Chapter - 4

Generating Permutations and Combinations

4.1 Generating permutations

Introduction: So far in chapter 2, we studied the number of n-permutations of the set containing 'n' elements. Next, question arises "Can we write down all such permutations". The question looks very devious but it could be very difficult to write down the all possible permutations.

How to generate a permutation

Let P_n = set of all permutations of a set with size n. Then if we know P_{n-1}, then we can generate all permutations of P_n. Similarly if we delete the nth object from P_n which will result P_{n-1}. In this manner if we know P_n, then we can delete the nth object from P_n only to get P_{n-1}.

This is summarized as follows.

Ex: Let $P_2 = \{12, 21\}$ where $S = \{1, 2, 3\}$

we can obtain P_3 by adding 3 to P_2

i.e. $P_3 = \{312, 132, 123, 321, 231, 213\}$

Now if P_3 is known to us, then we can delete '3' to get P_2.

So $P_2 = \{12, 21\}$

Ex: We consider one permutation of P_4 from the set $\{1, 2, 3, 4, 5\}$. Let it be 3412, then the corresponding permutations of P_5

out of 3412, can thus be written as

53412, 35412, 34512, 34152, 34125

If we delete 5 from the above permutation then we will get back 3412.

Generally, we can obtain a list of the $n!$ permutations of $\{1, 2, 3, ---, n\}$ by systematically inserting n to each permutation of $\{1, 2, 3 --, n-1\}$ in all possible cases.

Now we give an inductive description of such an algorithm. Let's consider all permutations of the set $\{1, 2, 3\}$.

Procedure $P_2 = \{12, 21\}$

write 12 thrice and 21 thrice and insert 3 as follows.

1 2 3
1 3 2
3 1 2
3 2 1
2 3 1
2 1 3

Similarly for $n = 4$, To generate the permutations of $\{1, 2, 3, 4\}$, write each of the permutations of 1, 2, 3 four times in the order generated above and interlace the 4 with them as follows.

1 2 3 4
1 2 4 3
1 4 2 3
4 1 2 3

4 1 3 2
1 4 3 2
1 3 4 2
1 3 2 4
3 1 2 4
3 1 4 2
3 4 1 2
4 3 1 2
4 3 2 1
3 4 2 1
3 2 4 1
3 2 1 4
2 3 1 4
2 3 4 1
2 4 3 1
4 2 3 1
4 2 1 3
2 4 1 3
2 1 4 3
2 1 3 4

Next we will write an algorithm to generate all permutations of $\{1, 2, \ldots, n\}$. Given an integer 'K', we assign a direction to it by writing an arrow above it pointing to the left or to the right $\overleftarrow{K}$ or $\overrightarrow{K}$.

Mobile

An integer K is called a mobile if its arrow points to a smaller integer adjacent to it.

Ex: $\vec{2}\,\vec{6}\,\vec{3}\,\overleftarrow{1}\,\vec{5}\,\vec{4}$

Here 6, 3, 5 are mobile.

* 1 can never be mobile since there is no integer in $\{1, 2, ---, n\}$ smaller than 1.

* The integer n is mobile, except in two cases:

(i) n is the first integer and its arrow points to the left as $\overleftarrow{n}$. ---.

(ii) n is the last integer and its arrow points to the right: . . . --- $\vec{n}$

Algorithm for generating the permutation of $\{1, 2, ---, n\}$.

Begin with $\overleftarrow{1}\,\overleftarrow{2}, ---, \overleftarrow{n}$

while there exists a mobile integer, follow the steps as given below

(1) Find the largest mobile integer m.

(2) Switch m and the adjacent integer to which its arrow points.

(3) Switch the direction of all the arrows above integers p with $p > m$.

Repeat steps 1, 2, 3 till you don't find a mobile integer.

Let us illustrate this algorithm for $n = 4$.

