Chapter 7

1 (a) $f_1 + f_3 = 1 + 2 = 3 = f_4$

$f_1 + f_3 + f_5 = 1+2+5 = 8 = f_6$

$f_1 + f_3 + f_5 + f_7 = 1+2+5+13 = 21 = f_8$

$f_1 + f_3 + f_5 + f_7 + f_9 = 1+2+5+13+34 = 55 = f_{10}$

Guess: $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$.

Basis: $f_1 = 1 = f_2 = f_{2.1}$ so result is true for $n=1$.

Ind. Step. Supp. that the result is true for n.
Then $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$. So

$$f_1 + f_3 + f_5 + \cdots + f_{2n+1} = (f_1 + f_3 + \cdots + f_{2n-1}) + f_{2n+1}$$
$$= f_{2n} + f_{2n+1}$$
$$= f_{2n+2} = f_{2(n+1)}$$

So if the result is true for n, it will be true for $n+1$.
Hence by the Principle of Mathematical Induction the result is true for all $n \geq 1$.

(b) $f_0 + f_2 + f_4 + \cdots + f_{2n} = -1 + f_{2n+1}$ for $n \geq 0$

(c) $f_0 - f_1 + f_2 - \cdots + (-1)^n f_n = -1 + (-1)^n f_{n-1}$, $n \geq 1$

(d) $f_0^2 + f_1^2 + f_2^2 + \cdots + f_n^2 = f_n \cdot f_{n+1}$ for $n \geq 0$

The proofs by induction of (b), (c) & (d) are very similar to that in (a).

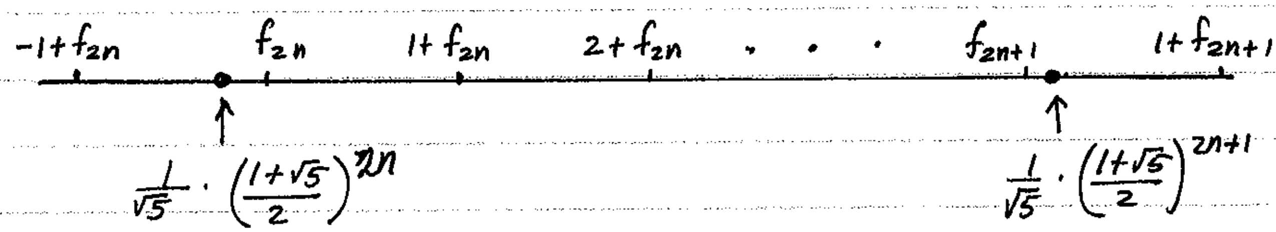
2. We know that from page 195 that

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

So $f_n = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^n + (-1)^n \cdot \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}-1}{2}\right)^n.$

Since $\frac{\sqrt{5}-1}{2} < 1$, $\frac{1}{\sqrt{5}} \cdot \left(\frac{\sqrt{5}-1}{2}\right)^n < \frac{1}{\sqrt{5}} \cdot (1)^n < \frac{1}{2}$

So f_n will be the integer nearest to $\frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^n$



3. (a) We have

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} \\ &= f_{n-2} + f_{n-3} + f_{n-2} \\ &= 2f_{n-2} + f_{n-3} \end{aligned}$$

So $f_n \equiv f_{n-3} \pmod 2$.

Now if $n \equiv 0 \pmod 3$, then

$$f_n \equiv f_{n-3} \equiv f_{n-6} \equiv \cdots \equiv f_3 \equiv f_0 \equiv 0 \pmod 2$$

because $f_0 = 0$.

Also if $n \equiv 1 \pmod 3$, then

$$f_n \equiv f_{n-3} \equiv f_{n-6} \equiv \cdots \equiv f_4 \equiv f_1 \equiv 1 \pmod 2$$

because $f_1 = 1$.

And if $n \equiv 2 \pmod 3$, then

3 (a) $f_n \equiv f_{n-3} \equiv f_{n-6} \equiv \dots \equiv f_5 \equiv f_2 \equiv 1 \pmod 2$
because $f_2 = 1$.

Hence $f_n \equiv 0 \pmod 2$ iff $n \equiv 0 \pmod 3$,
i.e., f_n is even iff n is divisible by 3.

(b) We have $f_n = 2f_{n-2} + f_{n-3}$ from part (a)

$$= 2(f_{n-3} + f_{n-4}) + f_{n-3}$$
$$= 3f_{n-3} + 2f_{n-4}$$

So $f_n \equiv 2f_{n-4} \pmod 3$

Now if $n = 4k$, we see as in part (a) that
$f_n \equiv f_{4k} \equiv 2f_{4(k-1)} \equiv 2^2 f_{4(k-2)} \equiv \dots \equiv 2^k f_0 \pmod 3$
Since $f_0 = 0$, $f_{4k} \equiv 0 \pmod 3$

And if $n = 4k+1$, then
$f_n \equiv f_{4k+1} \equiv 2 \cdot f_{4(k-1)+1} \equiv \dots \equiv 2^k f_1 \not\equiv 0 \pmod 3$
because $f_1 = 1$ & $2^k \cdot 1 \not\equiv 0 \pmod 3$.

Similarly $f_{4k+2} \equiv 2^k \cdot f_2 \equiv 2^k \cdot 1 \not\equiv 0 \pmod 3$
and $f_{4k+3} \equiv 2^k \cdot f_3 \equiv 2^k \cdot 2 \not\equiv 0 \pmod 3$
So $f_n \equiv 0 \pmod 3$ iff $n \equiv 0 \pmod 4$

(c) We have $f_n = 3f_{n-3} + 2f_{n-4}$ from part (b)

$$= 3(f_{n-4} + f_{n-5}) + 2f_{n-4}$$
$$= 5 \cdot f_{n-4} + 3 \cdot f_{n-5}$$
$$= 5 \cdot (f_{n-5} + f_{n-6}) + 3 f_{n-5}$$
$$= 8 \cdot f_{n-5} + 5 \cdot f_{n-6}$$

3 (c) So $f_n \equiv 5 f_{n-6} \pmod 4$
We can then show exactly as in part (b) that $f_n \equiv 0 \pmod 4$ iff $n \equiv 0 \pmod 6$.

