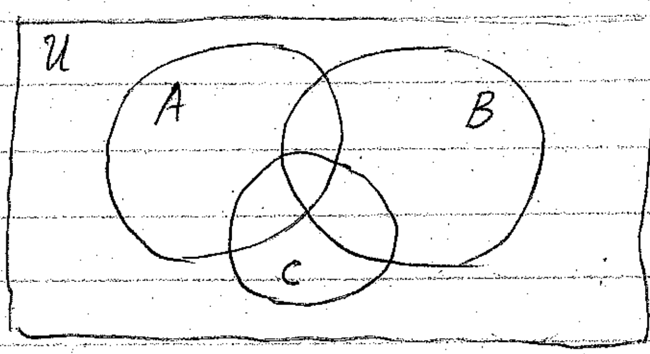
Ch.4 – The Inclusion-Exclusion Principle

§1. Two forms of the Inclusion-Exclusion Principle

Ex. 1 How many integers in the set $\mathcal{U} = \{1, 2, 3, \ldots, 1000\}$ are divisible by 3, 5, or 7.

Sol. Let $A = \{n \in \mathcal{U} : n \text{ is divisible by } 3\}$,
$B = \{n \in \mathcal{U} : n \text{ is divisible by } 5\}$,
and $C = \{n \in \mathcal{U} : n \text{ is divisible by } 7\}$.



Then the answer to our problem is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Now $A = \{3(1), 3(2), 3(3), \ldots, 3(k)\}$ where k is the largest integer $\leq 1000/3$. So $|A| = \left\lfloor \frac{1000}{3} \right\rfloor$. Similarly $|B| = \left\lfloor \frac{1000}{5} \right\rfloor$ & $|C| = \left\lfloor \frac{1000}{7} \right\rfloor$

Also $A \cap B = \{n \in \mathcal{U} : n \text{ is divisible by both } 3 \,\&\, 5\}$
$= \{n \in \mathcal{U} : n \text{ is divisible by } lcm(3,5)\}$
$= \{n \in \mathcal{U} : n \text{ is divisible by } 15\}$. So
$|A \cap B| = \left\lfloor \frac{1000}{15} \right\rfloor$. Similarly $|A \cap C| = \left\lfloor \frac{1000}{21} \right\rfloor$ & $|B \cap C| = \left\lfloor \frac{1000}{35} \right\rfloor$ and $|A \cap B \cap C| = \left\lfloor \frac{1000}{105} \right\rfloor$. Thus

$$|A \cup B \cup C| = \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{7} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{1000}{21} \right\rfloor - \left\lfloor \frac{1000}{35} \right\rfloor + \left\lfloor \frac{1000}{105} \right\rfloor = 333 + 200 + 142 - 66 - 47 - 28 + 9 = 543.$$

Ex. 2 Prove $n! = 1 + \sum_{k=0}^{n-1} k \cdot (k!)$ for $n \geq 0$. by (a) induction on n, (b) combinatorially.

Check, n=3 $\quad 1+0\cdot(0!)+1\cdot(1!)+2\cdot(2!) = 1+0+1+4 = 6. \quad 3! = 6.$ ✓

Def. Let $A_1, A_2, \ldots, A_n$ be subsets of a universal set U. A positive set w.r.t. U & $A_1, A_2, \ldots, A_n$ is any set of the form $U \cap A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$ where $\langle i_1, i_2, \ldots, i_k \rangle$ is any subsequence (including the empty subsequence $\langle\rangle$) of the sequence $\langle 1, 2, 3, \ldots, n \rangle$.

We usually leave out the U when $\langle i_1, \ldots, i_k \rangle$ is not the empty sequence – and we also leave out the intersection signs. So we will write

$$U \cap A_2 \cap A_4 \cap A_5 \quad \text{as} \quad A_2 A_4 A_5.$$

Note Recall that a sequence is just a function f with domain $\{1, 2, 3, \ldots, n\}$. The subsequences of f are obtained by restricting f to the different subsets of $\{1, 2, 3, \ldots, n\}$. So from the sequence $\langle f(1), f(2), \ldots, f(n) \rangle$ we can get

$\langle\ \rangle$ by restricting f to $\emptyset$

$\langle f(2), f(3) \rangle$ by restricting f to $\{2, 3\}$

$\langle f(1), f(3), f(4) \rangle$ by restricting f to $\{1, 3, 4\}$.