Consider the set $\{1, 2, 3, 4\}$

Let's start with $\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}\,\overleftarrow{4}$

$\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}\,\overleftarrow{4}$	$\overrightarrow{4}\,\overrightarrow{3}\,\overleftarrow{2}\,\overleftarrow{1}$
$\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{4}\,\overleftarrow{3}$	$\overrightarrow{3}\,\overrightarrow{4}\,\overleftarrow{2}\,\overleftarrow{1}$
$\overleftarrow{1}\,\overleftarrow{4}\,\overleftarrow{2}\,\overleftarrow{3}$	$\overrightarrow{3}\,\overleftarrow{2}\,\overrightarrow{4}\,\overleftarrow{1}$
$\overleftarrow{4}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}$	$\overrightarrow{3}\,\overleftarrow{2}\,\overleftarrow{1}\,\overrightarrow{4}$
$\overrightarrow{4}\,\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2}$	$\overleftarrow{2}\,\overrightarrow{3}\,\overleftarrow{1}\,\overleftarrow{4}$
$\overleftarrow{1}\,\overrightarrow{4}\,\overleftarrow{3}\,\overleftarrow{2}$	$\overleftarrow{2}\,\overrightarrow{3}\,\overleftarrow{4}\,\overleftarrow{1}$
$\overleftarrow{1}\,\overleftarrow{3}\,\overrightarrow{4}\,\overleftarrow{2}$	$\overleftarrow{2}\,\overleftarrow{4}\,\overrightarrow{3}\,\overleftarrow{1}$
$\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2}\,\overrightarrow{4}$	$\overleftarrow{4}\,\overleftarrow{2}\,\overrightarrow{3}\,\overleftarrow{1}$
$\overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{4}$	$\overrightarrow{4}\,\overleftarrow{2}\,\overleftarrow{1}\,\overrightarrow{3}$
$\overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{4}\,\overleftarrow{2}$	$\overleftarrow{2}\,\overrightarrow{4}\,\overleftarrow{1}\,\overrightarrow{3}$
$\overleftarrow{3}\,\overleftarrow{4}\,\overleftarrow{1}\,\overleftarrow{2}$	$\overleftarrow{2}\,\overleftarrow{1}\,\overrightarrow{4}\,\overrightarrow{3}$
$\overleftarrow{4}\,\overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{2}$	$\overleftarrow{2}\,\overleftarrow{1}\,\overrightarrow{3}\,\overrightarrow{4}$

Since no integer is mobile in $\overleftarrow{2}\,\overleftarrow{1}\,\overrightarrow{3}\,\overrightarrow{4}$, the algorithm stops. This algorithm generates all 24 permutations of $\{1,2,3,4\}$.

4.2 Inversions in Permutation

Let $i_1 i_2 \cdots i_n$ be a permutation of the set $\{1,2,\ldots,n\}$. The pair (i_k, i_l) is called an inversion if $k<l$ and $i_k > i_l$. For example, the permutation 31524 has four inversions namely $(3,1), (3,2), (5,2), (5,4)$. The only permutation of $\{1,2,\ldots,n\}$ with no inversions is $12\cdots n$. For a permutation $i_1 i_2 \cdots i_n$, we let a_j denote the number of inversions whose second component is j.

In other words, a_j equals the number of integers that precede j in the permutation but are greater than j.

The sequence of numbers $a_1, a_2, \ldots, a_n$ is called the inversion sequence of the permutation $i_1 i_2 \ldots i_n$. ~~Let a_j denote the number of inversions that~~

Ex:- The inversion sequence of the permutation 31524 is 1, 2, 0, 1, 0

The inversion sequence $a_1, a_2, \ldots, a_n$ of the permutation $i_1 i_2 \ldots i_n$ satisfies the conditions $0 \le a_1 \le n-1$, $0 \le a_2 \le n-2, \ldots$, $0 \le a_{n-1} \le 1$, $a_n = 0$

Thm 4.2.1

Let $b_1, b_2, \ldots, b_n$ be a sequence of integers satisfying $0 \le b_1 \le n-1$, $0 \le b_2 \le n-2, \ldots$, $0 \le b_{n-1} \le 1$, $b_n = 0$. Then there exists a unique permutation of $\{1, 2, \ldots, n\}$ whose inversion sequence is $b_1, b_2, \ldots, b_n$.

