Theory of Computation

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DMS

Data Structure

Introduction

Theory of Computation (TOC)

It is a branch of computer science that deals with how efficiently problem can be solved on a model of computation, using an algo.

Components of TOC

1. Computability
2. Complexity
3. Automata

Computability, Complexity, and Automata

“What are the fundamental capabilities and limitations of computers?”

Three traditional central areas of the TOC such as automata, computability, and complexity are linked.

Computability Theory

"Problem of determining whether a mathematical statement is true or false"

- Certain basic problems cannot be solved by computers
  - One example, the problem of determining whether a mathematical statement is true or false.

Complexity Theory

What makes some problems computationally hard and others easy?

- Computer problems come in different varieties; some are easy, and some are hard.
  - For example, the sorting problem is an easy one.
  - The scheduling problem seems to be much harder than the sorting problem.

Computability and Complexity (Cont.)

The theories of computability and complexity are closely related:

1. In complexity theory, the objective is to classify problems as easy ones and hard ones;
2. Whereas in computability theory, the classification of problems is by those that are solvable and those that are not.

Automata Theory

- Automata theory deals with the definitions and properties of mathematical models of computation.
- These models play a role in several applied areas of computer science.
  - One model, called the finite automaton, is used in text processing, compilers, and hardware design.
  - Another model, called the context-free grammar, is used in programming languages and artificial intelligence.

Algorithms

- Empirically, an algorithm is ...
  - A tool for solving a well-specified computational problem.
- Problem specification includes what the input is, what the desired output should be.
- A correct algorithm solves the given computational problem.

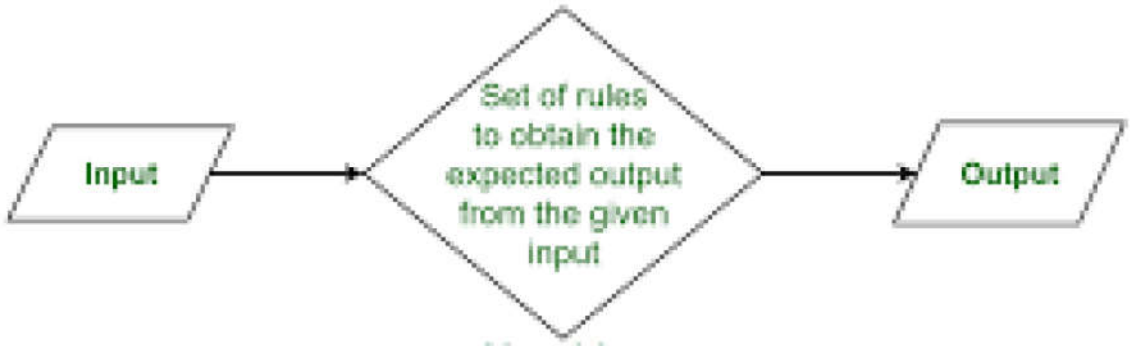

Figure: Algorithm

Mathematical Notions and terminology

1. Prerequisite knowledge.
2. Review as necessary.

Sets

$\{1, 1, 2, 2\} = \{1, 2\} = \{2, 1\} = \{2, 1, 2, 1\}$

- A collection of objects presented as a single unit is called a **set**
- Each object in a set is called a **set element** or **set member** and always written within a pair of braces.
- The order, sequence and repetition of set elements in a set doesn't matter.

$\{\text{anything}\}$

- A set is represented using the **set builder notation** or **roster notation**.
- The symbols $\in$ and $\notin$ denote set membership and nonmembership respectively.

Set Examples

Example 1.1:

A set of natural numbers equal to or less than 5 can be represented: in roster notation as $\{1, 2, 3, 4, 5\}$ and in set builder notation as $\{x | x \in \mathrm{N}, x \leq 5\}$

Example 1.2:

A set of even integer numbers greater than -3 and less than 5 can be represented: in roster notation as $\{-2, 0, 2, 4\}$ and in set builder notation as $\{x | x = 2n, -1 \leq n \leq 2\}$

Question 1.3:

Roster and Set builder notation for prime numbers less than 20?

