Ch. 7 – Generating functions & their applications

§1 Finding generating functions for sequences

Def. A generating function is a function that can be used to code a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$. We get different kinds of generating functions by using different methods of coding.

Def. The (standard) generating function of the sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ is the function $f(x)$ defined by

$$f(x) = \sum_{n=0}^{\infty} a_n x^k = a_0 + a_1 x + a_2 x^2 + \cdots$$

provided this power series has radius of convergence > 0.

The exponential generating function of $\langle b_n \rangle_{n\in\mathbb{N}}$ is the function $f_E(x)$ defined by

$$f_E(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = b_0 + b_1 \frac{x}{1!} + b_2 \frac{x^2}{2!} + \cdots$$

provided this power series has radius of convergence ≥ 0,

Ex. 1 Find the standard gen. func. of the sequence

(a) $\langle 1 \rangle_{n\in\mathbb{N}}$ (b) $\langle 2^n \rangle_{n\in\mathbb{N}}$ (c) $\left\langle \frac{(-1)^n}{3^n} \right\rangle_{n\in\mathbb{N}}$

Sol (a)
$$f(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + 1\cdot x + 1\cdot x^2 + \cdots$$
$$= 1 + x + x^2 + \cdots = (1-x)^{-1} = \frac{1}{1-x}$$

(b)
$$f(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + 2\cdot x + 2^2 \cdot x^2 + \cdots$$
$$= 1 + (2x) + (2x)^2 + \cdots = [1-(2x)]^{-1} = \frac{1}{1-2x}$$

(c)
$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} \cdot x^n = 1 + \left(\frac{-1}{3}\right)x + \left(\frac{-1}{3}\right)^2 x^2 + \cdots$$
$$= 1 + \left(-\frac{x}{3}\right) + \left(\frac{-x}{3}\right)^2 + \cdots = \frac{1}{1-(-x/3)} = \frac{3}{3+x}$$

Ex.2 Find the sequence coded by the standard generating functions: (a) $\frac{1}{(1+\alpha x)}$ (b) $\frac{20}{5+x}$

Sol (a) $\frac{1}{1+\alpha x} = \frac{1}{1-(-\alpha x)} = 1 + (-\alpha x)^1 + (-\alpha x)^2 + \dots + (-\alpha x)^n + \dots$

$= 1 + (-\alpha)x + (-\alpha)^2 x^2 + \dots + (-\alpha)^n x^n + \dots$

$\therefore a_n =$ coeff. of x^n in the exp. of $\frac{1}{1+\alpha x} = (-\alpha)^n$.

So $\langle(-\alpha)^n\rangle_{n\in\mathbb{N}}$ is the sequence coded by $\frac{1}{1+\alpha x}$.

(b) $\frac{20}{5+x} = \frac{20}{5(1+x/5)} = 4 \cdot \frac{1}{[1-(-x/5)]} = 4 \cdot \sum_{n=0}^{\infty} \left(\frac{-x}{5}\right)^n$

$= \sum_{n=0}^{\infty} 4 \cdot \left(\frac{-1}{5}\right)^n \cdot x^n = 4 + 4\left(\frac{-1}{5}\right)^1 x + \dots + 4\left(\frac{-1}{5}\right)^n x^n + \dots$

$\therefore a_n =$ coeff. of x^n in the exp. of $\frac{20}{5+x} = 4 \cdot \left(\frac{-1}{5}\right)^n$.

$\therefore \langle 4 \cdot \left(\frac{-1}{5}\right)^n\rangle_{n\in\mathbb{N}}$ is the sequence coded by $\frac{20}{5+x}$.

Note All of our solutions were based on the fact that the standard generating function $\frac{1}{1-\alpha x}$ codes the sequence $\langle\alpha^n\rangle_{n\in\mathbb{N}}$.

Ex.3 Find the standard generating functions of the seq.

(a) $\langle n\rangle_{n\in\mathbb{N}}$ (b) $\langle n^2\rangle_{n\in\mathbb{N}}$ (c) $\langle\frac{1}{n+1}\rangle_{n\in\mathbb{N}}$.

Sol. (a) $(1-x)^{-1} = 1 + x + x^2 + \dots + x^n + \dots$

$\therefore \frac{d}{dx}(1-x)^{-1} = \frac{d}{dx}(1 + x + x^2 + \dots + x^n + \dots)$

$\therefore (-1).(1-x)^{-2}.(-1) = 0 + 1.x^0 + 2x^1 + \dots + nx^{n-1} + \dots$

$\therefore (1-x)^{-2} = 0 + 1 + 2x^1 + \dots + nx^{n-1} + \dots$

$\therefore x.(1-x)^{-2} = 0 + x + 2x^2 + \dots + n.x^n + \dots$

(a) So standard gen. func. of $\langle n \rangle_{n\geq 0}$ is $\frac{x}{(1-x)^2}$.