(d) In the process of show that $f_n = 8f_{n-5} + 5f_{n-6}$ in part (c) we showed that $f_n = 5f_{n-4} + 3f_{n-5}$.
So $f_n \equiv 3 f_{n-5} \pmod 5$
As in part (b) we can now easily see that
$$f_n \equiv 0 \pmod 5 \quad \text{iff} \quad n \equiv 0 \pmod 5$$

4. Use induction as in #3

6(a) We will first show by induction on b that
$$f_{a+b} = f_{a-1} \cdot f_b + f_a \cdot f_{b+1}$$

We have
$$f_{a+0} = f_a = f_{a-1} \cdot 0 + f_a \cdot 1$$
$$= f_{a-1} \cdot f_0 + f_a \cdot f_1$$
So the result is true for $b = 0$

Now suppose the result is true for $0, 1, 2, \ldots,$ and b
Then $f_{a+c} = f_{a-1} \cdot f_c + f_a \cdot f_{c+1}$ for all $0 \leq c \leq b$
So
$$f_{a+(b+1)} = f_{(a+b)+1} = f_{a+b} + f_{a+(b-1)}$$
$$= (f_{a-1} \cdot f_b + f_a \cdot f_{b+1}) + (f_{a-1} f_{b-1} + f_a \cdot f_{b-1+1})$$
$$= f_{a-1} \cdot (f_b + f_{b-1}) + f_a \cdot (f_{b+1} + f_b)$$
$$= f_{a-1} \cdot f_{b+1} + f_a \cdot f_{b+2}$$
$$= f_{a-1} \cdot f_{b+1} + f_a \cdot f_{(b+1)+1}$$

So if the result is true for $0, 1, 2, \ldots, b$; then it will be true for $b+1$. By the 2nd principle of Math. Induction it follows that the result is true for all b.

6 (b) Now we will show by induction on k that $f_{m.k}$ is divisible by f_m

Since $f_{m.1} = f_m$ is divisible by f_m, the result is true for $k=1$.
Now suppose the result is true for k
Then $f_{m.k}$ is divisible f_m.

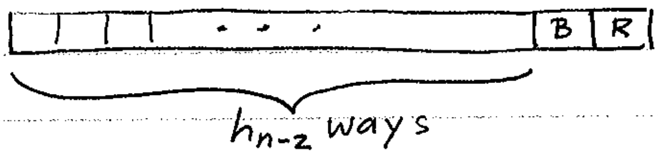
But $f_{m.(k+1)} = f_{mk+m} = f_{mk-1} \cdot f_m + f_{mk} \cdot f_{m+1}$
by part (a), and since f_{mk} is divisible by f_m
it follows that $f_{m.(k+1)}$ is divisible by f_m.
So if the result it true for k, it will be true for $k+1$. By the First Principle of Math. Induction, it follows that the result is true for all k.

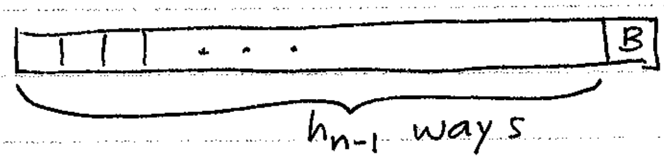
So if n is divisible by m, i.e. if $n = m.k$, then $f_n = f_{mk}$ will be divisible by f_m.

8 h_n = No. of ways of coloring the $1 \times n$ chessboard with no two red squares being adjacent
= No. of ways with nth square being red
+ No. of ways with nth square being blue
= No. of ways with n-th red & $(n-1)$th blue & $1 \times (n-2)$ arb.
+ No. of ways with n-th blue & $1 \times (n-1)$ colored arb.
= $h_{n-1} + h_{n-2}$.

So $h_n = h_{n-1} + h_{n-2}$. A more detailed solution follows.

8. Consider the $1\times n$ chessboard. In any legal coloring of it, the n-th square must be red or blue. If n-th square is red, then the $(n-1)$th square must be blue. And if the n-th square is blue, then the $(n-1)$th square can be anything (i.e., red or blue). So we can color the first $n-2$ squares in h_{n-2} ways & then the $(n-1)$th blue & the nth red, or we can color the first $n-1$ squares in h_{n-1} ways & the nth blue.

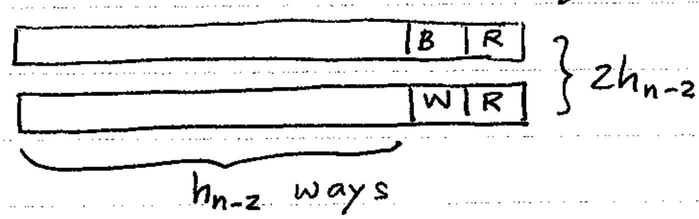




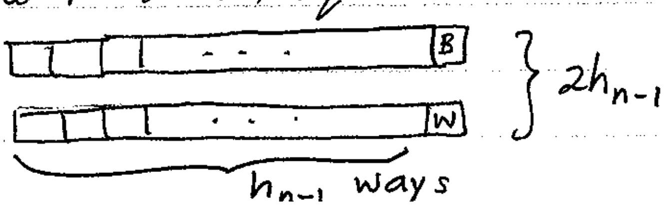
Thus

$$h_n = h_{n-1} + h_{n-2}$$

9. Consider the $1\times n$ square. In any legal coloring of it the last square is red, blue or white. If last square is red, then 2nd to last square be blue or white. We can color the first $n-2$ squares in h_{n-2} ways



If last square is blue or white, then we can color the first $n-1$ squares in h_{n-1} ways. So $h_n = 2h_{n-1} + 2h_{n-2}$.