Answer 1.3:

Roster notation: $\{2, 3, 5, 7, 11, 13, 17, 19\}$ Set builder notation: $\{x |\ x$ is prime, $x < 20\}$

Continue..

If $A \subset B$ and $B \subset A$ then $\underline{\underline{A = B}}$

- For two sets A and B,
  - We say that A is a subset of B, written $A \subseteq B$, if every member of A also is a member of B.
  - We say that A is a proper subset of B, written $A \subset B$, if A is a subset of B and not equal to B.

Example 1.4:

Let as assume, three sets, $A = \{1, 2, a, b\}$, $B = \{1, 2, a, b\}$, $C = \{1, 2, 3, a, b, c\}$ → extra elements
where, A is a subset of B and written as $A \subseteq B$, whereas A is a proper subset of C and written as $A \subset C$.

$A \subset C$ All elements of A are in C but the reverse is not true $C \not\subset A$

Different type of Sets

Empty set is a finite set.

- A set that contains finite number of elements is called a **finite set**
  - E.g., $\{1, 2, 3, 4, 5\}$, $\{0, 1, 1, 2, 3\}$, $\{1, 2, 4, 8, 16\}$ are examples of some finite set
    5 4 5
- A set that contains infinitely many elements is called an **infinite set**
  - Set of **natural numbers** $N = \{1, 2, 3, \ldots\}$ ∞, Set of **integers** $Z =$ $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ ∞ are examples of some infinite sets.
- A set with zero members or no member is called an **empty set**
  - Empty sets are denoted as $\emptyset$ or $\{\ \}$ $\{\phi\}$.
- A set with exactly one member is called a **singleton set**
  - $\{1\}$ and $\{a\}$ are two examples of singleton sets
- A set with exactly two members is called an **unordered pair**.
  - $\{2, 4\}$ and $\{a, b\}$ are two examples of unordered pair
  $\{2,4\} = \{4,2\}$ $\{4,2\} = \{2,\ldots$
- Repetition of a set element in a set is called **multiset**
  - $\{3\}$ is a set but $\{3, 3\}$ is a multiset

$\{\phi\}$ what type of set is it?

Operation of Sets

Order pair: (a,b) → written with a pair of parenthesis
→ Order and repetation matters

If we have two sets $A = \{1, 2, 3\}$ and set $B = \{3, 4, 5\}$,

- Union ($\cup$): The union of A and B, written A $\cup$ B, is the set we get by combining all the elements in A and B into a single set.
  - $A \cup B = \{\ x \mid x \in a$ or $x \in B\ \} = \{1, 2, 3, 4, 5\}$
- Intersection ($\cap$): The intersection of A and B, written A $\cap$ B, is the set of elements that are in both A and B.
  - $A \cap B = \{x \mid x \in A$ and $x \in B\} = \{3\}$.
- Cartesian Product or Cross product ($\times$): The set of all possible order pairs in which the first element is in A and the second element is in B
  - $A\times$ B$= \{($ a, b $)|\ a \in A$ and $b \in B\} =$
    $\{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5)\}$

$(a,b) \neq (b,a) \neq (a,a) \neq (a,a,a)$

Operation of Sets (cont..)

$A = \{1, 2, 3\}$ $B = \{3, 4, 5\}$

Present only in B But not in A

- Set difference (-): The set of all elements that are in A but not in B
  - $A - B = \{x \mid x \in A \text{ and } x \notin B\} = \{1, 2\}$ where as $B - A = \{4, 5\}$

not in B, only in A

- Complement(' or -): The complement of A, written $\overline{\mathrm{A}}$, is the set of all elements under consideration that are not in A.

NOT → $\bar{A}$ = {All elements except 1, 2 and 3}

- Cardinality ($\mid A \mid$): Number of elements present in a set is called its **cardinality**

$|A| = 3$ $|A - B| = 2$ $|A \cap B| = 1$ $|A \times B| = 9$

$|A \times B| = |A| \times |B|$

- Power set (P(A)): The set of all possible subsets of A
  - If A is the set $\{0, 1\}$, then the power set of A denoted as $P(A)$ or 2^A is the set $\{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
  - The cardinality of power set of a set A is $2^{|A|}$. As in this example, $\mid A \mid$ is 2, $P(A)$ is 2^2 i.e. 4.

