なので, 不变量 $F_{\mathcal{C}}(L_c) \in \text{End}(1)$ が得られる.
- 特に, \mathcal{C} を量子群 $U_q(\mathfrak{g})$ の有限次元表現の圏とすると $F_{\mathcal{C}}(L_c)$ として色付き \mathfrak{g} ジョーンズ多項式が得られる.

▶ 線形スケイン理論を用いた量子不变量の構成 (概要)

- 結び目 L の結び目図式 D_L を描き, "Jones-Wenzl 射影子" を乗せる.
- "スケイン関係式" によって交点を持たない図式の線形和 $D_L = \sum_i D_i$ に分解する.
- 各 D_i の"elliptic face" をスケイン関係式で消していくと空の図式 \emptyset となる.
- 以上の操作で $D_L = f_L(q)\emptyset$ となり, 得られた多項式 $f_L(q)$ が色付き \mathfrak{g} ジョーンズ多項式が得られる.

表現圏を用いた量子不变量の構成

a (k, l) -tangle diagram

\Leftrightarrow a generic immersion of arcs & loops
into $\mathbb{R} \times [0, 1]$

s.t. • intersection point =

$$\bullet \partial\{\text{arcs}\} = \{(1,0), (2,0), \dots, (k,0)\}$$

$$\cup \{(1,1), (2,1), \dots, (l,1)\}$$



$(5,3)$ -tangle diagram

the set of framed (k, l) -tangles

$$:= \left\{ (k, l)\text{-tangle diagram} \right\} / \begin{cases} (\Omega_0), (\Omega_2), \\ (\Omega_3), \text{isotopy} \end{cases}$$

Fix a strict ribbon category ℓ

$$\left[\begin{array}{ll} \text{braiding} & c_{V,W}: V \otimes W \rightarrow W \otimes V \\ \text{duality} & \left\{ \begin{array}{l} *: V \rightarrow V^* \\ b_V: 1 \rightarrow V \otimes V^* \\ d_V: V^* \otimes V \rightarrow 1 \end{array} \right. \\ \text{twist} & \theta_V: V \rightarrow V, \text{ isom.} \end{array} \right]$$

a category of ℓ -colored framed tangles, \mathcal{T}_ℓ

$\Leftrightarrow \text{Obj } \mathcal{T}_\ell : \text{a finite sequence of } \text{Obj } \ell \times \{+, -\}$
 $\circlearrowleft \emptyset : \text{the empty sequence.}$

$$\text{Obj } \mathcal{T}_\ell \ni \eta_1 = ((V_1, \varepsilon_1), (V_2, \varepsilon_2), \dots, (V_k, \varepsilon_k))$$

$$\eta_2 = ((W_1, \delta_1), (W_2, \delta_2), \dots, (W_\ell, \delta_\ell))$$

$\text{Mor}(\eta_1, \eta_2)$: an oriented framed (k, l) -tangle

s.t. • each component is colored by $\text{Obj } \ell$

$$\bullet \quad \begin{array}{c} \xrightarrow{\varepsilon_i, +} \\ \downarrow \\ V_i \end{array} \quad \text{if } \varepsilon_i = +, \quad \begin{array}{c} \xrightarrow{\varepsilon_i, -} \\ \uparrow \\ V_i \end{array} \quad \text{if } \varepsilon_i = -$$

$$\begin{array}{c} \xrightarrow{W_i, +} \\ \downarrow \\ \dashv \end{array} \quad \text{if } \delta_i = +, \quad \begin{array}{c} \xrightarrow{W_i, -} \\ \downarrow \\ \dashv \end{array} \quad \text{if } \delta_i = -$$

A category of \mathcal{C} -colored framed tangles, $\mathcal{T}_{\mathcal{C}}$

- $\text{Obj } \mathcal{T}_{\mathcal{C}}$: a finite sequence of $\text{Obj } \mathcal{C} \times \{+, -\}$
 \emptyset : the empty sequence.

