Ch.8 - Stirling numbers and partition numbers

§1. The Difference Operator

Recall that if $\langle x_n \rangle_{n\in\mathbb{N}}$ was sequence, we defined the operator E by $E(\langle x_n\rangle) = \langle x_{n+1}\rangle_{n\in\mathbb{N}}$.

Def. We define the difference operator Δ by $\Delta = E - I$.
So $\Delta(\langle x_n\rangle) = (E-I)(\langle x_n\rangle) = \langle x_{n+1} - x_n\rangle_{n\in\mathbb{N}}$.

Ex. 1 Let $\langle x_n\rangle = \langle n^2+3n+1\rangle_{n\in\mathbb{N}}$. Then

$$\langle x_n\rangle = 1,\ 5,\ 11,\ 19,\ 29,\ 41,\ \ldots$$
$$\langle \Delta x_n\rangle = 5-1,\ 11-5,\ 19-11,\ 29-19,\ 41-29,\ 55-41,\ \ldots$$
$$= 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ \ldots$$

We can also write Δx_n just as $\langle x_{n+1} - x_n\rangle$.
So $\Delta x_n = x_{n+1} - x_n = \{(n+1)^2+3(n+1)+1\} - \{n^2+3n+1\}$
$= (n^2+2n+1+3n+3) - (n^2+3n) = 2n+4$.

Def We define Δ^k by recursion as follows.

$$\Delta^0 = I$$
$$\Delta^{k+1} = \Delta(\Delta^k), \quad \text{for } k \geq 0.$$

In particular $\Delta^2 = \Delta(\Delta) = (E-I)(E-I)$
$= E^2 - 2E + I$.

$\Delta^3 = (E-I)^3 = E^3 - 3E^2 + 3E - I$.

$\therefore \langle \Delta^2(x_n)\rangle = \langle x_{n+2} - 2x_{n+1} + x_n\rangle$

$\langle \Delta^3(x_n)\rangle = \langle x_{n+3} - 3x_{n+2} + 3x_{n+1} - x_n\rangle$.

Ex. 2 Let $\langle x_n\rangle = \langle n^2+3n+1\rangle_{n\in\mathbb{N}}$. Find $\langle\Delta x_n\rangle$, $\langle\Delta^2 x_n\rangle$ and $\langle\Delta^3 x_n\rangle$.

$\langle n \rangle$	0	1	2	3	4	5	6 ...
Ex. 2 $\langle X_n \rangle =$	1,	5,	11,	19,	29,	41,	55 ...
$\langle \Delta X_n \rangle =$	4	6	8	10	12	14	16 ...
$\langle \Delta^2 X_n \rangle =$	2	2	2	2	2	2	2 ...
$\langle \Delta^3 X_n \rangle =$	0	0	0	0	0	0	0 ...

Prop. 1 Let $X_n = P_k(n)$, where $P_k(n)$ is a polynomial in n of degree k. Then $\langle \Delta^{k+1} X_n \rangle = \langle 0 \rangle_{n \in \mathbb{N}}$.

Proof: We will prove the result by induction on k.

(a) If $k=0$, then $P_k(n) = a_0$, where a_0 is a constant. So $\Delta X_n = X_{n+1} - X_n = a_0 - a_0 = 0$. Hence the result is true when $k=0$.

(b) Suppose the result is true for k where $k \geq 0$. Then $\langle \Delta^{k+1} P_k(n) \rangle = \langle 0 \rangle_{n \in \mathbb{N}}$ for any polynomial in n $P_k(n)$ of degree k. Now let $P_{k+1}(n)$ be any polynomial in n of degree $k+1$. Then

$$P_{k+1}(n) = a_{k+1} n^{k+1} + a_k n^k + \dots + a_1 n^1 + a_0.$$

So

$$\begin{aligned}
\Delta P_{k+1}(n) &= a_{k+1}\left[(n+1)^{k+1} - n^{k+1}\right] + a_k\left[(n+1)^k - n^k\right] + \dots \\
&\quad \dots + a_1\left[(n+1) - n\right] + \left[a_0 - a_0\right] \\
&= a_{k+1}\left[n^{k+1} + \binom{k+1}{1} n^k + \dots + \binom{k+1}{k} n^1 + 1 - n^{k+1}\right] \\
&\quad + a_k\left[(n+1)^k - n^k\right] + \dots + a_1 \\
&= Q_k(n), \text{ a polynomial of degree } k
\end{aligned}$$

Thus

$$\Delta^{k+2}\left(P_{k+1}(n)\right) = \Delta^{k+1}\left(\Delta P_{k+1}(n)\right) = \Delta^{k+1}\left(Q_k(n)\right) = \langle 0 \rangle$$

because of the induction hypothesis. So if the result is true for k, it will be true for $k+1$.

(c) By the Principle of Math Induction, it follows that the result is true for all k.

Def. The zero-column of an infinite sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ is the sequence $\langle \Delta^k x_0 \rangle_{k\in\mathbb{N}}$. (Some authors say zero-diagonal instead of zero-column).

Ex 3 Let $\langle x_n\rangle_{n\in\mathbb{N}} = \langle n^2+3n+1\rangle_{n\in\mathbb{N}}$. Then the zero-column of $\langle x_n\rangle$ is $\langle 1, 4, 2, 0, 0, \cdots\rangle$ as shown in Ex. 2. The interesting thing is that

$$1\cdot\binom{n}{0} + 4\cdot\binom{n}{1} + 2\binom{n}{2} = 1 + 4n + 2\frac{n(n-1)}{2}$$
$$= 1 + 4n + (n^2 - n)$$
$$= 1 + 3n + n^2 = x_n.$$

This fact is true whenever x_n is a polynomial in n.

Prop 2 Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be an infinite sequence and suppose that $\langle \Delta^k x_0\rangle_{k\in\mathbb{N}} = \langle c_0, c_1, \cdots, c_p, 0, 0, \cdots\rangle$ where all the terms after c_p are all zeros. Then

$$x_n = c_0\binom{n}{0} + c_1\binom{n}{1} + c_2\binom{n}{2} + \cdots + c_p\binom{n}{p} = \sum \frac{(\Delta^k x_0)}{k!}[n]_k$$

Proof. See textbook. Here $[n]_k = n(n-1)(n-2)\cdots(n-(k-1))$.

It is nice to know that if x_n is a polynomial of degree p, then x_n can be expressed as

$$x_n = c_0\binom{n}{0} + c_1\binom{n}{1} + c_2\binom{n}{2} + \cdots + c_p\binom{n}{p}.$$

But the main reason for us to express x_n in this form is to be able to find $\sum_{k=1}^{n} x_k$.

