The Dihedral Group

Notation $\longrightarrow$ D_n or D_{2n}

order of D_n : $O(D_{2n}) = 2n$

$n \rightarrow$ reflection

$n \rightarrow$ rotation

$$D_4 = \begin{cases} a,\ a^2 = e = b^4 \\ ab,\ ab^2,\ ab^3 \\ b,\ b^2,\ b^3 \end{cases}$$

reflection

$$D_6 = \begin{cases} a,\ a^2 = e = b^6 \\ ab,\ ab^2,\ ab^3,\ ab^4,\ ab^5 \\ b,\ b^2,\ b^3,\ b^4,\ b^5 \end{cases}$$

$\rightarrow$ rotation

In general,

$$D_{2n} = \{x^i y^i : i = 0, 1,\ j = 0, 1, \ldots, n-1,\ x^2 = e = y^n,\ xy = y^{-1}x\}$$

is the dihedral group $(n \geq 3)$.

or Representation of elements of D_n

$$D_{2n} = \begin{cases} a,\ a^2 = e \\ b,\ b^2,\ b^3, \ldots, b^{n-1} \\ ab,\ ab^2,\ ab^3, \ldots, ab^{n-1} \end{cases}$$

Rotation and Reflection of D_n.

Rotation and reflection of D_n:

Rotation: There are $n-1$ rotation like $b, b^2, \dots, b^{n-1}$

Reflection: There are n reflection like $a, ab, ab^2, \dots, ab^{n-1}$

Example: Analyse rotation and reflection in D_4.

$$D_4 = \begin{cases} a,\ a^2 = e = b^4 \\ ab,\ ab^2,\ ab^3 \\ b,\ b^2,\ b^3 \end{cases}$$

angle of rotation in D_n is $\frac{2\pi}{n}$.

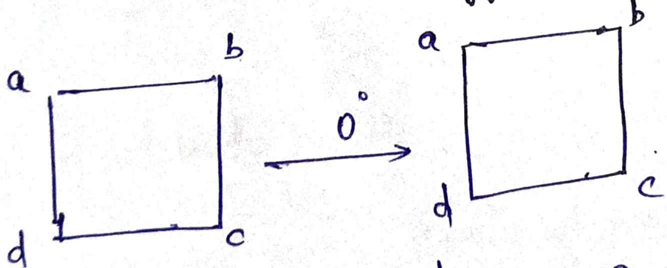
angle of rotation in D_4 is $\frac{360^\circ}{4} = 90^\circ$

Rotation:

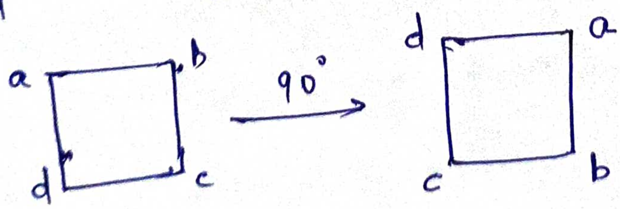
$D_3 \to$ Polygon of three sides △

$D_4 \to$ Polygon of four sides □

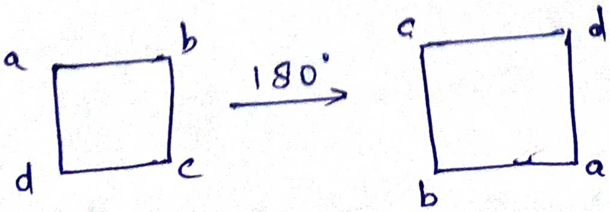
$D_5 \to$ Polygon of five sides

(1)


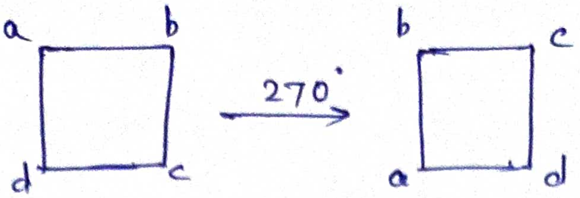
This is identity.

(2)


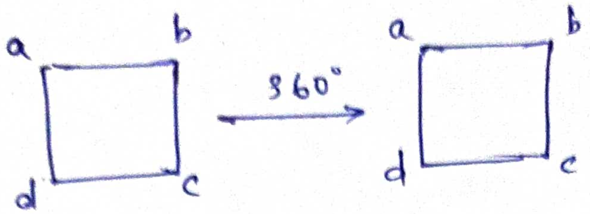
This is b.

(3)


This is b^2.

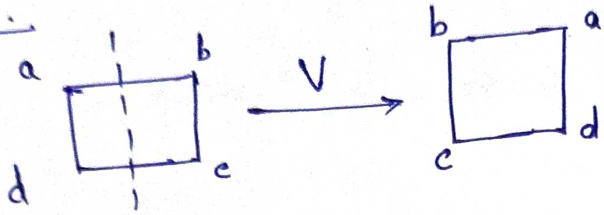
(4) 

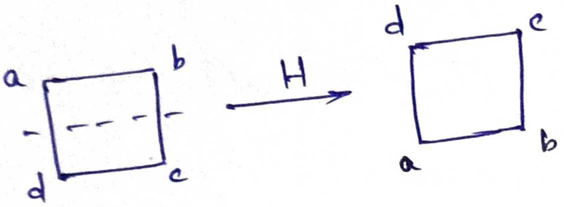
This is b^3.

