The Formal Semantics of Programming Languages / Chapter1-3

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Diagonal Argument

Claim

A set X and $\mathcal{P}ow(X)$ are never in 1-1 correspondence for any X.

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Example

- $X = \{1, 2, 3\}$
- $\mathcal{P}ow(X) = \{\phi,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$

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dose claim still hold?

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answer is YES

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Proof.

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In either case, we have contradiction. $\square$

Why it's Called "Diagonal" Arugument

consider following table where ith row and jth column is placed 1 if $x_i \in \theta(x_j)$ and 0 otherwise

	$\theta(x_0)$	$\theta(x_1)$	$\theta(x_2)$	$\cdots$	$\theta(x_j)$	$\cdots$
x_0	0	1	1	$\cdots$	1	$\cdots$
x_1	1	1	0	$\cdots$	0	$\cdots$
x_2	0	0	1	$\cdots$	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
x_i	0	1	0	$\cdots$	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	

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x_2	0	0	$\boxed{1}$	$\cdots$	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
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$$Y = \{x_0\}$$

Direct and Inverse Image of Relation

Definition (direct and inverse image)

Let be $R : X \times Y$ is a relation, A is a subset of X and B is a subset of Y, a set RA and $R^{-1}B$ are defined as follows;

- $RA = \{y \in Y \mid \exists x \in A((x, y) \in R)\}$,
- $R^{-1}B = \{x \in X \mid \exists y \in B((x, y) \in R)\}$.

The set RA is called direct image of A under R, and the set $R^{-1}B$ is called inverse image of B under R.

Example

- $X = \{0, 1, 2, 3, 4\}$, $Y = \{5, 6, 7, 8, 9\}$, $R = \{(0, 9), (1, 5), (2, 6), (3, 6)\}$, $A = \{1, 3, 4\}$, $B = \{5, 7, 9\}$
- $RA = \{5, 6\}$, $R^{-1}B = \{0, 1\}$

Equivalence Relation

Definition (equivalence relation and equivalence class)

An equivalence relation is a relation $R \subseteq X \times X$ on a set X which is

- reflexive: $\forall x \in X(xRx)$,
- symmetric: $\forall x, y \in X(xRy \rightarrow yRx)$ and
- transitive: $\forall x, y, z \in X((xRy \wedge yRz) \rightarrow xRz)$.

Example

- $R = \{(x, y) \in B \times B \mid x \in A \wedge y \in A\}$ where $A \subseteq B$
- congruence of figures
- congruence of integers

Equivalence Class

Definition (equivalence class)

If R is an equivalence relation on a set X then $(R-)$equivalence class $\{x\}_R$ of an element $x \in X$ is defined as follows;

$$\{x\}_R = \{y \in X \mid yRx\}.$$

Example

- $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \equiv y \pmod 3\}$
- $\{1\}_R = \{1, 4, 7, 10, \cdots\}$