Ex. 4 Let $x_n = 1+3n+n^2$. Then from Example 3

$$x_n = 1\cdot\binom{n}{0} + 4\cdot\binom{n}{1} + 2\cdot\binom{n}{2}.$$

So $$\sum_{k=0}^{n} x_k = \sum_{=1}^{n}\left\{1\cdot\binom{k}{0} + 4\binom{k}{1} + 2\cdot\binom{k}{2}\right\}$$

Ex. 4

$$= 1\cdot\sum_{k=0}^{n}\binom{k}{0} + 4\cdot\sum_{k=0}^{n}\binom{k}{1} + 2\cdot\sum_{k=0}^{n}\binom{k}{2}$$
$$= 1\cdot\binom{n+1}{1} + 4\cdot\binom{n+1}{2} + 2\cdot\binom{n+1}{3}$$
$$= (n+1) + 4\cdot\frac{(n+1)(n)}{2} + \cancel{2}\cdot\frac{(n+1)(n)(n-1)}{3\cdot\cancel{2}}$$
$$= (n+1)\left\{1 + 2n + \frac{n(n-1)}{3}\right\}$$
$$= \frac{n+1}{3}\left\{3 + 6n + n^2 - n\right\} = (n+1)(n^2+5n+3)/3$$

Let us check: $1 = 1$, $\quad (0+1)(0^2+5(0)+3)/3 = 1$ ✓

$1+5 = 6$, $\quad (1+1)(1^2+5(1)+3)/3 = 2(9)/3 = 6$ ✓

$1+5+11 = 17$, $\quad (2+1)(2^2+5(2)+3)/3 = 3(17)/3 = 17$ ✓

$1+5+11+19 = 36$, $\quad (3+1)(3^2+5(3)+3)/3 = 4(27)/3 = 36$ ✓

Theorem 3 Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be an infinite sequence with zero-column $\langle c_0, c_1, c_2, \dots, c_p, 0, 0, \dots\rangle$. Then

$$\sum_{k=0}^{n} x_k = c_0\binom{n+1}{1} + c_1\binom{n+1}{2} + \cdots + c_p\binom{n+1}{p+1}.$$

Proof: We know from Prop. 9 from the Binomial Coeff. Chapter,

$$\sum_{k=0}^{n}\binom{k}{i} = \binom{0}{i} + \binom{1}{i} + \binom{2}{i} + \cdots + \binom{n}{i} = \binom{n+1}{i+1}. \quad \text{So}$$

$$\sum_{k=0}^{n} x_k = \sum_{k=0}^{n}\left\{c_0\binom{k}{0} + c_1\binom{k}{1} + \cdots c_p\binom{k}{p}\right\}$$
$$= c_0\sum_{k=0}^{n}\binom{k}{0} + c_1\sum_{k=0}^{n}\binom{k}{1} + \cdots + c_p\sum_{k=0}^{n}\binom{k}{p}$$
$$= c_0\binom{n+1}{1} + c_1\binom{n+1}{2} + \cdots + c_p\binom{n+1}{p+1}$$

Ex. 5 Let $\langle x_n\rangle = \langle n^4\rangle$. Find $\sum_{k=0}^{n} x_k$.

see next page how to get these numbers

Sol. The zero-column of $\langle x_n\rangle$ is $\langle 0, 1, 14, 36, 24, 0, 0, \dots\rangle$

$$\text{So } \sum_{k=0}^{n} n^4 = 0\cdot\binom{n+1}{1} + 1\binom{n+1}{2} + 14\cdot\binom{n+1}{3} + 36\cdot\binom{n+1}{4} + 24\binom{n+1}{5}$$
$$= \cdots = n(n+1)(2n+1)(3n^2+3n-1)/30$$
$$= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

§2. The Stirling Numbers of the First & Second kinds

The numbers that occur in the zero column of the sequence $\langle n^p \rangle_{n \in N}$ have combinatorial significance

$\langle n^0 \rangle$	1, 1, 1, 1, 1, 1 ...
$\langle \Delta(n^0) \rangle$	0, 0, 0, 0 0 ...
$\langle n^1 \rangle$	0, 1, 2, 3, 4, 5, ...
$\langle \Delta(n^2) \rangle$	1, 1, 1, 1, 1
$\langle \Delta^2(n^1) \rangle$	0, 0, 0 0,
$\langle n^2 \rangle$	0, 1, 4, 9, 16, 25, 36
$\langle \Delta(n^2) \rangle$	1, 3, 5, 7, 9, 11,
$\langle \Delta^2(n^3) \rangle$	2, 2, 2, 2, 2,
	0, 0, 0, 0, ...
$\langle n^3 \rangle$	0, 1, 8, 27, 64, 125, 216, ...
$\langle \Delta(n^3) \rangle$	1, 7, 19, 37, 61, 91,
$\langle \Delta^2(n^3) \rangle$	6, 12, 18, 24, 30, ...
$\langle \Delta^3(n^3) \rangle$	6, 6, 6, 6, ...
$\langle \Delta^4(n^3) \rangle$	0, 0, 0, ...
$\langle n^4 \rangle$	0, 1, 16, 81, 256, 625, 1296, ...
$\langle \Delta(n^4) \rangle$	1, 15, 65, 175, 369, 671, ...
$\langle \Delta^2(n^4) \rangle$	14, 50, 110, 194, 302, ...
$\langle \Delta^3(n^4) \rangle$	36, 60 84, 108, ...
$\langle \Delta^4(n^4) \rangle$	24, 24, 24, ...
$\langle \Delta^5(n^4) \rangle$	0, 0, ...

$\Delta^k(n^p)_0 = k! \left\{ {p \atop k} \right\}$ — where $\left\{ {p \atop k} \right\}$ = the Stirling coeff. of the 2nd kind

Def. For each $k \in \mathbb{N}$, we define the polynomial $[n]_k$ in n of degree k by $[n]_k = n(n-1)\cdots(n-(k-1))$.
So $[n]_0 = 1$, $[n]_1 = n$, $[n]_2 = n(n-1)$, & $[n]_k = \frac{n!}{(n-k)!}$

Def. For each $k, p \in \mathbb{N}$, we define the Stirling numbers (or coefficients) of the Second kind as the unique numbers $\left\{ {p \atop k} \right\}$ such that

$$n^p = \sum_{k=0}^{p} \left\{ {p \atop k} \right\} [n]_k .$$

Note: Recall that $\binom{p}{k}$ were the unique numbers such that $(1+n)^p = \sum_{k=0}^{p} \binom{p}{k} \cdot n^k$. So there is a certain amount of similarity between $\left\{ {p \atop k} \right\}$ & $\binom{p}{k}$.

