Chapter – 6

sub – ADM
Introductory combinatorics

The Inclusion – Exclusion Principle and Applications

In this chapter, we derive an important counting formula called Inclusion–exclusion principle. Recall that the addition principle gives a formula for counting the number of objects in a union of sets provided that the sets do not overlap. The Inclusion–exclusion principle gives a formula for the most general of circumstances in which the sets are free to overlap without restriction.

The Inclusion–exclusion principle

Let's explain the above principle by means of an example.

Ex: Count the number of permutations $i_1 i_2 \cdots i_n$ of $\{1,2,\cdots,n\}$ in which 1 is not in first position. (i.e. $i_1 \neq 1$)

Sol^n Direct Method

1st	2nd	3rd		
$n-1$	$n-2$	$n-3$	$\cdots\cdots$	1

We observe that the permutations with 1 not in the first position can be divided into $n-1$ parts according to which $n-1$ integers k from $\{2,3,\cdots,n\}$ is in the first position. A permutation with k in the first position consists of k followed by a permutation of the $(n-1)$ elements set $\{1,2,\cdots,k-1,k+1,\cdots,n\}$. Hence there are $(n-1)!$ permutations of $\{1,2,\cdots,n\}$ with k in the first position. By addition principle, there are $(n-1).(n-1)!$ permutations of $\{1,2,\cdots,n\}$ with 1 not in the first position.

Exclusion

We could use the subtraction principle by observing that the number of permutations of $\{1, 2, --, n\}$ with 1 in the first position is same as the number $(n-1)!$ permutations of $\{2, 3, --, n\}$. Since the total number of permutations of $\{1, 2, --, n\}$ is $n!$, so the number of permutations of $\{1, 2, --, n\}$ in which 1 is not in the first position is $n! - (n-1)!$ $= (n-1)(n-1)!$

The subtraction principle is the simplest instance of the inclusion-exclusion principle.

Generalization of the subtraction principle as inclusion-exclusion principle

Let S be a finite set of objects and let P_1 and P_2 be two properties that each object in S may or may not possess. We want to count the number of objects in S that have neither of the properties P_1 and P_2. Let A_1 be the subset of objects of S that have property P_1 and let A_2 be the subset of objects of S that have property P_2. Then $\overline{A_1}$ consists of those objects of S not having property P_1 and $\overline{A_2}$ consists of those objects of S not having property P_2. The objects of the set $\overline{A_1} \cap \overline{A_2}$ are those having neither property P_1 nor property P_2. Then we have $|\overline{A_1} \cap \overline{A_2}| = |\overline{(A_1 \cup A_2)}|$

$\Rightarrow |\overline{A_1} \cap \overline{A_2}| = |S| - |A_1 \cup A_2|$

$= |S| - \{|A_1| + |A_2| - |A_1 \cap A_2|\}$

$\Rightarrow |\overline{A_1} \cap \overline{A_2}| = |S| - |A_1| - |A_2| + |A_1 \cap A_2|$

Ex: Count the number of integers between 1 and 600, inclusive, which are not divisible by 6.

Sol: Inclusion: Total number of integers $= 600$ and total number of integers divisible by 6

$= \frac{600}{6} = 100$.

Exclusion: Total number of integers from the set $\{1, 2, ---, 600\}$ which are not divisible by 6 is given by $600 - 100 = 500$ Ans

Theorem-6.1.1

Let $P_1, P_2, ---, P_m$ be m properties referring to the objects in S and let $A_i = \{x : x \text{ in } S \text{ and } x \text{ has property } P_i\}$ $(i = 1, 2, --, m)$

Then $A_i \cap A_j$ is the subsets of objects that have both properties P_i and P_j. $A_i \cap A_j \cap A_k$ is the subset of objects that have properties P_i, P_j and P_k and so on. The subset of objects having none of the properties $\overline{A_1} \cap \overline{A_2} \cap --- \cap \overline{A_m}$. Then the number of objects of the set S that have none of the properties $P_1, P_2, ---, P_m$ is given by

$$|\overline{A_1} \cap \overline{A_2} \cap --- \cap \overline{A_m}| = |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + ----+ (-1)^m |A_1 \cap A_2 \cap --- \cap A_m|$$

For $n=3$, Th^m 6.1.1 reduces to

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |S| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|$$

For $n=4$, Th^m 6.1.1 reduces to

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = |S| - (|A_1| + |A_2| + |A_3| + |A_4|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|) - (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) + |A_1 \cap A_2 \cap A_3 \cap A_4|$$

Corollary - 6.1.2

The number of objects of S which have at least one of the properties $P_1, P_2, \dots, P_n$ is given by

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \dots + (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n|$$

where the summations are as specified in Th^m - 6.1.1.