Since there are 2^n subsets of $\{1, 2, 3, \ldots, n\}$ there will be 2^n different subsequences of $\langle f(1), f(2), \ldots, f(n) \rangle$. This immediately tells us that there will be 2^n positive sets because there are 2^n subsequences $\langle i_1, \ldots, i_k \rangle$ of $\langle 1, 2, 3, \ldots, n \rangle$.

Let us analyze the positive sets in more details.

Def. The order of a positive set is the number of A_i's it contains. In other words, it is the length of the subsequence $\langle i_1, \dots, i_k \rangle$ from which it came.

Prop. 1 (a) There are $\binom{n}{k}$ positive sets of order k.
(b) Consequently there are (again) 2^n positive sets.

Proof (a) The number of positive sets of order k is the number of subsequences $\langle i_1, \dots, i_k \rangle$ of $\langle 1, 2, 3, \dots, n \rangle$ with k terms. But this is just the number of k-subsets of $\{1, 2, \dots, n\}$ because $\langle i_1, \dots, i_k \rangle$ has to be in increasing order. Since there are $\binom{n}{k}$ k-subsets of $\{1, 2, \dots, n\}$, there will be $\binom{n}{k}$ positive sets of order k.

(b). The number of positive sets is equal to $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$ by a previous result.

Positive sets of Order 0: $\mathcal{U}$. $\binom{n}{0}$

Order 1: $A_1, A_2, \dots, A_n$. $\binom{n}{1}$

Order 2: $A_1A_2, A_1A_3, \dots A_1A_n, \dots A_{n-1}A_n$.

$\vdots$

Order n: $A_1A_2A_3 \cdots A_n$ $\binom{n}{n}$

Order 2 (in more details): $A_1A_2, A_1A_3, A_1A_4, \dots, A_1A_n$, $[n-1]$

$[n-2] \rightarrow A_2A_3, A_2A_4, \dots, A_2A_n$, $A_3A_4, A_3A_5, \dots, A_3A_n$, $[n-3]$

$[n-4] \rightarrow A_4A_5, A_4A_6, \dots, A_4A_n, \dots, \underbrace{A_{n-2}A_{n-1}, A_{n-2}A_n}_{2}, \underbrace{A_{n-1}A_n}_{1}$.

So the number of positive sets of order 2 will be $(n-1)+(n-2)+(n-3)+\cdots+2+1 = \frac{n(n-1)}{2} = \binom{n}{2}$ as indicated before.

<u>Theorem 2</u> (Inclusion-Exclusion Theorem - First version)
Let $\mathcal{U}$ be a universal set and $A_i = \{x \in \mathcal{U} : x \text{ has property } P_i\}$ for $i = 1, 2, \ldots, n$. Then the number of elements of $\mathcal{U}$ with none of the properties P_i is given by

positive sets of order k.

$$(*) \quad |A_1^c A_2^c A_3^c \cdots A_n^c| = \sum_{k=0}^{n} (-1)^k \left\{ \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} A_{i_2} \cdots A_{i_k}| \right\}$$

$$= |\mathcal{U}| - \sum_{1 \le i \le n} |A_i| + \sum_{1 \le i < j \le n} |A_i A_j| - \cdots$$

$$+ (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cdots A_{i_k}| + \cdots + (-1)^n |A_1 A_2 \cdots A_n|.$$

<u>Proof</u>: We shall prove $(*)$ by showing that an element of $\mathcal{U}$ with none of the properties $P_1, \ldots, P_n$ is counted once in the RHS$(*)$ and that an element of $\mathcal{U}$ with at least one of the properties $P_1, \ldots, P_n$ is counted zero times in the RHS$(*)$.

Now if x has none of the properties $P_1, \ldots, P_n$ then x will be counted exactly once in $|\mathcal{U}|$ and zero times in all the other terms of the RHS $(*)$.

Also if x has the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$

then x will be counted in the RHS(*)

$$\binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \cdots + (-1)^k \binom{k}{k} = 0 \text{ times.}$$

No. of times x is counted in $\mathcal{U}$.

No. of times x is counted in the sets of order 1

No. of times x is counted in the sets of order 2

No. of times x is counted in the sets of order k.

$\therefore$ LHS(*) = RHS(*). So the result follows.

