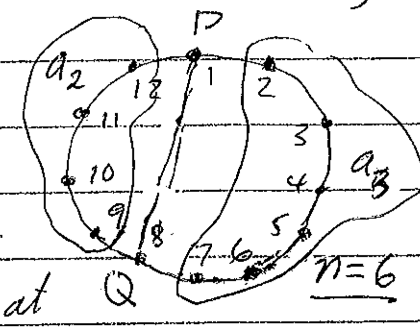
Chapter 8 – (Fifth Edition)

1. Let a_n = number of ways of joining $2n$ equally spaced points on a circle, in pairs, so that the resulting line segments do not intersect.
Choose one point & call it P. Then P must be joined to a point Q with an even no. of points on both sides of $\overline{PQ}$. So from this we can see that



$$a_n = a_0 a_{n-1} + a_1 a_{n-2} + \cdots + a_{n-1} a_0. \quad \text{Also } a_0 = 1$$

Since the Catalan numbers C_n satisfy the same difference equation with the same initial conditions, $a_n = C_n$.

6. $h_n = 2n^2 - n + 3$. The difference table for $\langle h_n \rangle$ is thus

	$n=0$	$n=1$	$n=2$	$n=3$	.	.
$\langle h_n \rangle$	3	4	9	18	31	46
$\langle \Delta h_n \rangle$	1	5	9	13	17	
$\langle \Delta^2 h_n \rangle$	4	4	4	4		
$\langle \Delta^3 h_n \rangle$	0	0	0			

So by Theorem 8.2.2, $h_n = 3\cdot\binom{n}{0} + 1\cdot\binom{n}{1} + 4\cdot\binom{n}{2}$
because $3, 1, 4, 0, 0 \cdots$ is the zero diagonal (column).

By theorem 8.2.3, it now follows that

$$\sum_{k=0}^{n} h_k = \sum_{k=0}^{n} \left\{ 3\cdot\binom{k}{0} + 1\cdot\binom{k}{1} + 4\cdot\binom{k}{2} \right\}$$

$$= 3\cdot\binom{n+1}{0+1} + 1\cdot\binom{n+1}{1+1} + 4\cdot\binom{n+1}{2+1}$$

$$= 3(n+1) + \frac{(n+1)(n)}{2} + 4\cdot\frac{(n+1)(n)(n-1)}{3!}$$

$$= (n+1)\left[3 + \frac{n}{2} + \frac{2n^2 - 2n}{3}\right] = \frac{(n+1)}{6}\left(18 - n + 4n^2\right).$$

#7. The 0th row is $1, -1, 3, 10$. So we start calculating the differences from this

$\langle h_n \rangle$	1	-1	3	10
$\langle \Delta h_n \rangle$	-2	4	7	$c = 7$
$\langle \Delta^2 h_n \rangle$	6	3	$b = 0$	$e = -3$
$\langle \Delta^3 h_n \rangle$	-3	$a = -3$	$d = -3$	$f = -3$
$\langle \Delta^4 h_n \rangle$	0	0	0	0

These are not needed to find h_n. It is just to make the display bigger.

$$a - 3 = 0 \Rightarrow a = -3$$
$$b - 3 = a = -3 \Rightarrow b = 0$$
$$c - 7 = b = 0 \Rightarrow c = 7$$
$$d - a = 0 \Rightarrow d = -3$$
$$e - b = d \Rightarrow e - 0 = -3 \Rightarrow e = -3$$
$$f - d = 0 \Rightarrow f = -3$$

$\langle \Delta^4 h_n \rangle = 0, 0, 0 \ldots$ bec. h_n is a polynomial of degree 3.

So zero-diagonal is $1, -2, 6, -3, 0, 0, 0, \ldots$

$$\therefore \quad h_n = 1 \cdot \binom{n}{0} - 2 \cdot \binom{n}{1} + 6\binom{n}{2} - 3 \cdot \binom{n}{3}$$

So
$$\sum_{k=0}^{n} h_k = \sum_{k=0}^{n} 1 \cdot \binom{k}{0} - 2 \cdot \binom{k}{1} + 6\binom{k}{2} - 3\binom{k}{3}$$
$$= 1 \cdot \binom{n+1}{0+1} - 2 \cdot \binom{n+1}{1+1} + 6\binom{n+1}{2+1} - 3\binom{n+1}{3+1}$$
$$= \binom{n+1}{1} - 2 \cdot \binom{n+1}{2} + 6\binom{n+1}{3} - 3\binom{n+1}{4}.$$