$$\text{Obj } \mathcal{T}_{\mathcal{C}} \ni \eta_1 = ((V_1, \varepsilon_1), (V_2, \varepsilon_2), \dots, (V_k, \varepsilon_k))$$

$$\eta_2 = ((W_1, \delta_1), (W_2, \delta_2), \dots, (W_\ell, \delta_\ell))$$

- $\text{Mor}(\eta_1, \eta_2)$: an oriented framed (k, ℓ) -tangle

s.t. • each component is colored by $\text{Obj } \mathcal{C}$

$$\begin{array}{c} \xrightarrow{\varepsilon_i=+} \\ \downarrow V_i \end{array} \quad \text{if } \varepsilon_i = +, \quad \begin{array}{c} \xleftarrow{\varepsilon_i=-} \\ \downarrow V_i \end{array} \quad \text{if } \varepsilon_i = -$$

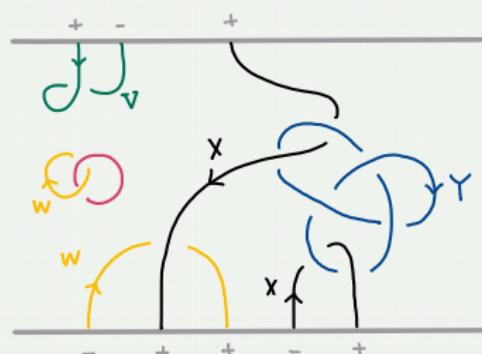
$$\begin{array}{c} \xrightarrow{\delta_j=+} \\ \downarrow W_j \end{array} \quad \text{if } \delta_j = +, \quad \begin{array}{c} \xleftarrow{\delta_j=-} \\ \downarrow W_j \end{array} \quad \text{if } \delta_j = -$$

• Composition

$$\boxed{T_1} \circ \boxed{T_2} := \boxed{\frac{T_1}{T_2}}$$

• tensor product

$$\boxed{T_1} \otimes \boxed{T_2} := \boxed{T_1} \boxed{T_2}$$



\mathcal{C} -colored $(5,3)$ -tangle diagram

Theorem[Reshetikhin-Turaev, 1990]

$\exists!$ $F_e: \mathcal{T}_e \rightarrow \mathcal{C}$, a \otimes -preserving functor
s.t. $F((v,+)) = V, F((v,-)) = V^*$

$$\begin{array}{c} w \\ \swarrow \searrow \end{array} v = c_{v,w} \quad \begin{array}{c} w \\ \nearrow \searrow \end{array} v = c_{w,v}^{-1} \quad \begin{array}{c} w \\ \nearrow \swarrow \end{array} v = c_{w,v^*}^{-1} \quad \begin{array}{c} w \\ \nearrow \nearrow \end{array} v = c_{v^*,w}$$

$$\begin{array}{c} w \\ \swarrow \nearrow \end{array} v = c_{w^*,v}^{-1} \quad \begin{array}{c} w \\ \swarrow \nearrow \end{array} v^* = c_{v,w^*} \quad \begin{array}{c} w \\ \nearrow \nearrow \end{array} v = c_{v^*,w^*} \quad \begin{array}{c} w \\ \nearrow \swarrow \end{array} v^* = c_{w^*,v^*}$$

$$v \downarrow = \text{id}_V \quad v \uparrow = \text{id}_{V^*} \quad v \downarrow \rho = \theta_v \quad v \uparrow \theta = \theta_v^{-1}$$

$$\begin{array}{c} v \\ \cap \end{array} = d_V \quad \begin{array}{c} v \\ \cup \end{array} = b_V$$

$\rightsquigarrow L$: an \mathcal{C} -colored framed link.

$$\Rightarrow F_e(L) \in \text{End}(1)$$

e.g. (the colored \mathfrak{g} Jones polynomial)

\mathfrak{g} : a simple Lie algebra

$\text{Rep}_f U_q(\mathfrak{g})$: the category of finite dimensional representations of the quantum group $U_q(\mathfrak{g})$. ($q = "q^2"$: a formal variable)

$L = L_1 \sqcup L_2 \sqcup \dots \sqcup L_e$: a framed link

$V_i \in \text{Rep}_f U_q(\mathfrak{g})$ is a coloring of L_i

the $(\mathfrak{g}, (V_1, \dots, V_e))$ -colored Jones polynomial

$J_L^{\mathfrak{g}}(V_1, V_2, \dots, V_e) \in \mathbb{C}(q^{\frac{1}{20}})$ is defined by