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$$\begin{aligned} h_n &= 4 \cdot h_{n-2} &&= 2^2 \cdot h_{n-2} \\ &= 2^2 \cdot 4 h_{n-4} &&= 2^4 \cdot h_{n-4} \\ &= 2^4 \cdot 4 h_{n-6} &&= 2^6 \cdot h_{n-6} \\ &\vdots && \vdots \\ &= 2^{2k-2} \cdot 4 h_{n-2k} &&= 2^{2k} \cdot h_{n-2k} \\ &\vdots \end{aligned}$$

$$= \begin{cases} 2^{2 \cdot n/2} \cdot h_0 & \text{if } n \text{ is even} \\ 2^{2 \cdot (n-1)/2} \cdot h_1 & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} 2^n \cdot h_0 \\ 2^{n-1} \cdot h_1 \end{cases} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2^{n-1} \cdot 1 & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \quad h_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2^{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

We can also solve this by the E-Method.
We have $h_n - 4h_{n-2} = 0$. So the auxiliary equation is $(E^2 - 4) = 0$ $\quad \therefore (E-2)(E+2) = 0$

$\therefore \quad h_n = A \cdot (2)^n + B \cdot (-2)^n$

But $\quad 0 = h_0 = A \cdot (2)^0 + B \cdot (-2)^0$

$\quad\quad 1 = h_1 = A \cdot (2)^1 + B (-2)^1$

$\therefore \quad A + B = 0 \implies B = -A$

$\quad\quad 2A - 2B = 1 \longrightarrow 4A = 1 \implies A = 1/4$

$\quad\quad\quad\quad \implies B = -1/4$

$\therefore \quad h_n = \frac{1}{4}(2)^n - \frac{1}{4} \cdot (-2)^n = \frac{1}{4}\left[(2)^n - (-2)^n\right]$

<u>Note:</u> The two answers look different, but they are the same.

32 $h_n = (n+2) \cdot h_{n-1}$

$= (n+2) \cdot (n+1)\, h_{n-2}$

$= (n+2) \cdot (n+1) \cdot (n) h_{n-3}$

$\vdots$

$= (n+2) \cdot (n+1) \cdot (n) \ldots (4)(3) \cdot h_{n-n}$

$= \frac{(n+2)!}{2} \cdot h_0 = \frac{(n+2)!}{2} \cdot 2 = (n+2)!$

33 We have $h_n - h_{n-1} - 9h_{n-2} + 9h_{n-3} = 0$

So the auxiliary equation is

$f(E) = E^3 - E^2 - 9E + 9 = 0$ $\qquad f(1) = 0 \Rightarrow$

$(E-1)(E^2-9) = 0$ $\qquad (E-1)$ is a factor

$(E-1)(E-3)(E+3) = 0$

$\therefore \quad h_n = A \cdot (1)^n + B \cdot (3)^n + C \cdot (-3)^n$

$0 = h = A + B + C$

$1 = h_1 = A + 3B - 3C$

$2 = h_2 = A + 9B + 9C$

Now solve for A, B & C to get the answer.

34. We have $h_n - 8h_{n-1} + 16h_{n-2} = 0$

So the auxiliary equation is

$E^2 - 8E + 16 = 0$

$(E-4)^2 = 0$

$\therefore \quad h_n = (A + Bn) \cdot (4)^n$

$-1 = h_0 = (A + B \cdot 0) \cdot 4^0 \Rightarrow A = -1$

$0 = h_1 = (A + B) \cdot 4 \Rightarrow B = -A = 1$

$\therefore \quad h_n = (n-1) \cdot (4)^n$

35. We have $h_n - 3h_{n-2} + 2h_{n-3} = 0$. So the auxiliary equation is

$f(E) = E^3 - 3E + 2 = 0$

$(E-1)(E^2+E-2) = 0$ $\quad$ $f(1)=0 \Rightarrow (E-1)$ is a factor

$(E-1)(E-1)(E+2) = 0$

$(E-1)^2 . (E+2) = 0$

$\therefore \quad h_n = (A+Bn).(1)^n + C.(-2)^n$

$1 = h_0 = A + B.0 + C$

$0 = h_1 = A + B.1 + C(-2)$

$0 = h_2 = A + B.2 + C(-2)^2$

Solving for A, B & C will give us the solution

$$h_n = \frac{8}{9} - \frac{2}{3}n + \frac{1}{9}(-2)^n$$

36. We have $h_n - 5h_{n-1} + 6h_{n-2} + 4h_{n-3} - 8h_{n-4} = 0$

So the auxiliary equation is

$f(E) = E^4 - 5E^3 + 6E^2 + 4E - 8 = 0$

and this problem seems to be intended for masochists

$(E+1)(E^3 - 6E^2 + 12E - 8) = 0$ $\quad$ $f(-1) = 0$ & $f(2) = 0$

$(E+1)(E-2)(E^2 - 4E + 4) = 0$ $\quad$ $\Rightarrow (E+1)$ & $(E-2)$ are factors

$(E+1)(E-2)^3 = 0$

$\therefore \quad h_n = A.(-1)^n + (B + Cn + Dn^2)(-2)^n$

$0 = A + B$

$1 = -A - 2B - 2C - 2D$

$1 = A + 4B + 8C + 16D$

$2 = -A - 8B - 24C - 72D$

Solving for A, B, C & D will give us the answer.

38 You don't have to use induction. All you have to do is to compute the first few values & then guess a solution. If you try your guess and it works – That's it! Your are done

(a) $h_0 = 1$

$h_1 = 3h_0 = 3.1 = 3$

$h_2 = 3h_1 = 3.3 = 3^2$

$h_3 = 3h_2 = 3.3^2 = 3^3$

<u>Guess</u>: $h_n = 3^n$. So $h_{n-1} = 3^{n-1}$

<u>Check</u>: $h_n - h_{n-1} = 3^n - 3.3^{n-1} = 0$

So your guess is correct

$\therefore$ solution is $h_n = (3)^n$.

(b) There is a misprint in this problem. The equation should be $h_n = h_{n-1} + 3$. (If you use the equation $h_n = h_{n-1} - n + 3$, you can guess "'til thy kingdom come" and still not get the solution.)

$h_0 = 2$

$h_1 = h_0 + 3 = 3 + 2 = 3.1 + 2$

$h_2 = h_1 + 3 = 3.1 + 2 + 3 = 3.2 + 2$

$h_3 = h_2 + 3 = 3.2 + 2 + 3 = 3.3 + 2$

<u>Guess</u>: $h_n = 3.n + 2$ $\qquad h_{n-1} = 3(n-1) + 2$

<u>Check</u>:
$$\begin{aligned} h_n - h_{n-1} - 3 &= (3n+2) - [3(n-1)+2] - 3 \\ &= 3n + 2 - (3n - 3 + 2) - 3 \\ &= 3n + 2 - 3n + 3 - 2 - 3 = 0 \checkmark \end{aligned}$$

By the way, the solution for $h_n = h_{n-1} - n + 3$ with $h_0 = 2$ is $h_n = (4 + 5n - n^2)/2$. YOU GUESSED IT?

