Chapter 5

1. $\binom{n-1}{k-1} + \binom{n-1}{k} = \dfrac{(n-1)!}{(k-1)!\,(n-k)!} + \dfrac{(n-1)!}{k!\,(n-1-k)!}$

$$= \frac{(n-1)!}{k!\,(n-k)!} \cdot \left[\frac{k}{1} + \frac{n-k}{1} \right]$$

$$= \frac{n.\,(n-1)!}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

2. (a) 1 9 36 84 126 126 84 36 9 1

(b) 1 10 45 120 210 252 210 120 45 10 1

3. $a_0 = 1$, $a_1 = 1$
$a_2 = 1+1 = 2$
$a_3 = 1+2 = 3$
$a_4 = 2+3 = 5$
$a_5 = 3+5 = 8$

In general
$a_n = a_{n-1} + a_{n-2}$
for $n \geq 2$.

$\langle a_n \rangle$ is the Fibonacci sequence

4. $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$
$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$.

5 $(2x-y)^7 = [(2x) + (-y)]^y$
$= (2x)^7 + 7.(2x)^6.(-y) + 21.(2x)^5(-y)^2 + 35(2x)^4(-y)^3$
$\quad + 35(2x)^3.(-y)^4 + 21(2x)^2(-y)^5 + 7.(2x).(-y)^6 + (-y)^7$
$= 128x^7 - 448x^6y + 672x^5y^2 - 560x^4y^3 + 28x^3y^4$
$\quad - 84x^2y^5 + 14xy^6 - y^7$

6 (a) Look at $\binom{18}{5}.(3x)^5.(-2y)^{13}$. Ans: $-\binom{18}{5}.3^5.2^{13}$

(b) 0. [The term x^8y^9 does not appear in the expansion so its coefficient is naturally 0.]

7 (a) $\displaystyle\sum_{k=0}^{n}\binom{n}{k} 2^k = \sum_{k=0}^{n}\binom{n}{k}\cdot 2^k\cdot 1^{n-k}$

$= (2+1)^n$ by the Binomial theorem

$= 3^n$

(b) $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\cdot r^k = \sum_{k=0}^{n}\binom{n}{k}\cdot r^k\cdot 1^{n-k} = (r+1)^n$

8. $\displaystyle\sum_{k=0}^{n}(-1)^k\cdot\binom{n}{k}\cdot 3^{n-k} = \sum_{k=0}^{n}\binom{n}{k}\cdot(-1)^k\cdot 3^{n-k}$

$= (-1+3)^n = 2^n.$

9 $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}\cdot 10^k = \sum_{k=0}^{n}\binom{n}{k}\cdot(-10)^k\cdot 1^{n-k}$

$= (-10+1)^n = (-9)^n = (-1)^n\cdot 9^n$

10(a) Number of ways of choosing a team of k from n players with a designated captain $= \binom{n}{k}\cdot k$, $\binom{n}{k}$ teams, k choices for captains.

(b) Choose captain, then choose the other $k-1$ players. $\Big\} \to n\cdot\binom{n-1}{k-1}$. $\therefore\ k\cdot\binom{n}{k} = n\cdot\binom{n-1}{k-1}$.

11. Let $S = \{a_1, a_2, a_3, a_4, \ldots, a_n\}$ be a set with n distinct elements. Then

$\binom{n}{k}$ = No. of k- subsets of S, and

$\binom{n-3}{k}$ = No. of k-subsets of $\{a_4, a_5, \ldots, a_n\}$.

So $\binom{n}{k} - \binom{n-3}{k}$ = No. of k-subsets of S which contains at least one of the three elements a_1, a_2, a_3.

Let A_1 = collection of all k-subsets of S which contains a_1,

11. A_2 = Collection of all k-subsets of S which contains a_2 but does not contain a_1,
& A_3 = Collection of all k-subsets of S which contains a_3 but contains neither a_1 nor a_2.