Algorithm - 1

Construction of a permutation from its inversion sequence

n: Write down n

$n-1$: Consider b_{n-1}. We are given that $0 \le b_{n-1} \le 1$. If $b_{n-1} = 0$, then $n-1$ must be placed before n. If $b_{n-1} = 1$ then $n-1$ must be placed after n.

$n-2$: Consider b_{n-2}. For $0 \le b_{n-2} \le 2$. If $b_{n-2} = 0$, then $n-2$ must be placed before the two numbers from step $n-1$. If $b_{n-2} = 1$, then $n-2$ must be placed between the two numbers from step $n-1$. If $b_{n-2} = 2$, then $n-2$ must be placed after the two numbers from step $n-1$.

⋮

$n-k$: Consider b_{n-k}. For $0 \le b_{n-k} \le k$. In steps n through $n-k+1$, the k numbers $n, n-1, \ldots, n-k+1$ have already been placed in the required order. If $b_{n-k} = 0$ then $n-k$ must be placed before all the numbers from step $n-k+1$. If $b_{n-k} = 1$, then $n-k$ must be placed between the first two numbers. ... If $b_{n-k} = k$, then $n-k$ must be placed after all the numbers.

⋮

1: We must place 1 after b_1st number in the sequence constructed in step 2.

Steps $n, n-1, n-2, \ldots, 1$, when carried out, determine the unique permutation of $\{1, 2, \ldots, n\}$ whose inversion sequence is $b_1, b_2, \ldots, b_n$.

Algorithm - II

Construction of a permutation from its inversion sequence

We begin with n empty locations, which we label $1, 2, \ldots, n$ from left to right.

1 : Since there are to be b_1 Integers that precede 1 in the permutation, we must put 1 in location number b_1+1.

2 : Since there are to be b_2 Integers that precede 2 and are larger than 2 in the permutation, and since these Integers have not yet been inserted, we must leave exactly b_2 empty locations for them. Thus counting from the left, we put 2 in the (b_2+1)st empty location.

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|
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n : We put n in the one remaining location.

Ex :- Determine the permutation of {1, 2, 3, 4, 5, 6, 7, 8} whose inversion sequence is 5, 3, 4, 0, 2, 1, 1, 0.

Algorithm – I

8 : 8

7 : 8 7

6 : 8 6 7

5 : 8 6 5 7

4 : 4 8 6 5 7

3 : 4 8 6 5 3 7

2 : 4 8 6 2 5 3 7

1 : 4 8 6 2 5 1 3 7

Hence the permutation is 48625137.

Algorithm-II

1 :						1		
2 :				2		1		
3 :				2		1	3	
4 :	4			2		1	3	
5 :	4			2	5	1	3	
6 :	4		6	2	5	1	3	
7 :	4		6	2	5	1	3	7
8 :	4	8	6	2	5	1	3	7
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)

Hence the required permutation is 48625137.

Ex :- Bring the permutation 361245 to 123456 by successive switches of adjacent.

Sol The inversion sequence of the above permutation is 2, 2, 0, 1, 1, 0. Sum $b_1+b_2+\cdots+b_6=6$.

3 6 1 2 4 5
3 1 6 2 4 5
1 3 6 2 4 5
1 3 2 6 4 5
1 2 3 6 4 5
1 2 3 4 6 5
1 2 3 4 5 6

4.6 <u>Exercise</u>

③ Generate first 50 permutations of $\{1,2,3,4,5\}$ starting with $\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}\,\overleftarrow{4}\,\overleftarrow{5}$.

Sol:

$\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}\,\overleftarrow{4}\,\overleftarrow{5}$

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(1) Which permutation of {1, 2, 3, 4, 5} follows 31524 in using algorithm described in section 4.1? Which permutation comes before 31524?

Soln From the solution of Q.(3), It is clear that 35124 follows 31524 and 31254 comes before 31524.

(10) Bring the permutations 256143 and 436251 to 123456 by successive switches of adjacent numbers.

Soln The inversion sequence of 256143 and 436251 are 3, 0, 3, 2, 0, 0 and 5, 3, 1, 0, 1, 0 respectively.

Now by applying successive switches

256143	436251
251643	436215
215643	436125
125643	431625
125634	413625
125364	143625
123564	143265
123546	142365
123456	124365
	123465
	123456