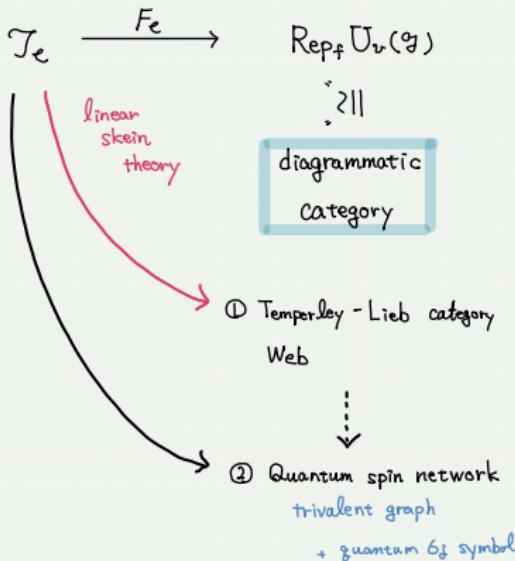
$$F_e(L)(1) = J_L^{\mathfrak{g}}(V_1, V_2, \dots, V_e) \cdot 1$$

($\because D$ is determined by \mathfrak{g})

線形スケイン理論を用いた量子不变量の構成

$$\mathcal{L} = \text{Rep}_f U_q(\mathfrak{g})$$

$\mathcal{T}_{\mathcal{L}}$ = \mathcal{L} -colored framed tangles



"Linear skein theory"

= a functor from $\mathcal{T}_{\mathcal{L}}$ to
a diagrammatic representation

of $\text{Fun} \text{Rep}_f U_q(\mathfrak{g})$

or $\text{Kar}(\text{Fun} \text{Rep}_f U_q(\mathfrak{g}))$

the Kauffman bracket ($\mathfrak{g} = sl_2$)

$$\boxed{} = \boxed{1} = \boxed{\nabla_a \otimes \nabla_a^*}$$

the 2-dim.
irreducible rep.

$$\boxed{n} = \underbrace{\boxed{\dots}}_n = \boxed{\nabla_1 \otimes \dots \otimes \nabla_n}$$

the Kauffman bracket ($\mathfrak{g} = \text{sl}_2$)

skein relation $\left\{ \begin{array}{l} \times = g^{\frac{1}{2}} (\quad + g^{-\frac{1}{2}} \quad) \\ \circ = -[2] \emptyset = (g^{\frac{1}{2}} + g^{-\frac{1}{2}}) \emptyset \end{array} \right.$

construction of the color $|_{V_{n+1}}$

= the Jones-Wenzl projector $\boxed{\vdash^n}$

i.e. $\boxed{\vdash^n} = \boxed{\frac{1 \cdots}{1 \cdots}} : V_2^{\otimes n} \xrightarrow{\sim} \text{Sym}^n V_2 \hookrightarrow V_2^{\otimes n}$

Diagrammatic definition

$$[n] = \frac{g^{\frac{n}{2}} - g^{-\frac{n}{2}}}{g^{\frac{1}{2}} - g^{-\frac{1}{2}}}$$

$V_{n+1} : \boxed{\vdash^n} = \boxed{\vdash^{n-1}} | + \frac{[n-1]}{[n]} \boxed{\begin{smallmatrix} & & \\ \nearrow & \searrow & \\ \vdash^{n-1} & & \vdash^{n-1} \\ \searrow & \nearrow & \\ & & \vdash^{n-1} \end{smallmatrix}}$

$\leadsto k = \text{?}$

then, $\boxed{\text{?}} = J_{k,n}^{\text{sl}_2}(\mathfrak{g}) \emptyset$

the A_2 bracket ($\mathfrak{g} = \text{sl}_3$) $\uparrow = \uparrow_{V_{(0,0)}} = \uparrow_{V_{(0,1)}}$

skein relation $\left\{ \begin{array}{l} \times = g^{\frac{1}{3}} \uparrow - g^{-\frac{1}{3}} \times \\ \times = g^{-\frac{1}{3}} \uparrow - g^{\frac{1}{3}} \times \\ \times = \circ +) (\\ \circ = [2] \uparrow, \quad \circ = \circ = [3] \emptyset \end{array} \right.$

$V_{(n,0)} : \boxed{\uparrow^n} = \boxed{\uparrow^{n-1}} \uparrow - \frac{[n-1]}{[n]} \boxed{\begin{smallmatrix} & & & \\ \nearrow & \searrow & & \\ \vdash^{n-1} & & \vdash^{n-1} & \\ \searrow & \nearrow & & \\ & & \vdash^{n-1} & \\ & & \uparrow & \\ & & 1 & \\ & & n-1 & \\ & & n-2 & \\ & & \vdash^{n-2} & \\ & & \uparrow & \\ & & 1 & \\ & & n-1 & \\ & & \vdash^{n-1} & \\ & & \uparrow & \\ & & 1 & \end{smallmatrix}}$