Then

$$|A_1| = \underbrace{\binom{n-1}{k-1}}_{a_1 + (k-1) \text{ elements from } \{a_2,\dots,a_n\}} \quad |A_2| = \underbrace{\binom{n-2}{k-1}}_{a_2 + (k-1) \text{ elements from } \{a_3,\dots,a_n\}} \quad \text{and } |A_3| = \underbrace{\binom{n-3}{k-1}}_{a_3 + (k-1) \text{ elements from } \{a_4,\dots,a_n\}}$$

Also A_1, A_2 & A_3 are mutually disjoint. So

$$\begin{aligned}\binom{n}{k} - \binom{n-3}{k} &= \text{No. of k-subsets of } S \text{ which contains at least one of } a_1, a_2, a_3 \\ &= |A_1| + |A_2| + |A_3| \\ &= \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}\end{aligned}$$

and we are done.

13 Let $S = \{a,b,c,a_1,a_2,\dots,a_n\}$. Put

B_0 = set of all k-subsets of S with none of a,b,&c
B_1 = " " " with one of a,b,&c
B_2 = " " " with two of a,b,&c
B_3 = " " " with three of a,b,&c

Then

B_0, B_1, B_2 & B_3 are all disjoint and
$B_0 \cup B_1 \cup B_2 \cup B_3$ = set of all k-subsets of S

Also

$$|B_0| = \underbrace{\binom{n}{k}}_{k \text{ elements from } \{a_1,a_2,\dots,a_n\}}, \quad |B_1| = \underbrace{3.\binom{n}{k-1}}_{a,b,\text{or } c + (k-1) \text{ from } \{a_1,\dots,a_n\}}, \quad B_2 = \underbrace{3.\binom{n}{k-2}}_{\text{two of } a,b,c + (k-2) \text{ from } \{a_1,\dots,a_n\}} \ \& \ B_3 = \underbrace{\binom{n}{k-3}}_{a,b\&c \text{ plus } (k-3) \text{ from } \{a_1,\dots,a_k\}}$$

13 So $\binom{n}{k} + 3\cdot\binom{n}{k-1} + 3\cdot\binom{n}{k-2} + \binom{n}{k-3}$

$= |B_0| + |B_1| + |B_2| + |B_3|$

$= |B_0 \cup B_1 \cup B_2 \cup B_3|$

$=$ No. of k-subsets of $S = \{a, b, c, a_1, \ldots, a_n\} = \binom{n+3}{k}$

14. $\binom{r}{k} = r.(r-1).(r-2) \cdots [r-(k-1)]/k! \to (r-k)$

$$= \frac{r}{(r-k)} \cdot \frac{(r-1).[(r-1)-1] \cdots [(r-1)-(k-2)]}{k!} \cdot \frac{(r-1)-(k-1)}{1} = \frac{r}{r-k}\cdot\binom{r-1}{k}.$$

15. From the Binomial theorem we have

$$\binom{n}{0}x^0y^n + \binom{n}{1}xy^{n-1} + \cdots + \binom{n}{n}x^ny^0 = (x+y)^n$$

Putting $y=1$, we get

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = (x+1)^n$$

Differentiating w.r.t x gives

$$0 + 1.\binom{n}{1}x^0 + 2\binom{n}{2}x + 3.\binom{n}{3}x^2 + \cdots + n\binom{n}{n}x^{n-1} = n.(x+1)^{n-1}$$

Finally by putting $x=-1$, we get

$$1.\binom{n}{1} - 2.\binom{n}{2} + 3.\binom{n}{3} - \cdots + (-1)^{n-1}.n.\binom{n}{n} = 0$$

16 As in problem #15 we get

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = (x+1)^n$$

Integrating both sides w.r.t. x gives us

$$\binom{n}{0}\cdot x + \binom{n}{1}\cdot\frac{x^2}{2} + \binom{n}{2}\cdot\frac{x^3}{3} + \cdots + \binom{n}{n}\frac{x^{n+1}}{n+1} = \frac{(x+1)^{n+1}}{n+1} + C$$

16. Putting $x=0$, give us

$$0+0+0+\cdots+0=\frac{(1)^{n+1}}{n+1}+C$$

So $C=\frac{-1}{n+1}$. Thus

$$\binom{n}{0}x+\frac{1}{2}\binom{n}{1}x^2+\frac{1}{3}\binom{n}{2}x^3+\cdots+\frac{1}{n+1}\binom{n}{n}x^{n+1}=\frac{(x+1)^{n+1}}{n+1}-\frac{1}{n+1}$$

