Ch 1 – Combinations & Permutations of Sets

§0. What is Combinatorics?

Combinatorics is the study of the properties of Discrete Structures. These structures are usually finite – but they can also be infinite. So in a certain sense Combinatorics includes Graph Theory, Number Theory, Optimization Theory, & even Abstract Algebra. In this course, MAD 4203 we will avoid, as far as possible, topics in these other fields – for the simple reason that they will be covered in those courses. There are 5 basic kinds of problems that are usually asked in Combinatorics.

1. Existence Problems: In these kinds of problems one is being asked if an object, with a prescribed property, exists.

Ex.1 Is there a positive integer n such that the sum of all the divisors of n equals (i) n, (ii) $2n$, (iii) $3n$?

(i) It is easy to see that $n=1$ is the only number with the sum of its divisors equal to n, because every integer $k>1$ has at least two divisors, 1 & k, and these two add up to more than k already.

(ii) 28 is an example of a positive integer n

with the sum of its divisors equal to $2n$ because the divisors of 28 are 1, 2, 4, 7, 14 & 28 and $1+2+4+7+14+28 = 56 = 2(28)$.

(iii') We will leave the problem of determining whether or not an integer n, with the sum of its divisors equal to $3n$, exists as a question for our kind reader to investigate.

2. Construction problems: In these kinds of problems, one is being asked to construct 1 or more objects with a prescribed property in a systematic way. Naturally these kinds of problems are related to Existence problems - because once we construct one object we will have shown that such kinds of objects exists. Sometimes we also have a proof by contradiction that such objects exist but we need to find an actual object. A construction will produce one or more such objects.

Ex.2 Find a systematic way of finding positive integers n with the sum of the divisors of n equal to $2n$. Such integers are called perfect

Sol. We can easily verify that if 2^p-1 is prime, then $2^{p-1}(2^p-1)$ is perfect because

$$T + 2^0\cdot(2^p-1) + 2^1\cdot(2^p-1) + \cdots + 2^{p-1}(2^p-1) = 2^{p-1}(2^p-1)\cdot 2$$

So we have a systematic way of finding

Here $T = 2^0 + 2^1 + 2^2 + \cdots + 2^{p-1} = 2^p - 1$.

lots of perfect numbers. Let's give some.

$2^2-1=3$ is prime, so $2^{2-1}(2^2-1)=6$ is perfect.

$2^3-1=7$ is prime, so $2^{3-1}(2^3-1)=28$ is perfect.

$2^5-1=31$ is prime, so $2^{5-1}(2^5-1)=496$ is perfect.

$2^7-1=127$ is prime, so $2^{7-1}(2^7-1)=64(127)$ is perfect.

$2^{11}-1$ is not prime, so we can't say that $2^{11-1}(2^{11}-1)$ is perfect. By the way we only looked at 2^p-1 when p is prime because if k is not prime, then 2^k-1 is not prime.

3. Optimization problems In these kinds of problems one is asked to find all "best" (in a certain sense) objects which have a prescribed property. Usually there is only one "best" object with a given property but sometimes there can be more than one "best" object.

Ex. 3 Find the best approximation to π which is of the form p/q with $p,q\in\mathbb{Z}^+$ and p & q having at most 3 digits.

Sol. We know that there are several approximations to π.

$$\frac{31}{10}=3.1,\quad \frac{63}{20}=3.15,\quad \frac{157}{50}=3.14,\ \dots,\ \frac{355}{113}=3.1415929$$

Now $\pi=3.14159265\ldots$ and from the Theory of continued fractions, we can say that $355/113$ will be the best approximation. ($355/113$ agrees with π to 6 decimal places.)

4. <u>Counting Problems</u>: In these kinds of problems, one is being asked to count the number of objects with a particular property

<u>Ex. 4</u> How many numbers in the set $\{1, 2, 3, \ldots, n\}$ are divisible by 6?

<u>Sol.</u> The answer is $\lfloor n/6 \rfloor$ because the subset of all the numbers in $\{1, 2, 3, \ldots, n\}$ that are divisible by 6 is $\{6(1), 6(2), 6(3), \ldots, 6(k)\}$ where $k = \lfloor n/6 \rfloor$. Here $\lfloor n/6 \rfloor$ = largest integer $\leq n/6$.