$V_{(m,n)} : \boxed{\begin{smallmatrix} m & n \\ \uparrow & \downarrow \end{smallmatrix}} = \sum_{k=0}^{\min(m,n)} (-1)^k \frac{[m][n]}{[k]} \boxed{\begin{smallmatrix} m & n \\ \uparrow & \downarrow \\ \vdash^{n-k} & \vdash^{m-k} \\ \uparrow & \downarrow \\ m & n \end{smallmatrix}}$

$\leadsto \boxed{\text{?}} = J_{k,(m,n)}^{\text{sl}_3}(\mathfrak{g}) \emptyset$

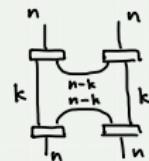
■ Example (時間があれば計算例)

ツイスト公式 ($\mathfrak{g} = \mathfrak{sl}_2$ の場合)

Theorem[Yamada, 1989]

$$\text{Diagram: Two strands labeled } n \text{ with caps at top and bottom. The left strand has a twist symbol. The right strand has a cap at top and a twist symbol at bottom. They are connected by a crossing.}$$

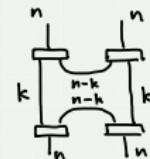
$$= \sum_{k=0}^n q^{\frac{1}{4}(-n^2+2k)} \frac{(\mathfrak{g})_n}{(\mathfrak{g})_k (\mathfrak{g})_{n-k}}$$



Theorem[Masbaum, 2003]

$$\text{Diagram: Two strands labeled } n \text{ with caps at top and bottom. The left strand has a twist symbol. The right strand has a cap at top and a twist symbol at bottom. They are connected by a crossing. The strands are labeled k and n-k. The crossing is a standard crossing (over-under).}$$

$$= \sum_{k=0}^n (-1)^{n-k} q^{\frac{1}{2}(-n^2-n+2k^2+k)} \frac{(\mathfrak{g})_n^2}{(\mathfrak{g})_k^2 (\mathfrak{g})_{n-k}}$$



Theorem[Y. 2017]

$$\begin{array}{c}
 \text{Diagram: } m \left\{ \begin{array}{c} \diagup \\ \times \\ \vdots \\ \diagdown \end{array} \right. \rightarrow \begin{array}{c} n \\ | \\ \square \text{ labeled } m \\ | \\ n \end{array} \quad \text{Diagram: } \begin{array}{c} n \\ | \\ \square \text{ labeled } m \\ | \\ n \end{array} \rightarrow \begin{array}{c} n \\ | \\ \square \text{ labeled } m \\ | \\ n \end{array}
 \end{array}
 = (-1)^{n-k_m} q^{\frac{n-k_m}{2}} \sum_{\substack{n \geq k_1 \geq k_2 \geq \dots \geq k_m \geq 0}} (-1)^{\sum_{i=1}^m k_i} q^{\frac{1}{2} \sum_{i=1}^m (k_i^2 + k_i)} \\
 \times \frac{(q)_n}{(q)_{n-k_1} (q)_{k_1-k_2} \cdots (q)_{k_{m-1}-k_m} (q)_{k_m}}$$

$$\begin{array}{c}
 \text{Diagram: } \begin{array}{c} n \\ | \\ \square \text{ labeled } 2m \\ | \\ n \end{array} \quad \text{Diagram: } \begin{array}{c} n \\ | \\ \square \text{ labeled } 2m \\ | \\ n \end{array}
 \end{array}
 = q^{-\frac{m}{2}(n^2+2n)} \sum_{\substack{n \geq k_1 \geq k_2 \geq \dots \geq k_m \geq 0}} (-1)^{n-k_m} q^{\frac{n-k_m}{2}} q^{\sum_{i=1}^m (k_i^2 + k_i)} \\
 \times \frac{(q)_n^2}{(q)_{n-k_1} (q)_{k_1-k_2} \cdots (q)_{k_{m-1}-k_m} (q)_{k_m}^2}$$

ツイスト公式 ($\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = (n, 0)$ の場合)