Finally, putting $x=1$, gives us

$$\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\cdots+\frac{1}{n+1}\binom{n}{n}=\frac{(1+1)^{n}}{n+1}-\frac{1}{n+1}$$

$$=\frac{2^n-1}{n+1}.$$

18. From problem #16 we have

$$\binom{n}{0}x+\frac{1}{2}\binom{n}{1}x^2+\frac{1}{3}\binom{n}{2}x^3+\cdots+\frac{1}{n+1}\binom{n}{n}x^{n+1}=\frac{(x+1)^{n+1}-1}{n+1}$$

Dividing both sides by x gives us

$$\binom{n}{0}+\frac{1}{2}\binom{n}{1}x+\frac{1}{3}\binom{n}{2}x^2+\cdots+\frac{1}{n+1}\cdot\binom{n}{n}x^{n}=\frac{(x+1)^{n+1}-1}{x\,(n+1)}$$

Finally by putting $x=-1$, we get

$$1-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\cdots+\frac{(-1)^n}{n+1}\binom{n}{n}=\frac{(0)^{n+1}-1}{(-1)(n+1)}$$

$$=\frac{1}{n+1}.$$

19. $$\sum_{m=1}^{n}m^2=\sum_{m=1}^{n}2\cdot\binom{m}{2}+\binom{m}{1}=2\cdot\sum_{m=1}^{n}\binom{m}{2}+\sum_{m=1}^{n}\binom{m}{1}$$

$$=2\cdot\binom{n+1}{2+1}+\binom{n+1}{1+1}=2\cdot\binom{n+1}{3}+\binom{n+1}{2}$$

$$=2\cdot\frac{(n+1).\,n.(n-1)}{3\cdot2\cdot1}+\frac{(n+1)\cdot n}{2\cdot1}$$

$$=\frac{1}{6}(n+1).n.\left[2(n-1)+3\right]=\frac{n(n+1)(2n+1)}{6}.$$

23. The student has to walk 24 blocks E or N. Now each block corresponds to an E or N, so a typical walk will be a permutation of 10 E's and 14 N's.

(a) No. of different walks

$$= \text{No. of permutations of } [10.E,\ 14.N]$$

$$= \frac{24!}{10!\,24!} \quad \text{by the theorem on permutations of multi-sets.}$$

(b) No. of walks

$$= \begin{pmatrix}\text{No. of walks} \\ \text{to 4th E \& 5th N}\end{pmatrix} \cdot \begin{pmatrix}\text{No. of walks from 4th E \&} \\ \text{5th N to 10th E \& 14th N}\end{pmatrix}$$

$$= \begin{pmatrix}\text{No. of perm.} \\ \text{of } [4.E,\ 5N]\end{pmatrix} \cdot \begin{pmatrix}\text{No. of perm.} \\ \text{of } [6.E,\ 9N]\end{pmatrix}$$

$$= \frac{9!}{4!\,5!} \cdot \frac{15!}{6!\,9!} = \frac{15!}{4!\,5!\,6!}$$

(c) $$\text{Ans} := \frac{9!}{4!\,5!} \cdot \frac{9!}{3!\,6!} \cdot \frac{6!}{3!\,3!} = \frac{(9!)^2}{4!\,5!\,(3!)^3}$$

(d) Answer in (b) − Answer in (c)

$$= \frac{15!}{4!\,5!\,6!} - \frac{(9!)^2}{4!\,5!\,(3!)^3}$$

$$= \frac{9!}{4!\,5!\,6}\left[\frac{15.14.13.12.11.10}{5!} - \frac{9!}{(3!)^2}\right]$$

$$= \frac{9!}{4!\,5!\,6} \cdot \left(15.14.13.11 - 9.8.7.5.4\right)$$

24. Ans. = No. of perm. of $[10E, 15N, 20B] = 45!/(10!\,15!\,20!)$.

25. a) Let $S = \{a_1, a_2, \cdots, a_{m_1}, b_1, \cdots, b_{m_2}\}$.