(b) From part (a), we know that

$$x.(1-x)^{-2} = 0 + 1.x + 2.x^2 + \dots + n.x^n + \dots$$

$$\therefore \frac{d}{dx}\left[x.(1-x)^{-2}\right] = 0 + 1^2.x^0 + 2^2.x^1 + \dots + n^2.x^{n-1} + \dots$$

$$\therefore 1.(1-x)^{-2} + x.2.(1-x)^{-3} = 0^2 + 1^2.x^0 + 2^2.x^1 + \dots + n^2.x^{n-1} + \dots$$

$$\therefore \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3} = 0^2 + 1^2.x^1 + 2^2.x^2 + \dots + n^2.x^n + \dots$$

So standard gen. func. of $\langle n^2 \rangle_{n\geq 0}$ is $\frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}$

$$= \frac{x(1-x) + 2x^2}{(1-x)^3} = \frac{x(x+1)}{(1-x)^3}.$$

(c) We know that by find the anti-derivative of

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots, \text{ we get}$$

$$-\ln(1-x) = \left(\frac{x^1}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^{n+1}}{n+1} + \dots\right) + C$$

But when $x=0$, $-\ln(1-0) = 0 + C \Rightarrow C = 0$.

$$\therefore \ln\left(\frac{1}{1-x}\right) = \frac{x^1}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^{n+1}}{n+1} + \dots$$

$$\therefore \frac{1}{x}.\ln\left(\frac{1}{1-x}\right) = \frac{1}{0+1} + \frac{x}{1+1} + \frac{x^2}{2+1} + \dots + \frac{x^n}{n+1} + \dots$$

So standard generating function of $\langle \frac{1}{n+1} \rangle_{n\geq 0}$ is $\frac{1}{x}.\ln\left(\frac{1}{1-x}\right)$.

<u>Note</u> We can get codes of more sequences by using Newton's Binomial Theorem, viz.

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n. \quad \text{So} \quad (1-x)^\alpha = \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} x^n$$

In particular, if we take $\alpha = -k$ and remember that $\binom{-k}{n} = (-1)^n \binom{n+k-1}{n}$ we get

$$(1-x)^{-k} = \sum_{k=0}^{\infty} (-1)^n \cdot (-1)^n \binom{n+k-1}{n} \cdot x^k = \sum_{k=0}^{\infty} \binom{n+k-1}{n} \cdot x^k.$$

So $(1-x)^{-k}$ codes the sequence $\left\langle \binom{n+k-1}{n} \right\rangle_{n \in \mathbb{N}}$.
In particular, if we put $k=-2$ & $k=-3$, we get $(1-x)^{-2}$ codes the seq. $\left\langle \binom{n+1}{n} \right\rangle = \langle n+1 \rangle_{n \in \mathbb{N}}$ and $(1-x)^{-3}$ codes the seq. $\left\langle \binom{n+2}{n} \right\rangle = \left\langle \frac{(n+1)(n+2)}{2} \right\rangle_{n \in \mathbb{N}}$.

<u>Prop 1</u> If $\langle a_n \rangle_{n \in \mathbb{N}}$ & $\langle b_n \rangle_{n \in \mathbb{N}}$ are two different sequences then the <u>standard</u> generating functions of $\langle a_n \rangle_{n \in \mathbb{N}}$ and $\langle b_n \rangle_{n \in \mathbb{N}}$ are different.

<u>Proof:</u> Suppose $\langle a_n \rangle$ & $\langle b_n \rangle$ have the same standard generating function, $f(x)$ say. Then for each $n \in \mathbb{N}$,

$$a_n = \text{coefficient of } x^n \text{ in the expansion of } f(x)$$
$$= b_n$$

So $\langle a_n \rangle_{n \in \mathbb{N}} = \langle b_n \rangle_{n \in \mathbb{N}}$. So if $\langle a_n \rangle$ & $\langle b_n \rangle$ are different then their standard gen. func. must be different, by the contrapositive law.