Theorem[Y. 2017] (anti-parallel case)

$$\begin{array}{c}
 \text{Diagram: } \begin{array}{c} n \\ \downarrow \\ \square \text{ (labeled 2m)} \\ \uparrow \\ n \end{array} \quad \begin{array}{c} n \\ \downarrow \\ \square \\ \uparrow \\ n \end{array} \\
 = q^{-\frac{2m}{3}(n^2+3n)} \sum_{n \geq k_1 \geq k_2 \geq \dots \geq k_m \geq 0} q^{n-k_m} q^{\sum_{i=1}^m (k_i^2 + 2k_i)} \\
 \times \frac{(q)_n^2}{(q)_{n-k_1} (q)_{k_1-k_2} \dots (q)_{k_{m-1}-k_m} (q)_{k_m}^2} \quad \begin{array}{c} n \\ \downarrow \\ \square \\ \uparrow \\ n \end{array} \quad \begin{array}{c} n \\ \downarrow \\ \square \\ \uparrow \\ n \end{array} \\
 \end{array}$$

Theorem[Y. 2020]

$$\begin{array}{c}
 \text{Diagram: } \begin{array}{c} n \\ \downarrow \\ \square \text{ (labeled 2m)} \\ \uparrow \\ n \end{array} \quad \begin{array}{c} n \\ \downarrow \\ \square \\ \uparrow \\ n \end{array} \\
 = q^{-\frac{m}{3}(n^2+3n)} \sum_{n \geq k_1 \geq k_2 \geq \dots \geq k_m \geq 0} q^{\frac{n-k_m}{2}} q^{\sum_{i=1}^m (k_i^2 + k_i)} \\
 \times \frac{(q)_n^2}{(q)_{n-k_1} (q)_{k_1-k_2} \dots (q)_{k_{m-1}-k_m} (q)_{k_m}^2} \quad \begin{array}{c} n \\ \downarrow \\ \square \\ \uparrow \\ n \end{array} \quad \begin{array}{c} n \\ \downarrow \\ \square \\ \uparrow \\ n \end{array} \\
 \end{array}$$

$$\left(\begin{array}{c} \xrightarrow{\alpha} \xrightarrow{\beta} \\ \nwarrow \nearrow \\ \square \end{array} \right) = \underbrace{\begin{array}{c} \xrightarrow{\alpha} \xrightarrow{\beta} \\ \vdots \\ \xrightarrow{\alpha} \xrightarrow{\beta} \end{array}}_{\mathcal{L}} \quad \left(\begin{array}{c} \xrightarrow{\alpha} \xrightarrow{\beta} \\ \vdots \\ \xrightarrow{\alpha} \xrightarrow{\beta} \end{array} \right)$$

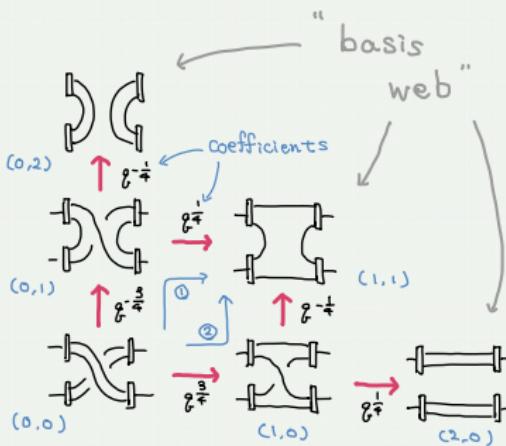
ツイスト公式の格子の経路を用いた証明方法

e.g.

$$= q^{\frac{3}{4}} \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + q^{-\frac{3}{4}} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}$$

$$= q^{\frac{3}{4}} \left(q^{\frac{1}{4}} \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + q^{-\frac{1}{4}} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right) + q^{-\frac{3}{4}} \left(q^{\frac{1}{4}} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} + q^{-\frac{1}{4}} \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right)$$

skein tree



coefficient of basis web on (1,1)

