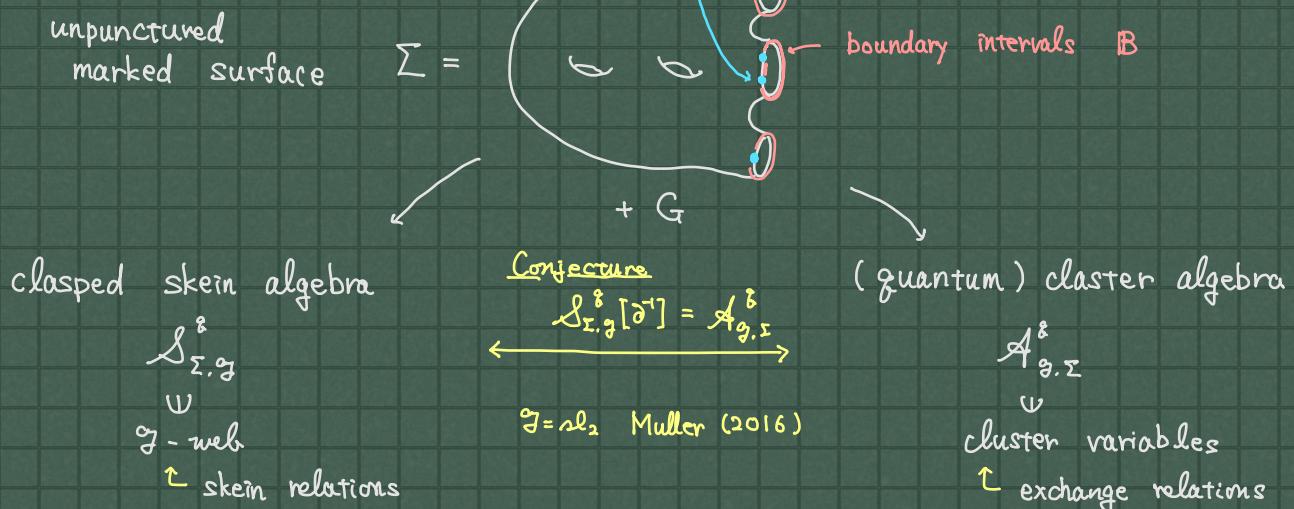


曲面のスケイン代数と量子クラスター代数

湯浅 亘 OCAMI · RIMS

- 石橋 典 (東北大)との共著 arXiv: 2101.00643 (sl_3)
 arXiv: 2207.01540 (sp_4)
 + in preparation (state-clasp)
 に基づく

§ Introduction



When $\mathfrak{s} = \text{sl}_3, \text{sp}_4, \#M \geq 2$

Theorem (Ishibashi-Y.) $\mathcal{S}_{\Sigma, g}^{\mathfrak{s}}[\delta] \subseteq \mathcal{A}_{g, \Sigma}^{\mathfrak{s}} \subsetneq \text{Frac } \mathcal{S}_{\Sigma, g}^{\mathfrak{s}}$

Corollary (Ishibashi-Y. + Ishibashi-Oya-Shen)

an open subspace of decorated twisted
G-local systems on Σ

$$\mathcal{S}_{\Sigma, g}^{\mathfrak{s}}[\delta] = \mathcal{A}_{g, \Sigma}^{\mathfrak{s}} = \mathcal{O}(\mathcal{A}_{g, \Sigma}^{\times})$$

§ clasped skein algebra $\mathcal{S}_{\mathfrak{g}, \Sigma}^{\natural}$

• \mathfrak{g} -webs \Leftrightarrow tangled uni-trivalent graphs on Σ



(ω_i : fundamental weights)

- Skein relations
 - internal : diagrammatic relations in FundRep $_{\mathfrak{g}}$
 - \mathfrak{sl}_2 : Kauffman
 - rank 2 : Kuperberg etc.
 - clasped : diagrammatic relations at Jones-Wenzl projectors
 - \mathfrak{sl}_2 : Muller
 - \mathfrak{sl}_3 : Frohman-Sikora
 - \mathfrak{sp}_4 : Ishibashi-Y.