Ex. 1 From the zero-column of $\langle n^4 \rangle_{n \in \mathbb{N}}$ and Prop. 2, we know that

$$\begin{aligned} n^4 &= 0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 14 \cdot \binom{n}{2} + 36 \binom{n}{3} + 24 \cdot \binom{n}{4} \\ &= \frac{0}{0!} [n]_0 + \frac{1}{1!} [n]_1 + \frac{14}{2!} [n]_2 + \frac{36}{3!} [n]_3 + \frac{24}{4!} [n]_4 \\ &= 0 \cdot [n]_0 + 1 \cdot [n]_1 + 7 \cdot [n]_2 + 6 [n]_3 + 1 \cdot [n]_4 \end{aligned}$$

So $\left\{ {4 \atop 0} \right\} = 0$, $\left\{ {4 \atop 1} \right\} = 1$, $\left\{ {4 \atop 2} \right\} = 7$, $\left\{ {4 \atop 3} \right\} = 6$, & $\left\{ {4 \atop 4} \right\} = 1$.

Note: We can say $\left\{ {4 \atop k} \right\} = 0$ for $k > 4$, since they do not appear

Prop. 4 For any $k, p \in \mathbb{Z}^+$ with $1 \le k \le p-1$.

$$\left\{ {p \atop k} \right\} = \left\{ {p-1 \atop k-1} \right\} + k \cdot \left\{ {p-1 \atop k} \right\} \quad \text{(Sterling's Second identity)}.$$

Note This is very similar to the Pascal's identity,

$$\binom{p}{k} = \binom{p-1}{k-1} + \binom{p-1}{k} .$$

Proof: We have $n^p = \sum_{k=0}^{p} \left\{ {p \atop k} \right\} . [n]_k \quad --- (*)$

So. $n^{p-1} = \sum_{k=0}^{p-1} \left\{ {p-1 \atop k} \right\} . [n]_k$. Thus

$$n^p = n . n^{p-1} = \sum_{k=0}^{p-1} \left\{ {p-1 \atop k} \right\} . [n]_k . n$$

$$= \sum_{k=0}^{p-1} \left\{ {p-1 \atop k} \right\} . [n]_k . \{(n-k) + k\}$$

$$= \sum_{k=0}^{p-1} \left\{ {p-1 \atop k} \right\} . [n]_k . (n-k) + \sum_{k=0}^{p-1} \left\{ {p-1 \atop k} \right\} . [n]_k . k$$

$$= \sum_{k=0}^{p-1} \left\{ {p-1 \atop k} \right\} . [n]_{k+1} + \sum_{k=0}^{p-1} k . \left\{ {p-1 \atop k} \right\} [n]_k$$

$$= \sum_{k=1}^{p} \left\{ {p-1 \atop k-1} \right\} . [n]_k + 0 + \sum_{k=1}^{p-1} k . \left\{ {p-1 \atop k} \right\} . [n]_k$$

$$= 0 + \sum_{k=1}^{p-1} \left(\left\{ {p-1 \atop k-1} \right\} + k . \left\{ {p-1 \atop k} \right\} \right) [n]_k + \left\{ {p-1 \atop p-1} \right\} [n]_p \quad (**)$$

Comparing the coefficients of $(*)$ & $(**)$ we get

$$\left\{ {p \atop k} \right\} = \left\{ {p-1 \atop k-1} \right\} + k . \left\{ {p-1 \atop k} \right\}, \quad \text{for } 1 \le k \le p-1$$

and $\left\{ {p \atop p} \right\} = \left\{ {p-1 \atop p-1} \right\}$ and $\left\{ {p \atop 0} \right\} = 0^p$ for $p \in \mathbb{N}$. $\square$

Ex. 2 (a) $n^0 = 1 . [n]_0$ & $n^0 = \left\{ {0 \atop 0} \right\} [n]_0 \Rightarrow \left\{ {0 \atop 0} \right\} = 1 = 0^0$.

(b) $n^1 = 0 . [n]_0 + 1 . [n]_1 \Longrightarrow \left\{ {1 \atop 0} \right\} = 0$ & $\left\{ {1 \atop 1} \right\} = 1$

(c) $n^2 = 0 . [n]_1 + 1 . [n]_1 + 1 . [n]_2 \Rightarrow \left\{ {2 \atop 0} \right\} = 0, \left\{ {2 \atop 1} \right\} = 1,$ & $\left\{ {2 \atop 2} \right\} = 1$

(d) $n^3 = 0 . [n]_1 + 1 . [n]_1 + 3 . [n]_2 + 1 . [n]_3 \Longrightarrow$

$\left\{ {3 \atop 0} \right\} = 0, \left\{ {3 \atop 1} \right\} = 1, \left\{ {3 \atop 2} \right\} = 3, \left\{ {3 \atop 3} \right\} = 1$.

Using Stirling's Second identity, we can compute the values of $\left\{ {p \atop k} \right\}$ for all $p, k \in \mathbb{N}$ much more easily.

First of all let us note that for all p,
$\left\{ {p \atop p} \right\} = 1$ for $p \in \mathbb{N}$ & $\left\{ {p \atop 0} \right\} = 0$ for $p \in \mathbb{Z}^+$.

$p \backslash k$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	0	1	$\boxed{1}$	$\boxed{0}$	0	0	0	0
3	0	1	3	1	0	0	0	0
4	0	1	7	6	1	0	0	6
5	0	1	15	25	10	1	0	0
6	0	1	$\boxed{31}$	$\boxed{90}$	65	15	1	0
7	0	1	63	301	350	140	21	1

$1 + 3(0) = \left\{ {2 \atop 2} \right\} + 3\left\{ {2 \atop 3} \right\} = \left\{ {3 \atop 3} \right\} = 1.$

$31 + 3(90) = \left\{ {6 \atop 2} \right\} + 3 \cdot \left\{ {6 \atop 2} \right\} = \left\{ {7 \atop 3} \right\} = 301.$ $\quad \left\{ {p \atop p-1} \right\} = \binom{p}{2}.$

<u>Note:</u> If we fill out the table as shown above, we can see that for all $k, p \in \mathbb{Z}^+$, $\left\{ {p \atop k} \right\} = \left\{ {p-1 \atop k-1} \right\} + k \cdot \left\{ {p-1 \atop k} \right\}$.

We now turn to Stirling's numbers of the first kind

<u>Def.</u> We define the Stirling numbers of the first kind as the unique integers $\left[{p \atop k} \right]$ such that

$$[n]_p = \sum_{k=0}^{p} (-1)^{p-k} \cdot \left[{p \atop k} \right] \cdot n^k \quad \text{for } 0 \le k \le p.$$

<u>Note:</u> 1. $\left[{p \atop k} \right] = (\text{coeff. of } n^k) / (-1)^{p-k}$ in the expansion of $[n]_k$ in terms of $n^0, n^1, \ldots, n^p$.