5. <u>Listing (or Enumeration) Problems</u>. In these kinds of problems, one is being ask to list (or enumerate) all the objects with a particular property. Naturally listing problems are closely related to counting problems — because if we can list all the objects with a given property, then we can easily count the number of such objects.

<u>Ex. 5</u> List all the numbers in the set $\{1, 2, 3, \ldots, n\}$ that are divisible by 12 and are also perfect squares.

<u>Sol.</u> The set of all such numbers is

$$\{[6(1)]^2, [6(2)]^2, [6(3)]^2, \ldots, [6(k)]^2\}$$

where $k = \lfloor \sqrt{n}/6 \rfloor = \lfloor \sqrt{n/36} \rfloor$.

<u>Note:</u> Most of the problems in Combinatorics are of type 4 or type 5 — so Combinatorics is called the "Art of Counting."

§1. Basic counting techniques:

There are three basic principles which we use in counting. The first one is the Equivalence principle.

Notation: If A is a finite set, then we shall use the notation $|A|$ to denote the size of A (no. of elements of A). So "$|\cdot|$" is a function from the finite sets to $\mathbb{N} = \{0,1,2,\ldots\}$. Note that $|\emptyset| = 0$. Recall also that a function $f: A \to B$ is a bijection if f is an injection (i.e., one-to-one) and if f is also a surjection (i.e., onto).

The Equivalence Principle: If we can find a bijection from A to B, then $|A| = |B|$. In particular if we can find a bijection from A to $\{1,2,3,\ldots,n\}$ then $|A| = n$, because by definition $|\{1,2,3,\ldots,n\}| = n$.

Ex. 1 How many elements of $U = \{1,2,3,\ldots,1000\}$ are

(a) divisible by 12?

(b) perfect squares?

(c) divisible by both 10 and 12?

(d) perfect squares which are divisible by 12?

Sol. Let $A = \{x \in U : x \text{ is divisible by } 12\}$

$B = \{x \in U : x \text{ is a perfect square}\}$

and $C = \{x \in U : x \text{ is divisible by } 10\}$.

(a) Then $A = \{12(1), 12(2), 12(3), \ldots, 12(\lfloor \frac{1000}{12} \rfloor)\}$
So number of elements of $\mathcal{U}$ that are divisible by 12 $= |A| = \lfloor \frac{1000}{12} \rfloor = 83$ by the equivalence principle because $f: A \to \{1,2,3,\ldots,83\}$, $f(k) = k/12$ is a bijection.

(b) Also $B = \{(1)^2, (2)^2, (3)^2, \ldots, (\lfloor \sqrt{1000} \rfloor)^2\}$.
So no. of elements of $\mathcal{U}$ that are perfect squares $= |B| = \lfloor \sqrt{1000} \rfloor = 31$ by the equivalence principle bec. $g: B \to \{1,2,3,\ldots,31\}$, $f(k) = \sqrt{k}$ is a bijection.

(c) No. of elements of $\mathcal{U}$ that are divisible by both 10 & 12 $= |A \cap C|$. Now $A \cap C = \{x \in \mathcal{U} : x \text{ is divisible by the l.c.m.} (12,10)\} = \{x \in \mathcal{U} : x \text{ is divisible by } 60\}$
$= \{60(1), 60(2), \ldots, 60(\lfloor \frac{1000}{60} \rfloor)\}$
So our answer will be
$|A \cap C| = \lfloor \frac{1000}{60} \rfloor = \lfloor \frac{100}{6} \rfloor = 16$.

(d) Our answer will be $|A \cap B|$. Now
$A \cap B = \{x \in \mathcal{U} : x \text{ is divisible by 12 \& } x \text{ is a perfect sq.}\}$
$= \{x \in \mathcal{U} : x \text{ is a perfect sq. \& } x = (2^2 \cdot 3) \cdot k \text{ with } k \in \mathbb{Z}^+\}$
$= \{x \in \mathcal{U} : x = 2^2 \cdot 3^2 \cdot l^2 \text{ for some } l \in \mathbb{Z}^+\}$
$= \{(6l)^2 : (6l)^2 \leq 1000 \text{ \& } l \in \mathbb{Z}^+\}$
$= \{[6(1)]^2, [6(2)]^2, [6(3)]^2, \ldots, (6\lfloor \sqrt{\frac{1000}{36}} \rfloor)^2\}$
So answer $= |A \cap B| = \lfloor \sqrt{\frac{1000}{36}} \rfloor = \lfloor \frac{\sqrt{1000}}{6} \rfloor = 5$.

