Sub — ADM

Ch-2

Permutations and Combinations

In this chapter, we explore four general principle of counting and some of the counting formulas with application to different problems.

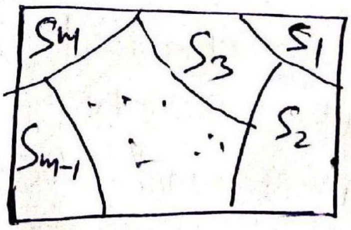
Partition of a set

Let S be a given set. A partition of S is a collection of subsets $S_1, S_2, S_3, \ldots, S_n$ of S such that each element of S is exactly one of those subsets:

$$\text{(i)}\ S = S_1 \cup S_2 \cup \cdots \cup S_n = \bigcup_{i=1}^{n} S_i$$

$$\text{(ii)}\ S_i \cap S_j = \phi,\ i \neq j$$

Here the sets $S_1, S_2, \ldots, S_n$ are pairwise disjoint sets and their union is S and intersection is empty.



Basic counting principle

1. Addition principle

Let $S_1, S_2, \ldots, S_n$ form a partition on S, Then the number of objects in S can be determined by finding the number of objects in each of the parts and adding those elements i.e. $|S| = |S_1| + |S_2| + \cdots + |S_m|$

Ex:- suppose a student is to allowed to take either a mathematics or a biology course but not both. If there are 4 mathematics courses and three biology courses then in how many ways is this possible?

Ans:- $|S| = |S_1| + |S_2| = 4 + 3 = 7$

Multiplication Rule

Let 'S' be a set of ordered pairs (a, b) where $a \in S_1$ with $|S_1| = p$, $b \in S_2$, $|S_2| = q$

Then the size of 'S' will be

$$|S| = |S_1||S_2| = pq.$$

Ex:- How many two digited numbers are possible?

soln $\boxed{9 \mid 10}$, $|S| = |S_1||S_2| = 9 \times 10 = 90$.

Ex:- A student is to take two courses. The first meets at any of 3 hrs in the morning and the 2nd at anyone 4 hrs in the afternoon. How many number of schedules are possible for the student?

soln $3 \times 4 = 12$.

Ex:- Chalk comes in three different lengths, eight different colors and four different diameters. How many different kinds of chalk are there?

Solⁿ We have to choose a length, color and a diameter to select a chalk. So by multiplication principle, the different kinds of chalk possible are $3 \times 8 \times 4 = 96$.

Ex:- How many ways a man, woman, boy and girl can be selected from five men, six woman, two boys and four girls?

Solⁿ Required ways $= 5 \times 6 \times 2 \times 4 = 240$.

Ex:- Determine the number of positive integers that are factors of $3^4 \times 5^2 \times 11^7 \times 13^8$.

Solⁿ By fundamental thm of arithmetic

If $x = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ where $p_1, p_2, \ldots, p_n$ are primes, $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{Z}^+$.

then the number of factors of x are $(\alpha_1+1)(\alpha_2+1) \cdots (\alpha_n+1)$.

Hence by multiplication principle the number of factors $= 5 \times 3 \times 8 \times 9$

$= 1080$ Ans

Ex:- How many two digit numbers have distinct and non zero digits?

Solⁿ

T	U
8	9

$8 \times 9 = 72$

Subtraction Principle

Let A be the set and U be a larger set containing A. Then

$\overline{A} = U \setminus A = \{x \in U : x \notin A\}$

i.e. $|\bar{A}| = |U| - |A|$ or $|A| = |U| - |\bar{A}|$.

Ex: Computer passwords are to consist of a string of six symbols taken from the digits 0, 1, 2, --- 9 and the lowercase letters a, b, c, ---, z. How many computer passwords have a repeated symbol?