Then

$$\sum_{k=0}^{n} \binom{m_1}{k}\cdot\binom{m_2}{n-k} = \underbrace{\binom{m_1}{0}\cdot\binom{m_2}{n}}_{\text{picking } 0\ a_i\text{'s and } n\ b_i\text{'s}} + \underbrace{\binom{m_1}{1}\cdot\binom{m_2}{n-1}}_{\text{picking } 1\ a_i \text{ and } n-1\ b_i\text{'s}} + \cdots + \underbrace{\binom{m_1}{n}\cdot\binom{m_2}{0}}_{\text{picking } n\ a_i\text{'s and } 0\ b_i\text{'s}}$$

= No. of n- subsets of S

$= \binom{m_1+m_2}{n}$ and we are done.

b) Replacing m_1 & m_2 by n in the formula above

$$\sum_{k=0}^{n} \binom{n}{k}\cdot\binom{n}{n-k} = \binom{n+n}{n}$$

Since $\binom{n}{n-k} = \binom{n}{k}$, we get $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

29. $$\sum_{\substack{r,s,t\geq 0\\ r+s+t=n}} \binom{m_1}{r}\cdot\binom{m_2}{s}\cdot\binom{m_3}{t} = \binom{m_1+m_2+m_3}{n}$$

30. Look at the 6 chains into which we partitioned the 2^4 subsets of $\{1,2,3,4\}$.

$\{1,2,3,4\}$					
$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$		$\{2,3,4\}$	
$\{1,2\}$	$\{1,4\}$	$\{1,3\}$	$\{3,4\}$	$\{2,3\}$	$\{2,4\}$
$\{1\}$	$\{4\}$	$\{3\}$		$\{2\}$	
$\emptyset$					

To get a clutter of size 6, we must pick one subset from each chain. Now $\{3,4\}$ & $\{2,4\}$ are the only subsets in their chains – so we have

30. to pick $\{2,4\}$ from the 6-th chain and $\{3,4\}$ from the 4-th chain.

Now $\{2,4\}$ eliminates $\{2\}$ & $\{2,3,4\}$ in the 5-th chain, $\{4\}$ & $\{1,2,4\}$ in the 2nd-chain, and ϕ & $\{1,2,3,4\}$ in the 1st-chain from further consideration. So we have to pick $\{1,4\}$ in the 2nd-chain and $\{2,3\}$ from the 5-th chain.

Also $\{3,4\}$ eliminates $\{3\}$ & $\{1,3,4\}$ in the 3rd chain from further consideration, so we must pick $\{1,3\}$ from the 3rd-chain. And since $\{1,3\}$ eliminates $\{1\}$ & $\{1,2,3\}$ in the 1-st chain from further consideration, we must pick $\{1,2\}$ from the first chain. Thus any clutter of size 6 must be

$$\{\{1,2\}, \{1,4\}, \{1,3\}, \{3,4\}, \{2,3\}, \{2,4\}\}.$$

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$\{1,2,3,4,5\}$				
$\{1,2,3,4\}$	$\{1,2,3,5\}$	$\{1,2,4,5\}$		$\{1,3,4,5\}$
$\{1,2,3\}$	$\{1,2,5\}$	$\{1,2,4\}$	$\{1,4,5\}$	$\{1,3,4\}$
$\{1,2\}$	$\{1,5\}$	$\{1,4\}$	$\{4,5\}$	$\{1,3\}$
$\{1\}$	$\{5\}$	$\{4\}$		$\{3\}$
ϕ				

→			$\{2,3,4,5\}$		
→	$\{1,3,5\}$	$\{3,4,5\}$	$\{2,3,4\}$	$\{2,3,5\}$	$\{2,4,5\}$
→	$\{3,5\}$	$\{3,4\}$	$\{2,3\}$	$\{2,5\}$	$\{2,4\}$.
			$\{2\}$		

31. Look at the 4-th chain. If we want a clutter of size 10, we must pick $\{4,5\}$ or $\{1,4,5\}$. If we pick $\{4,5\}$ we will be forced to pick the 2-subsets of $\{1,2,3,4,5\}$ and if we pick $\{1,4,5\}$ we will be forced to pick the 3-subsets of $\{1,2,3,4,5\}$. The proof is very similar to that in #30.