Next, we have our second fundamental counting principle, the Addition Principle.

The Addition Principle: If a set A can be partitioned into k disjoint non-empty subsets $A_1, \dots, A_k$ then $|A| = |A_1| + |A_2| + \cdots + |A_k|$. In particular, if all the A_i's are all of the same size, then

$$|A| = k \cdot |A_1|.$$

Ex.2 How many elements of $\mathcal{U} = \{1, 2, 3, \dots, 1000\}$ are

(a) not divisible by 12?

(b) divisible by 12 or 10?

(c) divisible by 12 or are perfect squares?

(d) divisible by neither 12 nor 10?

Sol. (a) Our answer is $|A^c|$. Now $\mathcal{U} = A \cup (A^c)$ is a partition of $\mathcal{U}$ into disjoint subsets. So $|\mathcal{U}| = |A| + |A^c|$. $\therefore |A^c| = |\mathcal{U}| - |A| = 1000 - 83 = 917$. So our answer will be 917.

(b) Our answer is $|A \cup C|$. Now

$$A \cup C = (A - A\cap C) \cup (A \cap C) \cup (C - (A \cap C)).$$

$$\therefore |A \cup C| = [|A| - |A \cap C|] + |A \cap C| + [|C| - |A \cap C|]$$

$$= |A| + |C| - |A \cap C|$$

$$= \left\lfloor \frac{1000}{12} \right\rfloor + \left\lfloor \frac{1000}{10} \right\rfloor - \left\lfloor \frac{1000}{60} \right\rfloor = 83 + 100 - 16 = 167.$$

(c) Our answer will be $|A \cup B|$. And as in part (b)

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= \left\lfloor \frac{1000}{12} \right\rfloor + \left\lfloor \sqrt{1000} \right\rfloor - \left\lfloor \frac{\sqrt{1000}}{6} \right\rfloor = 83 + 31 - 5 = 109.$$

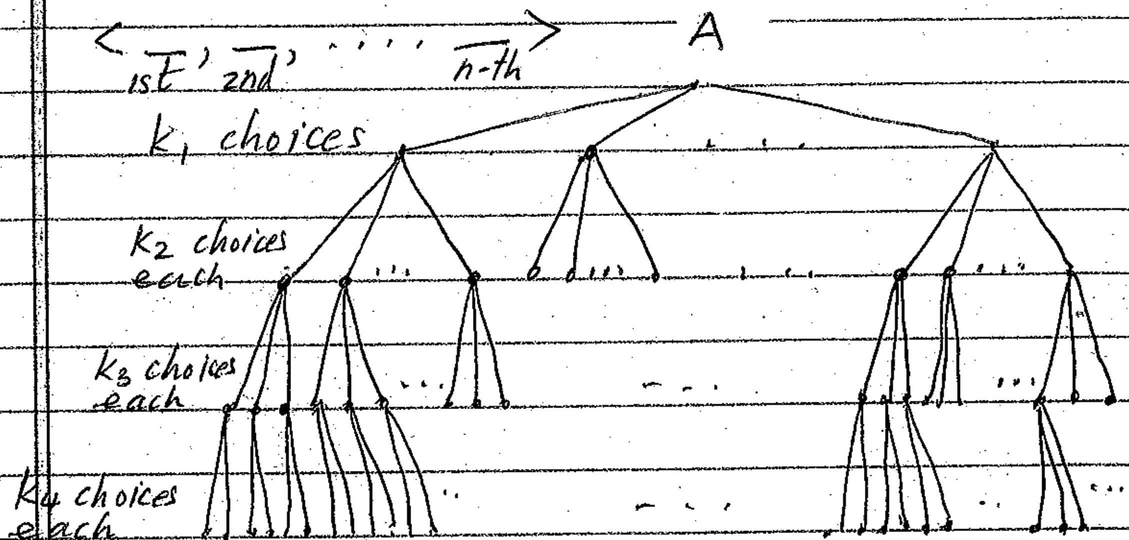
(d) Our answer will be $|A^c \cap C^c| = |(A \cup C)^c| = |\mathcal{U}| - |A \cup C| = 833$

or $|\mathcal{U}| - |A| - |C| + |A \cap C| = 1000 - 83 - 100 + 16 = 1000 - 167 = 833.$