Ex: Find the number of integers between 1 and 1000, inclusive that are not divisible by 5, 6 and 8.

Sol^n: $A_1 = \{x \mid x$ is divisible by 5 and $1 \le x \le 1000\}$

$$\Rightarrow |A_1| = \left\lfloor \frac{1000}{5} \right\rfloor = 200$$

$A_2 = \{y \mid y$ is divisible by 6 and $1 \le y \le 1000\}$

$$|A_2| = \left\lfloor \frac{1000}{6} \right\rfloor = 166$$

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$A_3 = \{z \mid z$ is divisible by 8, $1 \le z \le 1000\}$

$$|A_3| = \left\lfloor \frac{1000}{8} \right\rfloor = 125$$

Let S be the set of all integers from 1 to 1000 both inclusive. Now have to find $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$. So by using Th^m-6.1.1,

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|$$

Now $|A_1 \cap A_2|$ = integers divisible by 5 and 6 lying between 1 and 1000 inclusive.
= integers divisible by 30

$$= \left\lfloor \frac{1000}{30} \right\rfloor = 33 \quad //$$

$|A_2 \cap A_3|$ = integers divisible by 6 and 8 lying between 1 and 1000 inclusive.
= integers divisible by 24

$$= \left\lfloor \frac{1000}{24} \right\rfloor = 41 \quad //$$

$|A_1 \cap A_3|$ = integers divisible by 40 lying between 1 and 1000 inclusive.

$$= \left\lfloor \frac{1000}{40} \right\rfloor = 25$$

similarly $|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{1000}{120} \right\rfloor = 8$

Hence the required numbers of integers is equal to

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = 1000 - (200 + 166 + 125) + (33 + 41 + 25) - 8$$

$$= 600 \text{ Ans}$$

Ex:- How many permutations of the letters M, A, T, H, I, S, F, U, N are there such that none of the words MATH, IS, FUN occur as consecutive letters?

Soln:- Let S be the set of all permutations of the 9 letters given. Let P_1 be the property that a permutation in S contains the word MATH as consecutive letters, let P_2 be the property that a permutation contains the word IS as consecutive letters and let P_3 be the property that a permutation contains the word FUN as consecutive letters. For $i = 1, 2, 3$, let A_i be the set of those permutations in S satisfying property P_i. We wish to find the number of permutations in $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$. We have $|S| = 9! = 362,880$

The permutation in A_1 can be bought of permutations of six symbols: MATH, I, S, F, U, N treating MATH as one symbol. So $|A_1| = 6! = 720$

Similarly the permutations in A_2 are permutations of the eight symbols M, A, T, H, IS, F, U, N

So $|A_2| = 8! = 40,320$

The permutations in A_3 are permutations of the seven symbols M, A, T, H, I, S, FUN.

So $|A_3| = 7! = 5040$

The permutations in $A_1 \cap A_2$ are permutations of the five symbols MATH, IS, F, U, N.

$|A_1 \cap A_2| = 5! = 120$

The permutations in $A_1 \cap A_3$ are permutations of the four symbols MATH, I, S, FUN.

So $|A_1 \cap A_3| = 4! = 24$

Again the permutations in $A_2 \cap A_3$ are permutations of the six symbols M, A, T, H, IS, FUN. So $|A_2 \cap A_3| = 6! = 720$

Finally, $A_1 \cap A_2 \cap A_3$ consists of the permutations of the three symbols MATH, IS, FUN. Therefore $|A_1 \cap A_2 \cap A_3| = 3! = 6$

Hence $|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}| = |S| - \{|A_1| + |A_2| + |A_3|\} + \{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\} - |A_1 \cap A_2 \cap A_3|$