2. The term $(-1)^{p-k}$ is just there to ensure that $\left[{p \atop k} \right]$ is a non-negative integer.

Ex. 3 $[n]_4 = n(n-1)(n-2)(n-3)$

$$= n(n^2-3n+2)(n-3)$$

$$= n(n^3-6n^2+11n-6)$$

$$= n^4 - 6\cdot n^3 + 11\cdot n^2 - 6\cdot n^1 + 0\cdot n^0$$

So $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 0$, $\begin{bmatrix} 4 \\ 1 \end{bmatrix} = -6/(-1)^{4-1} = 6$, $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 11/(-1)^{4-2} = 11$

$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = -6/(-1)^{4-3} = 6$, and $\begin{bmatrix} 4 \\ 4 \end{bmatrix} = 1/(-1)^{4-4} = 1.$

Prop. 5 For any $k, p \in \mathbb{Z}^+$ with $1 \le k \le p-1$,

$$\begin{bmatrix} p \\ k \end{bmatrix} = \begin{bmatrix} p-1 \\ k-1 \end{bmatrix} + (p-1)\cdot\begin{bmatrix} p-1 \\ k \end{bmatrix}. \quad \text{(Stirling's First identity)}$$

Proof: We know that $[n]_p = \sum_{k=0}^{p} (-1)^{p-k}\cdot\begin{bmatrix} p \\ k \end{bmatrix}\cdot n^k \ldots (*)$

So

$$[n]_{p-1} = \sum_{k=0}^{p-1} (-1)^{p-1-k}\cdot\begin{bmatrix} p-1 \\ k \end{bmatrix}\cdot n^k. \quad \text{Thus}$$

$$[n]_p = [n]_{p-1}\cdot\{n-(p-1)\} = \sum_{k=0}^{p-1} (-1)^{p-1-k}\begin{bmatrix} p-1 \\ k \end{bmatrix}\cdot n^k\cdot\{n-(p-1)\}$$

$$= \sum_{k=0}^{p-1} (-1)^{p-1-k}\cdot\begin{bmatrix} p-1 \\ k \end{bmatrix}\cdot n^k\cdot n - \sum_{k=0}^{p-1} (-1)^{p-1-k}\cdot\begin{bmatrix} p-1 \\ k \end{bmatrix}\cdot n^k\cdot(p-1)$$

$$= \sum_{k=0}^{p-1} (-1)^{p-(k+1)}\begin{bmatrix} p-1 \\ k \end{bmatrix}\cdot n^{k+1} + \sum_{k=0}^{p-1} (-1)^{p-k}\cdot(p-1)\begin{bmatrix} p-1 \\ k \end{bmatrix}\cdot n^k$$

$$= \sum_{k=1}^{p} (-1)^{p-k}\begin{bmatrix} p-1 \\ k-1 \end{bmatrix} n^k + (-1)^p(p-1)\cdot\begin{bmatrix} p-1 \\ 0 \end{bmatrix} + \sum_{k=1}^{p-1} (-1)^{p-k}(p-1)\begin{bmatrix} p-1 \\ k \end{bmatrix} n^k$$

$$= \begin{bmatrix} p-1 \\ p-1 \end{bmatrix} n^p + \sum_{k=1}^{p-1} (-1)^{p-k}\left\{\begin{bmatrix} p-1 \\ k-1 \end{bmatrix} + (p-1)\cdot\begin{bmatrix} p-1 \\ k \end{bmatrix}\right\}\cdot n^k + (-1)^p(p-1)\begin{bmatrix} p-1 \\ 0 \end{bmatrix} \quad (**)$$

Comparing the coefficients of $(*)$ & $(**)$ we get

$$\begin{bmatrix} p \\ k \end{bmatrix} = \begin{bmatrix} p-1 \\ k-1 \end{bmatrix} + (p-1)\cdot\begin{bmatrix} p-1 \\ k \end{bmatrix} \quad \text{for } 1 \le k \le p-1.$$

Also $\begin{bmatrix} p \\ p \end{bmatrix} = \begin{bmatrix} p-1 \\ p-1 \end{bmatrix}$ & $\begin{bmatrix} p \\ 0 \end{bmatrix} = 0$ for $p \in \mathbb{Z}^+$.

Ex. 4 (a) $[n]_0 = 1$ & $[n]_0 = (-1)^{0-0}\left[{0 \atop 0}\right] n^0 = 1 \Rightarrow \left[{0 \atop 0}\right] = 1$

(b) $[n]_1 = n$ & $[n]_1 = (-1)^{1-0}\left[{1 \atop 0}\right] n^0 + (-1)^{1-1}\cdot\left[{1 \atop 1}\right] n^1$

$\Rightarrow \left[{1 \atop 0}\right] = 0$ & $\left[{1 \atop 1}\right] = 1$

(c) $[n]_2 = n(n-1) = n^2 - n^1 + 0.n^0$

$\Rightarrow \left[{2 \atop 0}\right] = 0$, $\left[{2 \atop 1}\right] = 1$, and $\left[{2 \atop 2}\right] = 1$.

(d) $[n]_3 = n(n-1)(n-2) = n(n^2 - 3n + 2) = n^3 - 3n^2 + 2n^1 + 0.n^0$

$\Rightarrow \left[{3 \atop 0}\right] = 0$, $\left[{3 \atop 1}\right] = 2$, $\left[{3 \atop 2}\right] = 3$, $\left[{3 \atop 3}\right] = 1$.

By using Sterling's First identity, we can compute the values of $\left[{p \atop k}\right]$ for all $p, k \in \mathbb{N}$ much more easily:

p \ k	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0
3	0	2	3	1	0	0	0	0
4	0	6	11	6	1	0	0	0
5	0	24	$\boxed{50}$	$\boxed{35}$	10	1	0	0
6	0	120	274	225	85	15	1	0
7	0	720	1764	1624	735	175	21	1

$$50 + 5(35) = \left[{5 \atop 2}\right] + (6-1)\cdot\left[{5 \atop 3}\right] = \left[{6 \atop 3}\right] = 225$$

Observe also that $\left[{p \atop p-1}\right] = \binom{p}{2}$ for all $p \in \mathbb{Z}^+$.

Note: If we fill out the table as shown above, we can see that for all $k, p \in \mathbb{Z}^+$, $\left[{p \atop k}\right] = \left[{p-1 \atop k-1}\right] + (p-1)\cdot\left[{p-1 \atop k}\right]$.

§3. Combinatorial significance of the Stirling Numbers.

Our aim in this section is to show that the Stirling numbers of the Second & First kind have combinatorial meanings just as the Binomial coefficients $\binom{p}{k}$ is also the number of k-subsets of $\{1,2,3,\dots,p\}$.

Def Let $k, p \in \mathbb{N}$ with $k \leq p$. We define $S(p,k)$ by $S(p,k)$ = no. of partitions of $\{1,2,\dots,p\}$ into k parts.