Soln Let A be the set of computer passwords with a repeated symbol. Let U be the set of all computer passwords. $\bar{A}$ be the set of computer passwords with no repeated symbol.

$$|U| = 36^6$$

$$|\bar{A}| = 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31$$

$$|A| = |U| - |\bar{A}| = 36^6 - 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31$$
$$= 774,372,096 \text{ Ans}$$

Ex: How many three digited numbers are possible such that at least one of their digit is repeated?

Soln $|U| = 9 \times 10^2 = 900$ (Total no. of 3 digited numbers)

Let A be the set of all 3 digit nos such that none of their digit is repeated.

So $|A| = 9 \times 9 \times 8 = 648$

Hence $|\bar{A}| = |U| - |A| = 900 - 648 = 252$ Ans

Division Principle

Let S be a finite set that is partitioned into K parts in such a way that each part contains the same number of objects.

Then the number of parts in the partition is given by $K = \dfrac{|S|}{\text{Number of objects in a part}}$

Ex: There are 740 pigeons in a collection of pigeonholes. If each pigeonhole contains 5 pigeons, the number of pigeonholes equals to $\dfrac{740}{5} = 148$.

Ex: How many 3 digit even numbers are possible?

Sol^n S_1 = Set of 3 digit even numbers that ends with 0 only.

S_2 = set of all 3 digit even numbers that ends with 2, 4, 6, 8.

Then $|S_1| = 9 \times 10 \times 1 = 90$ $\quad \boxed{9}\boxed{10}\boxed{1}$

$|S_2| = 9 \times 10 \times 4 = 360$ $\quad \boxed{9}\boxed{10}\boxed{4}$

Required three digit even numbers

$= 90 + 360 = 450$ Ans

Ex: How many 5 digit numbers can be formed out of the digits 0, 1, 2, 3, 4 and 5 such that the number is divisible by 3 and no digit can be repeated more than once.

Sol^n Case-I when '0' is excluded

$|S_1| = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Case - II '3' is excluded (0, 1, 2, 4, 5)

using the digits 0, 1, 2, 4, 5

Number of 5 digit numbers $= 4 \times 4 \times 3 \times 2 \times 1$

$= 96$

4	4	3	2	1

No other choice is possible.

Hence the total number of 5 digit numbers which is divisible by 3 using the digits 0, 1, 2, 3, 4, 5 is

$120 + 96 = 216$ Ans

Ex: How many odd numbers between 1000 and 9999 have distinct digits?

Soln The number between 1000 and 9999 is a four digited number and odd. so its unit digit can be any of the 1, 3, 5, 7, 9 i.e. 4 ways. Thousand place can be filled by 8 ways (since 0 excluds and one out of 1, 3, 5, 7, 9 is placed in unit place).

For hundredth place, it can be filled by 8 ways (since 8 is included) and tenth place can by 7 ways. since the digits are distinct. Hence the required odd numbers formed $= 5 \times 8 \times 8 \times 7 = 2240$.

8	8	7	5

Ex: How many different five-digit numbers can be constructed out of the digits 1, 1, 1, 3, 8? Ans: $5 \times 4 = 20$

2.2 Permutation of Sets

Let r be a positive Integer. By an r-permutation of a set S of n elements, we understand an ordered arrangement of r of the n elements. If $S = \{a, b, c\}$, then the 1-permutation of S are

a b c

2-permutations of S are

ab ac ba bc ca cb

3-permutations of S are

abc, acb, bac, bca, cab, cba

$P(n, r)$ = The number of r-permutations of an n-element set. If $r > n$, then $P(n, r) = 0$

* An n-Permutation of an n-element set S is simply called permutation of S or a permutation of n elements.

Thm 2.2.1

For n and r positive integers with $r \leq n$,

$$P(n, r) = n(n-1)(n-2) - - - - - - - - (n-r+1).$$

Pf:- In constructing an r-permutation of an n-element set, We can choose the first item in n ways, the second item in $(n-1)$ ways, and continuing in this way the rth item in $n-(r-1)$ ways what ever the choice of the first $r-1$ items. By the multiplication principle r items can be choosen in

$n(n-1)(n-2) \cdots (n-r+1)$ ways.