e.g. \mathfrak{sl}_2 $\left\{ \begin{array}{l} \text{X} = q \quad (+ q^{-1}) \\ \text{O} = (-q^2 - q^{-2}) \phi \end{array} \right. \quad \left\{ \begin{array}{l} q^{\frac{1}{2}} \text{Y} = \text{Y} = q^{\frac{1}{2}} \text{Y} \\ \text{O} = 0 \end{array} \right.$

e.g. \mathfrak{sl}_3

e.g. \mathfrak{sp}_4

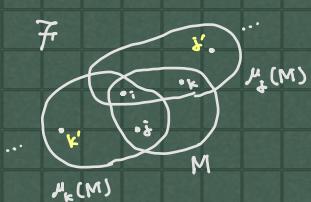
$$\begin{aligned}
 \textcircled{O} &= -\frac{[2][6]}{[3]} \textcircled{}, & \textcircled{O} &= \frac{[5][6]}{[2][3]} \textcircled{}, & \textcircled{O} &= 0 \\
 \textcircled{O} &= -[2] \textcircled{}, & \textcircled{O} &= 0, \\
 \textcircled{O} - [2] \textcircled{H} &= \textcircled{O} - [2] \textcircled{H}, \\
 \textcircled{X} &= \frac{v^2}{[2]} \textcircled{} + v^{-1} \textcircled{} + \textcircled{H}, \\
 &= v \textcircled{} + \frac{v^{-2}}{[2]} \textcircled{} + \textcircled{H}, \\
 \textcircled{X} &= v \textcircled{} + v^{-1} \textcircled{H}, \\
 \textcircled{X} &= v \textcircled{H} + v^{-1} \textcircled{H}, \\
 \textcircled{X} &= v^2 \textcircled{} + v^{-2} \textcircled{} + \textcircled{H}.
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{X} &= v \textcircled{}, & \textcircled{X} &= v^2 \textcircled{}, \\
 \textcircled{X} &= v \textcircled{}, & \textcircled{X} &= v \textcircled{}, \\
 \textcircled{X} &= \textcircled{}, & \textcircled{X} &= \textcircled{}, \\
 \textcircled{X} &= \frac{1}{[2]} \textcircled{}, & \textcircled{X} &= 0, \\
 \textcircled{X} &= 0, & \textcircled{X} &= 0, & \textcircled{X} &= 0, & \textcircled{X} &= 0, & \textcircled{X} &= 0.
 \end{aligned}$$

§ quantum cluster algebra $\mathcal{A}_{g,\Sigma}$

\mathcal{F} : a skew-field, $I = I_{uf} \sqcup I_f$: index set $D = \text{diag}(d_i \in \mathbb{Z}_{>0} \mid i \in I)$

• $S = (B, \Pi, \Lambda, M)$: quantum seed



• $B = (b_{ij})_{i,j \in I} = \left(\begin{array}{c|c} I_{uf} & I_f \\ \hline \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \frac{1}{2}\mathbb{Z} \end{array} \right) I_{uf} \quad \text{s.t. } DB \text{ : skew-symmetric exchange matrix}$

• $\Pi = (\pi_{ij} \in \mathbb{Z})_{i,j \in I}$: skew-symmetric compatibility matrix

• $\Lambda = \bigoplus_{i \in I} \mathbb{Z} f_i$ with a skew-symmetric form $\Pi(f_i, f_j) = \pi_{ij}$

• $M : \Lambda \rightarrow \mathcal{F} \setminus \{0\}$ s.t. $M(\alpha)M(\beta) = \frac{\pi(\alpha, \beta)}{2} M(\alpha + \beta)$
toric frame $\text{Frac } M(\Lambda) = \mathcal{F}$