$= 362{,}880 - 720 - 40{,}320 - 5040 + 120 + 24 + 720 - 6$

$= 317{,}658$ Ans

Some more application of inclusion-exclusion principle

Assume that the size of the set $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$ that occurs in the inclusion-exclusion principle depends only on k and not on which k sets are used in the intersection. Thus, there are constants $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_m$ s.t

$$|S| = \alpha_0, \quad |A_1| = |A_2| = \cdots = |A_m| = \alpha_1$$

$$|A_1 \cap A_2| = |A_2 \cap A_3| = \cdots = |A_{m-1} \cap A_m| = \alpha_2$$

$$|A_1 \cap A_2 \cap A_3| = |A_4 \cap A_5 \cap A_6| = \cdots = |A_{m-2} \cap A_{m-1} \cap A_m| = \alpha_3$$

$$\vdots$$

$$|A_1 \cap A_2 \cap \cdots \cap A_m| = \alpha_m$$

In this case, the inclusion-exclusion principle simplifies to:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_m}| = \alpha_0 - \binom{m}{1}\alpha_1 + \binom{m}{2}\alpha_2 - \binom{m}{3}\alpha_3 + \cdots + (-1)^k \binom{m}{k}\alpha_k + \cdots + (-1)^m \alpha_m$$

Ex:- How many integers between 0 and 99,999 (inclusive) have among their digits each of 2, 5 and 8?

Soln Let S be the set of integers between 0 and 99,999. Each integer in S has 5 digits including possible leading 0s.

Let P_1 be the property that an integer does not contain the digit 2, let P_2 be the property that an integer does not contain the digit 5 and let P_3 be the property that an integer does not contain the digit 8. For $i = 1, 2, 3$ let A_i be the set consisting of those integers in S with property P_i.

We wish to count the number of integers in $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$.

$\alpha_0 = 10^5$

$\alpha_1 = 9^5$

$\alpha_2 = 8^5$

$\alpha_3 = 7^5$

Hence required number of integers $= \alpha_0 - \binom{3}{1}\alpha_1 + \binom{3}{2}\alpha_2 - \binom{3}{3}\alpha_3$

$= 10^5 - 3 \times 9^5 + 3 \times 8^5 - 7^5$ Ans

6.2 Combinations with Repetition

We know that the number of r-combinations of a multiset with k distinct objects, each with an infinite repetition number, equals to $\binom{r+k-1}{r}$.

In this section, with inclusion-exclusion principle, we give a method for finding the number of r-combinations of a multiset without any restriction on its repetition of numbers.

Suppose T is a multiset and an object x of T of a certain type has a repetition number that is greater than r.

The number of r-combinations of T equals the number of r-combinations of the multiset obtained from T by replacing the repetition number of x by r. The number of times x can be used in an r-combination of T cannot exceed r. Therefore any repetition numbers that is greater than r can be replaced by r.

For example, the number of 8-combination of the multiset $\{3 \cdot a, \infty \cdot b, 6 \cdot c, 10 \cdot d, \infty \cdot e\}$ equals to the number of 8-combinations of the multiset $\{3 \cdot a, 8 \cdot b, 6 \cdot c, 8 \cdot d, 8 \cdot e\}$.

We can summarize that the number of r-combinations of a multiset $T = \{n_1 \cdot a_1, n_2 \cdot a_2, ---, n_k \cdot a_k\}$ in two extreme cases

(i) $n_1 = n_2 = \cdots = n_k = 1$ (i.e T is a set)

(ii) $n_1 = n_2 = \cdots = n_k = r$

Let's consider the specific example to clear more in general.

Ex :- Determine the number of 10-combinations of the multiset $T = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$.

Sol Consider $T^* = \{10 \cdot a, 10 \cdot b, 10 \cdot c\}$

Let P_1 be the property that a 10-combination of T^* has more than 3 a's. Let P_2 be the property that a 10-combination of T^* has more than 4 b's. Let P_3 be the property that a 10-combination of T^* has more than 5 c's. The number of 10-combinations of T is then the number of 10-combination of T^* that have none of the property P_i ($i = 1, 2, 3$). We will find the size of $\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}$. By the Inclusion-exclusion principle,

$$|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}| = |S| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|$$

$$|S| = \binom{10+3-1}{10} = \binom{12}{10} = {}^{12}C_2 = \frac{12 \times 11}{1 \times 2} = 66$$

Let A_i consists of those 10-combinations of T^* which have property P_i ($i = 1, 2, 3$).

The set A_1 consists of all 10-combinations of T^* in which a occurs at least four times. If we take any of these 10-combinations in A_1 and remove four a's, we are left with a 6-combination of T^*. Conversely, If we take a 6-combination of T^* and four a's to it, we get a 10-combination of T^* in which a occurs at least 4 times. Thus the number of 10-combinations in A_1 equals the number of 6-combinations of T^*. Hence

$|A_1| = \binom{6+3-1}{6} = \binom{8}{6} = 28$. Similarly $|A_2| = \binom{5+3-1}{5} = \binom{7}{5} = 21$

$|A_3| = \binom{4+3-1}{4} = \binom{6}{4} = 15$

$|A_1 \cap A_2| = \binom{1+3-1}{1} = \binom{3}{1} = 3$

$|A_1 \cap A_3| = \binom{0+3-1}{0} = \binom{2}{0} = 1$

$|A_2 \cap A_3| = 0$, $|A_1 \cap A_2 \cap A_3| = 0$.