8. $\langle h_n \rangle = 0 \quad 1 \quad 32 \quad 243 \quad 1024 \quad 3125$

$\langle \Delta h_n \rangle = 1 \quad 31 \quad 211 \quad 781 \quad 2101$

$\langle \Delta^2 h_n \rangle = 30 \quad 180 \quad 570 \quad 1320$

$\langle \Delta^3 h_n \rangle = 150 \quad 390 \quad 750$

$\langle \Delta^4 h_n \rangle = 240 \quad 360$

$\langle \Delta^5 h_n \rangle = 120$

So $n^5 = h_n = 0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 30 \cdot \binom{n}{2} + 150 \cdot \binom{n}{3} + 240 \binom{n}{4} + 120 \binom{n}{5}$

$$\therefore \sum_{k=0}^{n} k^5 = \sum_{k=0}^{n} 0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 30 \cdot \binom{n}{2} + 150 \cdot \binom{n}{3} + 240 \cdot \binom{n}{4} + 120 \cdot \binom{n}{5}$$

$$= 0 \cdot \binom{n+1}{0+1} + 1 \cdot \binom{n+1}{1+1} + 30 \cdot \binom{n+1}{2+1} + 150 \binom{n+1}{3+1} + 240 \binom{n+1}{4+1} + 120 \binom{n+1}{5+1}$$

$$= \binom{n+1}{2} + 30 \binom{n+1}{3} + 150 \binom{n+1}{4} + 240 \binom{n+1}{5} + 120 \binom{n+1}{6}$$

11.

p \ k	0	1	2	3	4	5	6	7	8
7	0	1	63	301	350	140	21	1	
8	0	1	127	966	1701	1050	266	28	1

12. $S(p,k)$ = coefficient of $[n]_k$ in the expansion of n^k in terms of $[n]_0, [n]_1, \ldots, [n]_p$

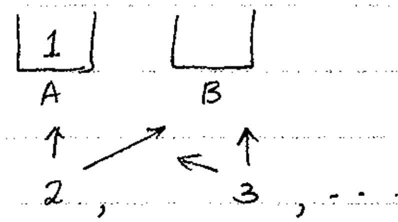
= no. of partitions of $\{1,2,3,\ldots,p\}$ into k ident. boxes with none being empty. by Thm 8.2.5

(a) So $S(p,1)$ = no. of partitions of $\{1,2,\ldots,p\}$ into 1 box

= 1 and we are done.

12 (b) $S(p,2)$ = No. of partitions of $\{1,2,\ldots,p\}$ into 2 identical boxes with none being empty

Now if we distribute $1,2,\ldots,p$ between the two boxes one of the boxes must get the element 1. Call this box A. and call the other box B. Then for each of the elements $\{2,\ldots,p\}$ we have 2 choices:



A or B. So there will be $2 \cdot 2 \cdots 2$ ($p-1$ times) $= 2^{p-1}$ ways of distributing $\{1,2,\ldots,p\}$ into the two boxes. But one of these ways will have B being empty. So there are $2^{p-1}-1$ ways of having both A & B non-empty.

$\therefore \quad S(p,2) = 2^{p-1}-1$

(c) $S(p,p-1)$ = No. of partitions of $\{1,2,\ldots,p\}$ into $p-1$ boxes with none being empty.

Now if none of the boxes are empty, then one box must contain 2 elements and the other $p-2$ boxes must contain one element each. Since the boxes are identical, as soon as we decide which two elements to put in the same box, this will determine the partition.

So $S(p,p-1)$ = No. of ways of pick 2 elements out of $\{1,2,\ldots,p\}$

$= \binom{p}{2}$ and we are done.

12 (d) $S(p, p-2)$ = No. of ways of partitioning $\{1, \ldots, p\}$ into $p-2$ identical boxes, with none empty.

Now if we partition $\{1, \ldots, p\}$ into $p-2$ identical boxes either

(i) one box gets 3 elements & the rest get 1 each

or (ii) two boxes get 2 elements & the rest get 2 each

So $S\{p, p-2\}$ = No. of ways of picking 3 elements out of $\{1, 2, 3, \ldots, p\}$

+

No. of ways of pick 4 elements out of $\{1, 2, 3, \ldots, p\}$ and distributing these 4 elements into 2 identical boxes with each box getting 2 elements

Now there are $\binom{p}{3}$ ways of picking 3 elements out of $\{1, 2, \ldots, p\}$ and $\binom{p}{4}$ ways of picking 4 elements out of $\{1, 2, \ldots, p\}$. And if we picked 4 elements, say $\{i_1, i_2, i_3, i_4\}$, we can distribute them in 3 ways into two identical boxes with each box getting 2.