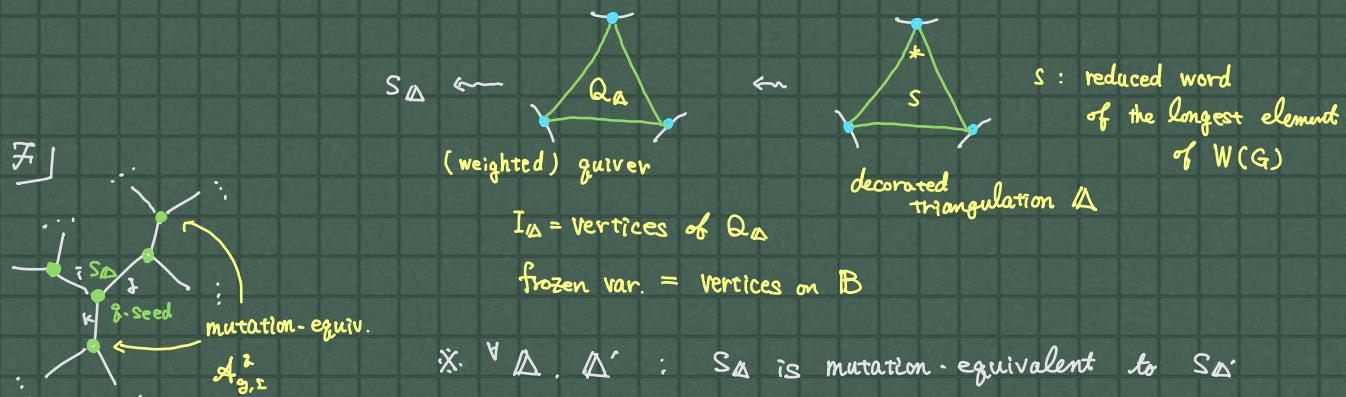
• $M(f_i) = A_i$: cluster variable $\begin{cases} \text{unfrozen if } i \in I_{uf} \\ \text{frozen if } i \in I_f \end{cases}$
 $\{A_i\}_{i \in I}$: cluster invertible

• quantum seed mutation at $k \in I_{uf}$

$$(B, \Pi, \Lambda, M) \xleftrightarrow{\mu_k} (B', \Pi', \Lambda', M')$$

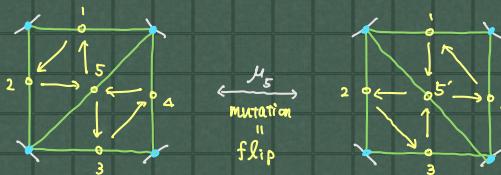
$\mathcal{A}_{g,\Sigma}^{\mathbb{Z}} : \Leftrightarrow \mathbb{Z}_g = \mathbb{Z}[\delta^{\pm\frac{1}{2}}]$ subalgebra of \mathcal{F} generated by

all quantum seeds mutation-equivalent to quantum seed
associated with decorated triangulation Δ of Σ .



⑩ Construct seeds S_{Δ} & adjacent ones in $\mathcal{F} = \text{Frac } \mathcal{A}_{g,\Sigma}^{\mathbb{Z}}$ } $\rightsquigarrow \mathcal{A}_{g,\Sigma}^{\mathbb{Z}} \subset \text{Frac } \mathcal{A}_{g,\Sigma}^{\mathbb{Z}}$
+ quantum Laurent phenomenon

e.g. sl_2 $\Delta = \Delta$: ideal triangulation
all seeds are associated with Δ 's



$$\mathcal{A}_{sl_2, \Delta}^{\mathbb{Z}} = \text{quantum torus}$$

$$\mathcal{A}_{sl_2, \square}^{\mathbb{Z}} = \langle A_1, \dots, A_4, A_5 \rangle \cup \langle A'_1, \dots, A'_4, A'_5 \rangle$$

$$A_5 A'_5 = A_2 A_4 + A_1 A_3$$

$$A_i = A'_i \quad (i \neq 5)$$

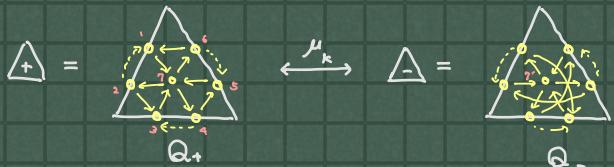
⑪ In $\mathcal{F} = \text{Frac } \mathcal{A}_{sl_2, \Sigma}^{\mathbb{Z}}$ A_i = simple arc along