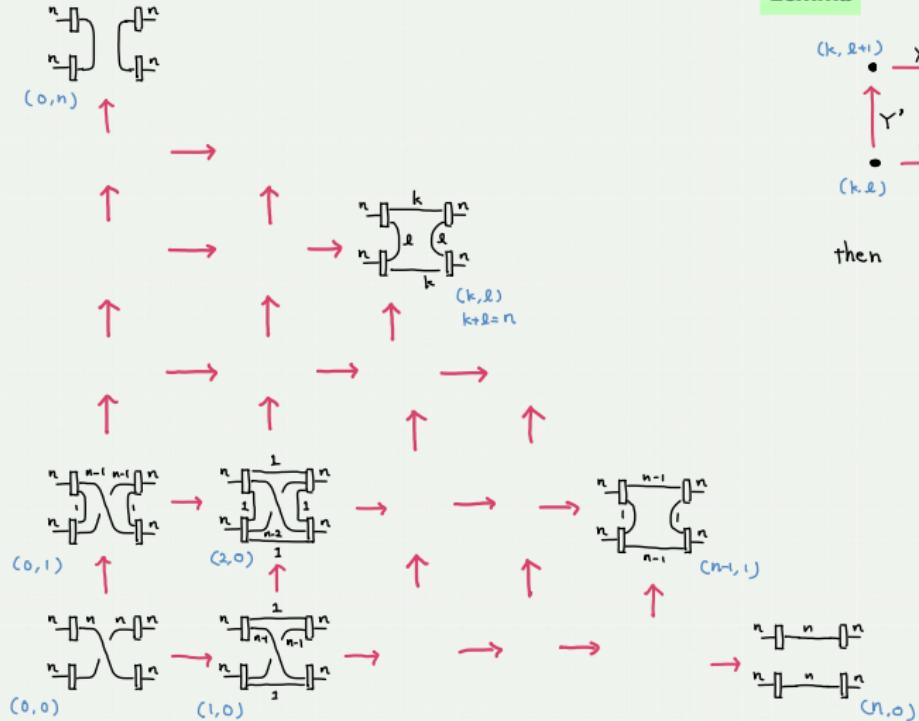
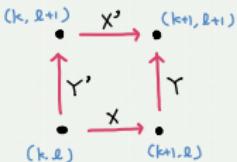
$$= \sum_{\gamma: \text{paths from } (0,0) \text{ to } (1,1)} \prod_{\delta} w_{\delta}$$

(0,0) \rightarrow (1,1)

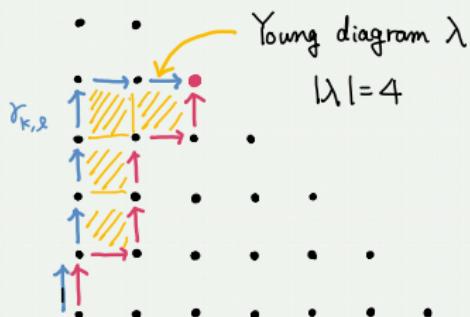
$$= \frac{q^{-\frac{3}{4}} q^{\frac{1}{4}}}{①} + \frac{q^{\frac{3}{4}} q^{-\frac{1}{4}}}{②}$$

$$\times q$$

(half twist formula)

**Lemma**

$$\text{then } XY = \varrho Y'X'$$



$$\text{coeff} \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right) = g^{\frac{|\lambda|}{4}} \left(\begin{array}{c} \nearrow \\ \uparrow \uparrow \uparrow \uparrow \end{array} \right)$$

④ coefficient of (k, l) ($k+l=n$)

$$= \sum_{\gamma: \text{path from } (0,0) \text{ to } (k,l)} \prod w$$

$w: \text{weight}$
 $\text{on } \gamma$

$$= \left(\prod_{w: \text{weight} \text{ on } \gamma_{k,l}} w \right) \left(\sum_{\lambda: \text{Young diagram}} g^{|\lambda|} \right)$$

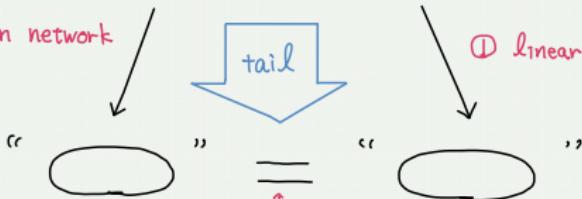
$\# \text{row} \leq k$
 $\# \text{column} \leq l$

$$= \left(\prod_{w: \text{weight} \text{ on } \gamma_{k,l}} w \right) \frac{(g)_n}{(g)_k (g)_{n-k}}$$