<u>Note:</u> Not every sequence will have a standard generating function. For example, consider the sequence $\langle n! \rangle_{n \in \mathbb{N}}$. The standard gen. function of this seq. would have to be

$$0! + (1!)x + (2!)x^2 + \ldots + (n!)x^n + \cdots = \sum_{n=0}^{\infty} (n!) x^n.$$

But since this power has radius of convergence 0, it is pretty much useless. So $\langle n! \rangle_{n \in \mathbb{N}}$ does not have a standard gen. function. This is partly why, exponential generating functions were introduced.

§2 Using standard gen. func. to solve recurrence equations

Ex. 1 Find the solution of the recurrence equation
$a_n - 5a_{n-1} + 6a_{n-2} = 0$ for $n \geq 2$ with $a_0 = 1$ & $a_1 = 7$

Sol. Let $f(x)$ = the standard generating function of $\langle a_n \rangle_{n \in \mathbb{N}}$.
Then
$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots,$$
$$-5x f(x) = \quad -5a_1 x - 5a_2 x^2 - \cdots - 5a_{n-1} x^n - \cdots,$$
&
$$6x^2 f(x) = \quad 6a_2 x^2 + \cdots + 6a_{n-2} x^n + \cdots.$$

$$\therefore (1 - 5x + 6x^2) f(x) = a_0 + (a_1 - 5)x + (a_2 - 5a_1 + 6a_0)x^2 + \cdots + (a_n - 5a_{n-1} + 6a_{n-2})x^n + \cdots$$
$$= a_0 + (a_1 - 5) = 1 + (7-5)x = 1 + 2x.$$

$$\therefore f(x) = \frac{1+2x}{(1-5x+6x^2)} = \frac{1+2x}{(1-3x)(1-2x)} = \frac{A}{1-3x} + \frac{B}{1-2x}$$

$$\therefore \quad 1 + 2x = A(1-2x) + B(1-3x).$$

Putting $x = 1/3$, gives us $1 + 2(1/3) = A(1 - 2/3) + 0$
$\therefore 5/3 = A/3 \Rightarrow A = 5.$

Putting $x = 1/2$ gives us $1 + 2(1/2) = 0 + B(1 - 3/2)$
$\therefore 2 = -B/2 \Rightarrow B = -4.$

$$\therefore f(x) = \frac{A}{1-3x} + \frac{B}{1-2x} = \frac{5}{1-3x} - \frac{4}{1-2x}$$
$$= 5\left[1 + (3x) + (3x)^2 + \cdots + (3x)^n + \cdots\right.$$
$$\left. - 4[1 + (2x) + (2x)^2 + \cdots + (2x)^n + \cdots\right]$$

$\therefore a_n$ = coeff. of x^n in the expansion of $f(x)$
$$= 5 \cdot (3)^n - 4 \cdot (2)^n.$$

Ex. 2 Find the solution of the recurrence equation
$a_n - 2a_{n-1} - 2 = 0$ for $n \geq 1$ with $a_0 = 1$.

Sol. First observe that the standard gen. func. of $\langle -2 \rangle_{n\in\mathbb{N}}$ is $(-2)/(1-x)$. Now let $f(x)$ = standard generating function of $\langle a_n \rangle_{n\in\mathbb{N}}$. Then

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots,$$
$$(-2x)f(x) = \quad -2a_0 x - 2a_1 x^2 - \cdots - 2a_{n-1}x^n - \cdots, \quad \&$$
$$\frac{-2}{1-x} = -2 - 2x - 2x^2 - \cdots - 2 \cdot x^n - \cdots.$$

$$\therefore (1-2x)f(x) - \frac{2}{1-x} = (a_0 - 2) + (a_1 - 2a_0 - 2)x + (a_2 - 2a_1 - 2)x^2 + \cdots + (a_n - 2a_{n-1} - 2)x^n + \cdots$$
$$= (1-2) + 0 \cdot x + 0 \cdot x^2 + \cdots + 0 \cdot x^n + \cdots = -1.$$

$$\therefore (1-2x)f(x) = \frac{2}{1-x} - 1 = \frac{2-(1-x)}{1-x} = \frac{1+x}{1-x}$$

$$\therefore f(x) = \frac{1+x}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x}$$

$$\therefore \quad 1+x = A(1-2x) + B(1-x)$$

Putting $x=1$, gives us $1+1 = A(1-2) + 0$
$$\therefore \quad 2 = -A \Rightarrow A = -2$$

