Binomial coefficients and Pascal's Triangle

The binomial coefficients $\binom{n}{k}$ for all nonnegative Integers k and n is defined as

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots\cdots(n-k+1)}{k(k-1)\cdots\cdots 1}$$

* $\binom{n}{k} = \binom{n}{n-k}$
* $\binom{n}{k} = 0$ If $k > n$ and $\binom{n}{0} = 1 \ \forall n.$
* $\binom{n}{n} = 1$

Pascal's Triangle

$(x+y)^0 = 1$

$(x+y)^1 = x+y$

$(x+y)^2 = x^2+2xy+y^2$

$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$

and so on.

n\k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

From the Pascal's triangle, we have noticed that the sum of the binomial coefficients is 2^n.

i.e. $\boxed{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+ \cdots + \binom{n}{n} = 2^n}$

Binomial Theorem

Let n be a positive integer. Then $\forall\, x, y$

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}x^0y^n$$

In summation notation

$$(x+y)^n = \sum_{K=0}^{n}\binom{n}{K}x^{n-K}y^K$$

The binomial thm can be written in several other equivalent forms

$$(x+y)^n = \sum_{K=0}^{n}\binom{n}{n-K}x^{n-K}y^K = \sum_{K=0}^{n}\binom{n}{n-K}x^K y^{n-K}$$

$$= \sum_{K=0}^{n}\binom{n}{K}x^K y^{n-K}$$

Ex: $(x+y)^2 = \binom{2}{0}x^2y^0 + \binom{2}{1}xy + \binom{2}{2}x^0y^2 = x^2+2xy+y^2$

$$(x+y)^3 = \binom{3}{0}x^3y^0 + \binom{3}{1}x^2y^1 + \binom{3}{2}x^1y^2 + \binom{3}{3}x^0y^3$$
$$= x^3+3x^2y+3xy^2+y^3$$

$$(x+y)^4 = \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}x^0y^4$$
$$= x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

Th^m – 5.2.2

Let n be a +ve integer. Then $\forall x$

$$(1+x)^n = \sum_{K=0}^{n} \binom{n}{K} x^K = \sum_{K=0}^{n} \binom{n}{n-K} x^K$$

$$= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \cdots + \binom{n}{n} x^n$$

Ex:- Prove that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$ $(n \geq 0)$

Sol^n – We know that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \cdots + \binom{n}{n} x^n \quad -(1)$$

put $x = 1$ in eq^n (1), We have

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n \quad \square$$

Ex:- Prove that $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.$

Sol^n Put $x = -1$ in eq^n(1) in the previous example

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0 \quad \square$$

Ex:- Prove that $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$

$$= \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}.$$

Sol^n We know that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0$$

$$\Rightarrow \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots \quad -\text{eq}^n(2)$$

Again we know

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+ \cdots\cdots+\binom{n}{n}=2^n \quad \text{— eq}^n(3)$$

using eq^n(2) and eq^n(3), we have

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+ \cdots\cdots = 2^{n-1}$$

$$\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+ \cdots\cdots = 2^{n-1} \quad \square$$

Ex: prove that

$$1\binom{n}{1}+2\binom{n}{2}+ \cdots\cdots + n\binom{n}{n} = n2^{n-1}, \; (n \geq 1).$$

Sol^n We know $(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+ \cdots+\binom{n}{n}x^n$ — eq^n(4)

Differentiating eq^n(4) w.r.t x at $x=1$ we have

$$n2^{n-1}=\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+ \cdots + n\binom{n}{n}$$

$$\Rightarrow 1\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+ \cdots\cdots + n\binom{n}{n} = n2^{n-1} \quad \square$$

Multinomial Th^m (Th^m - 5.4.1)

Let n be a positive integer. For all $x_1, x_2, \cdots, x_t$,

$$(x_1+x_2+ \cdots + x_t)^n = \sum \binom{n}{n_1\, n_2 \cdots n_t} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$$

where the summation extends over all non-negative integral solutions $n_1, n_2, \cdots n_t$ of $n_1+n_2+n_3+ \cdots + n_t = n$.

Ex: When $(x_1+x_2+x_3+x_4+x_5)^7$ is expanded then find the coefficient of $x_1^2 x_3 x_4^3 x_5$

Sol^n From multinomial th^m, the required coefficient is $\binom{7}{2\ 0\ 1\ 3\ 1} = \frac{7!}{2!\,0!\,1!\,3!\,1!} = 420.$

Ex: When $(2x_1-3x_2+5x_3)^6$ is expanded, the coefficient of $x_1^3 x_2 x_3^2$ is ______.