Now using Inclusion-exclusion principle

$|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}| = 66 - (28+21+15) + (3+1+0) - 0$
$= 6$

Ex: What is the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 18$$

that satisfy $1 \le x_1 \le 5$, $-2 \le x_2 \le 4$, $0 \le x_3 \le 5$, $3 \le x_4 \le 9$?

Soln Let $y_1 = x_1 - 1$, $y_2 = x_2 + 2$, $y_3 = x_3$ and $y_4 = x_4 - 3$

Now the given eqn reduces to

$$y_1 + y_2 + y_3 + y_4 = 16 \quad \text{— (1)}$$

$0 \le y_1 \le 4$, $0 \le y_2 \le 6$, $0 \le y_3 \le 5$, $0 \le y_4 \le 6$.

Let S be the set of all non negative integral solutions of eqn (1). Then

$|S| = \binom{16+4-1}{16} = \binom{19}{16} = 969$

Let P_1 be the property that $y_1 \geq 5$, P_2 be the property that $y_2 \geq 7$, P_3 be the property that $y_3 \geq 6$ and P_4 be the property that $y_4 \geq 7$. Let A_i denote the subset of S consisting of the solutions satisfying property P_i, ($i = 1, 2, 3, 4$). We want to find the size of $\bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cap \bar{A_4}$

The set A_1 consists of all those solutions in S for which $y_1 \geq 5$.

So let $z_1 = y_1 - 5$, $z_2 = y_2$, $z_3 = y_3$, $z_4 = y_4$

The number of solutions in A_1 is same as the number of nonnegative integral solutions of $z_1 + z_2 + z_3 + z_4 = 11$.

$z_1 \geq 0, z_2 \geq 0, z_3 \geq 0, z_4 \geq 0$

So $|A_1| = \binom{11+4-1}{11} = \binom{14}{11} = 364$

Similarly for A_2, $y_2 \geq 7$

So let $z_1 = y_1$, $z_2 = y_2 - 7$, $z_3 = y_3$, $z_4 = y_4$

Eq^n(1) reduces to

$$z_1 + z_2 + 7 + z_3 + z_4 = 16$$

$$z_1 \geq 0, z_2 \geq 0, z_3 \geq 0, z_4 \geq 0$$

$$\Rightarrow z_1 + z_2 + z_3 + z_4 = 9$$

$$z_1 \geq 0, z_2 \geq 0, z_3 \geq 0, z_4 \geq 0$$

$$\text{So } |A_2| = \binom{9+4-1}{9} = \binom{12}{9} = 220$$

Again for A_3, $y_3 \geq 6$.

Let $z_1 = y_1$, $z_2 = y_2$, $z_3 = y_3 - 6$, $z_4 = y_4$

Eq^n(1) reduces to $z_1 + z_2 + z_3 + z_4 = 10$

$$z_1 \geq 0, z_2 \geq 0, z_3 \geq 0, z_4 \geq 0$$

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$$|A_3| = \binom{10+4-1}{10} = \binom{13}{10} = 286$$

For A_4, $y_4 \geq 7$.