Hence $P(n,r) = n(n-1)(n-2) \cdots (n-r+1)$. □

* $P(n,n) = n(n-1)(n-2) \cdots 1$
$= n!$

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}}, \quad 0 \le r \le n$$

Ex:- What is the number of ways to order the 26 letters of the alphabet so that no two of the vowels a, e, i, o and u occur ~~so~~ consecutively?

Soln We see that there are 22 gaps between 21 consonents. So 5 vowels can be arranged in 22 places in $P(22,5)$. Again ~~vowels and~~ consonents can be arranged ~~altered~~ among themselves in $21!$ ways.

Hence the total number of ways for the ordered arrangements of the letters of the alphabet with no two vowels consecutive $= 21! \times \cancel{22!}\ P(22,5)$

$$= 21! \times \frac{22!}{17!} \quad \text{Ans}$$

Ex:- How many seven digit numbers are there such that the digits are distinct integers from $\{1, 2, \cdots, 9\}$ and such that the digit 5 and 6 do not appear consecutively in either order?

Soln Let T be the set of all seven digit distinct numbers formed using the digits from $\{1, 2, \text{---}, 9\}$. Then the size of $T = P(9,7) = \frac{9!}{(9-7)!} = \frac{9!}{2!} = 181,440$

Let S consist of those numbers in T in which 5 and 6 do not occur consecutively. So the complement $\bar{S}$ consists of those numbers in T in which 5 and 6 do occur consecutively. We want to determine the size of S. First find the size $\bar{S}$ and then apply subtraction principle.

Now size of $\bar{S} = 2 \times 6 \times P(7,5)$
$= 30,240$

(As there are six ways to position 5 followed by a 6 and six ways to position a 6 followed by a 5. The remaining digits constitute a 5-permutation of $\{1, 2, 3, 4, 7, 8, 9\}$).

Hence the size of S = size of T − size of $\bar{S}$
$= 181,440 - 30,240$
$= 151,200$ <u>Ans</u>

<u>Thm 2.2.2</u>

The number of circular r-permutations of a set of n elements is given by

$$\frac{P(n,r)}{r} = \frac{n!}{r(n-r)!}$$

In particular, the number of circular permutations of n elements is $(n-1)!$ (since all cyclic permutations of objects are equivalent because of the circle can be rotated)

Ex:- In how many ways 7 people be seated on a circular table?

Solⁿ No. of people = 7

No. of circular permutation

$= (7-1)! = 6!$

Ex:- Ten people, including two who do not wish to sit next to one another are to seated at a round table. How many circular seating arrangements are there?

Solⁿ We solve this problem using the subtraction principle. Let the 10 people be $P_1, P_2, P_3, \ldots, P_{10}$, where P_1 and P_2 are the two do not wish to sit together. First 10 people can be seated at a round table in $(10-1)! = 9!$ ways. When P_1, P_2 are to be together, think P_1P_2 as one. Now there are 9 people which can be arranged in a round table by $8!$ ways but P_1, P_2 can be interchanged among themselves by $2!$ ways. So the number of circular arrangements in which P_1, P_2 are to seat together out of 10 people is $8! \times 2!$ ways,

Hence the number of circular seating arrangements in which P_1 and P_2 are not together is $9! - 2!\,8!$

$= 8! \times 9 - 2! \times 8! = 8!\,(9-2) = 7 \times 8!$

Ex: How many number of ways can 12 different markings on a rotating drum can be done?

Sol^n $\frac{P(12,12)}{12} = \frac{12!}{12} = 11!$

Ex: What is the number of necklaces that can be made from 20 beads, each of a different color?

Sol^n There are 20! permutation of the 20 beads. Since each necklace can be rotated without changing the arrangement of the beads, the number of necklaces is at most $\frac{20!}{20} = 19!$. Since a necklace can also be turned over without changing the arrangement of the beads, the total number of necklaces, by the division principle is $\frac{19!}{2}$.