Sol^n Required coefficient is $\binom{6}{3\ 1\ 2} 2^3 (-3)^1 (5)^2$

$= -36{,}000.$

Th^m - 5.5.1 (Newton's Binomial Th^m)

Let α be a real number. Then $\forall\ x, y$ with $0 \le |x| < |y|$,

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k}$$

where $$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

If α is a +ve integer n, then for $k > n$ $\binom{n}{k} = 0$ and for $k \le n$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Some more problems from the exercise - Ch-5

Pb-8 Use the binomial theorem to prove that $2^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}$

Solⁿ $2^n = (3-1)^n = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (-1)^k$ (By using Binomial thm)

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k} \quad \square$$

(12) Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} 0 & \text{If } n \text{ is odd} \\ (-1)^m \binom{2m}{m} & \text{If } n = 2m. \end{cases}$$

Solⁿ When n is odd

The number of terms on L.H.S is even.

And using $\binom{n}{k} = \binom{n}{n-k}$ and grouping the terms, we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = (-1)^0 \binom{n}{0}^2 + (-1)^1 \binom{n}{1}^2 + \cdots + (-1)^{n-1} \binom{n}{n-1}^2 + (-1)^n \binom{n}{n}^2$$

$$= \binom{n}{0}^2 - \binom{n}{n}^2 - \binom{n}{1}^2 + \binom{n}{n-1}^2 + \cdots$$

$$= 0 + 0 + 0 - \cdots + 0$$

$$= 0 \quad \square$$

When n is even

$(1-x^2)^n = (1-x)^n (1+x)^n$

Expanding the L.H.S, We get

$$(1-x^2)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{2k}$$

The coefficient of x^n, we obtain when $k = m$ is given by $(-1)^m \binom{2m}{m}$.

Expanding the R.H.S, we get

$$(1-x)^n(1+x)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^k \sum_{j=0}^{n}\binom{n}{j}x^j$$

$$= \sum_{k=0}^{n}\sum_{j=0}^{n}(-1)^k\binom{n}{k}\binom{n}{j}x^{k+j}$$

The coefficient of x^n we obtain when for every fixed k we choose $j=n-k$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n}{k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}^2$$

since the coefficient of x^n must be equal we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = (-1)^m\binom{2m}{m}$$

Hence

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \begin{cases} 0 & \text{If } n \text{ is odd} \\ (-1)^m\binom{2m}{m} & \text{If } n=2m. \end{cases}$$

$\square$

Pb-15 Prove that for every integer $n>1$

$$\binom{n}{1}-2\binom{n}{2}+3\binom{n}{3}+ \cdots\cdots + (-1)^{n-1}n\binom{n}{n}=0.$$

Soln We know

$$(1+x)^n = 1+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+ \cdots +\binom{n}{n}x^n$$

Differentiating both the sides w.r.t x we have,

$$n(1+x)^{n-1} = \binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+ \cdots + n\binom{n}{n}x^{n-1}$$

By setting $x=-1$, we get

$$n(1-1)^{n-1}=\binom{n}{1}-2\binom{n}{2}+ \cdots + (-1)^{n-1}n\binom{n}{n}$$

$$\Rightarrow \binom{n}{1}-2\binom{n}{2}+3\binom{n}{3}- \cdots + (-1)^{n-1}n\binom{n}{n}=0$$

□

(16) Prove that for a +ve integer n,

$$1+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+ \cdots + \frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}.$$

Solⁿ We know

$$(1+x)^n = 1+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+ \cdots +\binom{n}{n}x^n$$

$$\int_0^1 (1+x)^n dx = \int_0^1 \left\{1+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+ \cdots +\binom{n}{n}x^n\right\}dx$$

$$\Rightarrow \left.\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left. x+\binom{n}{1}\frac{x^2}{2}+\binom{n}{2}\frac{x^3}{3}+ \cdots +\binom{n}{n}\frac{x^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow \frac{2^{n+1}-1}{n+1} = 1+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+ \cdots +\frac{1}{n+1}\binom{n}{n}$$

Hence $1+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+ \cdots \cdots +\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}$