Recall that a partition of $\{1,2,\dots,p\}$ is a collection of disjoint, non-empty subsets $\{A_i : i \in I\}$ of $\{1,2,\dots,p\}$ such that $\bigcup_{i \in I} A_i = \{1,2,\dots,p\}$. If $|I| = k$, then we say that the partition has k parts.

Ex.1 (a) $\{\{1,2\},\{3,4\}\}$ is a partition of $\{1,2,3,4\}$ with 2 parts

(b) $\{\{1,2\},\{3\},\{4\}\}$ and $\{\{1\},\{2\},\{3,4\}\}$ are both partitions of $\{1,2,3,4\}$ with 3 parts.

Ex.2 Let $p=4$. Then $\{\{1\},\{2,3,4\}\}$, $\{\{2\},\{1,3,4\}\}$, $\{\{3\},\{1,2,4\}\}$, $\{\{4\},\{1,2,3\}\}$, $\{\{1,2\},\{3,4\}\}$, $\{\{1,3\},\{2,4\}\}$ and $\{\{1,4\},\{2,3\}\}$ are all the possible partitions of $\{1,2,3,4\}$ into 2 parts. $\therefore S(4,2) = 7$.

Note: $S(p,p) = 1$ for $p \in \mathbb{N}$ & $S(p,0) = 0$ for $p \in \mathbb{Z}^+$ because $\{\{1\},\{2\},\dots,\{p\}\}$ is the only partition of $\{1,2,\dots,p\}$ into p parts.

Prop. 6 For each $k, p \in \mathbb{Z}^+$ with $1 \le k \le p-1$,

$$S(p,k) = S(p-1,k-1) + k \cdot S(p-1,k).$$

Proof. Let $\mathcal{A}$ = set of all partitions of $\{1,2,\dots,p\}$ into k parts, $\mathcal{B}$ = set of partitions in $\mathcal{A}$ with p in a part by itself, and $\mathcal{C}$ = set of partitions in $\mathcal{A}$ with p <u>not</u> in a part by itself. Then $\mathcal{B} \cap \mathcal{C} = \emptyset$ and $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$. So

$$|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}|.$$

Now if we remove the part $\{p\}$ from each of the partitions in $\mathcal{B}$, we will get a partition of $\{1,2,\dots,p-1\}$ into $k-1$ parts. And if we add $\{p\}$ to any partition of $\{1,2,\dots,p-1\}$ into $k-1$ parts, then we will get a partition of $\mathcal{B}$. So $|\mathcal{B}| = S(p-1,k-1)$

Also if we remove p from its part in a partition of $\mathcal{C}$, we will get a partition of of $\{1,2,\dots,p-1\}$ into k parts. And if we add p to each part, in turns, to a partition of $\{1,2,\dots,p-1\}$ into k parts, we will get k partitions of $\mathcal{C}$. So $|\mathcal{C}| = k \cdot S(p-1,k)$. Thus

$$\begin{aligned} S(p,k) = |\mathcal{A}| &= |\mathcal{B}| + |\mathcal{C}| \\ &= S(p-1,k-1) + k \cdot S(p-1,k). \end{aligned}$$

Corollary 7: For each $k, p \in \mathbb{N}$, $S(p,k) = \left\{ {p \atop k} \right\}$.

Proof: $S(p,k)$ & $\left\{ {p \atop k} \right\}$ satisfy the same recurrence equation with the same boundary conditions. Hence we must have $S(p,k) = \left\{ {p \atop k} \right\}$.

<u>Def.</u> Let S_p be the set of pemutation of $\{1,2,\dots,p\}$. We define the relation $\sim$ on S_p as follows. $\langle i_1,\dots,i_p\rangle \sim \langle j_1,\dots,j_p\rangle$ if we can find a non-negative integer k such that
$\langle i_1,\dots,i_p\rangle = \langle j_{k+1}, j_{k+2},\dots, j_p, j_1,\dots, j_k\rangle$

<u>Ex.3</u> Let $\langle i_1, i_2, i_3, i_4\rangle = \langle 1,3,2,4\rangle$ and $\langle j_1, j_2, j_3, j_4\rangle = \langle 2,4,1,3\rangle$. Then
$\langle i_1, i_2, i_3, i_4\rangle = \langle j_3, j_4, j_1, j_2\rangle$. So $\langle 1,3,2,4\rangle \sim \langle 2,4,1,3\rangle$.

<u>Fact 1</u> The relation $\sim$ is an equivalence relation on S_p and it partitions S_p into $(p-1)!$ equivalence classes, each with p elements. Each equivalence class will be called a <u>circular permutation</u>.

<u>Ex.4</u> The equivalence classes of S_3 are
$\{\langle 1,2,3\rangle, \langle 2,3,1\rangle, \langle 3,1,2\rangle\}$ & $\{\langle 1,3,2\rangle, \langle 3,2,1\rangle, \langle 2,1,3\rangle\}$

<u>Fact 2</u> Each of the $(p-1)!$ equivalence classes of S_p have an element which begins with "1". We shall use this permutation to represent the equivalence class from which it came. This permutation beginning with a "1" will also be called the representative of the circular permutation.

<u>Ex.5</u> The representative of the circular permutations of $\{1,2,3,4\}$ are
$\langle 1,2,3,4\rangle$, $\langle 1,2,4,3\rangle$, $\langle 1,3,2,4\rangle$, $\langle 1,3,4,2\rangle$
$\langle 1,4,2,3\rangle$, $\langle 1,4,3,2\rangle$

So $\{1,2,3,4\}$ has 6 circular permutations.

Note If A is any set, we usually pick a fixed element (usually the smallest element) to anchor the circular permutations. So the circular permutations of $\{2,3,5\}$ will be

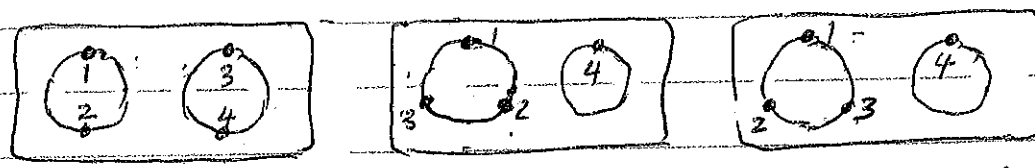
$$\langle 2,3,5\rangle \text{ and } \langle 2,5,3\rangle$$

Here 2 serves as the anchor.

Def. Let $k, p \in \mathbb{N}$ with $k \le p$. We define $s(p,k)$ by
$s(p,k)$ = no. of arrangements of $\{1,2,3,\ldots,p\}$ into k non-empty circular partitions.

In other words, $s(p,k)$ = no. of ways we can seat $1,2,3,\ldots,p$ at k indistinguishable circular tables with no table being empty.

Ex. 6 Let us count the number of different ways we can seat $\{1,2,3,4\}$ at 2 indistinguishable tables.