2.3 Combination of Sets

A Combination of a set S is usually defined as the selection of the elements of S irrespective of the order.

Let r be a non negative Integer. By an r-combination of a set S of n elements, we understand an unordered selection of r of the n-objects of S.

The result of an r-combination is an r-subset of S which consists of r of the n objects of S.

The number of r-subsets of an n element set is denoted by ${}^nC_r$ or $\binom{n}{r}$

* ${}^nC_r = 0$ If $r > n$
* ${}^0C_r = 0$ If $r > 0$
* ${}^nC_0 = 1$, ${}^nC_1 = n$, ${}^nC_n = 1$

Thm – 2.3.1 For $0 \leq r \leq n$,

$$P(n,r) = r!\, {}^nC_r$$

Hence $${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

Pf:– Let S be an n-element set. Each r-permutation of S arises in exactly one way as a result of carrying out the following two tasks:

(i) Choose r elements from S.
(ii) Arrange the chosen r elements in some order

First choose r elements from S by ${}^nC_r$ ways. Then ~~arrange~~ the chosen r elements can be arranged in $r!$ ways. Now by the multiplication principle,

$$P(n,r) = r!\, {}^nC_r.$$

Hence $${}^nC_r = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!} \quad \square$$

Ex: Twenty-five points are chosen in the plane so that no three of them are collinear. How many straight lines do they determine? How many triangles do they determine?

Sol^n Since no three of the points lie on a line, every pair of points determines a unique straight line. Thus the number of straight lines determined is equal to the number of 2 combinations of a 25-element set and is given by

$$^{25}C_2 = \frac{25!}{2!\,23!} = \frac{25 \times 24}{1 \times 2} = 300$$

Similarly every three points determines a unique triangle, so that the number of triangles formed is $^{25}C_3 = \frac{25!}{3!\,22!}$.

Ex: There are 15 people enrolled in a mathematics course, but exactly 12 attend on any given day. If there are 25 seats in the classroom, in how many ways an instructor might see the 12 students in the classroom?

Sol^n The number of different ways that 12 students can be chosen is $^{15}C_{12}$ ways. If there are 25 seats in the classroom, the 12 students could seat themselves in $P(25,12) = \frac{25!}{13!}$ ways. Hence the number of ways an instructor might see the 12 students in the classroom is $^{15}C_{12} \times P(25,12)$

$$= \frac{15!}{12!\,3!} \times \frac{25!}{13!}$$ Ans

Ex:- How many eight-letter words can be constructed by using 26 letters of the alphabet if each word contains three, four or five vowels? It is understood that there is no restriction on the number of times a letter can be used in a word.

Soln First consider words with three vowels. Now the three positions occupied by the vowels can be chosen in $^8C_3$ ways and other five positions are occupied by consonants. The vowels position can be completed in 5^3 ways and the consonant positions in 21^5 ways. So the number of words with four vowels is $^8C_3\, 5^3\, 21^5$

Similarly the number of words with four vowels is $^8C_4\, 5^4\, 21^4$. And the number of words with five vowels is $^8C_5\, 5^5\, 21^3$. Hence the total number of words is $\frac{8!}{3!\,5!}\, 5^3\, 21^5 + \frac{8!}{4!\,4!}\, 5^4\, 21^4 + \frac{8!}{5!\,3!}\, 5^5\, 21^3$

Corollary 2.3.2 For $0 \le r \le n$

$$^nC_r = {}^nC_{n-r}$$

Thm 2.3.3 (Pascal's formula)

For all integers n and k with $1 \le k \le n-1$,

$$^nC_k = {}^{n-1}C_k + {}^{n-1}C_{k-1}.$$

Thm 2.3.4 for $n \geq 0$,

$$^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n.$$

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2.4 Permutation of Multisets

If S is a multiset, an r-permutation of S is an ordered arrangement of r of the objects of S. If the total number of objects of S is n (counting repetitions), then an n-permutation of S will also be called a permutation of S.