$$A_5 A'_5 = A_2 A_4 + A_1 A_3$$

$\mathcal{A}_{sl_2, \Sigma}^{\mathbb{Z}} [\partial']$ is generated by simple arcs $\rightsquigarrow \mathcal{A}_{sl_2, \Sigma}^{\mathbb{Z}} [\partial'] \subseteq \mathcal{A}_{sl_2, \Sigma}^{\mathbb{Z}}$

\forall cluster variables correspond to simple arcs $\rightsquigarrow \mathcal{A}_{sl_2, \Sigma}^{\mathbb{Z}} \subseteq \mathcal{A}_{sl_2, \Sigma}^{\mathbb{Z}} [\partial']$

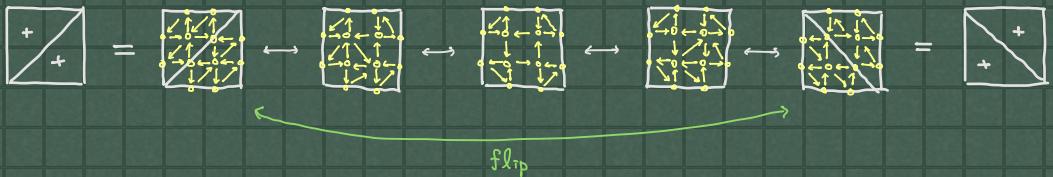
e.g. rl_3 decorated triangles



$$\mathcal{A}_{\text{rl}_3, \Delta}^{\pm} = \langle A_1, \dots, A_6, A_7 \rangle \cup \langle A'_1, \dots, A'_6, A'_7 \rangle$$

$$\begin{cases} A'_i = A_i & (i \neq 7) \\ A_7 A'_7 = 8^\circ A_1 A_3 A_5 + 8^\circ A_2 A_4 A_6 \end{cases}$$

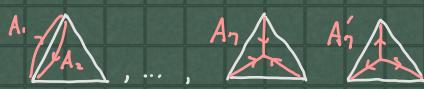
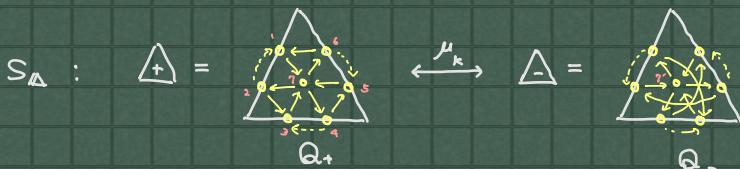
* mutation \neq flip



* $\mathcal{A}_{\text{rl}_3, \square}^{\pm}$ has 50 clusters (type D_4)

in general, $\mathcal{A}_{\text{rl}_3, \Sigma}^{\pm}$ is infinite mutation type

- In $\mathcal{F} = \text{Frac } \mathcal{A}_{\text{rl}_3, \Sigma}^{\pm}$



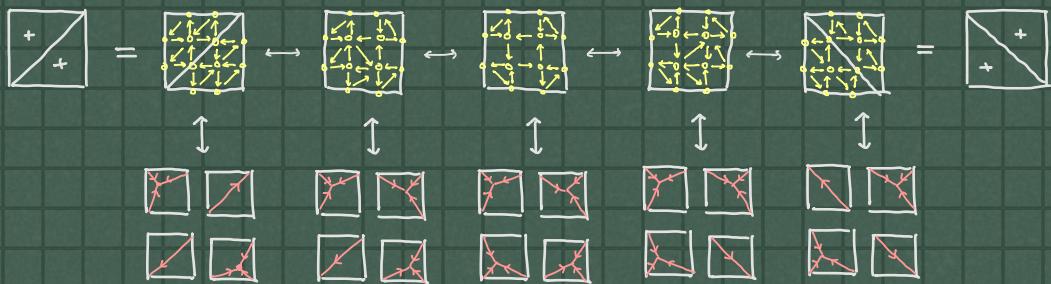
$$A_7 A'_7 = 8^\circ A_1 A_3 A_5 + 8^\circ A_2 A_4 A_6$$

from exchange relation.