35. What we want here is the size of the largest clutter of $\{1,2,3,\ldots,10\}$. Each element of the clutter will correspond to the set of jokes used on a given night. Now the largest clutter of $\{1,2,\ldots,10\}$ is of size $\binom{10}{5} = \frac{10.9.8.7.6.}{5.4.3.2}$

$$= 9.4.7 = 252.$$

36. $$\binom{m_1+m_2}{n} = \text{Coefficient of } x^n \text{ in the exp. of } (1+x)^{m_1+m_2}$$
$$= \text{Coeff. of } x^n \text{ the exp. of } (1+x)^{m_1}\cdot(1+x)^{m_2}$$
$$= \text{coeff. of } x^n \text{ in } \left[\binom{m_1}{0}+\binom{m_1}{1}x+\cdots+\binom{m_1}{m_1}x^{m_1}\right]\cdot\left[\binom{m_2}{0}+\binom{m_2}{1}x+\cdots+\binom{m_2}{m_2}x^{m_2}\right]$$
$$= \binom{m_1}{0}\cdot\binom{m_2}{n}+\binom{m_1}{1}\cdot\binom{m_2}{n-1}+\cdots+\binom{m_1}{n}\cdot\binom{m_2}{0}$$
$$= \sum_{k=0}^{n}\binom{m_1}{k}\cdot\binom{m_2}{n-k}.$$

37. Just put $x_1 = x_2 = \cdots = x_t = 1$ in the multinomial formula and we'll get
$$\sum\binom{n}{n_1,\ldots,n_t} = (1+1+\cdots+1)^n = t^n.$$

39. Ans. $= \binom{10}{3,1,4,0,2} = \dfrac{10!}{3!\,1!\,4!\,0!\,2!} = \dfrac{10!}{3!\,4!\,2!}$.

40. Look at the term $(x_1)^3 \cdot (-x_2)^3 \cdot (2x_3)^1 \cdot (-2x_4)^2$

Ans: $\binom{9}{3,3,1,2} \cdot (-1)^3 \cdot 2^1 \cdot (-2)^2 = \dfrac{(-1)^5 \cdot 2^3 \cdot 9!}{3!\,3!\,1!\,2!} = \dfrac{-8 \cdot 9!}{3!\,3!\,2!}$

41. $(x_1+x_2+x_3)^n = [(x_1+x_2)+x_3]^n$

$$= \sum_{k=0}^{n} \binom{n}{k} \cdot (x_1+x_2)^k \cdot x_3^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \cdot \sum_{l=0}^{k} \binom{k}{l} \cdot x_1^{l} \cdot x_2^{k-l} \cdot x_3^{n-k}$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k} \cdot \binom{k}{l} \cdot x_1^{l} \cdot x_2^{k-l} \cdot x_3^{n-k}$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{n!}{k!(n-k)!} \cdot \frac{k!}{l!(k-l)!} \cdot x_1^{l} \cdot x_2^{k-l} \cdot x_3^{n-k}$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{n!}{(n-k)!\,l!\,(k-l)!} x_1^{l} x_2^{k-l} \cdot x_3^{n-k}$$

$$= \sum_{\substack{n_1+n_2+n_3=n \\ n_1,n_2,n_3 \geq 0}} \binom{n}{n_1,n_2,n_3} x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3} \quad \text{where}$$

$n_1 = l,\ n_2 = k-l,\ n_3 = n-k$.