結び目の tail と q -級数の恒等式

$\{ V_\lambda \text{ colored link } L \}_\lambda$

② quantum spin network



① linear skein theory (twist formula)

because the tail is invariant of L

③ quantum spin network ($\mathfrak{g} = sl_2$)

$$\begin{array}{c} a \\ | \\ b \quad c \end{array} := \begin{array}{c} a \\ | \\ b \quad c \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ b \quad c \end{array}$$

$\frac{b+c-a}{2}$

$$\begin{array}{c} b \\ | \\ i \\ \diagup \quad \diagdown \\ c \quad d \end{array} = \sum_i \left\{ \begin{array}{ccc} a & b & i \\ c & d & i \end{array} \right\}_q \begin{array}{c} b \\ | \\ i \\ \diagup \quad \diagdown \\ a \quad d \\ \diagup \quad \diagdown \\ c \end{array}$$

quantum
 $6j$ -symbol

$$\begin{array}{c} c \\ | \\ b \\ \diagup \quad \diagdown \\ a \end{array} = \gamma_{a}^{bc} \begin{array}{c} c \\ | \\ b \\ \diagup \quad \diagdown \\ a \end{array}$$

(false) theta series に関する Andrews-Gordon 型の恒等式

$$\textcircled{1} \quad f(a, b) := \sum_{i=0}^{\infty} a^{\frac{i(i+1)}{2}} b^{\frac{i(i-1)}{2}} + \sum_{i=1}^{\infty} a^{\frac{i(i-1)}{2}} b^{\frac{i(i+1)}{2}} : \text{the theta series}$$

$$\textcircled{2} \quad \psi(a, b) := \sum_{i=0}^{\infty} a^{\frac{i(i+1)}{2}} b^{\frac{i(i-1)}{2}} - \sum_{i=1}^{\infty} a^{\frac{i(i-1)}{2}} b^{\frac{i(i+1)}{2}} : \text{the false theta series}$$

Theorem[Armond-Dasbach, 2011]

$$f(-g^{2m}, -g) / (1-g) = J_{T(2, 2m+1)}^{ab}(g) = \frac{(g)_\infty}{1-g} \sum_{k_1 \geq k_2 \geq \dots \geq k_{m-1} \geq 0} \frac{g^{\sum_{i=1}^{m-1} k_i^2 + k_i}}{(g)_{k_1-k_2} (g)_{k_2-k_3} \dots (g)_{k_{m-1}-k_m} (g)_{k_m}}$$

Theorem[Hajij, 2015]

$$\Psi(g^{2m-1}, g) / (1-g) = J_{T(2, 2m)}^{ab}(g) = \frac{(g)_\infty}{1-g} \sum_{k_1 \geq k_2 \geq \dots \geq k_m \geq 0} \frac{g^{-k_m} g^{\sum_{i=1}^m k_i^2 + k_i}}{(g)_{k_1+k_2} (g)_{k_2+k_3} \dots (g)_{k_{m-1}+k_m} (g)_{k_m}^2}$$

Theorem[Y. 2018]

$$\sum_{i=0}^{\infty} q^{-2i} q^{m(i^2+2i)} \frac{(1-q^{i+1})^3 (1+q^{i+1})}{(1-q)^2 (1-q^2)} = \mathcal{T}_{T_{\neq}(2,2m)}^{ab_3}(q)$$

$$= \frac{(q)_{\infty}}{(1-q)(1-q^2)} \sum_{k_1 \geq k_2 \geq \dots \geq k_m \geq 0} \frac{q^{-2k_m} q^{\sum_{i=1}^m k_i^2 + 2k_i}}{(q)_{k_1-k_2} (q)_{k_2-k_3} \dots (q)_{k_{m-1}-k_m} (q)_{k_m}^2}$$

Theorem[Y. 2020]

$$\mathcal{T}_{T_{\neq}(2,2m+1)}^{ab_3}(q) = \frac{f(-q^{2m}, -q)}{(1-q)^2 (1-q^2)}$$

$$\mathcal{T}_{T_{\neq}(2,2m)}^{ab_3}(q) = \frac{\Psi(q^{2m-1}, q)}{(1-q)^2 (1-q^2)}$$

$\leadsto \mathcal{T}_{T_{\neq}(2,m)}^{ab_3}(q) = \frac{1}{(1-q)(1-q^2)} \mathcal{T}_{T(2,m)}^{ab_1}(q)$

Fin.