By skein relation :

$$\begin{array}{ccccccccc}
 \text{triangle} & = q^{-\frac{1}{2}} & \text{triangle} & = q^{\frac{3}{2}} & \text{triangle} & + q^{-\frac{3}{2}} & \text{triangle} & = q^{\frac{3}{2}} & \text{triangle} & + q^{-\frac{3}{2}} \\
 A_7 A'_7 & & & & & & & & & A_1 A_3 A_5 & & A_2 A_4 A_6
 \end{array}$$

⑩ a flip sequens in $\text{Frac } \mathcal{S}_{\text{ab}, \Sigma}^{\delta}$



⑪ other cluster variables in \square



Theorem (Ishibashi - Y. '21)

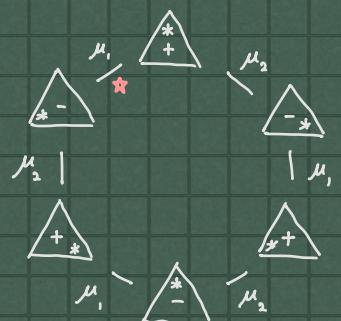
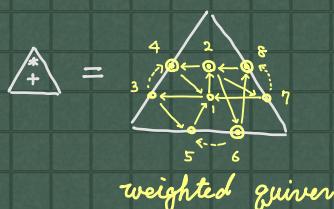
matrix elements of
simple Wilson lines

$\mathcal{S}_{\text{ab}, \Sigma}^{\delta}[\delta^*]$ is generated by the above cluster variables in ideal quadrilaterals

proof By the sticking trick.

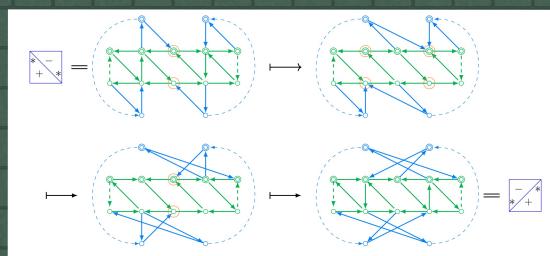
$$\rightsquigarrow \mathcal{S}_{\text{ab}, \Sigma}^{\delta}[\delta^*] \subseteq \mathcal{A}_{\text{ab}, \Sigma}^{\delta}$$

e.g. A_{P_4} decorated triangulations



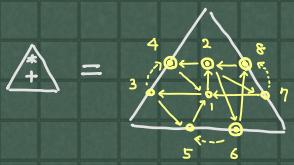
* a flip is realized by 8 mutations

$$M_i : A_i A'_i = g^* A_2 A_3 + g^* A_4 A_5 A_7$$

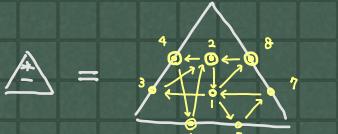


* $A_{\text{op}_4, \square}^{\delta}$ is infinite mutation type

$$\text{In } \mathcal{F} = \text{Frac } \mathcal{S}_{\text{ap}, \Sigma}^{\delta}$$



$$A_1 = \begin{array}{c} \text{triangle} \\ \text{with arrows} \end{array}$$



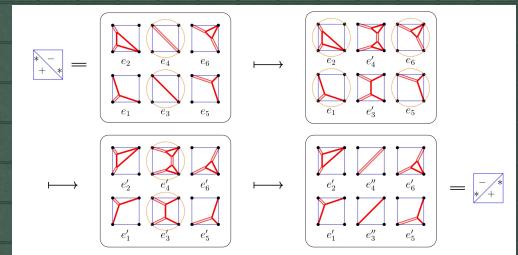
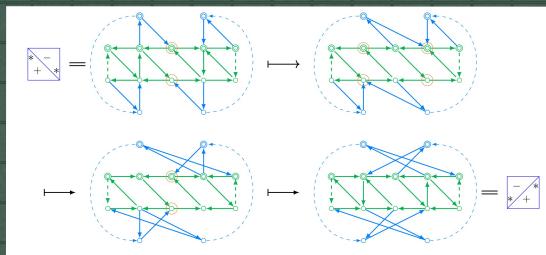
$$A_1 = \begin{array}{c} \text{triangle} \\ \text{with arrows} \end{array}$$