$\langle 1,2\rangle$ & $\langle 3,4\rangle$	$\langle 1,2,3\rangle$ & $\langle 4\rangle$	$\langle 1,3,2\rangle$ & $\langle 4\rangle$
$\langle 1,3\rangle$ & $\langle 2,4\rangle$	$\langle 1,2,4\rangle$ & $\langle 3\rangle$	$\langle 1,4,2\rangle$ & $\langle 3\rangle$
$\langle 1,4\rangle$ & $\langle 2,3\rangle$	$\langle 1,3,4\rangle$ & $\langle 2\rangle$	$\langle 1,4,3\rangle$ & $\langle 2\rangle$
	$\langle 2,3,4\rangle$ & $\langle 1\rangle$	$\langle 2,4,3\rangle$ & $\langle 1\rangle$.

So $s(4,2) = 11$.

Note: For each $p \in \mathbb{N}$, $s(p,0) = 0^p$ and $s(p,p) = 1$.
Remember that $0^0 = 1$ and $0^p = 0$ for $p > 0$.

Prop. 8 For each $k, p \in \mathbb{Z}^+$ with $1 \le k \le p-1$,

$$s(p,k) = s(p-1,k-1) + (p-1)\cdot s(p-1,k).$$

Proof: Let $\mathcal{A}$ = set of all seating arrangements of $\{1,2,\ldots,p\}$ at k indistinguishable tables with no tables empty. Put $\mathcal{B}$ = set of seatings in $\mathcal{A}$ with p at a table by itself & $\mathcal{C}$ = set of seatings in $\mathcal{A}$ with p not by itself at a table. Then $\mathcal{B}\cap\mathcal{C} = \emptyset$ & $\mathcal{A} = \mathcal{B}\cup\mathcal{C}$. So $|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}|$.

Now if we remove the table with p from each seating in $\mathcal{B}$, then we will get a seating of $\{1,2,\ldots,p-1\}$ at $k-1$ non-empty tables. And if we p at a new table to any seating of $\{1,2,\ldots,p-1\}$ at $k-1$ non-empty tables, we will get a seating of $\mathcal{B}$. So $|\mathcal{B}| = s(p-1,k-1)$.
Also if we remove p from its table in a seating of $\mathcal{C}$, we will get a seating of $\{1,2,3,\ldots,p-1\}$ at k non-empty tables. And if we seat p to the left of each of $1,2,\ldots,p-1$; in turns, at the respective tables of a seating of $\{1,2,\ldots,p-1\}$ at k non-empty tables, we will get $p-1$ seatings of $\mathcal{C}$. So $|\mathcal{C}| = (p-1)\cdot s(p-1,k)$. Hence $s(p,k) = |\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}| = s(p-1,k-1) + (p-1)\cdot s(p-1,k)$.

Corollary 9: For each $k, p \in \mathbb{N}$, $s(p,k) = \left[{p \atop k}\right]$.

Proof: $s(p,k)$ & $\left[{p \atop k}\right]$ satisfies the same recurrence equation with the same boundary conditions. Hence we must have $s(p,k) = \left[{p \atop k}\right]$.

§4. Partitions of a non-negative integer

<u>Ex.1</u> In how many ways can the multiset $[4.a]$ be partitioned into non-empty sub-multisets?

<u>Sol.</u> First of all remember that a <u>join</u> of multisets is not the same as a union of sets. So for example $[2.a, 1.b] + [1.a, 3.b] = [3.a, 4.b]$. A partition of the multiset M is a collection of sub-multisets $[A_1, \dots, A_k]$ such that $M = A_1 + \cdots + A_k$. Two collections $[A_1, \dots, A_k]$ & $[B_1, \dots, B_k]$ are considered the same if $[A_1, \dots, A_k] = [B_1, \dots, B_k]$ as multi-multisets. Now the partitions of $[4.a]$ are

$$\begin{array}{ll} [[4.a]] & 4 = 4 \\ [[3.a], [1.a]] & 4 = 3+1 \\ [[2.a], [2.a]] & 4 = 2+2 \\ [[2.a], [1.a], [1.a]] & 4 = 2+1+1 \\ [[1.a], [1.a], [1.a], [1.a]]. & 4 = 1+1+1+1. \end{array}$$

This problem is equivalent to the number of ways of writing 4 as a sum of positive integers as shown above on the right. So our answer is 5. Notice also that there are 2 ways of partitioning $[4.a]$ into 2 sub-multisets.

<u>Def.</u> Let $k, n \in \mathbb{N}$. We define $p(n,k)$ to be the number of partitions of $[n.1]$ into k non-empty parts. We also define $p(n)$ to be the total number of partitions of $[n.1]$. So $p(n) = p(n,1) + \cdots + p(n,n)$.

Ex. 2 Find $p(6,2)$ & $p(n,0)$.

Sol. We have $6=5+1$, $6=4+2$, $6=3+3$. So $p(6,2)=3$. Also $0=$ empty sum of pos. integ., and since there is only one way to write this $p(0,0)=1=0^0$. And if $n>0$, then n cannot be expressed as an empty sum of positive integers. So in this case $p(n,0)=0=0^n$. Thus for <u>any</u> $n\in\mathbb{N}$, $p(n,0)=0^n$.

Prop. 10 For any $k,n\in\mathbb{N}$ with $1\leq k\leq n$,

$$p(n,k) = p(n-1,k-1)+p(n-k,k)$$

Proof: Let $\mathcal{A}$ be the collection of all partitions of n into exactly k non-empty parts. Put
$\mathcal{B}=$ collection of partitions of $\mathcal{A}$ with a part of size 1 &
$\mathcal{C}=$ collection of partitions of $\mathcal{A}$ with no part of size 1.
Then $|\mathcal{A}|=|\mathcal{B}|+|\mathcal{C}|$.
Now if we remove a part of size 1 from a partition of $\mathcal{B}$, we will get a partition of $n-1$ into $k-1$ non-empty parts. And if we add a part of size 1 to a partition of $n-1$ into $k-1$ non-empty parts, we will get a partition of $\mathcal{B}$.
Hence $|\mathcal{B}|=p(n-1,k-1)$.
Also if we remove a "1" from each of the parts of a partition of $\mathcal{C}$, we will get a partition of $n-k$ into k non-empty parts. And if we add a "1" to each part of a partition of $n-k$ into k non-empty parts, we will get a partition of $\mathcal{C}$.

Proof: Hence $|\mathcal{C}| = p(n-k,k)$. So

$$p(n,k) = |\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}| = p(n-1,k-1) + p(n-k,k).$$

Using Proposition 10, we can compute the values of $p(n,k)$, for all $k, n \in \mathbb{N}$.