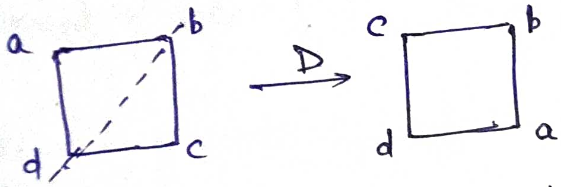
(5) 

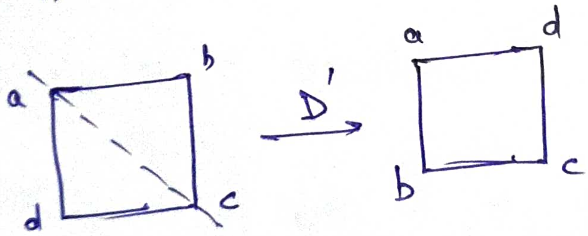
This is also identity. b^4.

Reflection:

(1) 

(2) 

(3) 

(4) 

Order of elements in D_n:

(i) $\exists$ an identity element of order 1.

(ii) $\exists$ an element of order 2.

(iii) $\exists$ an element of order d if $d \mid n$.

Solution: D_3 have an elements of order 1, 2 and 3.

Note: Order of every reflection are 2.

Number of elements in D_n

(1) Number of element of order 1 is 1.

(2) Number of elements of order 2 are

$$\begin{cases} n & \text{if } n \text{ is odd} \\ (n+1) & \text{if } n \text{ is even} \end{cases}$$

(3) Number of elements of order d ($d \neq 2$ and $d \mid n$) are $\phi(d)$

Example: Let $G = D_3$, then $D_3 = \begin{cases} a, a^2 = e \\ ab, ab^2 \\ b, b^2 \end{cases}$

(i) Element of order 1 is identity.

(ii) Elements of order 2 are 3.

$o(a) = 2$, $o(ab) = 2$, $o(ab^2) = 2$

(iii) Elements of order 3 are $\phi(3) = 2$.

$o(b) = 3$, $o(b^2) = 3$

D_3

$1 \longrightarrow 1$

$2 \longrightarrow 3$

$3 \longrightarrow \phi(3) = 3^1 - 3^0 = 2$

Example: $G = D_4$
↳ even $(n+1)$

$1 \longrightarrow 1$

$2 \longrightarrow 5$

$4 \longrightarrow \phi(4) = \phi(2^2)$

$= 2^2 - 2^1 = 2$

$$D_4 = \begin{cases} a, a^2 = e = b^4 \\ ab, ab^2, ab^3 \\ b, b^2, b^3 \end{cases}$$

(i) Element of order 1 is 1

(ii) Elements of order 2 are 5

$o(a) = o(ab) = o(ab^2)$

$= o(ab^3) = 2$

This all are reflection and

$o(b^2) = 2$. This is rotation.

(iii) Element of order 4

are $\phi(4) = 2$

$o(b) = o(b^3) = 4$

<u>Que 1</u>: Number of elements of order 2 in D_5.

<u>Sol</u>: We know that 5 is odd number.

No. of elements of order 2 in D_5 are 5.

<u>Que 2</u>: Number of elements of order 6 in D_7

<u>Sol</u>: We know that $6 \nmid 7$

so $\nexists$ any element of order 6.

D_7

1

2

7

<u>Que 3</u>: Number of elements of order 3 in D_6

We know that $3 | 6$

So $\phi(3) = 2$ elements have order 3.

$D_6 \rightarrow$ even

$1 \rightarrow 1$

$2 \rightarrow 7$

$3 \rightarrow \phi(3) = 2$

$6 \rightarrow \phi(6)$

$= \phi(3) \cdot \phi(2)$

$= 2 \cdot 1$

$= 2$

If p is prime.

$\phi(p) = p - 1$

$\phi(p^n) = p^{n-1}(p-1)$

Que 1: Compute the order of each of the elements in the following groups:

(a) D_6. (b) D_8 (c) D_{10}

$$D_{2n} = \{ r, s \mid r^n = s^2 = 1, \; rs = sr^{-1} \}$$

Elements of $D_6 = D_{2n} =$

$$D_6 = \begin{cases} a, & a^2 = e = b^3 \\ ab, & ab^2 \\ b, & b^2 \end{cases}$$

$o(e) = 1$

$o(a) = 2$

$o(ab) = ?$ $\quad (ab)(ab) = a(ba)b = aab^{-1}b$

$= a^2 = e$

$o(ab) = 2$

$o(b) = ?$ $\quad o(b) = 3$

$o(b^2) = ?$ $\quad b^2 . b^2 . b^2 = e$