let $z_1 = y_1$, $z_2 = y_2$, $z_3 = y_3$, $z_4 = y_4 - 7$

Eq^n(1) reduces to

$$z_1 + z_2 + z_3 + z_4 = 9$$

$$z_1 \geq 0,\ z_2 \geq 0,\ z_3 \geq 0,\ z_4 \geq 0.$$

so $|A_4| = \binom{9+4-1}{9} = \binom{12}{9} = 220$

Similarly for $A_1 \cap A_2$, $y_1 \geq 5$, $y_2 \geq 7$,

let $z_1 = y_1 - 5$, $z_2 = y_2 - 7$, $z_3 = y_3$, $z_4 = y_4$

so eq^n(1) reduces to

$$z_1 + z_2 + z_3 + z_4 = 4$$

$$z_1 \geq 0,\ z_2 \geq 0,\ z_3 \geq 0,\ z_4 \geq 0.$$

so $|A_1 \cap A_2| = \binom{4+4-1}{4} = \binom{7}{4} = 35$

For $A_2 \cap A_3$, $y_2 \geq 7$, $y_3 \geq 6$

let $z_1 = y_1$, $z_2 = y_2 - 7$, $z_3 = y_3 - 6$, $z_4 = y_4$

so eq^n(1) reduces to

$$z_1 + z_2 + z_3 + z_4 = 3$$

$$z_1, z_2, z_3, z_4 \geq 0$$

$$|A_2 \cap A_3| = \binom{3+4-1}{3} = \binom{6}{3} = 20$$

For $A_2 \cap A_4$, $y_2 \geq 7$, $y_4 \geq 7$

let $z_1 = y_1$, $z_2 = y_2 - 7$, $z_3 = y_3$, $z_4 = y_4 - 7$

so eq^n(1) reduces to

$$z_1 + z_2 + z_3 + z_4 = 2$$

$$z_1, z_2, z_3, z_4 \geq 0$$

so $|A_2 \cap A_4| = \binom{4+2-1}{2} = \binom{5}{2} = 10$

For $A_1 \cap A_3$, $y_1 \geq 5$, $y_3 \geq 6$

let $z_1 = y_1 - 5$, $z_2 = y_2$, $z_3 = y_3 - 6$, $z_4 = y_4$

So eqn(1) reduces to

$$z_1 + z_2 + z_3 + z_4 = 5$$

$$z_1 \geq 0,\ z_2 \geq 0,\ z_3 \geq 0,\ z_4 \geq 0.$$

$$|A_1 \cap A_3| = \binom{4+5-1}{5} = \binom{8}{5} = 56$$

For $A_1 \cap A_4$, $y_1 \geq 5$, $y_4 \geq 7$

let $z_1 = y_1 - 5$, $z_2 = y_2$, $z_3 = y_3$, $z_4 = y_4 - 7$

so eqn(1) reduces to $z_1 + z_2 + z_3 + z_4 = 4$

$$z_1 \geq 0,\ z_2 \geq 0,\ z_3 \geq 0,\ z_4 \geq 0.$$

$$|A_1 \cap A_4| = \binom{4+4-1}{4} = \binom{7}{4} = 35$$

For $A_3 \cap A_4$, $y_3 \geq 6$, $y_4 \geq 7$

Let $z_1 = y_1$, $z_2 = y_2$, $z_3 = y_3 - 6$, $z_4 = y_4 - 7$

so eqn(1) reduces to

$$z_1 + z_2 + z_3 + z_4 = 3$$

$$z_1 \geq 0,\ z_2 \geq 0,\ z_3 \geq 0,\ z_4 \geq 0$$

$$|A_3 \cap A_4| = \binom{4+3-1}{3} = \binom{6}{3} = 20$$

since the intersection of any of the three sets A_1, A_2, A_3, A_4 is empty. Now by applying inclusion-exclusion principle

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = 969 - (364 + 220 + 286 + 220) + (35 + 56 + 35 + 20 + 10 + 20)$$

$$= 55 \text{ Ans}$$

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Exercise - 6.7

(2) Find the number of Integers between 1 and 10,000 Inclusive that are not divisible by 4, 6, 7 or 10.

Soln Let S be the set of all Integers between 1 and 10,000. $|S| = 10,000$

Let $A \subset S$ and $A = \{x \mid x$ is divisible by $4\}$

$$|A| = \left\lfloor \frac{10,000}{4} \right\rfloor = 2500$$

$B \subset S$ and $B = \{x \mid x$ is divisible by $6\}$

$$\Rightarrow |B| = \left\lfloor \frac{10,000}{6} \right\rfloor = 1666$$

$C \subset S$ and $C = \{x \mid x$ is divisible by $7\}$

$$|C| = \left\lfloor \frac{10,000}{7} \right\rfloor = 1428$$

$D \subset S$ and $D = \{x \mid x$ is divisible by $10\}$

$$|D| = \left\lfloor \frac{10,000}{10} \right\rfloor = 1000$$

$A \cap B = \{x \mid x$ is divisible by both 4 and $6\}$

$$= \left\lfloor \frac{10,000}{12} \right\rfloor = 833$$

$B \cap C = \{x \mid x$ is divisible by both 6 and $7\}$

$$= \left\lfloor \frac{10000}{42} \right\rfloor = 238$$

$A \cap C = \{x \mid x$ is divisible by both 4 and $7\}$

$$= \left\lfloor \frac{10000}{28} \right\rfloor = 357$$

$A \cap D = \{x \mid x$ is divisible by both 4 and $10\}$

$$= \left\lfloor \frac{10,000}{20} \right\rfloor = 500$$

$B \cap D = \{x \mid x$ is divisible by both 6 and $10\}$

$$= \left\lfloor \frac{10,000}{30} \right\rfloor = 333$$

$C \cap D = \{x \mid x$ is divisible by both 7 and $10\}$

$$= \left\lfloor \frac{10,000}{70} \right\rfloor = 142$$

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$A\cap B\cap C = \{x \mid x \text{ is divisible by } 4, 6 \text{ and } 7\}$