$$\mu: A_i A'_i = g^{\circ} A_2 A_3 + g^{\circ} A_4 A_5 A_7, \quad \text{from exchange relation}$$

$$\begin{array}{c} \text{triangle} \\ \xrightarrow{\mu} \\ \text{triangle} \end{array}$$

$$\begin{aligned} A_i A'_i &= \begin{array}{c} \text{triangle} \\ \text{with arrows} \end{array} = g^{-\frac{1}{2}} \begin{array}{c} \text{triangle} \\ \text{with arrows} \end{array} = g^{-\frac{1}{2}} \left(g \begin{array}{c} \text{triangle} \\ \text{with arrows} \end{array} + \frac{g^{\circ}}{[2]} \begin{array}{c} \text{triangle} \\ \text{with arrows} \end{array} + \begin{array}{c} \text{triangle} \\ \text{with arrows} \end{array} \right) \\ &= g^{\frac{1}{2}} \begin{array}{c} \text{triangle} \\ \text{with arrows} \end{array} + g^{-\frac{1}{2}} \begin{array}{c} \text{triangle} \\ \text{with arrows} \end{array} \\ &\qquad \qquad \qquad A_4 A_5 A_7 \qquad \qquad \qquad A_2 A_3 \end{aligned}$$

④ a flip sequence in $\text{Frac } \mathcal{S}_{\text{ap}, \Sigma}^{\delta}$



④ other cluster variables in $\text{Frac } \mathcal{S}_{\text{ap}, \square}^{\delta}$:



Theorem (Ishibashi - Y, '22)

$\mathcal{S}_{\text{ap}, \Sigma}^{\delta} [\delta']$ is generated by the above cluster variables in ideal quadrilaterals

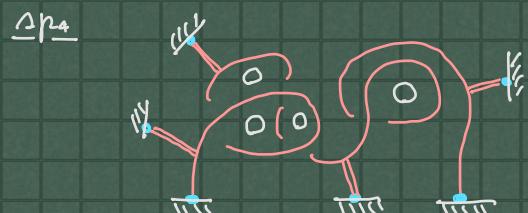
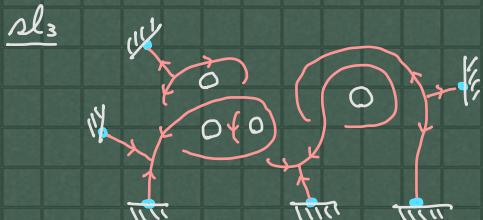
proof By the sticking trick.

$$\rightsquigarrow \mathcal{S}_{\text{ap}, \Sigma}^{\delta} [\delta'] \subseteq \mathcal{A}_{\text{ap}, \Sigma}^{\delta}$$

\S generators of $\mathcal{S}_{g,\Sigma}^{\mathfrak{sl}}$

Theorem (descending generators for $\mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sp}_4$)

$\mathcal{S}_{g,\Sigma}^{\mathfrak{sl}}$ is generated by descending curves with / without legs.



descending generators $\xrightarrow[\text{sticking trick}]{\text{in } \mathcal{S}_{g,\Sigma}^{\mathfrak{sl}}[\delta]}$ simple Wilson lines