$\{i_1, i_2\} + \{i_3, i_4\}$ $\quad$ $\{i_1, i_3\} + \{i_2, i_4\}$ $\quad$ $\{i_1, i_4\} + \{i_2, i_3\}$

So $S(p, p-2) = \binom{p}{3} + 3 \cdot \binom{p}{4}$ and we are done

14. $$\sum_{k=0}^{n} k^p = \sum_{k=0}^{n} \sum_{t=0}^{p} t!\, S(p,t) \binom{k}{t} = \sum_{t=0}^{p} t!\, S(p,t) \cdot \sum_{k=0}^{n} \binom{k}{t}$$

$$= \sum_{t=0}^{p} t!\, S(p,t) \cdot \binom{n+1}{t+1}.$$ (compare with Qu. #8.)

13. Without loss of generality we may assume that $X = \{1, 2, 3, \dots, p\}$ and $Y = \{1, 2, 3, \dots, k\}$.
Let $\mathcal{S}$ = set of all <u>surjective</u> functions from X to Y.
and $\mathcal{O}$ = set of all ordered partitions of $\{1, 2, \dots, p\}$ into k parts with each part being non-empty.

We will show that there is a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{O}$. Since $|\mathcal{O}| = S^{\#}(p,k)$, it will follow that $|\mathcal{S}| = S^{\#}(p,k)$.
Note $S^{\#}(p,k) = k!\, S(p,k)$ from 8.18 page 275

Given $f \in \mathcal{S}$, define for each $y \in Y$
$$f^{-1}[y] = \{x \in Y : f(x) = y\}$$
Then $f \in \mathcal{S}$ will correspond to the ordered partition
$$\langle f^{-1}[1], f^{-1}[2], \dots, f^{-1}[k] \rangle$$

And given an ordered partition
$$\langle A_1, A_2, \dots, A_k \rangle$$
of $\{1, 2, \dots, p\}$ into k non-empty parts, this will correspond to the function f which has
$$f^{-1}[1] = A_1, \quad f^{-1}[2] = A_2, \quad \dots, \quad f^{-1}[k] = A_k$$

15. $(k)^n$ = No. of partitions of $\{1, 2, 3, \dots, n\}$ into k distinguishable boxes with empty boxes being allowed $\quad B_1 \; B_2 \; \dots \; B_k$

because there are k choices for 1
k choices for 2
$\vdots$
k choices for n.

15. Now each partition of $\{1,2,3,\cdots,n\}$ into the boxes $B_1,\ldots,B_k$ corresponds to an ordered partition of $\{1,2,\cdots,n\}$ into k parts with some parts allowed to be empty. Note that if there are i non-empty parts, each partition will correspond to a choices of i of the k boxes & an ordered partition of $\{1,2,\cdots,n\}$ into i non-empty parts. Since there $\binom{k}{i}$ ways of choosing the non-empty boxes & $S^{\#}(n,i)$ ordered partitions of $\{1,2,\cdots,n\}$ into i non-empty parts

$$k^n = \sum_{i=1}^{k} \binom{k}{i} \cdot \begin{pmatrix} \text{No. of ordered partitions of} \\ \{1,2,\cdots,n\} \text{ into } i \text{ non-empty boxes} \end{pmatrix}$$

$$= \sum_{i=1}^{k} \binom{k}{i} S^{\#}(n,i) = \sum_{i=1}^{k} \binom{k}{i} \cdot i! \cdot S(n,i)$$

$$= \binom{k}{1} \cdot 1!\, S(n,1) + \binom{k}{2} \cdot 2!\, S(n,2) + \cdots + \binom{k}{k} \cdot k!\, S(n,k)$$

Note: There is a slight misprint in the textbook.

16 $B_p = S(p,0) + S(p,1) + S(p,2) + \cdots + S(p,p)$, p.277

$B_7 = 0 + 1 + 63 + 301 + 350 + 140 + 21 + 1 = 877$

$B_8 = 0 + 1 + 127 + 966 + 1701 + 1050 + 266 + 28 + 1$

$= 4140.$

$$B_8 = \binom{8-1}{0} \cdot B_0 + \binom{8-1}{1} \cdot B_1 + \binom{8-1}{2} B_2 + \cdots + \binom{8-1}{8-1} \cdot B_7$$

$$= 1.1 + 7.1 + 21.2 + 35.5 + 35.15 + 21.52 + 7.203 + 877$$

$$= 1 + 7 + 42 + 175 + 525 + 1092 + 1421 + 877 = 4140 \checkmark$$

18.

