Chapter-3

The Pigeonhole Principle (Dirichlet drawer principle) / Shoebox principle.

Pigeonhole Principle : Simple form

Thm - 3.1.1

If $n+1$ objects are distributed into n boxes, then at least one box contains two or more of the objects.

Pf :- This can be proved by method of contradiction. If each of the n boxes contains at most one of the objects, then the total number of objects is at most $1+1+ \cdots +1$ (n 1's) $= n$. But we are having $n+1$ objects. So the boxes can not contain at most one object. Hence if we distribute $n+1$ objects, some box contains at least two of the objects. □

Application 1

Among 13 people there are 2 who have their birthdays in the same month.
(It is a direct application of pigeon-hole principle).

Application 2

There are n married couples. How many of the $2n$ people must be selected to guarantee that a married couple has been selected?

Soln To apply the pigeonhole principle in this case, think of n boxes, one corresponding to each of the n couples.

If we select $n+1$ people and put each of them in the box corresponding to the couple to which they belong, then some box contains two people, i.e. we have selected a married couple. Two of the ways to select n people without getting a married couple are to select all the husbands or all the wives. Therefore $n+1$ is the smallest number that will guarantee that a married couple has been selected.

Application 3

Given m integers $a_1, a_2, \ldots, a_m$, there exist integers k and l with $0 \le k < l \le m$ such that $a_{k+1} + a_{k+2} + \cdots + a_l$ is divisible by m. Less formally, there exist consecutive a's in the sequence $a_1, a_2, \ldots, a_m$ whose sum is divisible by m.

Pf: Let's consider the m sums

$$\underbrace{a_1}_{S_1},\ \underbrace{a_1+a_2}_{S_2},\ \underbrace{a_1+a_2+a_3}_{S_3},\ \ldots,\ \underbrace{a_1+a_2+a_3+\cdots+a_m}_{S_m}.$$

Now if any of the sums is divisible by m, then the conclusion holds. Again we consider that each of these sums has a nonzero remainder when divided by m and so a remainder equals to one of $1, 2, \ldots, m-1$. Since there are m sums and only $m-1$ remainders, two of the sums have the same remainder when divided by m. Therefore there are integers k and l with $k < l$ such that $a_1 + a_2 + \cdots + a_k$, $a_1 + a_2 + \cdots + a_l$ have the same remainder r when divided by n. | 2

$a_1+a_2+\cdots+a_k = bn+r$ —(1) , $a_1+a_2+\cdots+a_l = cn+r$ —(2)

Subtracting eq^n(1) from eq^n(2) we find.

$a_{k+1}+a_{k+2}+\cdots+a_l = (c-b)n$, thus $a_{k+1}+a_{k+2}+\cdots+a_l$ is divisible by n.

Example: let $n=7$, The integers are

2, 4, 6, 3, 5, 5, 6.

computing the sums, we get

2, 6, 12, 15, 20, 25, 31 whose remainders when divided by 7 are 2, 6, 5, 1, 6, 4, 3 respectively. Now we have two remainders = 6.

$\Rightarrow 6+3+5=14$ is divisible by 7.

Ex: let $n=5$, The integers are

2, 4, 5, 6, 7

computing the sums, we get

2, 6, 11, 17, 24 whose remainders when divided by 5 are 2, 1, 1, 2, 4.

Now have two remainders = 1.

So $a_1+a_2 = 2+4 = 6$

$a_1+a_2+a_3 = 2+4+5 = 11$

$\Rightarrow 11-6 = 5$ which is divisible by 5.

Ex: let $n=8$, The integers are

1, 2, 3, 4, 5, 6, 7, 8

Computing the sums 1, 3, 6, 10, 15, 21, 28, 36 whose remainders when divided by 8 are

1, 3, 6, 2, 7, 5, 4, 4. Now we have two remainders = 4.

So $(a_1+a_2+ \dots + a_8) - (a_1+a_2+ \dots + a_7) = 8$

is divisible by 8 □

Application 4

A chess master who has 11 weeks to prepare for a tournament decides to play at least one game everyday but to avoid tiring himself, he decides not to play more than 12 games during any calender week. Show that there exists a succession of days during which the chess master will have played exactly 21 games.

Pf:- Let a_1 be the number of games played on the first day, a_2 the total number of games played on the first and second days, a_3 the total number of games played on the first, 2nd and third days and so on. The sequence $a_1, a_2, \dots, a_{77}$ is a strictly increasing sequence since at least one game is played each day. More over $a_1 \geq 1$ and since at most 12 games are played during one week, $a_{77} \leq 12 \times 11 = 132$

Hence we have $1 \leq a_1 < a_2 < \dots < a_{77} \leq 132$.

The sequences $a_1+21, a_2+21, \dots < a_{77}+21$

Now each of the 154 numbers $\leq 132+21 = 153$

$a_1, a_2, \dots, a_{77}, a_1+21, a_2+21, \dots, a_{77}+21$

is an integer between 1 and 153.