42. First observe that No. of n-perm. of $[n_1 \cdot a_1, \dots, n_k \cdot a_k]$ beginning with a_1 = No. of $(n-1)$-perm. of $[(n_1-1) \cdot a_1, \dots, n_k \cdot a_k]$.
Here $n = n_1+n_2+\dots+n_k$ and $n_1, n_2, n_3, \dots, n_k \geq 1$. Now

$$\binom{n-1}{n_1-1,\dots,n_k} + \binom{n-1}{n_1,n_2-1,\dots,n_k} + \dots + \binom{n-1}{n_1,n_2,\dots,n_k}$$

= No. of $(n-1)$-perm. of $[(n_1-1) \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k]$
+ No. of $(n-1)$-perm. of $[n_1 \cdot a_1, (n_2-1) \cdot a_2, \dots, n_k \cdot a_k]$ +
$\dots$ + No. of $(n-1)$-perm. of $[n_1 \cdot a_1, n_2 \cdot a_2, \dots, (n_k-1) a_k]$
= No. of n-perm. of $[n_1 \cdot a_1, \dots, n_k \cdot a_k]$ beginning with a_1
+ No. of n-perm. of $[n_1 \cdot a_1, \dots, n_k \cdot a_k]$ beginning with a_2
+ $\dots$ + No. of n-perm. of $[n_1 \cdot a_1, \dots, n_k \cdot a_k]$ beginning with a_k
= No. of n-perm. of $[n_1 \cdot a_1, \dots, n_k \cdot a_k] = \binom{n}{n_1, n_2, \dots n_k}$.

Chapter 6

1. Let A = set of integers in $[1..10{,}000]$ div. by 4
B = " " " 5
C = " " " 6

Then
$$|A| = \left[\frac{10{,}000}{4}\right], \quad |B| = \left[\frac{10{,}000}{5}\right], \quad |C| = \left[\frac{10{,}000}{6}\right]$$
$$|A\cap B| = \left[\frac{10{,}000}{20}\right], \quad |B\cap C| = \left[\frac{10{,}000}{30}\right], \quad |C\cap A| = \left[\frac{10{,}000}{12}\right]$$
and $|A\cap B\cap C| = \left[\frac{10000}{60}\right]$

So no. of integers in $U = \{1..10{,}000]$ that are not divisible by 4, 5 or 6 $= |(A\cup B\cup C)^c|$
$$= |U| - |A| - |B| - |C| + |A\cap B| + |B\cap C| + |C\cap A| - |A\cap B\cap C|$$
$$= 10{,}000 - 2500 - 2000 - 1666 + 500 + 333 + 833 - 166 \qquad = 5{,}334$$

2. For Masochists only

3. Let U = set of integers between 1 & 10,000 (inclusive),
A = set of perfect squares in U
and B = set of " cubes in U. Then
$$|A| = [\sqrt{10{,}000}], \quad |B| = [\sqrt[3]{10{,}000}], \quad |A\cap B| = [\sqrt[6]{10{,}000}]$$

Ans: $= |(A\cup B)^c| = |U| - |A| - |B| + |A\cap B|$
$= 10{,}000 - 100 - 21 + 4$
$= 9{,}883$.

4. Let $U =$ set of all 12-comb. of $T = [\infty \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d]$

A = set of 12-comb. of T with ≥ 5 a's

B = " " T " ≥ 4 b's

C = " " T " ≥ 5 c's

D = " " T " ≥ 6 d's

Then $|U| = \binom{12+4-1}{4-1} = 455$

$|A| = |\text{set of 7-comb. of } T| = \binom{7+4-1}{4-1} = 120$

$|B| = |\text{set of 8-comb. of } T| = \binom{8+4-1}{4-1} = 165$

$|C| = |\text{set of 7-comb. of } T| = \binom{7+4-1}{4-1} = 120$

$|D| = |\text{set of 6-comb. of } T| = \binom{6+4-1}{4-1} = 84$

$|A \cap B| = |\text{set of 3-comb. of } T| = \binom{3+4-1}{4-1} = 20$

$|A \cap C| = |\text{set of 2-comb. of } T| = \binom{2+4-1}{4-1} = 10$

$|A \cap D| = |\text{" 1-comb. of } T| = \binom{1+4-1}{4-1} = 4$

$|B \cap C| = |\text{" 3-comb. "}| = \binom{3+4-1}{4-1} = 20$

$|B \cap D| = |\text{" 2-comb. "}| = \binom{2+4-1}{4-1} = 10$

$|C \cap D| = |\text{" 1-comb. "}| = \binom{1+4-1}{4-1} = 4$

$A \cap B \cap C = A \cap B \cap D = \cdots = A \cap B \cap C \cap D = \emptyset$