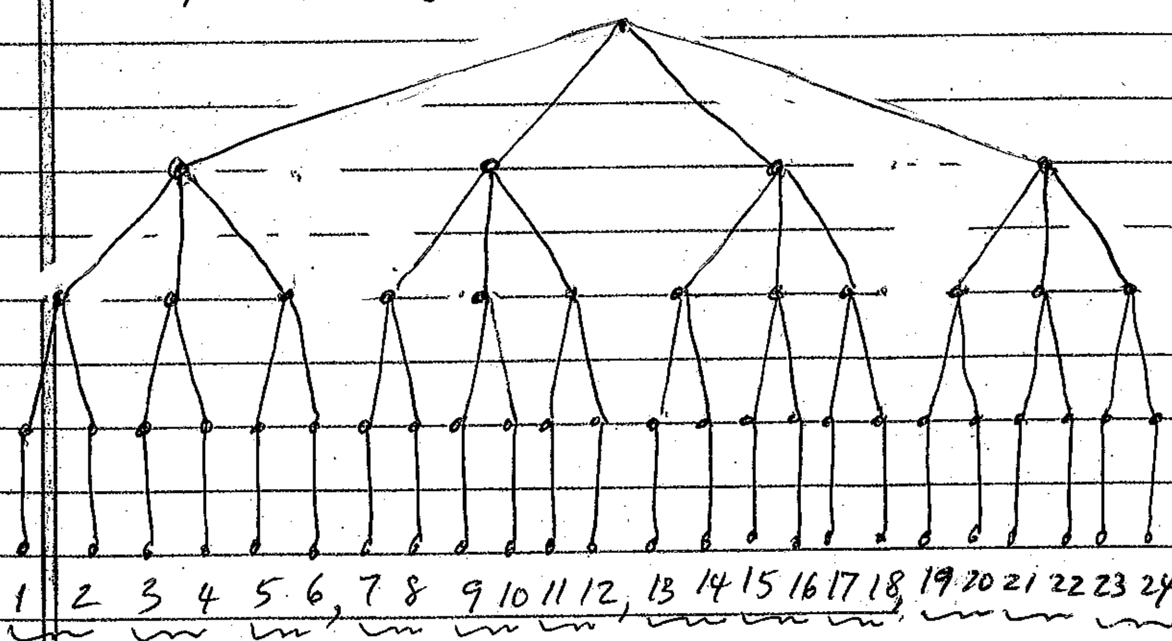
The Multiplication Principle

If A is a set of n-tuples and there are k_i ways of choosing the i-th component of the elements of A (for $i = 1, \dots, n$), then

$$|A| = k_1 \cdot k_2 \cdot k_3 \cdot \dots \cdot k_n.$$



Ex. 3 Let us illustrate how big the set A will be when $n = 4$ and $k_1 = 4$, $k_2 = 3$, $k_2 = 2$, and $k_4 = 1$.
As expected, we get $4(3)(2)(1) = 24$ elements in A



Ex 4 How many 3-digit (base 10) numerals
(a) have only even digits?
(b) begins with an even digit & ends with an odd digit?
(c) have all their digits distinct (different)?
(d) are even & have all their digits distinct?

Sol. (a) $\langle \underline{4}, \underline{5}, \underline{5} \rangle$

cannot be 0 — can be 0,2,4,6,8 — can be 0,2,4,6 or 8

So our answer is $4(5)(5) = 100$.

(b) $\langle \underline{4}, \underline{10}, \underline{5} \rangle$

can be 2,4,6, or 8 — can be any of the 10 — can be 1,3,5,7,9.

So our answer is $4(10)(5) = 200$.

(c) $\langle \underline{9}, \underline{9}, \underline{8} \rangle$

cannot be 0 — cannot be first digit — cannot be 1st or 2nd digit.

So our answer is $9(9)(8) = 81(8) = 648$.

(d) First observe that a numeral is even if and only if it ends in an even digit. Now split the problem into two cases.

Case (i): numeral ends in 0: $\langle \underline{9}, \underline{8}, \underline{1} \rangle$

cannot be 0 — cannot be 0 or the 1st digit. — only 0

Case (ii) numeral ends in 2,4,6 or 8: $\langle \underline{8}, \underline{8}, \underline{4} \rangle$

choose this 2nd; not 0 & not 3rd component. — choose this last – anything except 3rd & 1st comp. — choose this first

So our answer will be:

$9(8)(1) + 8(8)(4) = 72 + 256 = 328$.