Ex:- $S = \{2 \cdot a, 1 \cdot b, 3 \cdot c\}$ then acbc, cbcc, aabc, bccc, accc, aacc ~~bccc acc...bcc...~~ are 4-permutations of S.

aabc

Here S has no 7-permutation since

$$7 > 2+1+3 = 6 = \text{No of objects of } S.$$

Thm-2.4.1

Let S be a multiset with objects of k different types, where each object has an infinite repetition number. Then the number of r-permutations of S is k^r.

Pf:- In constructing an r-permutation of S, we can choose the first item to be an object of any of the k types. Similarly, the second item can be an object of any of the k-types and so on. Since all repetition numbers of S are infinite, the number of different choices for any item is k and it does not depend on the choices of any previous items. By the multiplication principle, the r items can be chosen in k^r ways.

Ex: What is the number of ternary numerals with at most four digits?

Sol: The number of 4-permutation of the multiset $\{\infty \cdot 0, \infty \cdot 1, \infty \cdot 2\}$ or the multiset $\{4 \cdot 0, 4 \cdot 1, 4 \cdot 2\}$. Hence the required number equals to $3^4 = 81$.

Thm 2.4.2

Let S be a multiset with objects of k different types with finite repetition numbers $n_1, n_2, ---, n_k$ respectively. Let the size of S be $n = n_1 + n_2 + --- + n_k$. Then the number of permutations of S equals

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Pf: We are given a multiset S having objects of k types say $a_1, a_2, ---, a_k$ with repetition numbers $n_1, n_2, ---, n_k$ respectively, for a total of $n = n_1 + n_2 + --- + n_k$ objects. We want to determine the number of permutation of these n objects. Think in this way. There are n places and we want to put exactly one of the objects of S in each of the places. We first decide which places are to be occupied by the a_1's. since there are n_1 a_1's in S, we must choose a subset of n_1 places from the n places. We can do this in $\binom{n}{n_1}$ ways. We next decide which places are to be occupied by the a_2's. There are $n - n_1$ places left, and we must choose n_2 of them. This can be done $\binom{n-n_1}{n_2}$ ways. We next find that there are $\binom{n-n_1-n_2}{n_3}$ ways to choose a_3's. Continuing in this manner and applying multiplication principle, find the number of permutations of S equals

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!\,(n-n_1)!}\;\frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!}\;\frac{(n-n_1-n_2)!}{n_3!\,(n-n_1-n_2-n_3)!}\cdots\cdots$$

$$\cdots\frac{(n-n_1-n_2-\cdots-n_{k-1})!}{n_k!\,(n-n_1-n_2-\cdots-n_k)!} = \frac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!\,0!}$$

$$= \frac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!}$$

Ex:- In how many ways the letters of the word MISSISSIPPI is arranged?

Solⁿ $\{1\cdot M, 4\cdot I, 4\cdot S, 2\cdot P\}$

Required number of ways $= \frac{11!}{1!\,4!\,4!\,2!}$

Thm 2.4.3

Let n be a positive integer and let $n_1, n_2, \ldots, n_k$ be positive integers with $n = n_1 + n_2 + \cdots + n_k$. The number of ways to partition a set of n objects into k labeled boxes in which Box 1 contains n_1 objects, Box 2 contains n_2 objects, ..., Box k contains n_k objects. $\frac{n!}{n_1!\,n_2!\cdots n_k!}$.

If the boxes are not labeled and $n_1 = n_2 = \cdots = n_k$, then the number of partitions equals to $\frac{n!}{k!\,n_1!\,n_2!\cdots n_k!}$.

Ex: How many possibilities are there for eight nonattacking rooks on an 8-by-8 chessboard?