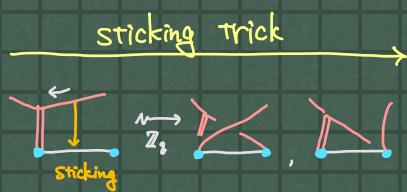
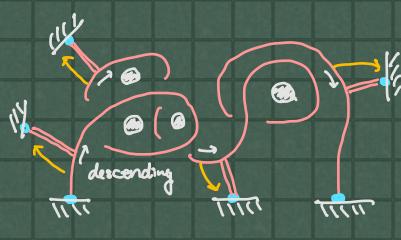
Lemma (the sticking trick) [Ishibashi - Y. '21 '22]

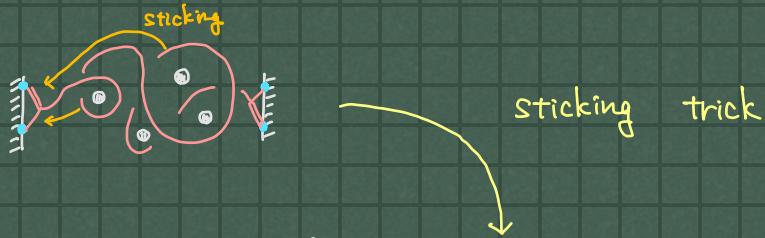
• \mathfrak{sl}_3

$$\begin{array}{c} \text{Diagram} \\ = A^6 - A^5 + A^2 \end{array}$$

• \mathfrak{sp}_4

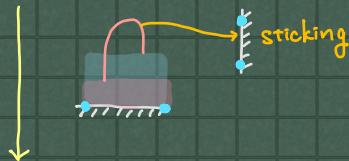
$$\begin{aligned} \text{Diagram} &= v - v^2 + v^3 - v^4, \\ \text{Diagram} &= v^2 - v^4 \\ &+ v^4[2] - v^4 + v^7. \end{aligned}$$





a \mathbb{Z}_8 -polynomial in

$$\left\{ \begin{array}{c} \text{Diagram 1} \\ | \\ \text{Diagram 2} \end{array} \right| = \left\{ \begin{array}{c} \text{Diagram 3} \\ | \\ \text{Diagram 4} \\ | \\ \text{Diagram 5} \end{array} \right\}$$



a \mathbb{Z}_8 -polynomial in

$$\left\{ \begin{array}{c} \text{Diagram 1} \\ | \\ \text{Diagram 2} \end{array} \right| = \left\{ \begin{array}{c} \text{Diagram 3} \\ | \\ \text{Diagram 4} \\ | \\ \text{Diagram 5} \end{array} \right\}$$

simple Wilson lines

{ state - clasp correspondence

⑩ the stated skein algebra $\mathcal{S}_{g,\Sigma}^{\mathfrak{s}}(\mathbb{B})$

\Leftrightarrow \mathfrak{s} -webs with  ($i \in \Lambda_\alpha$)

+ internal & stated skein relations

• \mathfrak{sl}_2 (Bonahon-Wong, Le)

$$\begin{aligned} \text{Diagram 1} &= U_{ij} \text{ Diagram 2}, \text{ where } (U_{ij}) = \begin{pmatrix} 0 & -A^{-\frac{5}{2}} \\ A^{-\frac{1}{2}} & 0 \end{pmatrix}, \\ \text{Diagram 3} &= A^{\frac{1}{2}} \text{ Diagram 4} - A^{\frac{5}{2}} \text{ Diagram 5} \end{aligned}$$

• \mathfrak{sp}_4 (Ishibashi-Y.)