$|A\cap B\cap C| = \left\lfloor \frac{10,000}{84} \right\rfloor = 119$

$A\cap B\cap D = \{x \mid x \text{ is divisible by } 4, 6 \text{ and } 10\}$

$|A\cap B\cap D| = \left\lfloor \frac{10,000}{60} \right\rfloor = 166$

$B\cap C\cap D = \{x \mid x \text{ is divisible by } 6, 7 \text{ and } 10\}$

$|B\cap C\cap D| = \left\lfloor \frac{10,000}{210} \right\rfloor = 47$

$A\cap C\cap D = \{x \mid x \text{ is divisible by } 4, 7 \text{ and } 10\}$

$|A\cap C\cap D| = \left\lfloor \frac{10,000}{140} \right\rfloor = 71$

$A\cap B\cap C\cap D = \{x \mid x \text{ is divisible by } 4, \cancel{6}, 7 \text{ and } 10\}$

$= \left\lfloor \frac{10,000}{420} \right\rfloor = 23$

Now $|\bar{A}\cap\bar{B}\cap\bar{C}\cap\bar{D}| = |S| - (|A|+|B|+|C|+|D|) + (|A\cap B|$
$+ |B\cap C| + |A\cap C| + |A\cap D| + |B\cap D| + |C\cap D|)$
$- (|A\cap B\cap C| + |A\cap B\cap D| + |B\cap C\cap D| + |A\cap C\cap D|)$
$+ |A\cap B\cap C\cap D|$

$= 10,000 - 2500 - 1666 - 1428 - 1000 + 833 + 357 + 500$
$+ 238 + 333 + 142 - 119 - 166 - 71 - 47 + 23$

$= 5429$ Ans

(4) Determine the number of 12-combination of the multiset $S = \{4\cdot a, 3\cdot b, 4\cdot c, 5\cdot d\}$

Soln $S = \{4\cdot a, 3\cdot b, 4\cdot c, 5\cdot d\}$

$S^* = \{12a, 12b, 12\cdot c, 12\cdot d\}$ or $\{\infty\cdot a, \infty\cdot b, \infty\cdot c, \infty\cdot d\}$

Let P be the set of all 12-Combination of S^*, $|P| = \binom{12+4-1}{12} = \binom{15}{12} = \binom{15}{3} = \frac{15\times14\times13}{1\times2\times3}$

$= 455$

Let A be the set of all 12-combination of S^* $7/5a$

~~PHP~~ For $|A|$,

$$x_1+x_2+x_3+x_4=12 \quad -(1)$$

$$x_1 \geq 5,\ x_2 \geq 0;\ x_3 \geq 0,\ x_4 \geq 0$$

Let $x_1-5=y_1,\ x_2=y_2,\ x_3=y_3,\ x_4=y_4$

Eqn(1) reduces to

$$y_1+y_2+y_3+y_4=7$$

$$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0$$

So $|A| = \binom{7+4-1}{7} = \binom{10}{7} = \binom{10}{3}$

$$= \frac{10 \times 9 \times 8}{1 \times 2 \times 3} = 120$$

Let B be the set of all 12-combination of $S^* \geq 4b$. For $|B|$, $x_1+x_2+x_3+x_4=12$

$$x_1 \geq 0,\ x_2 \geq 4,\ x_3 \geq 0,\ x_4 \geq 0$$

Let $x_1=y_1,\ x_2-4=y_2,\ x_3=y_3,\ x_4=y_4$

Eqn(1) reduces to

$$y_1+y_2+y_3+y_4=8$$

$$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0$$

$$|B| = \binom{8+4-1}{8} = \binom{11}{8} = \binom{11}{3} = 165$$

Let C be the set of all 12-combination of $S^* \geq 5c$. For $|C|$,

$$x_1+x_2+x_3+x_4=12$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 5,\ x_4 \geq 0$$