$\Rightarrow$ Exactly two of the above 154 numbers must be same, ~~that~~ since no two of the numbers $a_1, a_2, ---, a_{77}$ are equal and no two of the numbers $a_1+21, a_2+21, ------, a_{77}+21$ are equal, there must be an i and a j such that $a_i = a_j + 21$. Therefore on days $j+1, j+2, --- , i$, the Chess Master must have played 21 games.

$(\because a_i - a_j = 21)$.

Application 5

From the Integers $1, 2, ---, 200$, we choose 101 Integers. Show that, among the Integers chosen, there are two such that one of them is divisible by other.

Pf :- We know that any ~~number~~ Integer can be written in the form of $2^k \times a$ where $k \geq 0$ and a is odd. For any Integer between 1 and 200, a is one of the 100 numbers $1, 3, 5, --, 199$. Thus among the 101 Integers chosen, there are two having a's of equal value when written in this form.

Let $x = 2^r \times a$
$y = 2^s \times a$. If $r < s$ the second number is divisible by first. If $r > s$ then the first is divisible by the second □

Application 6 (Chinese remainder Thm)

Let m_i, m_j be pairwise co-prime i.e $(m_i, m_j) = 1$ whenever $i \neq j$. Then the system of n equations

$x \equiv a_1 \pmod{m_1}$, $x \equiv a_2 \pmod{m_2}$, $x \equiv a_3 \pmod{m_3}$

$\vdots$

$x \equiv a_n \pmod{m_n}$ has a unique solution.

* Let p, q be co-prime. Then the system of equations $x \equiv a \pmod{p}$, $x \equiv b \pmod{q}$ has a unique solution for x modulo pq.

Ex:- solve $x \equiv 2 \mod 3$
$x \equiv 3 \mod 5$
$x \equiv 2 \mod 7$,

Solⁿ $m_1 = 3$, $m_2 = 5$, $m_3 = 7$

$a_1 = 2$, $a_2 = 3$, $a_3 = 2$.

$x \equiv (a_1 M_1 M_1^{-1} + a_2 M_2 M_2^{-1} + a_3 M_3 M_3^{-1}) \mod M$

$M = m_1 m_2 m_3 = 3 \times 5 \times 7 = 105$

$M_1 = \frac{M}{m_1} = \frac{105}{3} = 35$, $M_2 = \frac{M}{m_2} = \frac{105}{5} = 21$,

$M_3 = \frac{M}{m_3} = \frac{105}{7} = 15$

$M_1 M_1^{-1} \equiv 1 \mod 3$, $M_1^{-1} = 2$

$35 \times \boxed{2} \equiv 1 \mod 3$

$M_2 M_2^{-1} \equiv 1 \mod 5$,

$21 \times \boxed{1} \equiv 1 \mod 5$, $M_2^{-1} = 1$

$M_3 M_3^{-1} \equiv 1 \mod 7$,

$15 \times \boxed{1} \equiv 1 \mod 7$, $M_3^{-1} = 1$

$x \equiv (2 \times 35 \times 2 + 3 \times 21 \times 1 + 2 \times 15 \times 1) \mod 105$

$\Rightarrow x \equiv 233 \mod 105$

$\Rightarrow \boxed{x = 23}$

Ex:- solve $x \equiv 3 \mod 5$
$x \equiv 1 \mod 7$
$x \equiv 6 \mod 8$.

Solⁿ $m_1 = 5$, $m_2 = 7$, $m_3 = 8$

$a_1 = 3$, $a_2 = 1$, $a_3 = 6$, $M_3 = \frac{M}{m_3} = \frac{280}{8} = 35$

$M = m_1 m_2 m_3 = 5 \times 7 \times 8 = 280$

$M_1 = \frac{M}{m_1} = \frac{280}{5} = 56$, $M_2 = \frac{M}{m_2} = \frac{280}{7} = 40$

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$M_1 M_1^{-1} \equiv 1 \mod 5$

$\Rightarrow 56 \times M_1^{-1} \equiv 1 \mod 5$, $M_1^{-1} = 1$

$\Rightarrow M_1^{-1} \equiv 1 \mod 5$

$M_2 M_2^{-1} \equiv 1 \mod 7$.

$40 M_2^{-1} \equiv 1 \mod 7$, $M_2^{-1} = 3$

$\Rightarrow 5 M_2^{-1} \cong 1 \mod \cancel{7}$

$M_3 M_3^{-1} \equiv 1 \mod 8$

$35 M_3^{-1} \equiv 1 \mod 8$, $M_3^{-1} = 3$

$\Rightarrow 3 M_3^{-1} \cong 1 \mod 8$

Now $x \equiv 3 \times 56 \times 1 + 1 \times 40 \times 3 + 6 \times 35 \times 3 \mod 280$

$\Rightarrow x \equiv 918 \mod 280$

$\Rightarrow x \cong 78 \mod 280$.