§2 Permutations & Combinations of sets.

Def. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of n elements and r be an integer with $0 \le r \le n$. An <u>r-permutation</u> of A is an r-tuple of r distinct elements of A. When $r = n$, we usually call an n-permutation of A just a <u>permutation</u> of A

Note: Since an r-tuple of elements of A is just a function from $\{1,2,3,\ldots,r\}$ to A, an r-tuple of distinct elements of A would be an injective function from $\{1,2,3,\ldots,r\}$ to A. So an r-permutation is just an injection from $\{1,2,\ldots,r\}$ to A and we can write it as: $\begin{pmatrix} 1 & 2 & 3 & \cdots & r \\ a_{i_1} & a_{i_2} & a_{i_3} & \cdots & a_{i_r} \end{pmatrix}$ where the a_{i_k}'s are distinct.

Def. An <u>r-combination</u> of A is a subset of A containing r elements. We sometimes call an r-combination of A an <u>r-subset</u> of A

Ex. 1 (a) Let $A = \{a,b,c\}$. Then the set of all 2-permutations of A is

$$\{\langle a,b\rangle, \langle a,c\rangle, \langle b,a\rangle, \langle b,c\rangle, \langle c,a\rangle, \langle c,b\rangle\}.$$

We usually abbreviate $\langle a,b\rangle$ by ab and so on. So this set is $\{ab, ac, ba, bc, ca, cb\}$.

(b) The set of all permutations of A is

$\{\langle a,b,c\rangle, \langle a,c,b\rangle, \langle b,a,c\rangle, \langle b,c,a\rangle, \langle c,a,b\rangle, \langle c,b,a\rangle\}$

$= \{abc, acb, bac, bca, cab, cba\}$.

Ex.1 (c) The set of all 2-combinations of A is $\{\{a,b\},\{a,c\},\{b,c\}\}$ and the set of all 3-combinations of A is $\{\{a,b,c\}\}$.

Ex.2 The set of all 0-permutations of A is $\{\langle\rangle\}$ where $\langle\rangle$ is the empty sequence. The set of all 0-combinations of A is $\{\emptyset\}$.

Prop. 1 Let $P(n,r)$ be the number of r-permutations of $\{1,2,3,\ldots,n\}$. Then $P(n,r) = \frac{n!}{(n-r)!}$

Proof: An r-permutation of $\{1,2,\ldots,n\}$ is an r-tuple of r distinct elements of $\{1,2,\ldots,n\}$. Now there are n ways of choosing the first component, $(n-1)$ ways of choosing the second component, $(n-2)$ ways of choosing the 3rd component

$\vdots$

$(n-(r-1))$ ways of choosing the r-th component.

So by the multiplication principle,

$$\langle \underset{n \text{ choices}}{\underline{1st}}, \underset{(n-1)\text{ choices}}{\underline{2nd}}, \underset{(n-2)\text{ ch}}{\underline{3rd}}, \ldots, \underset{(n-(r-1))\text{ ch.}}{\underline{r\text{-th comp}}} \rangle$$

$$\begin{aligned} P(n,r) &= n(n-1)(n-2)\cdots[n-(r-1)] \\ &= n(n-1)(n-2)\cdots[n-(r-1)].(n-r)!\,/\,(n-r)! \\ &= n!\,/\,(n-r)! \end{aligned}$$

Prop. 2 Let $C(n,r)$ be the number of r-combinations of $\{1,2,3,\ldots,n\}$. Then $C(n,r) = \frac{n!}{r!(n-r)!}$

Proof: Consider an r-combination of $\{1,2,3,\ldots,n\}$.

This r-combination can be ordered in $P(r,r) = r!$ ways to produce $r!$ r-permutations of $\{1,2,3,\ldots,n\}$. Since each r-permutation of $\{1,2,3,\ldots,n\}$ can be obtained by ordering a unique r-combination, it follows that

$$P(n,r) = (r!) \cdot C(n,r).$$

Thus $C(n,r) = \dfrac{P(n,r)}{r!} = \dfrac{n!}{r!(n-r)!} = \dbinom{n}{r}$

<u>Notation</u>: We shall use the expression $\binom{n}{r}$ to denote $\dfrac{n!}{r!(n-r)!}$ for any $n \in \mathbb{N}$ and r with $0 \le r \le n$.

When $r > n$, we will take $\binom{n}{r}$ to be 0. The expression $\binom{n}{r}$ is pronounced as "n choose r".