Let $x_1=y_1,\ x_2=y_2,\ x_3-5=y_3;\ x_4=y_4$

$$y_1+y_2+y_3+y_4=7$$

$$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0$$

$$|C| = \binom{7+4-1}{7} = \binom{10}{7} = \binom{10}{3} = 120$$

Let D be the set of all 12-combination of $S^* \geq 6d$. For $|D|$

$x_1+x_2+x_3+x_4=12$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 6.$

Let $x_1=y_1,\ x_2=y_2,\ x_3=y_3,\ x_4-6=y_4$

$y_1+y_2+y_3+y_4=6$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0$

So $|D| = \binom{6+4-1}{6} = \binom{9}{6} = \binom{9}{3} = 84$

Let $A \cap B$ be the set of all 12-combinations of S^* $\geq 5a$, $\geq 4b$.

So for $|A \cap B|$, $x_1+x_2+x_3+x_4=12$

$x_1 \geq 5,\ x_2 \geq 4,\ x_3 \geq 0,\ x_4 \geq 0$

Let $x_1-5=y_1,\ x_2-4=y_2,\ x_3=y_3,\ x_4=y_4$

Eqn(1) reduces to $y_1+y_2+y_3+y_4=3$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0$

Now $|A \cap B| = \binom{3+4-1}{3} = \binom{6}{3} = 20$

Let $B \cap C$ be the set of all 12-combinations of S^* $\geq 4b$, $\geq 5c$. For $|B \cap C|$,

$x_1+x_2+x_3+x_4=12$

$x_1 \geq 0,\ x_2 \geq 4,\ x_3 \geq 5,\ x_4 \geq 0$

Let $x_1=y_1,\ x_2-4=y_2,\ x_3-5=y_3,\ x_4=y_4$

Eqn(1) reduces to

$y_1+y_2+y_3+y_4=3$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0$

So $|B \cap C| = \binom{3+4-1}{3} = \binom{6}{3} = 20$

Let $C \cap D$ be the set of all 12-combinations of S^* $\geq 5c$, $\geq 6d$.

For $|C \cap D|$, $x_1+x_2+x_3+x_4=12$

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$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 5,\ x_4 \geq 6$

Let $x_1 = y_1,\ x_2 = y_2,\ x_3 - 5 = y_3,\ x_4 - 6 = y_4$

So eq^n(1) reduces to $y_1 + y_2 + y_3 + y_4 = 1$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0$

So $|C \cap D| = \binom{1+4-1}{1} = \binom{4}{1} = 4$

Let $A \cap D$ be the set of all 12-combinations of S^* $\geq 5a, \geq 6d$. For $|A \cap D|$,

$x_1 + x_2 + x_3 + x_4 = 12$

$x_1 \geq 5,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 6$

Let $x_1 - 5 = y_1,\ x_2 = y_2,\ x_3 = y_3,\ x_4 - 6 = y_4$

So eq^n(1) reduces to $y_1 + y_2 + y_3 + y_4 = 1$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0, y_4 \geq 0$

$|A \cap D| = \binom{1+4-1}{1} = \binom{4}{1} = 4$

Let $B \cap D$ be the set of all 12-combinations of S^* $\geq 4b, \geq 6d$. For $|B \cap D|$,

$x_1 + x_2 + x_3 + x_4 = 12$

$x_1 \geq 0,\ x_2 \geq 4,\ x_3 \geq 0,\ x_4 \geq 6$

Let $x_1 = y_1,\ x_2 - 4 = y_2,\ x_3 = y_3,\ x_4 - 6 = y_4$

Eq^n(1) reduces to

$y_1 + y_2 + y_3 + y_4 = 2$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0$

$|B \cap D| = \binom{2+4-1}{2} = \binom{5}{2} = 10$

Let $A \cap C$ be the set of all 12-combinations of S^* $\geq 5a, \geq 5c$. For $|A \cap C|$,

$x_1 + x_2 + x_3 + x_4 = 12$

$x_1 \geq 5,\ x_2 \geq 0,\ x_3 \geq 5,\ x_4 \geq 0$

Let $x_1 - 5 = y_1,\ x_2 = y_2,\ x_3 - 5 = y_3,\ x_4 = y_4$

Eq^n(1) reduces to $y_1+y_2+y_3+y_4=2$

$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0.$

$$|A \cap C| = \binom{2+4-1}{2} = \binom{5}{2} = 10$$

We observe that the intersection of 3 or more sets are empty.