<u>Corollary 3</u> (Inclusion-Exclusion Theorem – Version 2)

Let $A_1, A_2, \ldots, A_n$ be as in Theorem 2. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \cdot \left\{ \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} A_{i_2} \cdots A_{i_k}| \right\}$$

$$= \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i A_j| + \cdots + (-1)^{k-1} \cdot \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cdots A_{i_k}|$$

$$+ \cdots + (-1)^{n-1} \cdot |A_1 A_2 \cdots A_n|$$

<u>Proof:</u> We know that

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |\mathcal{U}| - |(A_1 \cup A_2 \cup \cdots \cup A_n)^c|$$

$$= |\mathcal{U}| - |A_1^c \cap A_2^c \cap \cdots \cap A_n^c|$$

$$= |\mathcal{U}| - |A_1^c A_2^c A_3^c \cdots A_n^c|$$

$$= |\mathcal{U}| - \text{RHS}(*) \text{ of Theorem 2}$$

$$= |\mathcal{U}| - \sum_{k=0}^{n} (-1)^k \cdot \left\{ \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} A_{i_2} \cdots A_{i_n}| \right\}$$

$$= \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i A_j| + \cdots + (-1)^{k-1} \cdot \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cdots A_{i_n}|$$

$$+ \cdots + (-1)^{n-1} \cdot |A_1 A_2 \cdots A_n|.$$

§2. Two forbidden position problems

Recall that a permutation of $\{1,2,3,\dots,n\}$ is a sequence of n distinct elements of $\{1,2,3,\dots,n\}$. In other words, a permutation of $\{1,2,3,\dots,n\}$ is a bijection (one-to-one correspondence) from $\{1,2,3,\dots,n\}$ to itself.

So the permutation $\langle 2,3,1\rangle$ is really the bijection $\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$, i.e. the function which sends 1 to 2, 2 to 3, and 3 to 1.

Def. A <u>derangement</u> of $\{1,2,\dots,n\}$ is a permutation of $\{1,2,\dots,n\}$ in which no element is in its natural position, i.e., in which no element goes to itself.

<u>Ex. 1</u> $\langle 2,3,1\rangle$ & $\langle 3,1,2\rangle$ are the derangements of $\{1,2,3\}$.
$\langle 2,1,3\rangle$, $\langle 1,3,2\rangle$, $\langle 3,2,1\rangle$ & $\langle 1,2,3\rangle$ are <u>not</u> derangements of $\{1,2,3\}$.

<u>Ex. 2</u> Let $\mathcal{D}_n$ = set of all derangements of $\{1,2,\dots,n\}$ and $D_n = |\mathcal{D}_n|$. Then

$\mathcal{D}_0 = \{\langle\,\rangle\}$ so $D_0 = 1$
$\mathcal{D}_1 = \emptyset$ $D_1 = 0$
$\mathcal{D}_2 = \{\langle 2,1\rangle\}$ $D_2 = 1$
$\mathcal{D}_3 = \{\langle 2,3,1\rangle, \langle 3,1,2\rangle\}$ and $D_3 = 2$.

We will later see that $\mathcal{D}_4 = 9$.

Theorem 4 The number of derangements of $\{1,2,\dots,n\}$ is given by

$$D_n = n!\left\{\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \cdot \frac{1}{n!}\right\}$$
$$= n!\left[\sum_{k=0}^{n} \{(-1)^k / k!\}\right]$$

Proof: Let U = set of all permutations of $\{1,2,\dots,n\}$

Put A_i = set of all permutations in U with i going to itself. $i = 1, \dots, n$.

Then $|U| = n!$

$|A_i| = (n-1)!$ $\qquad \begin{pmatrix} 1, 2, & \cdots & i & \cdots & n \\ \cdot \quad \cdot & \cdots & \downarrow\!\! i & \cdots & \cdot \end{pmatrix}$

Also $A_i A_j = A_i \cap A_j$
$=$ set of all permutations in U with i going to i & j going to j. So

$|A_i A_j| = (n-2)!$ for $1 \le i < j \le n$.