Putting $x = 1/2$, gives us $1 + 1/2 = 0 + B(1 - 1/2)$
$$\therefore \quad 3/2 = B/2 \Rightarrow B = 3.$$

$$\therefore f(x) = \frac{A}{1-x} + \frac{B}{1-2x} = \frac{3}{1-2x} - \frac{2}{1-x}$$
$$= 3\left[1 + (2x) + (2x)^2 + \cdots + (2x)^n + \cdots\right]$$
$$- 2\left[1 + x + x^2 + \cdots + x^n + \cdots\right]$$

$\therefore a_n$ = coefficient of x^n in the expansion of $f(x)$
$$= 3 \cdot (2)^n - 2.$$

Ex 3 Find the solution of the recurrence equation
$a_n - 6 \cdot a_{n-1} + 9a_{n-2} = 0$ for $n \geq 2$ with $a_0 = 2$ & $a_1 = 21$.

Sol. Let $f(x)$ = the standard generating function of $\langle a_n \rangle_{n\in\mathbb{N}}$.
Then $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$,
$-6x f(x) = \quad -6a_0 x - 6a_1 x^2 - \dots - 6a_{n-1}x^n - \dots$, &
$+9x^2 f(x) = \quad 9a_0 x^2 + \dots + 9a_{n-2}x^n - \dots$.

$$\therefore (1-6x+9x^2)f(x) = a_0 + (a_1 - 6a_0)x + (a_2 - 6a_1 + 9a_0)x^2 + \dots + (a_n - 6a_{n-1} + 9a_{n-2})x^n + \dots$$

$$\therefore (1-3x)^2 f(x) = 2 + (21 - 6(2))x + 0 + 0 + \dots = 2 + 9x$$

$$\therefore f(x) = \frac{2+9x}{(1-3x)^2} = \frac{A}{1-3x} + \frac{B}{(1-3x)^2}$$

$$\therefore \quad 2 + 9x = A(1-3x) + B$$

Putting $x = 1/3$, gives us $2 + 9/3 = 0 + B$
$\therefore B = 2 + 3 = 5$.
Putting $x = 0$ gives us $2 + 0 = A(1-0) + B$
$\therefore A = 2 - B = 2 - 5 = -3$

$$\therefore f(x) = \frac{A}{1-3x} + \frac{B}{(1-3x)^2} = \frac{-3}{1-3x} + \frac{5}{(1-3x)^2}$$

$$= -3\left[1 + (3x) + (3x)^2 + \dots + (3x)^n + \dots\right] + 5\left[1 + 2\cdot(3x) + 3\cdot(3x)^2 + \dots + (n+1)\cdot(3x)^n + \dots\right]$$

because $\dfrac{1}{(1-3x)^2} = \sum_{n=0}^{\infty} \binom{n+2-1}{n}(3x)^n = \sum_{n=0}^{\infty} (n+1)\cdot(3x)^n$.

$\therefore a_n$ = coeff. of x^n in the expansion of $f(x)$
$$= -3\cdot(3)^n + 5\cdot(n+1)\cdot(3)^n = (5n+2)\cdot(3)^n.$$

Ex.4 Find the solution of the recurrence equation $(n+1)a_{n+1} - 2a_n = 5 \cdot 3^n/(n!)$ for $n \geq 0$ with $a_0 = 2$

Sol. Let $f(x)$ = the generating function of $\langle a_n \rangle_{n\in N}$. Then

$$f'(x) = 1 \cdot a_1 + 2 \cdot a_2 x + 3 \cdot a_3 x^2 + \cdots + (n+1)a_{n+1} x^n + \cdots,$$

$$-2f(x) = -2a_0 - 2a_1 x - 2 \cdot a_2 x^2 - \cdots - 2a_n x^n - \cdots, \ \&$$

$$-5e^{3x} = -5(3x)^0 - 5\frac{(3x)}{1!} - 5 \cdot \frac{(3x)^2}{2!} - \cdots - 5 \cdot \frac{3^n}{n!} x^n - \cdots.$$

$$\therefore f'(x) - 2f(x) - 5e^{3x} = \left(1 \cdot a_1 - 2 \cdot a_0 - 5 \cdot \frac{3^0}{0!}\right) + \left(2 \cdot a_2 - 2a_1 - 5 \cdot \frac{3^1}{1!}\right)x + \cdots + \left[(n+1)a_{n+1} - 2a_n - 5 \cdot \frac{3^n}{n!}\right]x^n + \cdots$$