□

(18) Evaluate the sum

$$1-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\frac{1}{4}\binom{n}{3}+ \cdots +(-1)^n\frac{1}{n+1}\binom{n}{n}$$

Solⁿ $(1-x)^n = 1-\binom{n}{1}x+\binom{n}{2}x^2-\binom{n}{3}x^3+ \cdots +(-1)^n\binom{n}{n}x^n$

$$\Rightarrow \int_0^1 (1-x)^n dx = \int_0^1 \left\{1-\binom{n}{1}x+\binom{n}{2}x^2-\binom{n}{3}x^3+ \cdots +(-1)^n\binom{n}{n}x^n\right\}dx$$

$$\Rightarrow \left.\frac{(1-x)^{n+1}}{n+1}\right]_0^1 = \left. x-\binom{n}{1}\frac{x^2}{2}+\binom{n}{2}\frac{x^3}{3}-\binom{n}{3}\frac{x^4}{4}+ \cdots +(-1)^n\binom{n}{n}\frac{x^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow \frac{1}{n+1} = 1-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\frac{1}{4}\binom{n}{3}+ \cdots \frac{(-1)^n}{n+1}\binom{n}{n}$$

□

(25) Use a combinatorial argument to prove the Vandermonde convolution for the binomial coefficients: For all positive integers m_1, m_2 and n,

$$\sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k} = \binom{m_1+m_2}{n}$$

Sol^n Let S be a set with m_1+m_2 elements. Then the number of n-subsets of S is $\binom{m_1+m_2}{n}$. Now look from different perspective. We partition S into two subsets A and B such that A contains m_1 elements and B contains m_2 elements. Now each n-subset of S can contain k elements from A where $0 \le k \le n$ and the remaining $n-k$ elements comes from B. For a fixed k, the total number of n-subsets of S that contain exactly k elements of A is $\binom{m_1}{k}\binom{m_2}{n-k}$.
Therefore, the total number of n subsets of S is

$$\sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k}.$$

Hence

$$\sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k} = \binom{m_1+m_2}{n} \quad \square$$

(38) Expand $(x_1+x_2+x_3)^4$ by using multinomial th^m.

Sol^n By using Multinomial th^m

$$(x_1+x_2+x_3)^4 = \sum \binom{4}{n_1\ n_2\ n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3}$$

where $n_1+n_2+n_3=4$

$$= \binom{4}{4\ 0\ 0} x_1^4 x_2^0 x_3^0 + \binom{4}{0\ 4\ 0} x_1^0 x_2^4 x_3^0 + \binom{4}{0\ 0\ 4} x_1^0 x_2^0 x_3^4$$

$$+ \binom{4}{3\ 1\ 0} x_1^3 x_2^1 x_3^0 + \binom{4}{1\ 3\ 0} x_1^1 x_2^3 x_3^0 + \binom{4}{0\ 1\ 3} x_1^0 x_2^1 x_3^3$$

$$+ \binom{4}{3\ 0\ 1} x_1^3 x_2^0 x_3^1 + \binom{4}{1\ 0\ 3} x_1^1 x_2^0 x_3^3 + \binom{4}{0\ 3\ 1} x_1^0 x_2^3 x_3^1$$

$$+ \binom{4}{2\ 2\ 0} x_1^2 x_2^2 x_3^0 + \binom{4}{2\ 0\ 2} x_1^2 x_2^0 x_3^2$$

$$+ \binom{4}{0\ 2\ 2} x_1^0 x_2^2 x_3^2 + \binom{4}{2\ 1\ 1} x_1^2 x_2^1 x_3^1$$

$$+ \binom{4}{1\ 2\ 1} x_1^1 x_2^2 x_3^1 + \binom{4}{1\ 1\ 2} x_1^1 x_2^1 x_3^2$$

$$= x_1^4 + x_2^4 + x_3^4 + 4x_1^3 x_2 + 4x_1 x_2^3 + 4x_2 x_3^3$$

$$+ 4x_1^3 x_3 + 4x_1 x_3^3 + 4x_2^3 x_3 + 6x_1^2 x_2^2 + 6x_1^2 x_3^2$$

$$+ 6x_2^2 x_3^2 + 12x_1^2 x_2 x_3 + 12x_1 x_2^2 x_3 + 12x_1 x_2 x_3^2$$

Ans

$$\left(\text{Here } \binom{4}{n_1\ n_2\ n_3} = \frac{4!}{n_1!\, n_2!\, n_3!}\right)$$