In general $|A_{i_1} A_{i_2} \cdots A_{i_k}| = (n-k)!$ (for $1 \le i_1 < i_2 < \cdots < i_k \le n$) for each of the $\binom{n}{k}$ positive sets of order k. So by the Inclusion-Exclusion Theorem we get

$$D_n = |A_1^c \cap A_2^c \cap \cdots \cap A_n^c| = |A_1^c A_2^c \cdots A_n^c|$$
$$= \sum_{k=0}^{n} (-1)^k \left\{ \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} A_{i_2} \cdots A_{i_k}| \right\}$$
$$= \sum_{k=0}^{n} (-1)^k \cdot \left\{ \binom{n}{k} \cdot (n-k)! \right\}$$
$$= \sum_{k=0}^{n} (-1)^k \cdot \left\{ \frac{n!}{k!(n-k)!} (n-k)! \right\} = n! \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$
$$= n! \left\{ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \cdot \frac{1}{n!} \right\}.$$

Prop. 5 (a) For any $n \geq 1$, $D_n = \{n . D_{n-1}\} + (-1)^n$

(b) For any $n \geq 2$, $D_n = (n-1).(D_{n-1} + D_{n-2})$

Proof: (a) $D_n = n!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}\right]$

$= n.(n-1)!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\right] + n! . \frac{(-1)^n}{n!}$

$= n . D_{n-1} + (-1)^n$.

(b) $D_n = [n . D_{n-1}] + (-1)^n$

$= (n-1) . D_{n-1} + D_{n-1} + (-1)^n$

$= (n-1) . D_{n-1} + [(n-1) . D_{n-2} + (-1)^{n-1}] + (-1)^n$

$= (n-1) . D_{n-1} + (n-1) D_{n-2} + (-1)^{n-1}[1-1]$

$= (n-1) . [D_{n-1} + D_{n-2}]$.

Ex. 3 We have already seen that $D_3 = 2$. So

$D_4 = 4 . D_3 + (-1)^4 = 4(2) + 1 = 9$

$D_5 = 5 . D_4 + (-1)^5 = 5(9) + (-1) = 44$

$D_6 = 6 . D_5 + (-1)^6 = 5(44) + 1 = 265$

$D_7 = 7 . D_6 + (-1)^7 = 7(265) + (-1) = 1{,}854$.

Ex. 4 In how many ways can we return the watches of 3 men and 3 ladies so that

(a) no person gets their own watch

(b) no person gets their own watch and each lady receives a ladies watch.

Sol. (a) Answer $= D_6 = 265$ ways

(b) Answer $= (D_3)(D_3) = 2(2) = 4$ ways, bec. each lady will get a ladies watch & the men will get men's watches.

Ex.5 In how many ways can we return the cars of 5 super-models so that
(a) no supermodel gets her own car
(b) exactly one supermodel gets her own car
(c) exactly two supermodel gets her own car
(d) at most two super-models gets her own car
(e) at least two super-models gets her own car

(a) Answer $= D_5 = 44$

(b) There are $\binom{5}{1}$ ways to choose the one super-model who will get her own car. Then we derange the cars of the other 4 super-models in D_4 ways. So our answer $= \binom{5}{1} \cdot D_4 = 45$.

(c) There $\binom{5}{2}$ ways to choose the two super-models who will get their own cars. Then we derange the cars of the other 3 super-models in D_3 ways. Answer will be $\binom{5}{2} \cdot D_3 = \frac{5(4)}{2!} \cdot 2 = 20$.

(d) Answer $=$ Ans(a) + Ans(b) + Ans(c)
$= 44 + 45 + 20 = 109$

(e) Answer $=$ total no. of ways of permuting the cars $-$ the answers in (a) & (b)
$= 5! - [\text{Ans (a)} + \text{Ans (b)}]$
$= 120 - (44 + 45) = 31$.

(e′) We can also add the number of ways 2, 3, 4 & 5 super-models get their own cars to get

$$\text{Ans (e)} = \binom{5}{2} \cdot D_3 + \binom{5}{3} \cdot D_2 + \binom{5}{4} \cdot D_1 + \binom{5}{5} \cdot D_0$$
$$= 10(2) + 10(1) + 5(0) + 1(1)$$
$$= 20 + 10 + 1 = 31.$$

Def. A non-consecutive permutation of $\{1,2,\ldots,n\}$ is a permutation of $\{1,2,\ldots,n\}$ in which there is no pair of consecutive terms of the form $\langle i, i+1\rangle$. In other words, if we view the permutation as a bijection f, then there is no value of j such that $f(j+1) = f(j)+1$, for $j = 1,2,\ldots,n-1$.

Ex. 6 (a) $\langle 1,3,2\rangle$, $\langle 2,1,3\rangle$, and $\langle 3,2,1\rangle$ are non-consecutive permutations of $\{1,2,3\}$.
(b) $\langle \underline{1,2,3}\rangle$, $\langle \underline{2,3},1\rangle$, and $\langle 3,\underline{1,2}\rangle$ are not non-consecutive permutations of $\{1,2,3\}$.