$n \backslash k$	0	1	2	3	4	5	6	7	8	$p(n)$
0	1	0	0	0	0	0	0	0	0	1
1	0	1	0	0	0	0	0	0	0	1
2	0	1	1	0	0	0	0	0	0	2
3	0	1	1	1	0	0	0	0	0	3
4	0	1	2	1	1	0	0	0	0	5
5	0	1	2	$\boxed{2}$	1	1	0	0	0	7
6	0	1	3	3	2	1	1	0	0	11
7	0	1	$\boxed{3}$	4	3	2	1	1	0	15
8	0	1	4	5	5	3	2	1	1	22

Ex. 3 (a) $p(8-1, 3-1) + p(8-3, 3) = p(8,3)$

$$\boxed{3} + \boxed{2} = 5$$

(b) $p(8-1, 4-1) + p(8-4, 4) = p(8,4)$

$$4 + 1 = 5$$

Ex. 4 Let us now turn our attention to the collection of all partitions of a [n.1]. Consider the partition of 10.

5 = • • • • •

+ 3 = • • •

+ 1 = •

+ 1 = •

This diagram is called the Ferrer's diagram of the partition.

Ex. 4 If we interchange the rows & columns of the Ferrer's diagram of $5+3+1+1$, we will get the conjugate partition $4+2+2+1+1$.

$$
\begin{array}{ll}
4 = & \bullet \ \bullet \ \bullet \ \bullet \\
+2 = & \bullet \ \bullet \\
+2 = & \bullet \ \bullet \\
+1 = & \bullet \\
+1 = & \bullet
\end{array}
$$

Def. A partition of n is <u>self-conjugate</u> if it is the same as its conjugate.

$$
\begin{array}{l}
\bullet \ \bullet \ \bullet \\
\bullet \ \bullet \ \bullet \\
\bullet \ \bullet
\end{array}
$$

$3+2+2$ is a self-conjugate partition of 8

Prop. 11 Let $q(n,k)$ be the number of partitions of n in which the largest part is of size k. Then

$$q(n,k) = p(n,k)$$

Proof: Recall that $p(n,k)$ = no. of partitions of n into k non-empty parts. Let

$\mathcal{A}$ = collection of all partitions of n into k non-empty parts

& $\mathcal{B}$ = collection of all partitions of n in which the largest part has size k.

Then the conjugate of each partition of $\mathcal{A}$ is a partition of $\mathcal{B}$. And the conjugate of each partition of $\mathcal{B}$ is a partition of $\mathcal{A}$. So $|\mathcal{A}| = |\mathcal{B}|$. Hence $q(n,k) = p(n,k)$.

Corollary 12: $q(n,1) + q(n,2) + \cdots + q(n,k) = p(n,1) + p(n,2) + \cdots + p(n,k)$.

Ex.5 Let $\mathcal{A}_3$ = collection of partitions of 7 into 3 parts & $\mathcal{B}_3$ = collection of partitions of 7 with the largest part having size 3. Then

$\mathcal{A}_3 = \{5+1+1,\ 4+2+1,\ 3+3+1\}$ and

$\mathcal{B}_3 = \{3+1+1+1+1,\ 3+2+1+1,\ 3+2+2\}$. So $|\mathcal{A}_3| = |\mathcal{B}_3|$.

Def. Let $\mathcal{A}_{DIST}(n)$ = collection of all partitions of n in which each part of the partition is of different sizes, & $\mathcal{A}_{ODD}(n)$ = collection of all partitions of n in which each part is of odd size.

We define $P_{dist}(n) = |\mathcal{A}_{DIST}(n)|$ & $P_{odd}(n) = |\mathcal{A}_{ODD}(n)|$.

Ex.6 $\mathcal{A}_{DIST}(5) = \{5,\ 4+1,\ 3+2\}$

$\mathcal{A}_{ODD}(5) = \{5,\ 3+1+1,\ 1+1+1+1+1\}$

Notice that $|\mathcal{A}_{DIST}(5)| = |\mathcal{A}_{ODD}(5)|$.

Algorithm 1 (Distinct parts into Odd parts algorithm)

INPUT : A partition $\mathcal{P}$ of n into distinct parts.

OUTPUT : A partition Q of n into odd parts.

1. Let $i \leftarrow 0$ and $\mathcal{P}_i \leftarrow \mathcal{P}$.
2. If $\mathcal{P}_i$ has no part of even size, STOP; else split each even part of $\mathcal{P}_i$ to get $\mathcal{P}_{i+1}$.
3. Let $i \leftarrow i+1$ and go to step 2.

Ex.7 Find odd partitions of 10 corresponding to

(a) $5+3+2$ (b) $6+4$

(a) $\mathcal{P}_0 = 5+3+2$.

$\mathcal{P}_1 = 5+3+1+1$. This is our final answer.

Ex.7 (b) $P_0 = 6 + 4$

$P_1 = 3+3+2+2$

$P_2 = 3+3+1+1+1+1$ ← This is our final ans.

Algorithm 2 (Odd parts into Distinct parts algorithm)

INPUT: A partition Q of n into odd parts

OUTPUT: A partition P of n into distinct parts.

1. Let $i \leftarrow 0$ and $Q_i \leftarrow Q$.
2. If Q_i has no two parts of the same size, STOP; else group the parts of the same size in pairs (leave out 1 if there are an odd no. of parts of the same size) and union each pair to get a new partition Q_{i+1}.
3. Let $i \leftarrow i+1$ and go to step. 2.

Ex.8. Find the distinct partitions of 10 corresponding to
(a) $3+3+1+1+1+1$ (b) $5+1+1+1+1+1$

(a) $Q_0 = (3+3)+(1+1)+(1+1)$
$Q_1 = 6 + (2+2)$
$Q_2 = 6+4$ done

(b) $Q_0 = 5+(1+1)+(1+1)+1$
$Q_1 = 5+(2+2)+1$
$Q_2 = 5+4+1$ done.

Theorem 13: $P_{dist}(n) = P_{odd}(n)$.

(sketch of the)
Proof: Let $f: \mathcal{A}_{DIST}(n) \to \mathcal{A}_{ODD}(n)$ be defined by $f(P) =$ the unique partition of n produced by Algorithm 1. Also let $g: \mathcal{A}_{ODD}(n) \to \mathcal{A}_{DIST}(n)$ be defined by $g(Q) =$ the unique partition of n produced by Algorithm 2. Then $f \circ g =$ identity function & $g \circ f =$ identity function. So f is a bijection. $\therefore P_{dist}(n) = |\mathcal{A}_{DIST}(n)| = |\mathcal{A}_{ODD}(n)| = P_{odd}(n)$.

Ex. 5 Let $\mathcal{A}_3$ = collection of partitions of 7 into 3 parts & $\mathcal{B}_3$ = collection of partitions of 7 with largest part of size 3. Then

$\mathcal{A}_3 = \{5+1+1,\ 4+2+1,\ 3+3+1\}$

$\mathcal{B}_3 = \{3+1+1+1+1,\ 3+2+1+1,\ 3+2+2\}$.