So $|A \cap B \cap C| = |B \cap C \cap D| = |A \cap B \cap D| = |A \cap C \cap D|$

$= |A \cap B \cap C \cap D| = 0$

For Required answer,

$$|\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D}| = 455 - 120 - 165 - 120 - 84 + 20 + 10 + 4 + 20 + 10 + 4$$

$= 34$ Ans

(6) A bakery sells chocolate, cinnamon and plain doughnuts and at a particular time has 6 chocolate, 6 cinnamon and 3 plain. If a box contains 12 doughnuts, how many different options are there for a box of doughnuts?

Sol^n Let S be a multiset of 6 chocolates, 6 cinnamon and 3 plain doughnuts. Let a = chocolates, b = cinnamon and c = plain doughnuts then

$$S = \{6 \cdot a, 6 \cdot b, 3 \cdot c\}$$

Let P be the set of all 12-combination of

$$T = \{\infty \cdot a, \infty \cdot b, \infty \cdot c\} \text{ or } \{12 \cdot a, 12 \cdot b, 12 \cdot c\}$$

$$|P| = \binom{12+3-1}{12} = \binom{14}{12} = \binom{14}{2}, \quad x_1+x_2+x_3 = 12 \text{ -(1)}, \; x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Let A be a set of all 12-combination of T with at least $7a$.

$x_1 + x_2 + x_3 = 12$

$x_1 \geq 7, \; x_2 \geq 0, \; x_3 \geq 0$

Let $x_1 - 7 = y_1, \; x_2 = y_2, \; x_3 = y_3$

Eq^n(1) reduces to $y_1+y_2+y_3=5$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0$

so $|A| = \binom{5+3-1}{5} = \binom{7}{5} = \binom{7}{2} = \frac{7\times 6}{1\times 2} = 21$

Let B be the set of 12-combination of T with at least 7b.

For $|B|$, $x_1+x_2+x_3=12$

$x_1 \geq 0,\ x_2 \geq 7,\ x_3 \geq 0$

Let $x_1=y_1,\ x_2-7=y_2,\ x_3=y_3$

Now eq^n(1) reduces to $y_1+y_2+y_3=5$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0$

so $|B| = \binom{5+3-1}{5} = \binom{7}{5} = \binom{7}{2} = 21$

Let C be the set of all 12 combinations of T with at least 4c.

For $|C|$, $x_1+x_2+x_3=12$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 4$

Let $x_1=y_1,\ x_2=y_2,\ x_3-4=y_3$

Now eq^n(1) reduces to $y_1+y_2+y_3=8$

$y_1, y_2, y_3 \geq 0$

So $|C| = \binom{8+3-1}{8} = \binom{10}{8} = \binom{10}{2}$

$=45$

Let $A \cap B$ be the set of 12-combinations of T with at least 7a and 7b.

$A \cap B = \phi$, so $|A \cap B| = 0$

Let $B \cap C$ be the set of 12-combination of T with at least 7b and 4c.

For $|B \cap C|$, $x_1+x_2+x_3=12$

$x_1 \geq 0,\ x_2 \geq 7,\ x_3 \geq 4$

Let $x_1=y_1,\ x_2-7=y_2,\ x_3-4=y_3$

Now eq^n(1) reduces to $y_1+y_2+y_3=1$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0$

So $|B\cap C| = \binom{1+3-1}{1} = \binom{3}{1} = 3$

Let $A\cap C$ be the set of 12-combination of T with at least 7a and 4c.

For $|A\cap C|$, $x_1+x_2+x_3 = 12$

$x_1 \geq 7,\ x_2 \geq 0,\ x_3 \geq 4$

Let $x_1 - 7 = y_1,\ x_2 = y_2,\ x_3 - 4 = y_3$

Now eqn (1) reduces to $y_1+y_2+y_3 = 1$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0$

So $|A\cap C| = \binom{1+3-1}{1} = \binom{3}{1} = 3$

Let $A\cap B\cap C$ be the set of 12-combination of T with at least 7a, 7b and 4c.

$\therefore$ $A\cap B\cap C = \emptyset$, so $|A\cap B\cap C| = 0$

So $|\bar{A}\cap\bar{B}\cap\bar{C}| = |P| - (|A|+|B|+|C|) + (|A\cap B|+|B\cap C|+|A\cap C|) - |A\cap B\cap C|$

$= \frac{\cancel{14}^{7}\times 13}{1\times \cancel{2}} - (21+21+45) + (0+3+3) - 0$

$= 91 - 87 + 6$

$= 10$

Hence there are 10 boxes of doughnuts possible.