$$= 0$$

$$\therefore f'(x) - 2f(x) = 5e^{3x} \quad \& \quad f(0) = a_0 = 2$$

$$\therefore e^{-2x} f'(x) - 2e^{-2x} f(x) = 5e^{3x} \cdot e^{-2x} = 5e^x$$

$$\therefore \frac{d}{dx}\left[e^{-2x} f(x)\right] = 5e^x$$

$$\therefore e^{-2x} f(x) = \int 5e^x dx = 5e^x + C$$

$$\therefore f(x) = (5e^x + C)e^{2x} = 5e^{3x} + C \cdot e^{2x}$$

Since $f(0) = 2$, $\quad 2 = 5 + C \Rightarrow C = -3$

$$\therefore f(x) = 5e^{3x} + Ce^{2x} = 5e^{3x} - 3e^{2x}$$

$$= 5\left[1 + \frac{(3x)}{1!} + \frac{(3x)^2}{2!} + \cdots + \frac{(3x)^n}{n!} + \cdots\right]$$

$$- 3\left[1 + \frac{(2x)}{1!} + \frac{(2x)^2}{2!} + \cdots + \frac{(2x)^n}{n!} + \cdots\right]$$

$\therefore a_n$ = coeff. of x^n in the expansion of $f(x)$

$$= 5 \cdot \frac{3^n}{n!} - 3 \cdot \frac{2^n}{n!}$$

$$= \left[5 \cdot (3)^n - 3 \cdot (2)^n\right]/n!$$

§3 Other applications of generating functions.

We can use the standard generating functions to find the number of r-combinations of a multi-set. Now we already have ways of doing this from Ch.1 & Ch.3 — but if we have some way of easily extracting coefficients of x^n from a given expression, the current method will be of much use.

Ex.1 Find the no. of 5-combinations of the mult-set $[3.a, 2.b, 3.c]$

Sol. Let a_n = no. of n-comb. of $[3.a, 2.b, 3.c]$
Then a_n = coefficient of x^n in the expansion of $\underbrace{(1+x+x^2+x^3)}_{\text{no. of a's}}\underbrace{(1+x+x^2)}_{\text{no. of b's}}\underbrace{(1+x+x^2+x^3)}_{\text{no. of c's}}$.
To get a_5 we systematically look at how we can get the term with x^5.

$(1.x^2.x^3)$, $(x.x.x^3)$, $(x.x^2.x^2)$, $(x^2.1.x^3)$, $(x^2.x.x^2)$, $(x^2.x^2.x)$, $(x^3.1.x^2)$, $(x^3.x.x)$, $(x^3.x^2.1)$.
So $a_5 = 9$ and hence our answer is 9.

Ex.2 Find the no. of 5-combinations of the multi-set $[4.a, \infty.b, \infty.c]$ with an odd number of b's & an even number of c's

Sol. Let a_n = no. of n-comb. of $[4.a, \infty.b, \infty.c]$ with odd no. of b's & even no. of c's. Then a_n = coeff. of x^n in exp. of $\underbrace{(1+x+x^2+x^3+x^4)}_{\text{no. of a's}}\underbrace{(x+x^3+x^5+\cdots)}_{\text{no. of b's}}\underbrace{(1+x^2+x^4+\cdots)}_{\text{no. of c's}}$

Ex.2 To get a_5 we just have to look at the ways in which we can x^5.

$(1.x.x^4)$, $(1.x^3.x^2)$, $(1.x^5 1)$, $(x^2 x^1, x^2)$,
$(x^2 x^3, 1)$, $(x^4, x, 1)$

So $a_5 = 6$ and hence our answer is 6.

Ex.3 Find the number of integer solutions of $x_1 + x_2 + x_3 = 8$ with $1 \le x_1 \le 4$, $2 \le x_2 \le 5$ and $3 \le x_3 \le 6$.

Sol. Let a_n = no. of solutions of integer solutions of $x_1 + x_2 + x_3 = n$ with $1 \le x_1 \le 4$, $2 \le x_2 \le 5$ and $3 \le x_3 \le 6$. Then a_n = no. of 8-comb. of $[4.a, 5b, 6c]$ with at least $1a$, at least $2b$'s and least $3.c$'s = coeff. of x^n in the exp. of

$$\underbrace{(x+x^2+x^3+x^4)}_{\text{no. of } a\text{'s}}\underbrace{(x^2+x^3+x^4+x^5)}_{\text{no. of } b\text{'s}}\underbrace{(x^3+x^4+x^5+x^6)}_{\text{no. of } c\text{'s}}$$

Our answer is a_8 which is obtained by looking at the ways we can get x^8.