38 (c) $h_0 = 0$

$h_1 = -h_0 + 1 = 1$

$h_2 = -h_1 + 1 = 0$

$h_3 = -h_2 + 1 = 1$

$h_4 = -h_3 + 1 = 0$

<u>Guess</u>: $h_n = \frac{1}{2}\left[1 - (-1)^n\right] = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$

It works! (Check this)

(d) $h_0 = 1$

$h_1 = -h_0 + 2 = 1$

$h_2 = -h_1 + 2 = 1$

$h_3 = -h_2 + 2 = 1$ Something seems strange here!

<u>Guess</u>: $h_n = 1$ $h_{n-1} = 1$

<u>Check</u>: $h_n + h_{n-1} - 2 = 1 + 1 - 2 = 0$ ✓ Yep!

(e) $h_0 = 1 = 2^1 - 1$

$h_1 = 2h_0 + 1 = 3 = 2^2 - 1$

$h_2 = 2h_1 + 1 = 7 = 2^3 - 1$

$h_3 = 2h_2 + 1 = 15 = 2^4 - 1$

<u>Guess</u>: $h_n = 2^{n+1} - 1$ $h_{n-1} = 2^{(n-1)+1} - 1$

<u>Check</u>: $h_n - 2h_{n-1} - 1 = (2^{n+1} - 1) - 2(2^n - 1) - 1$

$= 2^{n+1} - 1 - 2^{n+1} + 2 - 1$

$= 0$ ✓ It works!

You should solve these problems again by using the method given for first order linear diff. eq. with variable coefficients.

43. $h_n = 4h_{n-1} + 3.2^n$ and $h_0 = 1$

So $h_n - 4h_{n-1} = 3.2^n$

The homogenous equation is $h_n - 4h_{n-1} = 0$

So auxiliary equation is $E - 4 = 0$

$\therefore \quad h_n^c = A.(4)^n$

Try as particular solution $h_n^p = B.2^n$

Then $h_{n-1}^p = B.2^{n-1}$. So

$h_n^p - 4h_{n-1}^p = 3.2^n$ becomes

$B.2^n - 4.B.2^{n-1} = 3.2^n$

$\therefore \quad (B - 2B).2^n = 3.2^n$

So $B - 2B = 3 \implies B = -3$

$\therefore \quad h_n^p = -3.2^n$

So $h_n = h_n^c + h_n^p = A.(4)^n - 3.2^n$

But $h_0 = 1$, so $1 = A - 3.2^0$

$\implies A = 4$

$\therefore \quad h_n = A.(4)^n - 3.(2)^n$

$= 4.(4)^n - 3.(2)^n$.

44. $h_n = 3h_{n-1} - 2$ and $h_0 = 1$

So $h_n - 3h_{n-1} = -2$

Homog. Eq. is $h_n - 3h_{n-1} = 0$

Aux. Eq. is $E - 3 = 0$

$\therefore \quad h_n^c = A.(3)^n$.

44 Try as a particular solution
$h_n^p = B$. Then $h_{n-1}^p = B$

So $h_n^p - 3h_n^p = -2$ becomes

$B - 3B = -2$

$\therefore -2B = -2 \Rightarrow B = 1$

$\therefore h_n^p = 1$

So $h_n = h_n^c + h_n^p = A.(3)^n + 1$

$1 = h_0 = A(3)^0 + 1 \Rightarrow A = 0$

$\therefore h_n = A.(3)^n + 1 = 1.$

That's it, there is nothing fishy here! $h_n = 1$!

45 We have $h_n - 2h_{n-1} = n$ and $h_0 = 1$

Homog. eq. is: $h_n - 2h_{n-1} = 0$

Auxiliary equation is: $E - 2 = 0$

$\therefore h_n^c = A.(2)^n$

Try $h_n^p = B + Cn$ as a particular solution

$h_{n-1}^p = B + C.(n-1)$

Then $h_n^p - 2h_{n-1}^p = n$ becomes

$(B + Cn) - 2[B + C(n-1)] = n$

$\therefore B + Cn - 2B - 2Cn + 2C = n$

$\therefore (-B + 2C) - Cn = n$

So $-C = 1$ (coeff. of n)

$-B + 2C = 0$ (const. terms)

$\Rightarrow C = -1$ and $B = -2$

$\therefore h_n^p = -n - 2$

45. So $h_n = h_n^c + h_n^p$

$= A.(2)^n - n - 2$

$h_0 = 1 \Rightarrow \quad 1 = A - 0 - 2 \quad \Rightarrow \quad A = 3$

$\therefore \quad h_n = 3.(2)^n - n - 2.$

46. We have $h_n - 6h_{n-1} + 9h_{n-2} = 2n$

Homog. Equation is $\quad h_n - 6h_{n-1} + 9h_n = 0$

Auxiliary eq. is $\quad E^2 - 6E + 9 = 0$

So $(E-3)^2 = 0 \quad \therefore \quad h_n^c = (A + Bn).(3)^n$

Try $h_n^p = C + Dn$. Then

$h_{n-1}^p = C + D(n-1) = C + Dn - D$

$h_{n-2}^p = C + D(n-2) = C + Dn - 2D$

So $\quad h_n - 6h_{n-1} + 9h_{n-2} = 2n \quad$ becomes

$C + Dn - 6(C + Dn - D) + 9(C + Dn - 2D) = 2n$

$\therefore \quad 4C - 12D + 4Dn = 2n$

$\therefore \quad 4D = 2 \quad$ (coeff. of n)

$4C - 12D = 0 \quad$ (constant term)

$\Rightarrow D = 1/2 \;\&\; C = 3D = 3/2$

$\therefore \quad h_n^p = \frac{3}{2} + \frac{n}{2}$.