No. of 12-comb. of $S = [4 \cdot a, 3 \cdot b, 4 \cdot c, 5 \cdot d]$

$= |(A \cup B \cup C \cup D)^c|$

$= |U| - |A| - |B| - |C| - |D| + |A \cap B| + |A \cap C|$
$\quad + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|$

$= 455 - 120 - 165 - 120 - 84 + 20 + 10$
$\quad + 4 + 20 + 10 + 4$

$= 34.$

5. Let $S' = [10.a, 4.b, 5.c, 7.d]$ and
$T = [\infty.a, \infty.b, \infty.c, \infty.d]$
Then No. of 10-comb. of S
= No. of 10-comb. of S'

Now let $\mathcal{U}$ = set of all 10-comb. of T.
A = set of all 10-comb. of T with ≥ 11 a's
B = " " 10-comb of T with ≥ 5 b's
C = " " 10-comb of T with ≥ 6 c's
D = " " 10-comb of T with ≥ 8 d's

Then $|\mathcal{U}| = \binom{10+4-1}{4-1} = 286$

$A = \emptyset = B \cap C = B \cap D = C \cap D = \cdots = A \cap B \cap C \cap D$

$|B| = |\text{set of 5-comb. of } T| = \binom{5+4-1}{4-1} = 56$

$|C| = |\text{set of 4-comb. of } T| = \binom{4+4-1}{4-1} = 35$

$|D| = |\text{set of 2-comb. of } T| = \binom{2+4-1}{4-1} = 10$

So No. of 10-comb. of $S' = |(A \cup B \cup C \cup D)^c|$

$= |\mathcal{U}| - |A| - |B| - |C| - |D| + |A \cap B| + \cdots - \cdots + |A \cap B \cap C \cap D|$

$= 286 - 0 - 56 - 35 - 10 + 0 + 0 + \cdots + 0$

$= 185$

6. Let a = chocolate doughnut
b = cinnamon doughnut
c = plain doughnut
No. of different boxes of a dozen doughnuts
= No. of 12-comb. of $[6.a, 6.b, 3.c]$
= ...
= ... use the method in #4.

7. No. of solutions of $x_1 + \cdots + x_4 = 14$ in non-negative integers not exceeding 8

$=$ No. of 14-comb. of $[8.a, 8.b, 8.c, 8.d]$

$=$ . . . (use method of #4)

8. No. of solutions of $x_1 + \cdots + x_4 = 14$ in positive integers not exceeding 8

$=$ No. of solutions of $y_1 + \cdots + y_4 = 10$ in non-neg. integers not exceeding 7 Put $y_i + 1 = x_i$, $0 \le y_i \le 7$

$=$ No. of 10-comb. of $[7.a, 7.b, 7.c, 7.d]$

$=$ . . . (use method of #4)

9. Put $y_1 + 1 = x_1$, $y_2 = x_2$, $y_3 + 4 = x_3$ & $y_4 + 2 = x_4$

Then

No. of solutions of $x_1 + \cdots + x_4 = 20$ with $1 \le x_1 \le 6$, $0 \le x_2 \le 7$, $4 \le x_3 \le 8$, $2 \le x_4 \le 6$

$=$ No. of solutions of $y_1 + y_2 + \cdots + y_4 = 13$ with $0 \le y_1 \le 5$, $0 \le y_2 \le 7$, $0 \le y_3 \le 4$, $0 \le x_4 \le 4$

$=$ No. of 13-comb. of $[5.a, 7.b, 4.c, 4.d]$

$=$ . . . (use method of #4)

11. Use the inclusion-exclusion principle

<u>Ans</u>: $\binom{4}{0} 8! - \binom{4}{1} \cdot 7! + \binom{4}{2} 6! - \binom{4}{3} \cdot 5! + \binom{4}{4} \cdot 4!$

Let $\mathcal{U}$ = set of all permutations of $\{1, 2, 3, \ldots, 8\}$ and A_{2i} = set of all perm. in $\mathcal{U}$ with $2i$ in its natural place.