$(x^1 x^2 x^5)$, $(x^1 x^3 x^4)$, $(x^1 . x^4 . x^3)$, $(x^2 . x^2 . x^4)$
$(x^2 . x^3 . x^3)$, $(x^3 . x^2 . x^3)$. So $a_8 = 6$

Hence our answer is 6.

Recall that the exponential generating function of the sequence $\langle b_n \rangle_{n\in\mathbb{N}}$ was defined by

$$f_E(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = b_0 + \frac{b_1}{1!}x + \frac{b_2}{2!}x^2 + \cdots$$

Ex.4 Find the exponential generating function of

(a) $\langle -2 \rangle_{n\in\mathbb{N}}$ (b) $\langle n! \rangle_{n\in\mathbb{N}}$ (c) $\langle (-3)^n . n! \rangle_{n\in\mathbb{N}}$

Ex. 4 (a) We know that $e^x = \sum_{n=0}^{\infty} 1 . \frac{x^n}{n!}$. So the exponential generating function of $\langle(-2)\rangle_{n\in N}$ is $(-2) + (-2).\frac{x}{1!} + (-2).\frac{x^2}{2!} + \dots + (-2).\frac{x^n}{n!} + \dots$

$$= \sum_{n=0}^{\infty} (-2).\frac{x^n}{n!} = (-2).\sum_{n=0}^{\infty} \frac{1}{n!} x^n = -2.e^x.$$

(b) Exponential generating function of $\langle(n!)\rangle_{n\in N}$ is given by $f_E(x) = \sum_{n=0}^{\infty} (n!).\frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n$

$$= \frac{1}{1-x}$$

(c) Exponential generating function of $\langle(-3)^n . n!\rangle_{n\in N}$ is given by $f_E(x) = \sum_{n=0}^{\infty} (-3)^n . n! \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-3)^n . x^n$

$$= \sum_{n=0}^{\infty} (-3x)^n = \frac{1}{1-(-3x)} = \frac{1}{1+3x}$$

Note: Sequences such as $(n!)^2$ and $2^{n!}$ do not even have exponential generating functions (much less standard generating functions).

We can use exponential generating functions to count the number of r-permutations of multi-sets.

Ex. 5 Find the number of 5-permutations of the multi-set $[3.a, 2.b, 2.c]$.

Sol. Let a_n = no. of n-permutations of $[3.a, 2.b, 2.c]$

Then a_n = coeff. of $\frac{x^n}{n!}$ in the exponential expansion of $\underbrace{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)}_{\text{no. of a's}} \underbrace{\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)}_{\text{no. of b's}} \underbrace{\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)}_{\text{no. of c's}}$

To get the no. of 5-perm., look at the term $\frac{x^5}{5!}$.

Ex.5 $\left(\frac{x^1}{1!},\frac{x^2}{2!},\frac{x^2}{2!}\right),\left(\frac{x^2}{2!},\frac{x^1}{1!},\frac{x^2}{2!}\right),\left(\frac{x^2}{2!},\frac{x^2}{2!},\frac{x^1}{1!}\right),$

$\left(\frac{x^3}{3!},1,\frac{x^2}{2!}\right),\left(\frac{x^3}{3!},\frac{x^1}{1!},\frac{x^1}{1!}\right),\left(\frac{x^3}{3!},\frac{x^2}{2!},1\right).$

So $a_5=$ coeff. of $\frac{x^5}{5!}$ in the expansion above

$$= 5!\left[\frac{1}{1!2!2!}+\frac{1}{2!1!2!}+\frac{1}{2!2!1!}+\frac{1}{3!2!}+\frac{1}{3!}+\frac{1}{3!2!}\right]$$

$$= 5!\left[3\cdot\frac{1}{4}+2\cdot\frac{1}{12}+\frac{1}{6}\right]=\frac{5!}{12}\left[9+2+2\right]$$

$$= \frac{120}{12}(13) = 130.$$

Ex.6 Find the number of 4-permutations of the multi-set $[2.a, \infty.b, 3.c]$

Sol. Let $a_n =$ no. of 4-perm. of $[2.a,\infty.b,3.c]$

Then $a_n =$ coeff. of $\frac{x^4}{4!}$ in the exponential exp. of

$$\underbrace{\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)}_{\text{no. of } a\text{'s}}\underbrace{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)}_{\text{no. of } b\text{'s}}\underbrace{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)}_{\text{no. of } c\text{'s}}$$