$\therefore \quad h_n = (A + Bn).(3)^n + \frac{3}{2} + \frac{n}{2}$

$\therefore 1 = h_0 = A. + \frac{3}{2} \quad \Rightarrow \quad A = -1/2$

$0 = h_1 = 3A + 3B + \frac{3}{2} + \frac{1}{2} \quad \Rightarrow \quad B = -1/6$

$\therefore \quad h_n = \left(-\frac{1}{2} - \frac{n}{6}\right)(3)^n + \frac{3}{2} + \frac{n}{2}.$

47. We have $h_n - 4h_{n-1} + 4h_{n-2} = 3n+1$

Homog. Eq. is $h_n - 4h_{n-1} + 4h_{n-2} = 0$

Auxiliary eq. is $E^2 - 4E + 4 = 0$

$\therefore \quad (E-2)^2 = 0$

$\therefore \quad h_n^c = (A+Bn)(2)^n$

Try $h_n^p = C + Dn$. Then

$h_{n-1}^p = C + D(n-1) = C + Dn - D$

$h_{n-2}^p = C + D(n-2) = C + Dn - 2D$

So $h_n - 4h_{n-1} + 4h_{n-2} = 3n+1$ becomes

$C + Dn - 4(C + Dn - D) + 4(C + Dn - 2D) = 3n+1$

$C - 4D + Dn = 3n + 1$

$\therefore \quad D = 3$

$C - 4D = 1 \quad \Rightarrow \quad C = 4D + 1 = 13$

$\therefore \quad h_n^p = 13 + 3n$

So $h_n = (A+Bn) \cdot 2^n + 13 + 3n$

$1 = A + 13 \quad \Rightarrow \quad A = -12$

$2 = 2A + 2B + 13 + 3 \quad \Rightarrow \quad B = 5$

$\therefore \quad h_n = (5n-12) \cdot 2^n + 3n + 13$.

13. (a) We know $1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$

$g_a(x) = 1 + cx + c^2x^2 + \cdots + c^n x^n + \cdots$

$= 1 + (cx) + (cx)^2 + \cdots + (cx)^n + \cdots$

$= \frac{1}{1-cx}$

$\therefore$ generating function of $\langle c^n \rangle_{n \geq 0}$ is $\frac{1}{1-cx}$

13 (b)
$$g_b(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$$
$$= 1 + (-x) + (-x)^2 + \cdots + (-x)^n + \cdots$$
$$= \frac{1}{1-(-x)} = \frac{1}{1+x}$$

(c)
$$g_c(x) = \binom{\alpha}{0} - \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 - \cdots + (-1)^n\binom{\alpha}{n}x^n + \cdots$$
$$= \binom{\alpha}{0} + \binom{\alpha}{1}(-x) + \binom{\alpha}{2}(-x)^2 + \cdots + \binom{\alpha}{n}(-x)^n + \cdots$$
$$= \sum_{n=0}^{\infty}\binom{\alpha}{n}(-x)^n = \left[1+(-x)\right]^\alpha$$ by Generalized Binomial Theorem.
$$= (1-x)^\alpha$$

(d)
$$g_d(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$
$$= e^x$$

(e)
$$g_e(x) = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n\frac{x^n}{n!} + \cdots$$
$$= 1 + \frac{(-x)}{1!} + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots + \frac{(-x)^n}{n!} + \cdots$$
$$= e^{-x}$$

14. (a)
$$g_a(x) = \underbrace{(x^1+x^3+x^5+\cdots)}_{\text{no. of } e_1\text{'s}}\underbrace{(x^1+x^3+x^5+\cdots)}_{\text{no. of } e_2\text{'s}}\underbrace{(x^1+x^3+x^5+\cdots)}_{\text{no. of } e_3\text{'s}}\underbrace{(x^1+x^3+\cdots)}_{e_4\text{'s}}$$
$$= (x+x^3+x^5+\cdots)^4$$
$$= x^4(1+x^2+x^4+\cdots)^4 = x^4\cdot\left(\frac{1}{1-x^2}\right)^4$$

$h_n =$ coefficient of x^n in the expansion of $(x^1+x^3+x^5+\cdots)(x^1+x^3+x^5+\cdots)(x^1+x^3+x^5+\cdots)(x^1+x^3+x^5+\cdots)$

14 (b) $h_n =$ coefficient of x^n in the expansion of

$$\underbrace{(x^0+x^3+x^6+\cdots)}_{\text{no. of } e_1\text{'s}}\underbrace{(x^0+x^3+x^6+\cdots)}_{\text{no. of } e_2\text{'s}}\underbrace{(x^0+x^3+\cdots)}_{e_3\text{'s}}\underbrace{(x^0+x^3+\cdots)}_{e_4\text{'s}}$$

$$\therefore g_b(x) = (1+x^3+x^6+\cdots)^4$$
$$= \left(\frac{1}{1-x^3}\right)^4 = (1-x^3)^{-4}$$

(c) $h_n =$ coefficient of x^n in the expansion of

$$\underbrace{(x^0)}_{\text{no. of } e_1\text{'s}}\cdot\underbrace{(x^0+x^1)}_{\text{no. of } e_2\text{'s}}\cdot\underbrace{(x^0+x^1+x^2+\cdots)}_{\text{no. of } e_3\text{'s}}\underbrace{(x^0+x^1+x^2+\cdots)}_{e_4\text{'s}}$$

$$\therefore g_c(x) = (1)(1+x)(1+x+x^2+\cdots)^2$$
$$= (1+x)\cdot\left(\frac{1}{1-x}\right)^2 = \frac{1+x}{(1-x)^2}$$

(d) $h_n =$ coefficient of x^n in the expansion of

$$(x^1+x^3+x^{11})(x^2+x^4+x^5)(x^0+x^1+\cdots)(x^0+x^1+\cdots)$$

$$\therefore g_d(x) = (x+x^3+x^{11})(x^2+x^4+x^5)(1+x+x^2+\cdots)^2$$
$$= x(1+x^2+x^{10})\cdot x^2(1+x^2+x^3)\left(\frac{1}{1-x}\right)^2$$
$$= \frac{x^3}{(1-x)^2}\cdot(1+x^2+x^3+x^4+x^5+x^{10}+x^{12}+x^{13})$$

(e) $h_n =$ coeff. of x^n in the expansion of

$$(x^{10}+x^{11}+\cdots)(x^{10}+x^{11}+\cdots)(x^{10}+x^{11}+\cdots)(x^{10}+x^{11}+\cdots)$$

$$\therefore g_e(x) = (x^{10}+x^{11}+x^{12}+\cdots)^4$$
$$= (x^{10})^4(1+x+x^2+\cdots)^4 = x^{40}\cdot\left(\frac{1}{1-x}\right)^4$$