This verifies that $|\mathcal{A}_3| = |\mathcal{B}_3|$.

Def. Let $\mathcal{A}_D(n)$ = collection of all partitions of n into parts which are all of different sizes, and

$\mathcal{B}_O(n)$ = collection of all partitions of n into parts which are all of odd sizes.

We define $P_{dist}(n) = |\mathcal{A}_D(n)|$ & $P_{odd}(n) = |\mathcal{B}_O(n)|$.

Ex. 6 $\mathcal{A}_D(5) = [5,\ 4+1,\ 3+2]$

$\mathcal{B}_O(5) = [1+1+1+1+1,\ 3+1+1,\ 5]$

$= [5.[1],\quad 2.[1]+1.[3],\quad 1.[5]]$

Theorem 13 For any $n \in \mathbb{N}$, $P_{dist}(n) = P_{odd}(n)$

Proof. Let us consider any partition of $\mathcal{B}_O(n)$. Then we can express that partition in the form

$$[n] = x_1.[1] + x_3.[3] + x_5.[5] + \cdots$$

as indicated in example 6. Now express each x_i as a binary numeral in reverse order.

$$x_i = 2^{a_i}\, 2^{b_i}\, 2^{c_i} \ldots \qquad \text{with } a_i < b_i < c_i < \cdots$$

Then we will have

$$\begin{aligned} n = {} & 1.2^{a_1} + 1.2^{b_1} + 1.2^{c_1} + \cdots \\ & + 3.2^{a_3} + 3.2^{b_3} + 3.2^{c_3} + \cdots \\ & + 5.2^{a_5} + 5.2^{b_5} + 5.2^{c_5} + \cdots \quad + \cdots \end{aligned}$$

(each term is one part of n)

(6 parts are listed followed by dots – but there could be less or more).

Proof But this gives us a partition of n in which all parts are of different sizes, because

$$2^{a_1} < 2^{b_1} < 2^{c_1} < \cdots$$
$$3.2^{a_3} < 3.2^{b_3} < 3.2^{c_3} < \cdots$$
$$5.2^{a_5} < 5.2^{b_5} < 5.2^{c_5} < \cdots$$

since $a_i < b_i < c_i < \cdots$ for each odd i and $2^a, 3.2^a, 5.2^a, \ldots$ are always different. So each partition of $\mathcal{B}_0(n)$ corresponds to a partition of $\mathcal{A}_D(n)$.

Now consider any partition of $\mathcal{A}_D(n)$. Write it as $n = y_1 + y_2 + y_3 + \cdots$ and express each y_i in the form $2^a.(2k+1)$. Then by adding together the the portions with the same odd part, $2k+1$, we will get

$$[n] = x_1.[1] + x_3.[3] + x_5.[5] + \cdots$$

which is a partition of $\mathcal{B}_0(n)$. Notice that

x_1 = coeff. of all the portions of the form $2^a.1$

x_3 = coeff. of all the portions of the form $2^a.3$

and so. So each partition of $\mathcal{A}_D(n)$ corresponds to a partition of $\mathcal{B}_0(n)$. Hence

$$P_{dist}(n) = |\mathcal{A}_D(n)| = |\mathcal{B}_0(n)| = P_{odd}(n).$$

Ex. 7 (a) $\mathcal{B}_0(7) = \{1+1+1+1+1+1+1,\ 1+1+1+1+3,\ 1+1+5,\ 7\}$

$= \{(2^0+2^1+2^2).1,\ (2^2.1+2^0.3),\ (2^1.1+2^0.5),\ (2^0.7)\}$

$\sim \{(1+2+4),\ (4+3),\ (2+5),\ (7)\} = \mathcal{A}_D(7)$.

(b) $\mathcal{A}_D(6) = \{2+4,\ 1+2+3,\ 6,\ 1+5\}$

$= \{(2^1+2^2).1,\ (2^0+2^1).1+2^0.3,\ 2^1.3,\ 2^0.1+2^0.5\}$

$\sim \{(6).1,\ 3.1+1.3,\ 2.3,\ 1.1+1.5\}$

$\sim \{1+1+1+1+1+1,\ 1+1+1+3,\ 3+3,\ 1+5\} = \mathcal{B}_0(6)$.

The Placement of balls into boxes.

Ex. 1 In how many ways can we distribute 4 balls into 2 boxes? The answer depends on the kinds of balls and the kind of boxes. Possible labels for balls: a, b, c, d.
possible labels for boxes: 1st, 2nd.

	4 BALLS LABELLED?	2 BOXES LABELLED?	EMPTY BOXES ALLOWED?	ANSWER	
1.	NO	NO	NO	2	3+1, 2+2
2.	NO	NO	YES	3	4, 3+1, 2+2
3.	NO	YES	NO	3	$\langle 3,1\rangle, \langle 1,3\rangle, \langle 2,2\rangle$
4.	NO	YES	YES	5	$\langle 4,0\rangle, \langle 0,4\rangle, \langle 3,1\rangle, \langle 1,3\rangle, \langle 2,2\rangle$
5.	YES	NO	NO	7	$\{abc, d\}, \{abd, c\}, \{acd, b\}$, $\{bcd, a\}, \{ab, cd\}, \{ac, bd\}, \{ad, bc\}$
6	YES	NO	YES	8	$\{abc, d\}, \{abd, c\}, \{acd, b\}$, $\{abcd\}, \{bcd, a\}, \{ab, cd\}, \{ac, bd\}, \{ad, bc\}$
7.	YES	YES	NO	14	Order the 7 partitions in 5. $\langle abc, d\rangle, \langle d, abc\rangle, \cdots$
8.	YES	YES	YES	16	Order the 8 partitions in 6. $\langle abcd, \emptyset\rangle, \langle \emptyset, abcd\rangle \ldots$

Fact:

	n BALLS LABELLED?	k BOXES LABELLED?	EMPTY BOXES ALLOWED?	ANSWER
1	NO	NO	NO	$p(n,k)$
2.	NO	NO	YES	$p(n,k) + \cdots + p(n,0)$
3.	NO	YES	NO	$\binom{n-1}{k-1}$
4.	NO	YES	YES	$\binom{n+k-1}{k-1} = \sum_{i=0}^{k} \binom{k}{i} \cdot \binom{n-1}{i-1}$
5.	YES	NO	NO	$\left\{ {n \atop k} \right\}$
6.	YES	NO	YES	$\left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\} + \cdots + \left\{ {n \atop 0} \right\}$
7.	YES	YES	NO	$k! \left\{ {n \atop k} \right\}$
8.	YES	YES	YES	$k^n = \sum_{i=0}^{k} [k]_i \left\{ {n \atop i} \right\}$