<u>Ex.3</u> How many 3-subsets of $\{1,2,3,\ldots,20\}$ have
(a) exactly one odd element?
(b) at most one odd element?
(c) at least one odd element?

<u>Sol.</u> (a) We need one odd element of $\{1,2,\ldots,20\}$ — and there are $\binom{10}{1}$ ways of choosing this odd element. We also need 2 even elements of $\{1,2,\ldots,20\}$ — and there are $\binom{10}{2}$ ways of choosing these 2 even elements. So our answer is $\binom{10}{1} \cdot \binom{10}{2} = 10 \cdot \frac{10 \cdot 9}{2 \cdot 1} = 10(45) = 450$.

(b) Answer = no. of 3-subsets with 0 odd elements + no. of 3-subsets with 1 odd element

$$= \binom{10}{0} \cdot \binom{10}{3} + \binom{10}{1} \cdot \binom{10}{2} = 1 \cdot \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} + 450 = 120 + 450 = 570.$$

Ex. 3(c) Answer = No. of 3-subsets with 1 odd element
+ No. of 3-subsets with 2 odd elements
+ No. of 3-subsets with 3 odd elements

$$= \binom{10}{1}\cdot\binom{10}{2} + \binom{10}{2}\cdot\binom{10}{1} + \binom{10}{3}\cdot\binom{10}{0} = 450 + 450 + 120 = 1020.$$

Alt. answer = No. of 3-subsets of $\{1,2,3,\ldots,20\}$
− No. of 3-subsets with 0 odd elements

$$= \binom{20}{3} - \binom{10}{0}\cdot\binom{10}{3} = \frac{20.19.18}{3.2.1} - 120 = 1140 - 120 = 1020.$$

Ex. 4. How many 3-permutations of $\{1,2,\ldots,20\}$ have
(a) exactly one odd term?
(b) at most one odd term?
(c) at least one odd term?

Sol. Since a 3-subset of $\{1,2,\ldots,20\}$ will produce $3! = 6$ 3-permutations of $\{1,2,3,\ldots,20\}$ our answers will just be 6 times the corresponding answers in Ex. 3. So our answers are:
(a) $6(450) = 2700$, (b) $6(570) = 3420$, (c) $6(1020) = 6120$.

Ex. 5 In how many ways can place 8 rooks on an 8×8 chess-board so that no two rooks attach each other (i.e., so that no two rooks are in the same row or same column)

Sol. Let $\langle i, c_i \rangle$ be the position of the single rook in row i. Here c_i denotes the column in which the rook is placed in row i. Now an arrangement of 8 rooks on the chessboard with no two rooks attacking each other will be an ~~8-tuple~~: injective function $f:\{1,2,\ldots,8\} \to \{1,2,\ldots,8\}$

$$f = \{\langle 1, c_1\rangle, \langle 2, c_2\rangle, \langle 3, c_3\rangle, \ldots, \langle 8, c_8\rangle\}$$

where $c_1, c_2, c_3, \ldots, c_8$ are all distinct. So $\langle c_1, c_2, \ldots, c_8\rangle$ will be a permutation of $\{1,2,3,\ldots,8\}$. Since there are $8!$ permutations of $\{1,2,\ldots,8\}$, there will be $8!$ ways of arranging the 8 rooks in mutually non-attacking positions.

Def. A subsequence of the sequence $\langle a_1, a_2, \ldots, a_n\rangle$ is any sequence of the form $\langle a_{i_1}, a_{i_2}, \ldots, a_{i_k}\rangle$ where $1 \leq i_1 < i_2 < i_3 < \cdots < i_k \leq n$.

Ex. 6 Find all the subsequences of $\langle a_1, a_2, a_3, a_4\rangle$ of lengths 0, 1, and 2 respectively

Sol. (a) There is only one subsequence of length 0, namely $\langle\,\rangle$.