48 (a) Let $g(x)$ be the standard gen. function of $\langle h_n \rangle$

Then

$$g(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_n x^n + \cdots$$
$$-4x^2 g(x) = -4h_0 x^2 - \cdots - 4h_{n-2} x^n - \cdots$$

$$\therefore (1-4x^2) g(x) = h_0 + h_1 x + 0.x^2 + 0.x^3 + \cdots + 0.x^n + \cdots$$
$$= 0 + x \qquad (\text{because } h_n - 4h_{n-2} = 0)$$

$$\therefore g(x) = \frac{x}{1-4x^2} = \frac{x}{(1-2x)(1+2x)}$$

Let $\dfrac{x}{(1-2x)(1+2x)} = \dfrac{A}{1-2x} + \dfrac{B}{1+2x} = \dfrac{A(1+2x) + B(1-2x)}{(1-2x)(1+2x)}$

Then $x = A(1+2x) + B(1-2x)$

Putting $x = 1/2$ gives $1/2 = A(1+2\cdot 1/2) \Rightarrow A = 1/4$

Putting $x = -1/2$ gives $-1/2 = B(1 - -2\cdot 1/2 \Rightarrow B = -1/4$

$$\therefore g(x) = \frac{1}{4}\cdot\left(\frac{1}{1-2x}\right) - \frac{1}{4}\left(\frac{1}{1+2x}\right)$$
$$= \frac{1}{4}\left[\frac{1}{1-2x}\right] - \frac{1}{4}\left[\frac{1}{1-(-2x)}\right]$$
$$= \frac{1}{4}\left[1 + 2x + (2x)^2 + \cdots + (2x)^n + \cdots\right]$$
$$- \frac{1}{4}\left[1 + (-2x) + (-2x)^2 + \cdots + (-2x)^n + \cdots\right]$$

So h_n = coefficient of x^n in the expansion of $g(x)$

$$= \frac{1}{4}\cdot 2^n - \frac{1}{4}(-2)^n.$$

(b) Let $g(x)$ be the standard gen. function of $\langle h_n \rangle$

Then
$$g(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_n x^n + \cdots$$
$$-x g(x) = -h_0 x - h_1 x^2 - \cdots - h_{n-1} x^n - \cdots$$
$$-x^2 g(x) = -h_0 x^2 - \cdots - h_{n-2} x^n - \cdots$$

48 (b) So $(1-x-x^2)g(x) = h_0 + (h_1-h_0)x + 0.x^2 + \cdots + 0x^n + \cdots$
$= 1 + (3-1)x$

$\therefore \quad g(x) = \dfrac{1+2x}{1-x-x^2}$

$1-x-x^2 = \left[1-\left(\frac{1+\sqrt{5}}{2}\right)x\right]\left[1-\left(\frac{1-\sqrt{5}}{2}\right)x\right]$

Let $\dfrac{1+2x}{1-x-x^2} = \dfrac{A}{1-\frac{1+\sqrt{5}}{2}x} + \dfrac{B}{1-\frac{1-\sqrt{5}}{2}x}$

Then $1+2x = A\left(1-\frac{1-\sqrt{5}}{2}x\right) + B\left(1-\frac{1+\sqrt{5}}{2}x\right)$

Putting $x = \frac{2}{1+\sqrt{5}}$ gives $1+\frac{4}{1+\sqrt{5}} = A\left(1-\frac{1-\sqrt{5}}{2}\cdot\frac{2}{1+\sqrt{5}}\right)$

$\therefore \quad \dfrac{5+\sqrt{5}}{1+\sqrt{5}} = A.\dfrac{1+\sqrt{5}-(1-\sqrt{5})}{1+\sqrt{5}} \Rightarrow A(2\sqrt{5}) = 5+\sqrt{5}$

$\Rightarrow A = \dfrac{\sqrt{5}+1}{2}$

Putting $x = \frac{2}{1-\sqrt{5}}$ gives $1+\frac{4}{1-\sqrt{5}} = B\left(1-\frac{1+\sqrt{5}}{2}\cdot\frac{2}{1-\sqrt{5}}\right)$

$\therefore \quad \dfrac{5-\sqrt{5}}{1-\sqrt{5}} = B.\dfrac{(1-\sqrt{5})-(1+\sqrt{5})}{1-\sqrt{5}} \Rightarrow B(-2\sqrt{5}) = 5-\sqrt{5}$

$\Rightarrow B = \dfrac{1-\sqrt{5}}{2}$

$\therefore \quad g(x) = A.\left(1-\frac{1+\sqrt{5}}{2}x\right)^{-1} + B.\left(1-\frac{1-\sqrt{5}}{2}x\right)^{-1}$

$= A.\left[1+\left(\frac{1+\sqrt{5}}{2}\right)x + \cdots + \left(\frac{1+\sqrt{5}}{2}\right)^n x^n + \cdots\right]$
$+ B\left[1+\left(\frac{1-\sqrt{5}}{2}\right)x + \cdots + \left(\frac{1-\sqrt{5}}{2}\right)^n x^n + \cdots\right]$

$\therefore \quad h_n =$ coeff. of x^n in the exp. of $g(x)$

$= A.\left(\frac{1+\sqrt{5}}{2}\right)^n + B.\left(\frac{1-\sqrt{5}}{2}\right)^n$

$= \frac{1+\sqrt{5}}{2}.\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1-\sqrt{5}}{2}.\left(\frac{1-\sqrt{5}}{2}\right)^n$

$= \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$ Gloria a dios!