Soln: Since the board is 8 by 8 and there are to be eight rooks on the board that cannot attack one another, there must be exactly one rook in each row. But there must also be exactly one rook in each column so that no two of the numbers $j_1, j_2, \ldots, j_8$ can be equal. More precisely $j_1, j_2, \ldots, j_8$ must be a permutation of $\{1, 2, \ldots, 8\}$. Conversely if $j_1, j_2, \ldots, j_8$ is a permutation of $\{1, 2, \ldots, 8\}$, then putting rooks in the squares with coordinates $(1, j_1), (2, j_2), \ldots, (8, j_8)$, we arrive at eight non attacking rooks on the board. Thus we have a one-to-one correspondence between 8 nonattacking rooks on the 8 by 8 board. Since there are $8!$ permutations of $\{1, 2, \ldots, 8\}$. Hence there are $8!$ ways to place eight rooks on an 8-by-8 board so that they are non attacking.

* The number of ways to have eight non attacking rooks of eight different colors on a 8 by 8 board is equal to $8! \times 8! = (8!)^2$

* $S = \{1 \cdot R, 3 \cdot B, 4 \cdot Y\}$, the number of permutations of the multiset $= \dfrac{8!}{1!\,3!\,4!}$

* The number of ways to place one red, three blue and four yellow rooks on a 8 by 8 board so that no rook can attack one another is $\dfrac{8! \times 8!}{1!\,3!\,4!} = \dfrac{(8!)^2}{1!\,3!\,4!}$

Th^m - 2.4.4

There are n rooks of k colors with n_1 rooks of the first color, n_2 rooks of the 2nd color, ---- and n_k rooks of the kth color. The number of ways to arrange these rooks on a n by n board so that no rook can attack another equals to

$$\frac{n! \times n!}{n_1! n_2! \cdots n_k!} = \frac{(n!)^2}{n_1! n_2! \cdots n_k!}$$

Ex: Consider the multiset $S = \{3\cdot a, 2\cdot b, 4\cdot c\}$ of nine objects of three types. Find the number of 8-permutations of S.

Sol^n The 8-permutations of S can be partitioned into three parts

(i) 8-permutations of $\{2\cdot a, 2\cdot b, 4\cdot c\}$ is

$$\frac{8!}{2!\,2!\,4!} = 420$$

(ii) 8-permutations of $\{3\cdot a, 1\cdot b, 4\cdot c\}$ is

$$\frac{8!}{3!\,1!\,4!} = 280$$

(iii) 8-permutations of $\{3\cdot a, 2\cdot b, 3\cdot c\}$ ~~of~~ ~~a~~ is $\frac{8!}{3!\,2!\,3!} = 560$

Hence the number of 8-permutations of S is $420 + 280 + 560 = 1260$ Ans

2.5 Combination of Multisets

If S is a multiset, then an r-combination of S is an unordered selection of r of the objects of S.

Ex: Let $S = \{2\cdot a, 1\cdot b, 3\cdot c\}$. Then the 3-combinations of S are $\{2\cdot a, 1\cdot b\}$, $\{2\cdot a, 1\cdot c\}$, $\{1\cdot a, 1\cdot b, 1\cdot c\}$, $\{1\cdot a, 2\cdot c\}$, $\{1\cdot b, 2\cdot c\}$, $\{3\cdot c\}$

Th^m - 2.5.1

Let S be a multiset with objects of k types, each with an infinite repetition number. Then the number of r-combinations of S is equal to $\binom{r+k-1}{r} = \binom{r+k-1}{k-1}$

(* The number of r-combinations of S is same as the number of solutions of the equation $x_1 + x_2 + \cdots + x_k = r$, where $x_1, x_2, \ldots, x_k$ are non negative integers.

* The number of r-combinations of k distinct objects, each available in unlimited supply is equal to $\binom{r+k-1}{r}$.

Ex: A bakery boasts eight varieties of doughnuts. If a box of doughnuts contains one dozen, how many different options are there for a box of doughnuts?