(b) There are 4 subsequences of length 1, which corresponds to $i_1=1$, $i_1=2$, $i_1=3$ & $i_1=4$. These are $\langle a_1\rangle$, $\langle a_2\rangle$, $\langle a_3\rangle$, and $\langle a_4\rangle$

(c) There are 6 subsequences of length 2 and these correspond to:

$i_1=1$ & $i_2=2$ $\quad$ $i_1=1$ & $i_2=3$ $\quad$ $i_1=1$ & $i_2=4$

$\langle a_1, a_2\rangle$ $\quad$ $\langle a_1, a_3\rangle$ $\quad$ $\langle a_1, a_4\rangle$

$i_1=2$ & $i_2=3$ $\quad$ $i_1=2$ & $i_2=4$ $\quad$ $i_1=3$ & $i_2=4$

$\langle a_2, a_3\rangle$ $\quad$ $\langle a_2, a_4\rangle$ $\quad$ $\langle a_3, a_4\rangle$.

Prop 3 $\langle a_1, a_2, \ldots, a_n \rangle$ has $\binom{n}{k}$ subsequences of length k.

Proof Each subsequence $\langle a_{i_1}, a_{i_2}, \ldots, a_{i_k} \rangle$ of length k of $\langle a_1, a_2, \ldots, a_n \rangle$ corresponds to a k-subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$ with the $i_1, \ldots, i_k$ listed in increasing order. Since there are $\binom{n}{k}$ k-subsets of $\{1, 2, \ldots, n\}$, it follows that there are $\binom{n}{k}$ subsequences of $\langle a_1, \ldots, a_n \rangle$ of length k.

Prop 4 The total number of subsets of $\{1, 2, \ldots, n\}$ is 2^n. Consequently there are 2^n possible subsequences of $\langle a_1, a_2, \ldots, a_n \rangle$.

Proof (a) A subset of $\{1, 2, 3, \ldots, n\}$ is determined by n choices we make: Is $i \in$ the subset for $i = 1, 2, \ldots, n$.

$$\langle \underset{\text{choices for 1}}{\underline{2}}, \underset{\text{choices for 2}}{\underline{2}}, \underline{2}, \ldots, \underset{\text{choices for } n}{\underline{2}} \rangle$$

Now there are 2 choices for 1 (either it is in the subset or it is out), 2 choices for 2, ..., and 2 choices for n. So by the multiplication principle, it follows that there are 2^n possible subsets.

(b) We know that a subsequence of $\langle a_1, \ldots, a_n \rangle$ corresponds to a subset of $\{1, 2, \ldots, n\}$. Since there are 2^n possible subsets of $\{1, 2, \ldots, n\}$ There will be 2^n possible subsequences of $\langle a_1, \ldots, a_n \rangle$.

§3. Inversion sequence of a permutation:

Ex 1. Consider the permutation $\sigma = \langle 3,2,5,1,4\rangle$ of $\{1,2,3,4,5\}$. The ordered pair $(3,1)$ is called an inversion of σ because in their natural order 1 precedes 3 — but in σ, 3 precedes 1. Similarly $(2,1)$, $(5,1)$, $(3,2)$, & $(5,4)$ are inversions of σ. $(3,4)$ is not an inversion in σ.

Def. Let $\sigma = \langle a_1, a_2, \ldots, a_n\rangle$ be a permutation of $\{1,2,\ldots,n\}$. An inversion of σ is any ordered pair (a_i, a_j) with $i<j$ & $a_i > a_j$

The number of inversions of σ with respect to the integer k $(1 \le k \le n)$ is defined by
$i_k(\sigma)$ = number of elements which precede k in σ and are bigger than k.

The inversion sequence of the permutation σ is the sequence $\langle i_1(\sigma), i_2(\sigma), \ldots, i_n(\sigma)\rangle$.

Ex. 2 Let $\sigma = \langle 3,2,5,1,4\rangle$. Then
$i_1(\sigma) = 3$, $i_2(\sigma) = 1$, $i_3(\sigma) = 0$, $i_4(\sigma) = 1$
and $i_5(\sigma) = 0$. So the inversion sequence of σ is $\langle 3,1,0,1,0\rangle$.

Note: Since there are only $n-k$ elements of $\{1,2,\ldots,n\}$ that are greater than k, it follows that
$0 \le i_k(\sigma) \le n-k$ for any permutation σ of $\{1,\ldots,n\}$.

Qu: Suppose $\langle a_1, a_2, \dots, a_n\rangle$ is a sequence of integers with $0 \le a_k \le n-k$ for $k=1,\dots,n$ is it always true that $\langle a_1,\dots,a_n\rangle$ is the inversion sequence of some permutation σ of $\{1,2,3,\dots,n\}$?

Ans: Yes.