48 (d) Let $g(x)$ be the standard gen. func. of x

Then

$$g(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_n x^n + \cdots$$
$$-8x\, g(x) = -8h_0 x - 8h_1 x^2 - \cdots - 8h_{n-1} x^n - \cdots$$
$$+16x^2 g(x) = 16h_0 x^2 - \cdots + 16 h_{n-2} x^n + \cdots$$

$$\therefore \left(1-8x+16x^2\right) g(x) = h_0 + h_1 x - 8h_0 x + 0 \cdot x^2 + \cdots$$
$$= -1 + 0x + 8x$$

$$\therefore \quad g(x) = \frac{8x-1}{1-8x+16x^2} = \frac{8x-1}{(1-4x)^2}$$

Let $\dfrac{8x-1}{(1-4x)^2} = \dfrac{A}{1-4x} + \dfrac{B}{(1-4x)^2}$. Then

$$8x-1 = A(1-4x) + B \quad \cdots (*)$$

Putting $x = 1/4$ gives $8 \cdot 1/4 - 1 = B \quad \therefore B = 1$

Differentiating both sides of $(*)$ gives

$$8 = A(0-4) + 0 \qquad \therefore A = -2$$

So $g(x) = \dfrac{-2}{1-4x} + \dfrac{1}{(1-4x)^2}$.

Now $\dfrac{1}{(1-x)^k} = \displaystyle\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^{k1}$. So

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} \binom{n+1}{1} x^n = \binom{1}{1} + \binom{2}{1}x + \binom{3}{1}x^2 + \cdots + \binom{n+1}{1}x^n + \cdots$$
$$= 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots$$

$$\therefore g(x) = -2\left(\frac{1}{1-4x}\right) + 1 \cdot \frac{1}{(1-4x)^2}$$
$$= -2\left[1 + (4x) + (4x)^2 + \cdots + (4x)^n + \cdots\right]$$
$$+ 1 \cdot \left[1 + 2(4x) + 3(4x)^2 + \cdots + (n+1)\cdot(4x)^n + \cdots\right]$$

$$\therefore h_n = \text{coeff. of } x^n = -2 \cdot 4^n + (n+1) \cdot 4^n = (n-1) \cdot 4^n.$$

15. We know that

$$1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x} = (1-x)^{-1}$$

Differentiating both sides we get

$$0 + 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = (1-x)^{-2}$$

Multiplying both sides by x gives us

$$x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots = x.(1-x)^{-2}$$

Differentiating both sides again gives us

$$1 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1} + \cdots = 1.(1-x)^{-2} + x.2.(1-x)^{-3}$$
$$= (1+x).(1-x)^{-3}$$

Multiplying both sides by x a second time gives

$$x + 2^2x^2 + 3^2x^3 + \cdots + n^2x^n + \cdots = (x+x^2)(1-x)^{-3}$$

Differentiating both sides a third time gives

$$1 + 2^3x + 3^3x^2 + \cdots + n^3x^{n-1} + \cdots = (1+2x).(1-x)^{-3} + 3.(x+x^2)(1-x)^{-4}$$
$$= (1+4x+x^2)(1-x)^{-4}$$

Multiplying both sides by x for a third time give

$$1^3x + 2^3x^2 + 3^3x^3 + \cdots + n^3x^n + \cdots = x(1+4x+x^2)(1-x)^{-4}$$

$\therefore$ generating function of $\langle n^3 \rangle_{n \geq 0}$ is

$$0^3 + 1^3.x + 2^3.x^2 + 3^3.x^3 + \cdots + n^3.x^n + \cdots$$
$$= x(1+4x+x^2).(1-x)^{-4}.$$

Soccorro!

18 $$2e_1 + 5e_2 + e_3 + 7e_4 = n$$

Let h_n = number of solutions of this equation in non-neg. integers.

Then h_n = No. of n-comb. of $[\infty.e_1, \infty.e_2, \infty.e_3, \infty.e_4]$
with e_1 being a mult. of 2,
with e_2 being a mult. of 5, and
with e_4 being a mult. of 7

(There is no restriction on the e_3's)

So h_n = coeff. of x^n in the expansion of

$$\underbrace{(x^0+x^2+x^4+\cdots)}_{e_1\text{'s}}\underbrace{(x^0+x^5+\cdots)}_{e_2\text{'s}}\underbrace{(x^0+x^1+\cdots)}_{e_3\text{'s}}\underbrace{(x^0+x^7+\cdots)}_{e_4\text{'s}}$$

$$\therefore\ g(x) = (1+x^2+x^4+\cdots)(1+x^5+x^{10}+\cdots)(1+x+x^2+\cdots)(1+x^7+\cdots)$$
$$= \frac{1}{1-x^2}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x}\cdot\frac{1}{1-x^7}$$

19. We know that
$$(1-x)^{-k} = \sum_{n=0}^{\infty}\binom{n+k-1}{k-1}x^n$$

So $$\underbrace{\binom{0}{2}}_{=0} + \underbrace{\binom{1}{2}}_{=0}x + \binom{2}{2}x^2 + \binom{3}{2}x^3 + \cdots + \binom{n}{2}x^n + \cdots$$
$$= x^2\left[\binom{2}{2} + \binom{3}{2}x + \binom{4}{2}x^2 + \cdots + \binom{n+2}{2}x^n + \cdots\right]$$
$$= x^2\cdot\sum_{n=0}^{\infty}\binom{n+2}{2}x^n = x^2\cdot\sum_{n=0}^{\infty}\binom{n+3-1}{3-1} = x^2\cdot(1-x)^{-3}$$

$\therefore$ generating function of $\binom{n}{2}$ is given by
$$g(x) = x^2\cdot(1-x)^{-3} = \frac{x^2}{(1-x)^3}$$

20. $\binom{0}{3}+\binom{1}{3}x+\binom{2}{3}x^2+\binom{3}{3}x^3+\cdots+\binom{n}{3}x^n+\cdots$

$= x^3\left\{\binom{3}{3}+\binom{4}{3}x+\binom{5}{3}x^2+\cdots+\binom{n+3}{3}x^n+\cdots\right\}$

$= x^3\cdot\sum_{n=0}^{\infty}\binom{n+3}{3}x^n = x^3\cdot\sum_{n=0}^{\infty}\binom{n+4-1}{4-1}x^n$

$= x^3\cdot(1-x)^{-4}$

$\therefore$ Generating function of $\binom{n}{3}$ is $x^3\cdot(1-x)^{-4}$

22. $h_n = n!$

$g_E(x) = h_0\cdot\frac{x^0}{0!}+h_1\frac{x^1}{1!}+h_2\frac{x^2}{2!}+\cdots+h^n\frac{x^n}{n!}+\cdots$

$= 0!\,\frac{x^0}{0!}+1!\,\frac{x^1}{1!}+2!\,\frac{x^2}{2!}+\cdots+n!\,\frac{x^n}{n!}+\cdots$

$= 1+x+x^2+\cdots+x^n+\cdots = \frac{1}{1-x}$

$\therefore$ the exponential gen. function of $\langle n!\rangle_{n\geq 0}$ is $\frac{1}{1-x}$.