$\Rightarrow \boxed{x = 78}$

3.2 Pigeonhole principle: strong form

Thm - 3.2.1

Let $q_1, q_2, ---, q_n$ be positive integers. If $q_1 + q_2 + --- + q_n - n + 1$ objects are distributed into n boxes, then either the first box contains at least q_1 objects or the 2nd box contains at least q_2 objects, ----, or the nth box contains at least q_n objects.

Pf: Let's assume that the ith box contains fewer than q_i objects for $i = 1, 2, -- n$, then the total number of objects in all boxes does not exceed $(q_1 - 1) + (q_2 - 1) + ---- + (q_n - 1)$

$= q_1 + q_2 + --- + q_n - n$.

Since this number is one less than the number of objects distributed $\Rightarrow$

the i-th box contains at least q_i objects.
Note If $q_1 = q_2 = \cdots = q_n = 2$, then simple form of pigeonhole principle is obtained from strong form. This theorem generalizes the simple form.

Corollary - 3.2.2

Let n and r be positive integers. If $n(r-1)+1$ objects are distributed into n boxes, then at least one of the boxes contains r or more of the objects.

Pf:- If the average of n non negative integers $m_1, m_2, \ldots, m_n$ is greater than $r-1$ or

i.e. $\frac{m_1+m_2+\cdots+m_n}{n} > r-1$ —(1) then at least one of the integers is greater than or equal to r. Let's put $n(r-1)+1$ objects into n boxes. Let m_i be the no. of objects in the ith box so from eq^n(1), we have

$$\frac{m_1+m_2+\cdots+m_n}{n} = \frac{n(r-1)+1}{n}$$

$$=(r-1)+\frac{1}{n}.$$

Since this average is greater than $r-1$, so one of m_i is at least r. In other words, one boxes contains at least r objects.

Application :- 7

A basket of fruit is being arranged out of apples, bananas and oranges. What is the smallest number of pieces of fruit that should be put in the basket to guarantee that either there are at least eight apples or at least six bananas or at least nine oranges?

Sol^n By applying the strong form of the pigeonhole principle, $8+6+9-3+1=21$ pieces of fruit no matter how selected, will guarantee a basket of fruit with the desired properties.

Exercise – 3.4

(3) Generalize Application 5 by choosing how many integers from the set $\{1, 2, --, 2n\}$

Pf The set given is $\{1, 2, 3, - - - , 2n\}$. Every element K of the set can be written as $K = 2^m p_i$, $m \geq 0$ and p_i is odd. For given m there are n different p_i from the set $\{1, 3, 5, - - - , 2n-1\}$

From the Th^m – 3.1.1, we know that if a total of $(n+1)$ integers have to chosen from this set ⟹ There are at least two elements $K_i = 2^{m_i} p_i$ and $K_j = 2^{m_j} p_j$ such that $p_i = p_j$. Now we have

$$p_i = p_j$$

$$\Rightarrow \frac{K_i}{2^{m_i}} = \frac{K_j}{2^{m_j}}$$

$$\Rightarrow \boxed{\frac{K_i}{K_j} = 2^{m_i - m_j}}$$

⟹ K_i is divisible by K_j. Hence atleast one of the set $\{1, 2, 3, -- , 2n\}$ is divisible by other and the application is generalized.

(4) Show that if $n+1$ integers are chosen from the set $\{1, 2, ---, 2n\}$, then there are always two which differ by 1.

Pf:- The above can be proved by pigeonhole principle. First partition the set $\{1, 2, ----, 2n\}$ into n-boxes with two numbers in each as follows $\{1, 2\}$, $\{3, 4\}$, ----, $\{2n-1, 2n\}$. If total of $n+1$ numbers are chosen from n boxes, by the pigeonhole principle it shows that at least two numbers coming from the same box. Then only gets two numbers that are differed by 1.

(10) A child watches TV at least one hour each day for seven weeks but, because of parental rules, never more than 11 hrs in any one week. Prove that there is some period of consecutive days in which the child watches exactly 20 hours of the TV. (It is assumed that the child watches TV for a whole number of hours each day).

Pf:- It is a direct consequence of Application-4.

Let a_1 = No. of hrs on 1st day the child played.

a_2 = No. of hrs on 1st day and 2nd day the child played.

a_3 = No. of hrs on the 1st day, 2nd day and 3rd day the child played and so on.

$$1 \le a_1 < a_2 < ---- < a_{49}$$

and $21 \le a_1+20 < a_2+20 < ---- < a_{49}+20 \le 11 \times 7 + 20 = 97$

Thus each of the 98 numbers $a_1, a_2, ---, a_{49}, a_1+20, a_2+20, --, a_{49}+20$ is an integer between 1 and 97. It follows that two of them are equal. Since no two of $a_1, a_2, --a_{49}$ ~~these~~ are equal and no two of $a_1+20, a_2+20, ---, a_{49}+20$ are equal, there must be an i and j such that $a_i = a_j + 20$. Therefore on days $j+1, j+2, ---, i$, the child watched TV for exactly 20 hrs.