1.3 Symmetric Group

Permutation group.

A permutation of a finite set is a bijection from S to itself.

$$f : S \xrightarrow{\text{1-1 onto}} S$$

Notation: Let S be a finite set of n elements $S = \{a_1, a_2, \ldots, a_n\}$

Then permutation f

$$f = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ f(a_1) & f(a_2) & \cdots & f(a_n) \end{pmatrix}$$

Let $S = \{1, 2, 3, 4\}$ and $f : S \to S$

$f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $f(4) = 1$

then $\Rightarrow f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

Equality of two permutation

Let f and g be two permutation then they are called equal iff $f(a) = g(a) \; \forall \; a \in S$.

i.e. image of every elements of S under both f and g are equal.

Example: $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$, $g = \begin{pmatrix} 4 & 2 & 3 & 1 \\ 1 & 4 & 2 & 3 \end{pmatrix}$

here $f(1) = 3 = g(1)$ $\quad f(3) = 2 = g(3)$

$f(2) = 4 = g(2)$ $\quad f(4) = 1 = g(4)$

$\Rightarrow f(a) = g(a) \; \forall \; a \in S = \{1, 2, 3, 4\}$

Identity permutation

Let S be a finite set of n elements then a permutation f is called identity permutation iff

$$f(a) = a \quad \forall a \in S$$

Ex: Let $S = \{a_1, a_2, \ldots, a_n\}$.

$$f = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ f(a_1) & f(a_2) & \cdots & f(a_n) \end{pmatrix}$$

and $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

Product of composition of permutation :

Let f and g be permutation of A then product of two permutation is also a composition of permutation

$$(f \cdot g)(x) = f \circ g(x) = f[g(x)].$$

Ex: $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$

$\to f = (1.234)$ $\quad$ $\to g = (1\ 3\ 2\ 4)$

Find $f \cdot g$ and $g \cdot f$

Soln: (i) $(f \cdot g)(x) = f \circ g(x) = f[g(x)]$

$(f \circ g)(x) = f[g(1)] = f(3) = 4$

$$f \cdot g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

$$g \cdot f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

$$f \cdot g \neq g \cdot f.$$

Cyclic permutation or cycles

A permutation σ of a set S is a cycle permutation or a cycle if $\exists$ a finite subset $(a_1, a_2, \dots, a_n)$ of S such that

$\sigma(a_1) = a_2$, $\sigma(a_2) = a_3$, ..., $\sigma(a_n) = a_1$

If $\sigma(x) = x$ and $x \in S$

Then $x \notin (a_1, a_2, \dots, a_n)$.

$$\begin{pmatrix} 1 & \boxed{2} & 3 & 4 \\ 3 & \boxed{2} & 4 & 1 \end{pmatrix}$$

$\times$

Then we can write cycle permutation σ $(1\ 3\ 4)$

$\sigma = (a_1, a_2, \dots, a_n)$

Ex1: Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$ be a permutation then we can write a cycle

$f(1) = 4$, $f(2) = 1$, $f(3) = 3$, $f(4) = 2$

then $\sigma = (1 4 2)$

$3 \notin \sigma = (1\ 4\ 2)$

Ex 2: Let $\sigma = (2\ 4\ 5\ 6) \in S_6$ be a cycle here.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 5 & 6 & 2 \end{pmatrix}$$

Length of cycle.

Number of elements in cycle is called length of cycle..

Ex1: $\sigma = (1\ 2\ 3\ 4)$, then length of cycle of S is 4.

If length of cycle is r then it is called r-cycle.

Ex1 is 4-cycle.

Ex 2: $(2\ 4\ 5)$ is 3-cycle.

Note: length of identity permutation is 1.

Order of cycle:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 5 & 6 & 2 \end{pmatrix} \to (2\ 4\ 5\ 6)$$

let $\sigma(2) = 4$

$\sigma(4) = 5$

$\sigma[\sigma(2)] = 5$

$\sigma[\sigma^2(2)] = 6$

$\sigma[\sigma^3(2)] = 2$

i.e. $\sigma^4(2) = 2$, $\sigma^4(4) = 4$, $\sigma^4(5) = 5$, $\sigma^4(6) = 6$

$\sigma^4(1) = 1$, $\sigma^4(3) = 3$

$$\Rightarrow \sigma^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = I$$

$\Rightarrow \sigma^4 = I$

$\Rightarrow o(\sigma) = 4$

$\Rightarrow$ A length of cycle is order of cycle.