Notation: Let $\mathcal{Q}_n$ = set of all non-consecutive permutations of $\{1,2,\ldots n\}$ and $Q_n = |\mathcal{Q}_n|$. Then

$\mathcal{Q}_0 = \{\langle\ \rangle\}$ so $Q_0 = 1$
$\mathcal{Q}_1 = \{\langle 1\rangle\}$ $Q_1 = 1$
$\mathcal{Q}_2 = \{\langle 2,1\rangle\}$ $Q_2 = 1$
$\mathcal{Q}_3 = \{\langle 1,3,2\rangle, \langle 2,1,3\rangle, \langle 3,2,1\rangle\}$ and $Q_3 = 3$

Later on we will see that that $Q_4 = 12$.

Theorem 6: The number of non-consec. permutations of $\{1,\ldots,n\}$ is

$$Q_n = \binom{n-1}{0} n! - \binom{n-1}{1}(n-1)! + \binom{n-2}{2}\cdot(n-2)! - \ldots + (-1)^{n-1}\binom{n-1}{n-1}\cdot 1!$$

$$= \sum_{k=0}^{n-1} (-1)^k \cdot \binom{n-1}{k} \cdot (n-k)!$$

Proof: Let $\mathcal{U}$ = set of all permutations of $\{1,2,\ldots,n\}$ and A_i = set of all permutations in $\mathcal{U}$ which contain $\langle i, i+1\rangle$ as consecutive terms.

Then A_1 = set of permutations in $\mathcal{U}$ with $\langle 1,2\rangle$ as a pair of consecutive terms
= set of permutations of $\{12, 3, 4, \dots, n\}$

So $|A_1| = (n-1)!$ Similarly, $|A_i| = (n-1)!$ for $i = 2, \dots, n-1$ as well.

Also A_1A_2 = set of permutations in $\mathcal{U}$ with both $\langle 1,2\rangle$ & $\langle 2,3\rangle$ as pairs of consecutive terms
= set of permutations of $\{\underbrace{123}_{1}, \underbrace{4, \dots, n}_{n-3}\}$

So $|A_1A_2| = (n-2)!$

And A_1A_3 = set of permutations in $\mathcal{U}$ with both $\langle 1,2\rangle$ & $\langle 3,4\rangle$ as pairs of consecutive terms
= set of permutations of $\{\underbrace{12, 34}_{2}, \underbrace{5, \dots, n}_{n-4}\}$

So $|A_1A_3| = (n-2)!$

From this we can see that for any i & j with $1 \le i < j \le n-1$, we have $|A_iA_j| = (n-2)!$

In general we can also see that for any $\langle i_1, \dots, i_k\rangle$ with $1 \le i_1 < i_2 < \dots < i_k \le n-1$, we have $|A_{i_1}A_{i_2}\dots A_{i_k}| = (n-k)!$ So

Q_n = set of all permutations in $\mathcal{U}$ with no pair of consecutive terms

$$= |A_1^cA_2^c \dots A_{n-1}^c| = \sum_{k=0}^{n-1} (-1)^k \left\{ \sum_{1 \le i_1 < \dots < i_k \le n-1} |A_{i_1} \dots A_{i_k}| \right\}$$

$$= \sum_{k=0}^{n-1} (-1)^k \cdot \binom{n-1}{k} \cdot (n-k)!$$

$$= \binom{n-1}{0} \cdot n! - \binom{n-1}{1} \cdot (n-1)! + \binom{n-1}{2} \cdot (n-2)! - \dots + (-1)^{n-1} \cdot \binom{n-1}{n-1} \cdot 1!$$

Prop. 7 For any $n \geq 1$, $Q_n = D_n + D_{n-1}$.