12. <u>Ans:</u> $\binom{8}{4} \cdot D_4 = \frac{8!}{4!\,4!} \cdot 4!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$

pick four integers which will go in their nat. pos.

Derange the other 4 integers

13. No. of perm. of $\{1,2,3,\dots,9\}$ in which at least one odd integer is in its natural position

= No. of perm. of $\{1,2,3,\dots,9\}$
  − No. of perm of $\{1,2,3,\dots,9\}$ in which no odd integer is in its natural position

$$= 9! - \underbrace{\left[\binom{5}{0}9! - \binom{5}{1}8! + \binom{5}{2}\cdot 7! - \binom{5}{3}\cdot 6! + \binom{5}{4}\cdot 5! - \binom{5}{5}\cdot 4!\right]}_{\text{This is similar to problem \# 11.}}$$

14. <u>Ans:</u> $\binom{n}{k} \cdot D_{n-k} = \frac{n!}{k!\,(n-k)!} \cdot k!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^{n-k} \cdot \frac{1}{(n-k)!}\right)$.

pick k integers to go into their nat. pos.

Derange the other n-k integers

15. (a) D_7 = No. of ways of deranging the 7 hats

(b) Ans: = No. of ways of returning hats
  − No. of ways in which no man receives his own hat
  $= 7! - D_7$

(c) Ans $= 7! - D_7 - \binom{7}{1} \cdot D_6$

Here $\binom{7}{1}D_6$ = no. of ways exactly 1 man got his own hat.

16. $$\underbrace{\binom{n}{0}}_{\substack{\text{No. of perm. of } \{1,2,\dots,n\} \text{ with} \\ \text{0 integers in} \\ \text{their nat. pos.}}} \cdot D_n + \underbrace{\binom{n}{1}}_{\substack{\text{No. of perm.} \\ \text{with 1 integer} \\ \text{in its nat.} \\ \text{position.}}} \cdot D_{n-1} + \cdots + \underbrace{\binom{n}{n-1}}_{\substack{\text{No. of perm} \\ \text{with } n-1 \\ \text{integers in} \\ \text{their nat pos.}}} \cdot D_1 + \underbrace{\binom{n}{n}}_{\substack{\text{No. of perm} \\ \text{with } n \\ \text{integers in} \\ \text{their nat. pos.}}} \cdot D_0$$

$=$ total number of permutations of $\{1,2,\dots,n\} = n!$

17. $$\frac{9!}{3!\,4!\,2!} - \left(\frac{7\cdot 6!}{4!\,2!} + \frac{6\cdot 5!}{3!\,2!} + \frac{8\cdot 7!}{3!\,4!}\right) + \left(\frac{4\cdot 3\cdot 2!}{1!\,1!\,2!} + \frac{6\cdot 5\cdot 4!}{1!\,4!\,1!} + \frac{5\cdot 4\cdot 3!}{3!\,1!\,1!}\right) - 3!$$

18. $$(n-1)\left[(n-2)! + (n-1)!\right] = (n-1)\left[(n-2)!\right]\left[1 + (n-1)\right]$$
$$= n\cdot(n-1)\cdot(n-2)! = n!$$

20. Hint: Use Mathematical Induction.

21. (a) Suppose n is odd. Then

$$D_n = \underbrace{(n-1)}_{\text{even}} \cdot (\underbrace{D_{n-1}}_{\text{integer}} + \underbrace{D_{n-2}}_{\text{integer}})$$

So D_n will be even, if n is odd.

(b) We will prove that D_{2k} is odd for each k by induction on k.

For $k=1$, $D_2 = 1$ and so the result is true for $k=1$.

Suppose that the result is true for k.

Then D_{2k} will be odd. So

$$D_{2(k+1)} = D_{2k+2} = \underbrace{(2k+2-1)}_{\text{odd}}\left[\underbrace{D_{2k+1}}_{\text{even by (a)}} + \underbrace{D_{2k}}_{\text{odd}}\right]$$

$\therefore$ $D_{2(k+1)}$ is odd. So by the Principle of Mathematical induction, D_{2k} is odd for each k.

22, 23 (maybe)