24 (a) $h_n =$ coeff. of $\frac{x^n}{n!}$ in the exponen. exp. of

$$\underbrace{\left(\frac{x^1}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)}_{e_1\text{'s}}\underbrace{\left(\frac{x^1}{1!}+\frac{x^3}{3!}+\cdots\right)}_{e_2\text{'s}}\cdots\underbrace{\left(\frac{x^1}{1!}+\frac{x^3}{3!}\right)}_{e_k\text{'s}}$$

$\therefore g_E(x) = \left(\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)^k$

$= [\sinh x]^k = \left(\frac{e^x-e^{-x}}{2}\right)^k$

Note: $\sinh x = \frac{e^x-e^{-x}}{2}$

24 (b) $h_n =$ coeff. of $\frac{x^n}{n!}$ in the exponential exp. of

$$\underbrace{\left(\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots\right)}_{e_1\text{'s}}\underbrace{\left(\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots\right)}_{e_2\text{'s}}\cdots\underbrace{\left(\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots\right)}_{e_k\text{'s}}$$

$$\therefore g_E(x) = \left(\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\cdots\right)^k$$
$$= \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \frac{x^3}{3!}\right)^k$$

(c) $h_n =$ coeff. of $\frac{x^n}{n!}$ in the exponential exp. of

$$\underbrace{\left(\frac{x^1}{1!}+\frac{x^2}{2!}+\cdots\right)}_{e_1\text{'s}}\underbrace{\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)}_{e_2\text{'s}}\cdots\underbrace{\left(\frac{x^k}{k!}+\frac{x^{k+1}}{(k+1)!}+\cdots\right)}_{e_k\text{'s}}$$

$$\therefore g_E(x) = \left(\frac{x^1}{1!}+\frac{x^2}{2!}+\cdots\right)\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\cdots\left(\frac{x^k}{k!}+\cdots\right)$$
$$= (e^x-1)\cdot\left(e^x-1-\frac{x}{1!}\right)\cdot\ \cdots\left(e^x-1-\frac{x}{1!}-\frac{x^2}{2!}-\cdots-\frac{x^{k-1}}{(k-1)!}\right)$$

(d) $h_n =$ coeff. of $\frac{x^n}{n!}$ in the exponen. exp. of

$$\left(\frac{x^0}{0!}+\frac{x^1}{1!}\right)\left(\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}\right)\cdots\left(\frac{x^0}{0!}+\frac{x^1}{1!}+\cdots+\frac{x^k}{k!}\right)$$

$$\therefore g_E(x) = \left(1+\frac{x}{1!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)\cdots\left(1+\frac{x}{1!}+\cdots+\frac{x^k}{k!}\right)$$

25. $h_n =$ no. of n-permutations of $[\infty.R, \infty.W, \infty.B, \infty.G]$ with an even number of R's and an odd number of W's

$$g(x) = \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)$$
$$= \frac{(e^x+e^{-x})}{2}\cdot\frac{(e^x+e^{-x})}{2}\cdot e^x\cdot e^x = \frac{e^{2x}-e^{-2x}}{4}\cdot e^{2x} = (e^{4x}-1)/4$$
$$= \frac{1}{4}\left(1+\frac{4x}{1!}+\frac{(4x)^2}{2!}+\cdots+\frac{(4x)^n}{n!}+\cdots-1\right)$$

$$\therefore h_n = \text{coeff. of } \frac{x^n}{n!} \text{ in expansion of } g(x) = \begin{cases} 4^{n-1} & \text{if } n \geq 0, \\ 0 & \text{if } n = 0. \end{cases}$$

42. $h_n - 4h_{n-1} = 4^n$ and $h_0 = 3$

Homog. Equation : $h_n - 4h_{n-1} = 0$

Auxiliary equation : $E - 4 = 0$

$\therefore \quad h_n^c = A.(4)^n$

Try $h_n^p = Bn.(4)^n$ $\quad$ [$h_n^p = B.(4)^n$ won't work!]

Then $h_{n-1}^p = B(n-1).(4)^{n-1}$

So $h_n - 4h_{n-1} = 4^n$ becomes

$$Bn.(4)^n - 4B.(n-1).4^{n-1} = 4^n$$

$$Bn.4^n - Bn.4^n + B.4^n = 4^n$$

$$\therefore \quad B.4^n = 4^n$$

So $B = 1$ $\quad \therefore \quad h_n^B = n.(4)^n$

So $h_n = h_n^c + h_n^p = A.(4)^n + Bn.(4)^n$

$$= A.(4)^n + n.(4)^n$$

$h_0 = 3 \Rightarrow 3 = A.(4)^0 + 0 \Rightarrow A = 3.$

$\therefore \quad h_n = 3.(4)^n + n.(4)^n = (n+3).4^n$

26. h_n = no. of n-perm. of $[\infty.R, \infty B, \infty.G, \infty.O]$ with even nos. of R's & G's

$$g(x) = \left(\frac{x^0}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(\frac{x^0}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)$$

$$= \frac{e^x + e^{-x}}{2} \cdot \frac{e^x + e^{-x}}{2} \cdot e^x \cdot e^x = \frac{(e^{2x}+1)^2}{4} = \frac{e^{4x} + 2e^{2x} + 1}{4}$$

h_n = coeff. of $(x^n/n!)$ in exp. of $g(x)$ $= \begin{cases} 4^{n-1} + 2^{n-1} & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}$

27. h_n = no. of n-perm. of $[\infty.1, \infty.3, \infty.5, \infty.7, \infty.9]$ with pos. even nos. of 1's & 3's.

$$g(x) = \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)^3 = \left(\frac{e^x + e^{-x}}{2} - 1\right)^2 \cdot e^{3x}$$

28. $$g(x) = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 \cdot \left(\frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)^2 \cdot \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2 \cdot (e^x - 1)^2 \cdot (e^x)^2$$