Proof:

$$Q_n = \sum_{k=0}^{n-1} (-1)^k \cdot \binom{n-1}{k} \cdot (n-k)!$$

$$= \sum_{k=0}^{n-1} (-1)^k \cdot \frac{(n-1)!}{k!((n-1)-k)!} \cdot (n-k)!$$

$$= \sum_{k=0}^{n-1} (-1)^k \cdot \frac{(n-1)!}{k!} \cdot (n-k)$$

$$= \sum_{k=0}^{n-1} (-1)^k \cdot \frac{(n-1)!}{k!} \cdot n - \sum_{k=0}^{n-1} (-1)^k \cdot \frac{(n-1)!}{k!} \cdot k$$

$$= \sum_{k=0}^{n-1} (-1)^k \cdot \frac{n!}{k!} - \left[\sum_{k=1}^{n-1} (-1)^k \cdot \frac{(n-1)!}{(k-1)!} + \underbrace{0}_{k=0}\right]$$

$$= \sum_{k=0}^{n-1} (-1)^k \cdot \frac{1}{k!} - \sum_{k=1}^{n-1} (-1)^k \frac{(n-1)!}{(k-1)!}$$

$$= \left[\sum_{k=0}^{n} (-1)^k \cdot \frac{n!}{k!}\right] - (-1)^n \frac{n!}{n!} - \sum_{k=0}^{n-2} (-1)^{k+1} \cdot \frac{(n-1)!}{k!}$$

replace $k-1$ by k

$$= n! \left[\sum_{k=0}^{n} (-1)^k \cdot \frac{1}{k!}\right] - (-1)^n + \sum_{k=0}^{n-1} (-1)^k \cdot \frac{(n-1)!}{k!} - (-1)^{n-1} \cdot \frac{(n-1)!}{(n-1)!}$$

$$= n! \left[\sum_{k=0}^{n} (-1)^n \cdot \frac{1}{k!}\right] + (n-1)! \left[\sum_{k=0}^{n-1} (-1)^k \cdot \frac{1}{k!}\right] - (-1)^{n-1} \cdot [-1+1]$$

$$= D_n + D_{n-1}. \qquad \square$$

Ex. 6 Five sisters walk to school in a straight line. In how many ways can they walk back home in a straight line so that no sister sees the same person in front of them again.

Sol. Answer $= Q_5 = D_5 + D_4 = 44 + 9 = 53$ ways.

§3. Solutions of $x_1 + \cdots + x_n = r$ with constraints. and r-combinations of finite multi-sets

Ex. 1 How many integer-solutions of the equation $x_1 + x_2 + x_3 = 17$ are there with $x_1 \geq 3$, $x_2 \geq 5$, and $x_3 \geq 2$?

Sol. Let $x_1 = y_1 + 3$, $x_2 = y_2 + 5$, and $x_3 = y_3 + 2$. Then our answer will be the same as the number of integer-solutions of the equation $(y_1+3) + (y_2+5) + (y_3+2) = 17$ with $y_1 + 3 \geq 3$, $y_2 + 5 \geq 5$, and with $y_3 + 2 \geq 2$.

This is the same as the number of integer-solutions of $y_1 + y_2 + y_3 = 7$ with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$. And we know that this is the same as the number of ways of arranging 8 1's & 2 +'s in a row, i.e. $\dfrac{(7+2)!}{7!\,2!} = \binom{7+3-1}{3-1} = \binom{9}{2}$

So our final answer is $\binom{9}{2} = \dfrac{9(8)}{2 \cdot 1} = 36$

Ex. 2 Let $M = [\infty \cdot a, \infty \cdot b, \infty \cdot c]$. How many 17-combinations of M are there with ≥ 3 a's, ≥ 5 b's, and ≥ 2 c's?

Sol. 1 Let x_1 = no. of a's in a 17-combination of M
x_2 = no. of b's in the same 17-comb. of M
and x_3 = no. of c's in the same 17-comb. of M.
Then our answer to the problem will be the number of integer-solutions of the equation

$x_1+x_2+x_3=17$ with $x_1 \geq 3$, $x_2 \geq 5$, and $x_3 \geq 2$.
And from example 1, we found that this is 36.

Sol. 2 Now there is another way to do this problem.
Let A = set of all 7-comb. of M and
A' = set of each 7-comb. in A plus $[3a, 5b, 2c]$
Then A' is a 17-comb. of M because each element of A' was obtained by adding a multi-set with 10 elements to a 7-comb. of M. Also each element of A' is a 17-comb. of M with ≥ 3 a's, ≥ 5 b's, and ≥ 2 c's.
Since there is an obvious bijection from A to A', it follows that

$$|A'| = \text{No. of 17-comb. of } M \text{ with } \geq 3 a\text{'s}, \geq 5 b\text{'s}, \geq 2 c\text{'s}$$
$$= \text{No of 7-comb. of } M = |A| = \binom{7+3-1}{3-1}.$$

So our final answer is $\binom{9}{2} = \frac{9.8}{2.1} = 36$ again.