$p \backslash k$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0
3	0	2	3	1	0	0	0	0
4	0	6	11	6	1	0	0	0
5	0	24	50	35	10	1	0	0
6	0	120	274	225	85	15	1	0
7	0	720	1764	1624	735	175	21	1

19. $$s(p,k) = \begin{pmatrix}\text{coeff. of } n^k \text{ in the expansion of } [n]_k \\ \text{in terms of } n^0, n^1, \ldots, n^p\end{pmatrix} \Big/ (-1)^{p-k}$$

(a) $$S(p,1) = \begin{pmatrix}\text{coeff. of } n^1 \text{ in the expansion} \\ \text{of } n(n-1)\cdots(n-(p-2))(n-(p-1))\end{pmatrix} \Big/ (-1)^{p-1}$$

$$= (-1)(-2)(-3)\cdots(-(p-1)) \big/ (-1)^{p-1}$$

$$= (p-1)!\,(-1)^{p-1} \big/ (p-1) = (p-1)!$$

(b) $$S(p,p-1) = \begin{pmatrix}\text{coeff. of } n^{p-1} \text{ in the exp.} \\ \text{of } n(n-1)(n-2)\cdots(n-(p-1))\end{pmatrix} \Big/ (-1)^{p-(p-1)}$$

$$= -\left[1+2+3+\cdots+(p-1)\right] \big/ (-1)^1$$

$$= \frac{(p-1)\,p}{2} = \binom{p}{2}.$$

20 (a) $$[n]_n = n(n-1)(n-2)\cdots(n-(n-1))$$

$$= n\cdot(n-1)(n-2)\cdots(2)(1) = n!$$

20 (b) $$[n]_p = \sum_{k=0}^{p} (-1)^{p-k}\, s(p,k)\cdot n^k$$

$$\therefore\quad n! = [n]_n = \sum_{k=0}^{n} (-1)^{n-k}\, s(n,k)\cdot n^k$$

$$6! = s(6,0) - s(6,1)\cdot 6 + s(6,2)\cdot 6^2 - s(6,3)\cdot 6^3 + s(6,4)\cdot 6^4 - s(6,5)6^5 + s(6,6)\cdot 6^6$$

$$= 0 - 120\cdot 6^1 + 274\cdot 6^2 - 225\cdot 6^3 + 85\cdot 6^4 - 15\cdot 6^5 + 6^6.$$

31. (a) $$h_n^{(k)} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}$$ by 8.24 p. 298

So $$h_{k-1}^{(k)} = \binom{k-1}{0} + \binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{k-1} + \underbrace{\binom{k-1}{k}}_{=0}$$

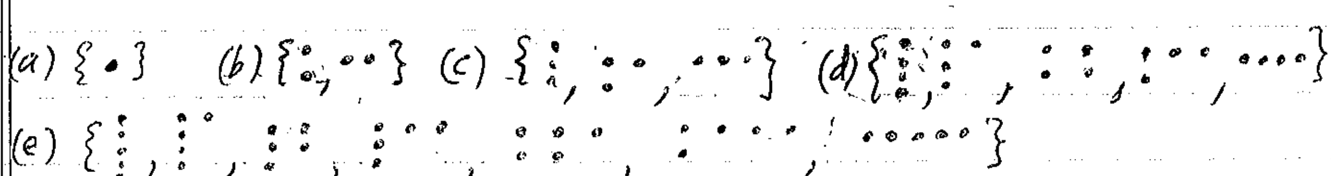
$$= \binom{k-1}{0} + \binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{k-1}$$

$$= (1+1)^{k-1} = 2^{k-1}$$ Biomialexpansion.

(b) If $n \le k$ then

$$h_n^{(k)} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} + \overbrace{\binom{n}{n+1} + \cdots + \binom{n}{k}}^{=0}$$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1+1)^n = 2^n$$

21.


26. (a) $12 = 4+3+2+2+1$ (b) $15 = 5+3+3+2+1+1$

(c) $20 = 4+4+4+4+2+2$ (d) $21 = 6+5+4+3+2+1$

27. (a) $2n-1 = n + \underbrace{1+\cdots+1}_{(n-1)\text{ times}}$ (b) $2n = n+2+\underbrace{1+\cdots+1}_{(n-2)\text{ times}}$