To get a_4, look at the term with $\frac{x^4}{4!}$

$\left(1,\frac{x^1}{1!},\frac{x^3}{3!}\right),\left(1,\frac{x^2}{2!},\frac{x^2}{2!}\right),\left(1,\frac{x^3}{3!},\frac{x}{1!}\right),\left(\frac{x}{1!},1,\frac{x^3}{3!}\right),$

$\left(\frac{x}{1!},\frac{x}{1!},\frac{x^2}{2!}\right),\left(\frac{x}{1!},\frac{x^2}{2!},\frac{x^1}{1!}\right),\left(\frac{x}{1!},\frac{x^3}{3!},1\right),\left(\frac{x^2}{2!},1,\frac{x^2}{2!}\right)$

$\left(\frac{x^2}{2!},\frac{x}{1!},\frac{x}{1!}\right),\left(\frac{x^2}{2!},\frac{x^2}{2!},1\right),\quad\left(1,\frac{x^4}{4!},1\right),$

$$\therefore a_4 = 4!\left[\frac{1}{1!3!}+\frac{1}{2!2!}+\frac{1}{3!1!}+\frac{1}{1!3!}+\frac{1}{1!1!2!}+\frac{1}{1!2!1!}+\frac{1}{1!3!}+\frac{1}{2!2!}+\frac{1}{2!1!1!}+\frac{1}{2!2!}+\frac{1}{4!}\right]$$

$$= 4+6+4+4+12+12+4+6+12+6+1 = 71.$$

Ch.7 §4. More applications of Generating Functions

In Chapter 2 we found the number of non-negative integer solutions of the equation $x_1 + \cdots + x_k = n$ by finding the number of ways of arranging n "1"'s and $(k-1)$ "+"'s in a row. Below is another way of solving this problem.

Ex.1 Find the no. of non-negative integer solutions of the equation $x_1 + x_2 + \cdots + x_k = n$.

Sol. No. of non-neg. integer solutions of the equation

$= $ no. of n-comb. of $[\infty.a_1, \infty.a_2, \ldots, \infty.a_k]$

$=$ coefficient of x^n in the expansion of

$$\underbrace{(1+x+x^2+\cdots)}_{\text{no. of } a_1\text{'s}}\underbrace{(1+x+x^2+\cdots)}_{\text{no. of } a_2\text{'s}} \cdots \underbrace{(1+x+x^2+\cdots)}_{\text{no. of } a_k\text{'s}}$$

$=$ coeff. of x^n in the exp. of $[1/(1-x)]^k$

$=$ coeff. of x^n in the exp. of $[1+(-x)]^{-k}$

$=$ coeff. of x^n in $\sum_{n=0}^{\infty} \binom{-k}{n} \cdot (-x)^n = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$

$= \binom{n+k-1}{n} = \binom{n+k-1}{k-1}$.

Ex.2 Find the no. of non-negative integer solutions of the equation $x_1 + 2x_2 + 4x_3 = 10$

Sol. Let $y_1 = x_1$, $y_2 = 2x_2$, and $y_3 = 4x_3$. Then

no. of non-neg. integer solutions of $x_1 + 2x_2 + 4x_3 = 10$

$=$ no. of non-neg. integer solutions of $y_1 + y_2 + y_3 = 10$ with y_2 being even & y_3 being a multiple of 4

$=$ coeff. of x^{10} in the expansion of

$$(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^4+x^8+\cdots)$$

$(x^{10}.1.1)$, $(x^8.x^2.1)$, $(x^6.x^4.1)$, $(x^6.1.x^4)$, $(x^4.x^6.1)$
$(x^4.x^2.x^4)$, $(x^2.x^8.1)$, $(x^2.x^4.x^4)$, $(x^2.1.x^8)$, $(1.x^{10}.1)$
$(1.x^6.x^4)$, $(1.x^2.x^8)$
So the answer is 12.

Ex.3 If we have large numbers of pennies, nickels, dimes, and quarters — in how many ways can make change for 40 cents.

Sol. Answer = no. of non-neg. integer solutions of

$$x_1 + 5x_2 + 10x_3 + 25x_4 = 40$$

= coefficient of x^{40} in the expansion of

$$(1+x+x^2+\cdots)(1+x^5+x^{10}+\cdots)(1+x^{10}+x^{20}+\cdots)(1+x^{25}+x^{50}+\cdots)$$

= coeff. of x^{40} in exp. of $\frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}} \cdot \frac{1}{1-x^{25}}$

This can be computed just as in Ex. 2 or we can use a computer program to extract the coefficient of x^{40} in the exp. of $(1-x)^{-1}(1-x^5)^{-1}(1-x^{10})^{-1}(1-x^{25})^{-1}$.