Ex. 3 How many 15-combinations of the finite multi-set $F = [4a, 6b, 20c]$ are there?

Sol. Let $M = [\infty.a, \infty.b, \infty.c]$ and put
$\mathcal{U}$ = set of all 15-combinations of M
A = set of all 15-comb. in $\mathcal{U}$ with > 4 a's,
B = set of all 15-comb. in $\mathcal{U}$ with > 6 b's,
& C = set of all 15-comb. in $\mathcal{U}$ with > 20 c's.
Then
A = set of all 10-comb. of M with 5 extra a's added
B = set of all 8-comb. of M with 7 extra b's added
& $C = \emptyset$, bec. a 15-comb. cannot have ≥ 21 c's.

So $|U| = \binom{15+3-1}{3-1}$, $|A| = \binom{10+3-1}{3-1}$, $B = \binom{8+3-1}{3-1}$ & $|C| = 0$.

Also $A \cap B$ = set of all 15-comb. of M with $\geq 5a$'s & $7b$'s
= set of all 3-comb. of M with $[5a, 7b]$ added
$\therefore$ $|A \cap B| = \binom{3+3-1}{3-1}$. Since we want ≤ 4 a's, ≤ 6 b's and ≤ 20 c's in our 15-comb. of M, our final answer would be $|A^c \cap B^c \cap C^c|$.
But by the Inclusion-Exclusion Theorem

$$|A^c \cap B^c \cap C^c| = U - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$$

$$= \binom{17}{2} - \binom{12}{2} - \binom{10}{2} - 0 + \binom{5}{2} + 0 + 0 - 0$$

$$= \binom{17}{2} + \binom{5}{2} - \binom{12}{2} - \binom{10}{2}$$ because C, $A \cap C$, $B \cap C$ and $A \cap B \cap C$ are all empty.

$$= \frac{17(16)}{2} + \frac{5(4)}{2} - \frac{12(11)}{2} - \frac{10 \cdot 9}{2} = 136 + 10 - 66 - 45 = 35.$$

Ex. 4 How many 26-comb. of the finite multi-set $F = [4.a, 6.b, 20.c]$ are there?

Sol. 1 Again let $M = [\infty.a, \infty.b, \infty.c]$ and put
U = set of all 26-comb. of M
A = set of all 26-comb. in U with > 4 a's (≥ 5 a's)
B = set of all 26-comb. in U with > 6 b's (≥ 7 b's)
& C = set of all 26-comb. in U with > 20 c's (≥ 21 c's)

Then
A = set of all 21-comb. of M with $[5.a]$ added to each 21-comb.
B = set of all 19-comb. of M with $[7.b]$ added to each 19-comb.
C = set of all 5-comb. of M with $[21c]$ added to each 5-comb.
$A \cap B$ = set of all 14-comb. of M with $[5a, 7b]$ added to each 14-comb.

$A \cap C$ = set of all 0-comb. of M with $[5a, 21c]$ added to each 0-comb.

$B \cap C = \emptyset$ and $A \cap B \cap C = \emptyset$. So

Number of 26-comb. of $F = |A^c \cap B^c \cap C^c|$

$= |U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$

$$= \binom{26+3-1}{3-1} - \binom{21+3-1}{3-1} - \binom{19+3-1}{3-1} - \binom{5+3-1}{3-1} + \binom{14+3-1}{3-1} + \binom{0+3-1}{3-1} + 0 - 0$$

$$= \binom{28}{2} + \binom{16}{2} + \binom{2}{2} - \binom{23}{2} - \binom{21}{2} - \binom{7}{2} = 499 - 484 = 15.$$

Sol. 2 But there is a much quicker way to do the same problem. We just have to observe that No. of 26-comb. of F = No. of 4-comb. of F because F has 30 elements. If we want to pick 26 elements out of F, we can just pick 4 elements to leave behind and get the same answer. So let $M = [\infty.a, \infty.b, \infty.c]$ & U = set of all 4-comb. of M. Put

A = set of all 4-comb. in U with > 4 a's,

B = set of all 4-comb. in U with > 6 b's, &

C = set of all 4-comb. in U with > 20 c's.