Ex: $\sigma = (3\ 4\ 6)$, then $o(\sigma) = 3$.

Inverse of cycle:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 3 & 2 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 4 & 5 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 2 & 3 \end{pmatrix}$$

$(2\ 5\ 3\ 4)$

let $\sigma = (2, 4, 3\ 5)$

Then $\sigma^{-1} = (5\ 3\ 4\ 2)$

Disjoint Cycle:

Two cycle are said to be disjoint if they have no common elements.

$\sigma_1 = (1\ 2\ 4) \in S_5$ They have no common elements

$\sigma_2 = (3\ 5) \in S_5$ $\Rightarrow \sigma_1$ and σ_2 are disjoint cycle.

Product of disjoint cycle.

Let σ_1 and σ_2 are disjoint cycle then they have no common elements.

$$\Rightarrow \sigma_1(x) \neq x \Rightarrow \sigma_2(x) = x$$

$$\Rightarrow \sigma_1 \sigma_2(x) = \sigma_1(x)$$

and $\sigma_2(x) = x$, then $\sigma_1(x) \neq x$

$$\sigma_2 \sigma_1(x) = \sigma_1(x)$$

$$\Rightarrow \sigma_1 \sigma_2(x) = \sigma_2 \sigma_1(x)$$

$$\Rightarrow \sigma_1 \sigma_2 = \sigma_2 \sigma_1$$

i.e. product of two disjoint cycle is commutative.

Ex: Let $\sigma = (3\ 4\ 6)$ and $\rho = (1\ 2\ 5)$; $\sigma, \rho \in S_6$

Then $\sigma\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 1 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 6 & 5 & 3 \end{pmatrix}$

$\rho\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 4 & 6 & 1 & 3 \end{pmatrix}$

$$\Rightarrow \sigma\rho = \rho\sigma$$

Note: Every permutation can be expressed as the product of disjoint cycle.

Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 6 & 3 \end{pmatrix}$

Then $S = (1\ 2)(3\ 4\ 5\ 6)$

Order of permutation

We know that every permutation can be written as product of disjoint cycle.

Let f be a permutation.

Then $O(f) =$ LCM of length of disjoint cycle.

Ex: $O(\sigma) = \text{LCM}\{2, 4\} = 4$

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 6 & 3 \end{pmatrix}$

Transposition

A cycle of length 2 is called transposition.

$\sigma = (1\ 2)$ is transposition.

Note:

(i) Order of every transposition is 2

(ii) Every permutation is a product of transposition.

Let a cycle $(a_1\ a_2\ \ldots\ a_{n-1}\ a_n)$

$= (a_1\ a_2)(a_1\ a_3)\ldots(a_1\ a_n)$

The set S_A of all permutations of a non-void set A is

6

Ex:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 6 & 2 & 5 & 1 & 3 & 8 & 7 \end{pmatrix}$$

$$\sigma = (1\,4\,5)(2\,6\,3)(7\,8)$$

$$= (1\,4)(1\,5)\,(2\,6)(2\,3)\,(7\,8)$$

Inversion: If σ be a permutation then the pair (i, j), $0 < i < j \leq n$ is an inversion for σ if $\sigma(i) > \sigma(j)$

let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$

Here $1 < 3$ but $f(1) > f(3) = 2 > 1$

$2 < 4$ but $f(2) > f(4) = 4 > 3$

$2 < 3$ but $f(2) > f(3) = 4 > 1$

$5 < 6$ but $f(5) > f(6) = 6 > 5$

Then the pair $(1,3)$, $(2,4)$, $(2,3)$, $(5,6)$ are called inversions.

The set S_A of all permutations of a non-void set A is a group for the product of permutation and it is denoted by $G = (S_A, *)$.

Proof:

(i) Closure property: Let $f \in S_A$ and $g \in S_A$ then $f \cdot g = f \circ g \in S_A$

(ii) Associativity: $f, g, h \in S_A$ then $(fg)h = f \circ g(h) = f \circ (g \circ h)$
$= f(gh)$

(iii) Existence of identity: Identity permutation is an identity element.

Let $I_A \in S_A$. Then $f \cdot I = f \circ I = f$

(iv) Inverse: Let $f \in S_A$ then $f^{-1} \in S_A$

such that $f \circ f^{-1} = I_A$

So $(S_A, \circ)$ is a group under the composition operation and it is also called permutation group.

Permutation Group or Symmetric Group.