Sol^n It is assumed that the bakery has on hand at least 12 of each variety. Since we assume the order of the doughnuts in a box is irrelevant for

the purchaser's purpose. The number of different options for boxes equals to 12-combination of a multiset with objects of 8-types, each having infinite repetition number is equal to $\binom{12+8-1}{12}$
$= \binom{19}{12}$.

Ex:- Let S be the multiset $\{10 \cdot a, 10 \cdot b, 10 \cdot c, 10 \cdot d\}$ with objects of four types a, b, c, d. What is the number of 10-combinations of S that have the property that each of the four types of objects occurs at least once?

Sol^n The answer is the number of positive integral solutions of $x_1+x_2+x_3+x_4 = 10$ where x_1 represents the number of a's, x_2 the number of b's, x_3 the number of c's and x_4 the number of d's in a 10-combination.

$x_1 \geq 1,\ x_2 \geq 1,\ x_3 \geq 1,\ x_4 \geq 1$

let $y_1 = x_1 - 1,\ y_2 = x_2 - 1,\ y_3 = x_3 - 1,$

$y_4 = x_4 - 1$

$y_1+y_2+y_3+y_4 = 6$

The number non negative integral solutions is $\binom{6+4-1}{6} = \binom{9}{6} = {}^9C_3 = \frac{\cancel{9}^3 \times \cancel{8}^4 \times 7}{1 \times \cancel{2} \times \cancel{3}}$

$= 84$ Ans

Ex:- What is the number of integral solutions of the eq^n $x_1+x_2+x_3+x_4 = 20$
$x_1 \geq 3,\ x_2 \geq 1,\ x_3 \geq 0$ and $x_4 \geq 5$?

Sol^n Let $y_1 = x_1 - 3,\ y_2 = x_2 - 1,\ y_3 = x_3,\ y_4 = x_4 - 5$
$y_1+y_2+y_3+y_4 = 11,\ y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0$

Required number of non-negative soln's

is $\binom{11+4-1}{11} = \binom{14}{11} = \binom{14}{3} = 364$ Ans

Exercise - 2.7

(6) How many integers greater than 5400 have both of the following properties?

(i) The digits are distinct

(ii) The digits 2 and 7 do not together

Soln Since we have a finite number of digits $\{0, 1, 3, 4, 5, 6, 8, 9\}$ that are allowed and distinct.

(i) 5 digit numbers

$7 \times 7 \times 6 \times 5 \times 4 = 5880$

(ii) 6 digit number

$7 \times 7 \times 6 \times 5 \times 4 \times 3 = 17640$

(iii) 7 digit number

$7 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 35280$

(iv) 8 digit number

$= 7 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 35280$

(v) 4 digit numbers

with 5 at Th place, $(5)1 \times 4 \times 6 \times 5 = 120$

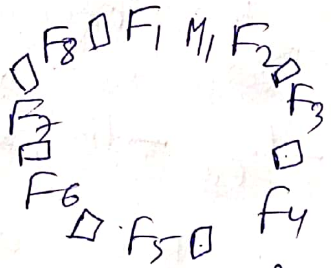
with 6, 8, 9 at Th place, $3 \times 7 \times 6 \times 5 = 630$

Hence the required integers formed $= 5880 + 17640 + 35280 + 35280 + 120 + 630$

$= 94830$ Ans

(7) In how many ways can four men and eight women be seated at a round table If there are to be two women between consecutive men around the table?

Sol^n Let us denote the eight women as $W_1, W_2, W_3, --, W_8$ and four men as M_1, M_2, M_3 and M_4. First eight women can be arranged around the table by $7! = 5040$ ways, After arranging the women, we can insert the first man M_1 in 8 different ways (because the 8 observed there are 8 places between the ladies when sitting at the circular table). Then there are three possible places where we must place the remaining men by $3!$.