Theorem 4: Let $\langle a_1,\dots,a_n\rangle$ be a sequence of integers with $0 \le a_k \le n-k$ for $k=1,2,\dots,n$. Then there is a unique permutation σ of $\{1,2,\dots,n\}$ such that $\langle i_1(\sigma),\dots,i_n(\sigma)\rangle = \langle a_1,\dots,a_n\rangle$.

Proof: We shall describe an algorithm for finding the unique permutation σ of $\{1,2,\dots,n\}$ such that $\langle i_1(\sigma),\dots,i_n(\sigma)\rangle = \langle a_1,\dots,a_n\rangle$.

Step 1: First write down n to get a sequence $\langle n\rangle$

Step 2: Then insert $n-1$ in this sequence so that there are a_{n-1} bigger terms in front of $n-1$

Step 3: Then insert $n-2$ in the sequence obtained from step 2, so that there are a_{n-2} bigger terms in front of $n-2$.

⋮

Step $n-k$: In general insert $k-1$ in the sequence so that there are a_k bigger terms in front of $k-1$.

Step n: In the last step we will insert 1 in the sequence so that there are a_1 bigger terms in front of 1.

If we proceed as in this algorithm we will get a permutation σ of $\{1,2,3,\ldots,n\}$ with $\langle i_1(\sigma), i_2(\sigma), \ldots, i_n(\sigma)\rangle = \langle a_1, a_2, \ldots, a_n\rangle$.

Ex. 3 Let $\langle a_1, a_2, a_3, a_4, a_5\rangle = \langle 3,1,0,1,0\rangle$. Then for each k, $0 \le a_k \le n-k$. Find the permutation σ of $\{1,2,\ldots,5\}$ which has inversion sequence $\langle 3,1,0,1,0\rangle$.

Sol. 1. Write down $\langle 5\rangle$

2. Insert 4, so that there are $a_4=1$ bigger terms in front of 4 to get $\langle 5,4\rangle$
3. Insert 3, so that there are $a_3=0$ bigger terms in front of 3 to get $\langle 3,5,4\rangle$
4. Insert 2, so that there are $a_2=1$ bigger terms in front of 2 to get $\langle 3,2,5,4\rangle$
5. Insert 1, so that there are $a_1=3$ bigger terms in front of 1 to get $\langle 3,2,5,1,4\rangle$.

So $\sigma = \langle 3,2,5,1,4\rangle$

Def. Let σ be a permutation of $\{1,2,3,\ldots,n\}$. The total no. of inversions in σ is defined by
$$I_T(\sigma) = i_1(\sigma) + i_2(\sigma) + \cdots + i_{n-1}(\sigma) + i_n(\sigma).$$

Ex. 4 Find the number of permutations of $\{1,2,3,\ldots,7\}$ in which the total no. of inversions is:

(a) 21, (b) 20, (c) 19, (d) 18.

Ex.4 (a) The maximum no. of inversions will come from the permutation of $\{1,2,\ldots,n\}$ with inversion sequence $\langle 6,5,4,3,2,1,0\rangle$. The permutation which corresponds to this inversion sequence is $\langle 7,6,5,4,3,2,1\rangle$. So the no. of permutations with 21 inversions is 1.

(b) To get a total no. of inversions of 20, we need to reduce exactly one of the first 6 terms of $\langle 6,5,4,3,2,1,0\rangle$ by 1. Since there are $\binom{6}{1}$ ways of doing this there will be $\binom{6}{1} = 6$ permutations of $\{1,2,\ldots,7\}$ with 20 inversions.

(c) To get a total no. of inversions of 19, we need to reduce exactly 2 of the first 6 terms of $\langle 6,5,4,3,2,1,0\rangle$ by 1 or to reduce exactly 1 of the first 5 terms of $\langle 6,5,4,3,2,1,0\rangle$ by 2. Since there are $\binom{6}{2}$ ways of choosing two of the first 6 terms and $\binom{5}{1}$ ways of choosing one of the first 5 terms, there will be $\binom{6}{2} + \binom{5}{1} = 15+5=20$ permutations of $\{1,2,\ldots,7\}$ with 19 inversions.

(d) Reduce 3 of the first 6 terms by 1; reduce 1 of the first 4 terms by 3; or reduce one of the first 5 terms by 2 & one of the other 5 (of 6 terms) by 1.
So answer $= \binom{6}{3} + \binom{4}{1} + \binom{5}{1}\cdot\binom{5}{1} = 20+4+25 = 49.$