$$\begin{aligned} \text{Diagram 1} &= U_{ij} \text{ Diagram 2}, \text{ where } (U_{ij}) = \begin{pmatrix} 0 & 0 & 0 & -v^{-\frac{3}{2}} \\ 0 & 0 & v^{-\frac{3}{2}} & 0 \\ 0 & -v^{-\frac{3}{2}} & 0 & 0 \\ v^{-\frac{1}{2}} & 0 & 0 & 0 \end{pmatrix}, \\ \text{Diagram 3} &= V_{ij} \text{ Diagram 4}, \text{ where } (V_{ij}) = \begin{pmatrix} 0 & -v^{-1} & -v^{-1} & -v^{-\frac{1}{2}}[2]^{-\frac{1}{2}} \\ 1 & 0 & -v^{-\frac{3}{2}}[2]^{-\frac{1}{2}} & -v^{-1} \\ 1 & v^{\frac{1}{2}}[2]^{-\frac{1}{2}} & 0 & -v^{-1} \\ v^{-\frac{1}{2}}[2]^{-\frac{1}{2}} & 1 & 1 & 0 \end{pmatrix}, \\ \text{Diagram 5} &= v \text{ Diagram 6} + \text{Diagram 7} \quad (i < j, i+j \neq 5) \\ \text{Diagram 8} &= v^2 \text{ Diagram 9} + v^{\frac{1}{2}}[2]^{-\frac{1}{2}} \text{ Diagram 10} + v^{-\frac{3}{2}}[2]^{-1} \text{ Diagram 11} \end{aligned}$$

• \mathfrak{sl}_3 (Higgins)

$$\begin{aligned} \text{Diagram 1} &= U_{ij} \text{ Diagram 2}, \text{ where } (U_{ij}) = \begin{pmatrix} 0 & 0 & A^{-7} \\ A^{-1} & -A^{-4} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \text{Diagram 3} &= V_{ij} \text{ Diagram 4}, \text{ where } (V_{ij}) = \begin{pmatrix} 0 & -A^{-\frac{7}{2}} & -A^{-\frac{7}{2}} \\ A^{-\frac{1}{2}} & 0 & A^{-\frac{7}{2}} \\ A^{-\frac{1}{2}} & A^{-\frac{1}{2}} & 0 \end{pmatrix}, \\ \text{Diagram 5} &= A^3 \text{ Diagram 6} + A^{-\frac{1}{2}} \text{ Diagram 7} \quad \text{for } i < j. \end{aligned}$$

② the reduced stated skein algebra $\mathcal{S}_{g,\Sigma}^{\text{st}}(\mathbb{B})_{\text{rd}}$

$$\mathcal{S}_{g,\Sigma}^{\text{st}}(\mathbb{B})_{\text{rd}} := \mathcal{S}_{g,\Sigma}^{\text{st}}(\mathbb{B}) / I_{\text{bad}}$$

$$I_{\text{bad}} := \text{Span}_{\mathbb{Z}_2} \left\{ \begin{array}{c} \text{Diagram} \\ i \quad j \end{array} \mid i < j \text{ for } i, j \in \Lambda_\sigma \right\}$$

* What is the stated arc $i \xrightarrow{\text{stated}} j$

\rightarrow the (i,j) -entry of a monodromy along $\xrightarrow{\text{Wilson line}} j$ of

the moduli space $\mathcal{A}_{G,\Sigma}$ of decorated twisted G -local systems

e.g. $\mathcal{O}(G) \cong \mathcal{S}_{g,0}^{\text{st}}(\mathbb{B})$ ($\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3, (\mathfrak{sp}_4)$)

Theorem (the state-clasp correspondence)

$$\mathcal{S}_{g,\Sigma}^{\text{st}}[\delta'] \cong \mathcal{S}_{g,\Sigma}^{\text{st}}(\mathbb{B})_{\text{rd}}$$

$\uparrow_{\text{clasped}}$ \uparrow_{stated}

$$\begin{array}{ccccc}
 & & \xrightarrow{\text{? generic}} & & \\
 & \mathcal{S}_{g,\Sigma}[\delta'] & \xrightarrow[\text{sticking}]{} & \mathcal{U}_{g,\Sigma} & \xrightarrow[\text{?}]{} \mathcal{O}(A_{g,\Sigma}) \\
 & \uparrow \cong & & \downarrow \cong & \\
 & \mathcal{S}_{M \rightarrow \mathbb{B}} & & \text{Ishibashi-Oya} & \\
 & \uparrow \cong & & - Shen & \uparrow \cong \\
 & \mathcal{S}_{g,\Sigma}^{\text{st}}(\mathbb{B})_{\text{rd}} & \xleftarrow[\text{(i,j)-entry of}]{\text{Wilson lines}} & &
 \end{array}$$