$\phi \subset A$
$\{0\} \subset A$
$\{1\} \subset A$
$\{0, 1\} \subset A$

* ϕ is subset of every set

* Every set is a subset of itself

Venn Diagram

- As is often the case in mathematics, a picture helps clarify a concept.
- For sets, we use a type of picture called a Venn diagram.
  - It represents sets as regions enclosed by circular lines.

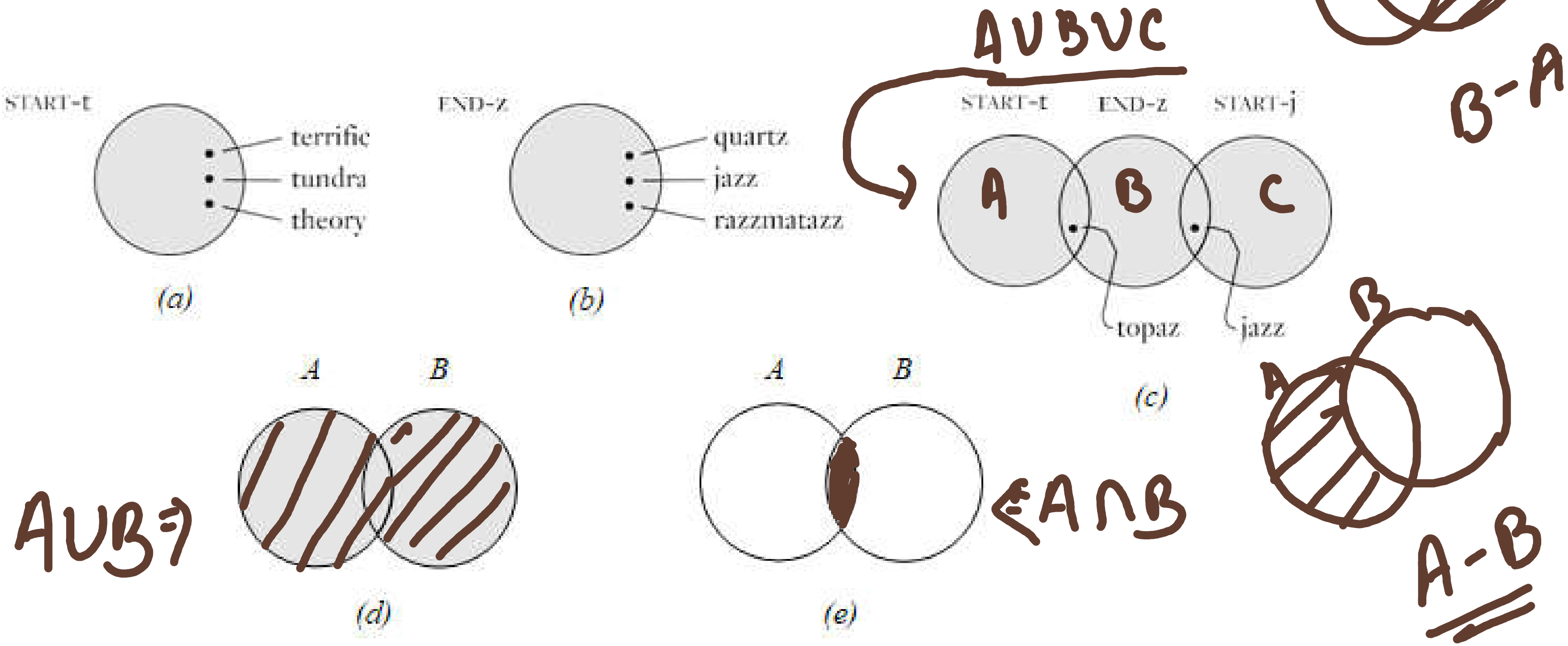

Figure: Venn diagram for (a) the set of English words starting with "t" (b) the set of English words ending with "z" (c) overlapping circles indicate common elements (d) $A \cup B$ (e) $A \cap B$.

Venn Diagram (Cont.)

- For example, the word topaz is in both sets.
- The figure also contains a circle for the set START-j.
- It doesn't overlap the circle for START-t because no word lies in both sets.