Hence the total number of ways for those 12 people to be seated is $7! \times 8 \times 3!$
$= 8!\,3!$ ways

(10) A committe of five people is to be chosen from a club that boasts a membership of 10 men and 12 women. How many ways can the committe be formed If it is to contain at least two women? How many ways if, in addition, one particular man and one particular woman who are members of the club

refuses to serve together on the committe?

Sol: We need that committee to have five members and at least two of them must be a women. We can choose 2, 3, 4 or all 5 female members. Let us consider the complementary case with one and no women in the committe and subtract from the total number of ways to choose 5 people out of the total 22. one ~~female~~ woman can be chosen from 12 women in the club $^{12}C_1$ and 4 men can be chosen in $^{10}C_4$ ways. Now by multiplication principle, the total number of committees that consist of one female and four male equals to $^{12}C_1 \times {}^{10}C_4$. Number of Committees with no women in the club is $^{10}C_5$. Hence the total number of ways the committee can be formed ~~if~~ if it contains at least two women is $^{22}C_5 - {}^{10}C_5 - {}^{12}C_1 \times {}^{10}C_4 = 23562$ ways

Again for the additional case

(i) Committe consists of 1 other women and 2 other men (one other woman and 2 other men)

$$^{11}C_1 \times {}^9C_2 = 396$$

(ii) The committe consist of 2 other women and 1 other men

$$^{11}C_2 \times {}^9C_1 = 495.$$

(iii) The committe consist of 3 other women and 0 man

$$^{11}C_3 \times {}^9C_0 = 165$$

Hence the required answer is $23562 - 396 - 495 - 165 = 22506$

(16) Using combinatorial argument prove that $\binom{n}{r} = \binom{n}{n-r}$.

Pf Observing that any subset of size r can be specified either by saying which r elements lie the subset or by saying which $n-r$ elements lie outside the subset. Suppose that A is a set with n elements. Any subset B with r elements completely determines a subset, $A-B$ with $n-r$ elements. Suppose that A has K subsets of sizes r: $B_1, B_2, \dots B_K$. Then each B_i can be paired up with exactly one set of size $n-r$, namely its complement $A-B_i$ as shown below.

subsets of sizes r		subsets of size $n-r$
B_1	$\longleftrightarrow$	$A-B_1$
B_2	$\longleftrightarrow$	$A-B_2$
$\vdots$		
B_K	$\longleftrightarrow$	$A-B_K$

The number of subsets of size r is equal to the number of subsets of size $n-r$. Hence $\binom{n}{r} = \binom{n}{n-r}$.

(32) Determine the number of 11-permutation of the multiset $S = \{3\cdot a, 4\cdot b, 5\cdot c\}$

Soln The 11-permutations of S can be partitioned into three parts

(i) 11-permutations of $\{2\cdot a, 4\cdot b, 5\cdot c\}$

is $\frac{11!}{2!\,4!\,5!} = 6930$

(ii) 11-permutations of $\{3\cdot a, 3\cdot b, 5\cdot c\}$ is

$\frac{11!}{3!\,3!\,5!} = 9240$.

(iii) 11-permutations of $\{3\cdot a, 4\cdot b, 4\cdot c\}$ is

$\frac{11!}{3!\,4!\,4!} = 11550$

Hence the required number of 11-permutations of the multiset is $6930 + 9240 + 11550$
$= 27720$ Ans

(11) How many sets of three integers between 1 and 20 possible if no two consecutive integers are to be in the set?

Soln The set of possible numbers is $\{2, 3, 4, 5, \ldots, 19\}$
Total number of 3 element subsets of a set of 18 elements is $^{18}C_3 = 816$
Now consider the complementary case 3 element subsets with at least two consecutive numbers.
Two consecutive numbers can be chosen in 17 ways and the third one can be chosen in 16 ways.
So the total of $17 \times 16 = 272$ ways of 3 element subsets of consecutive numbers. Hence our required answer by subtraction principle is $816 - 272 = 544$ Ans