Then $A = B = C = \emptyset$ and $A \cap B = A \cap C = B \cap C = A \cap B \cap C = \emptyset$ also. So

Number of 4-comb. of $F = |A^c \cap B^c \cap C^c|$

$= |U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$

$$= \binom{4+3-1}{3-1} = \binom{6}{2} = 15 \text{ (as before).}$$

So as you can see, it pays to be a little smart and think a little bit before trying to solve the problem. By the way, this trick would not have worked with Ex. 3.

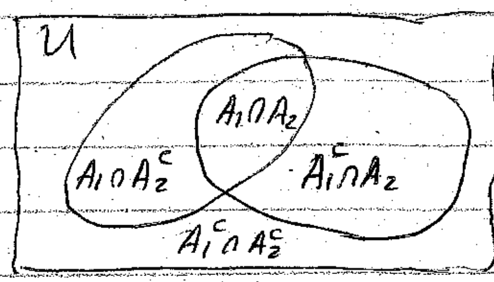
Def. Let U be a universal set and $A_1, \dots, A_n$ be subsets of U. An <u>ultimate set</u> with respect to $A_1, \dots, A_n$ is any set of the form $X_1 \cap X_2 \cap \dots \cap X_n$ where $X_i = A_i$ or A_i^c for $i = 1, \dots, n$.

Prop. 8 There are 2^n ultimate sets w.r.t. $A_1, \dots A_n$.

Proof. For each X_i we have 2 choices. Since there are n X_i's we will get 2^n choices & so 2^n ultimate sets.

Ex. 5 Let U = a universal and A_1 & A_2 be subsets of U. Find all the ultimate sets w.r.t. A_1 & A_2.

Sol. They are $A_1 \cap A_2$, $A_1^c \cap A_2$, $A_1 \cap A_2^c$, $A_1^c \cap A_2^c$



Prop. 9 The ultimate sets w.r.t. $A_1, \dots, A_n$ are all pairwise disjoint.

Proof: Suppose Z_1 and Z_2 are different ultimate sets. Let $Z_1 = X_1 \cap X_2 \cap \dots \cap X_n$ and $Z_2 = Y_1 \cap Y_2 \cap \dots \cap Y_n$ where $X_i = A_i$ or A_i^c; and $Y_i = A_i$ or A_i^c. Then for some i_0, X_{i_0} & Y_{i_0} must be different (because if $X_i = Y_i$ for each i, then Z_1 & Z_2 would be the same). Hence

$$Z_1 \cap Z_2 = (X_1 \cap X_2 \cap \dots \cap X_n) \cap (Y_1 \cap Y_2 \cap \dots \cap Y_n)$$
$$\subseteq X_{i_0} \cap Y_{i_0} = \emptyset \quad \text{bec. } X_{i_0} \cap Y_{i_0} = A_{i_0} \cap A_{i_0}^c$$

$\therefore Z_1 \cap Z_2 = \emptyset$. Hence any two ultimate sets are disjoint.

Consistency of Data

Ex. 6 Suppose we are told: among the Math majors
(a) 20 of them are taking Graph Theory, or Combinatorics, or both;
(b) 12 of them are taking Graph Theory
(c) 18 of them are taking Combinatorics
(d) 15 of them are not taking Combinatorics
Determine whether or not this data is consistent

Let U = set of Math majors, A = set of Math majors taking Graph Theory, and B = set of math majors taking Combinatorics. Put $x_1 = |A \cap B|$, $x_2 = |A \cap B^c|$, $x_3 = |A^c \cap B|$ & $x_4 = |A^c \cap B^c|$.
Then the data translates to the system of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 20 \quad (1) \\ x_1 + x_2 &= 12 \quad (2) \\ x_1 + x_3 &= 18 \quad (3) \\ x_2 + x_4 &= 15 \quad (4) \end{aligned}$$

Then there will be 3 possibilities.

I: The system has no non-neg. integer solution: In this case the data will be inconsistent.

IIA The system has a unique non-neg. integer solution: In this case the data is consistent & it determines the situation

IIB The system has more than one non-neg. integer solutions: In this case the data is consistent but it does not determine the situation.

$x_1 = 10$, $x_2 = 2$, $x_3 = 8$, $x_4 = 13$

In Ex. 6 the system has a solution. So the data is indeed consistent. (1) − (2) $\Rightarrow x_3 = 8$,
(3) + (4) − (1) $\Rightarrow x_4 = 13$, (3) $\Rightarrow x_1 = 18 - x_3 = 10$. (2) $\Rightarrow x_2 = 12 - x_1 = 2$.

END OF Ch. 4.