Ex.4 Find the number of permutations of $\{1,2,3,4,5\}$ which each have a total of 7 inversions.

Sol. Answer = no. of inversion seq. $\langle i_1, \ldots, i_5 \rangle$ with $i_1 + i_2 + \cdots + i_5 = 7$

= no. of non-neg. integer solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 7$ with $0 \le x_i \le 5-i$ for $i = 1, 2, \ldots, 5$.

= coeff. of x^7 in the expansion of

$$\underbrace{(1+x+x^2+x^3+x^4)}_{\text{value of } i_1}\underbrace{(1+x+x^2+x^3)}_{\text{value of } i_2}\underbrace{(1+x+x^2)}_{\text{val. of } i_3}(1+x)(1)$$

$(x^4 \cdot x^3 \cdot 1 \cdot 1 \cdot 1)$, $(x^4 \cdot x^2 \cdot x \cdot 1 \cdot 1)$, $(x^4 \cdot x \cdot x^2 \cdot 1 \cdot 1)$, $(x^4 \cdot x \cdot x \cdot x \cdot 1)$, $(x^4 \cdot 1 \cdot x^2 \cdot x \cdot 1)$
$(x^3 \cdot x^3 \cdot x \cdot 1 \cdot 1)$, $(x^3 \cdot x^3 \cdot 1 \cdot x \cdot 1)$, $(x^3 \cdot x^2 \cdot x^2 \cdot 1 \cdot 1)$, $(x^3 \cdot x^2 \cdot x \cdot x \cdot 1)$
$(x^3 \cdot x \cdot x^2 \cdot x \cdot 1)$, $(x^2 \cdot x^3 \cdot x^2 \cdot 1 \cdot 1)$, $(x^2 \cdot x^3 \cdot x \cdot x \cdot 1)$, $(x^2 \cdot x^2 \cdot x^2 \cdot x \cdot 1)$
$(x \cdot x^3 \cdot x^2 \cdot x \cdot 1)$. So the answer is 14.

Ex. 5 Find the number of ways to color the squares of a 1 by n chessboard with red, green, or blue if an even no. of squares must be colored red.

Sol. Answer = no. of n-perm. of $[\infty \cdot R, \infty \cdot G, \infty \cdot B]$ with R occurring an even no. of times

$$= \text{coeff. of } \frac{x^n}{n!} \text{ in } \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)$$

$$= \text{coeff. of } \frac{x^n}{n!} \text{ in exp. of } \frac{e^x+e^{-x}}{2} \cdot e^x \cdot e^x, \text{ i.e., } \frac{1}{2}\left(e^{3x}+e^x\right)$$

$$= \text{coeff. of } \frac{x^n}{n!} \text{ in } \frac{1}{2}\left(\sum_{n=0}^{\infty} 3^n \cdot \frac{x^n}{n!} + \sum_{n=0}^{\infty} 1 \cdot \frac{x^n}{n!}\right) = (3^n+1)/2.$$

Ex. 6 Find the number of ways to color the squares of a 1 by n chessboard with red, green, or blue if an even no. of squares must be colored red & at least one colored blue.

Sol. Answer = no. of n-perm. of $[\infty \cdot R, \infty \cdot G, \infty \cdot B]$ with R occurring an even no. of times & B at least once

$$= \text{coeff. of } \frac{x^n}{n!} \text{ in } \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)\left(x+\frac{x^2}{2!}+\cdots\right)$$

$$= \text{coeff. of } \frac{x^n}{n!} \text{ in exp. of } \frac{e^x+e^{-x}}{2} \cdot e^x \cdot (e^x-1)$$

$$= \text{coeff. of } \frac{x^n}{n!} \text{ in exp. of } \left(e^{3x}-e^{2x}+e^x-1\right)/2$$

$$= \text{coeff. of } \frac{x^n}{n!} \text{ in } \frac{1}{2}\left(\sum_{n=0}^{\infty} 3^n \cdot \frac{x^n}{n!} - \sum_{n=0}^{\infty} 2^n \frac{x^n}{n!} + \sum_{n=0}^{\infty} 1 \cdot \frac{x^n}{n!} - 1\right)$$

$$= \begin{cases} (3^n-2^n+1)/2, & \text{if } n \geq 1 \\ 0 & \text{if } n=0. \end{cases}$$

END.