The group of permutation of set $\{1, 2, \ldots, n\}$ is called the symmetric group of degree n and is denoted by S_n.

The order of this group $= n!$

$o(S_n) = n!$

* The group of permutation of set $\{1, 2, 3\}$ is called symmetric group of 3-symbol and is denoted by S_3.

If $A = \{1, 2, 3\}$, then there are 6 permutation of A

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

ρ_0: I; ρ_1: $(1\ 2\ 3)$; ρ_2: $(1\ 3\ 2)$

$$\mu_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

μ_0: $(2\ 3)$; μ_1: $(1\ 3)$; μ_2: $(1\ 2)$

$$\boxed{O(S_2) = 2 \quad \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}$$

Note:

(i) $S_3 = \{I, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$

(ii) $O(S_3) = 3! = 3 \times 2 \times 1 = 6$

(iii) Number of elements of order 2 = 3
$(1\ 2), (1\ 3), (2\ 3)$ are of order 2

(iv) Number of elements of order 3 = 2
$(1\ 3\ 2), (1\ 2\ 3)$ are of order 3.

Partitions of S_3

$1+1+1$

$1+2$

3

order 2 elements.

$$\frac{3!}{1^1\,1! \cdot 2^1(1!)} = \frac{6}{2} = 3$$

order 3 element

$$\frac{3!}{3^1(1!)} = 2$$

Permutation Group S_4:

S_4 is a set of permutation of 4-symbols and it is a group under product of permutation.

$S_4 = \{I, (1\,2), (1\,3), (1\,4), (2\,3), (2\,4), (3\,4),$
$(1\,2\,3), (1\,3\,2), (2\,3\,4), (2\,4\,3), (1\,2\,4),$
$(1\,4\,2), (1\,3\,4), (1\,4\,3), (1\,2\,3\,4), (1\,3\,2\,4),$
$(1\,4\,3\,2), (1\,4\,2\,3), (1\,2\,4\,3), (1\,3\,4\,2), (1\,2)(3\,4),$
$(1\,3)(2\,4), (1\,4)(2\,3)\}$

$\boxed{LCM}$. Partition

(i) Number of elements of order 3 = 8

(ii) No. of elements of order 4 = 6

(iii) No. of elements of order 1 = 1

(iv) No. of elements of order 2 = 6 + 3 = 9

$1+1+1+1$

$\frac{4!}{(1)^2 2! \cdot 2^1 1!} \leftarrow 1+1+2$] 2 ord

$\frac{4!}{2^2 2!}$ $\quad 2+2$

$\frac{4!}{1 \cdot 3^1 !}$ $\quad 1+3$] LCM → 3

4

Example: Express the following permutation as the product of disjoint cycles and find the order

(a) $\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 5 & 1 & 3 & 8 & 9 & 7 & 6 \end{pmatrix}$

(b) $\psi = (1\,2\,3\,4\,5)(1\,2\,3)(4\,5) \in S_5$

Sol:

(a) $\phi = (1\,2\,4)(3\,5)(6\,8\,7\,9)$

$o(\phi) = LCM\{3, 2, 4\} = 12$

Transposition of $\phi = (1\,2)(1\,4)(3\,5)(6\,8)(6\,7)(6\,9)$

1.3 Art

Que 4 Dummit & Foote.

Compute the order of each of the elements in the following groups. (a) S_3 (b) S_4.

Sol^n All elements in S_3 can be written as a single t-cycle, with t being the order of the elements:

Permutation	order in S_3
I	1
$(1\ 2)$	2
$(1\ 3)$	2
$(2\ 3)$	2
$(1\ 2\ 3)$	3
$(1\ 3\ 2)$	3

Que 5: Find the order of $G = (1\ 12\ 8\ 10\ 4)(2\ 13)(5\ 11\ 7)(6\ 9)$.

Sol^n: Since cycles are disjoints. Order of group = L.C.M of order of disjoint cycle.

$o(G) = \text{L.C.M.}\{5, 2, 3, 2\} = 30.$

Que 6: Write out the cycle decomposition of each element of order 4 in S_4.

Sol^n: $(1\ 2\ 3\ 4), (1\ 3\ 2\ 4), (1\ 4\ 3\ 2),$
$(1\ 4\ 2\ 3), (1\ 2\ 4\ 3), (1\ 3\ 4\ 2).$

Elements of order 4

$\frac{4!}{4(1!)} = \frac{24}{4} =$