Ch. 6 – Recurrence Equations

§1. Problems leading to recurrence equations

Def. A *recurrence equation* (or *recurrence relation*) is any equation that can be used to specify an infinite sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ by expressing x_n in terms of $x_{n-1}, x_{n-2}, \ldots, x_1, x_0$, and n.

Ex. 1 Let $x_n = x_{n-1} + 3$ for $n \geq 1$. Then

$$x_1 = x_0 + 3 = x_0 + 3(1)$$
$$x_2 = x_1 + 3 = (x_0 + 3(1)) + 3 = x_0 + 3(2)$$
$$x_3 = x_2 + 3 = (x_0 + 3(2)) + 3 = x_0 + 3(3)$$

It is not difficult to see that in general

$$x_n = x_0 + 3(n) = x_0 + 3n.$$

In order to uniquely define $\langle x_n \rangle_{n\in\mathbb{N}}$, we need to specify x_0. If we set $x_0 = 2$, then we get $x_n = 2 + 3n$. "$x_0 = 2$" is called an *initial condition*. Thus we can say that the recurrence equation

$$x_n = x_{n-1} + 3 \text{ for } n \geq 1 \ \& \ x_0 = 2$$

has the unique solution $x_n = 2 + 3n$.

Def. A recurrence equation of *order k* is one of the form

$$x_n = g(n, x_{n-1}, x_{n-2}, \ldots, x_{n-k}).$$

where g is a function of $k+1$ variables.

In order to get a unique solution for a recurrence equation of order k, we need initial conditions which specify the values of $x_0, x_1, \ldots, x_{k-1}$.

Ex. 2 (a) $x_n = 2x_{n-1} + 3n$ for $n \geq 1$ is of order 1

(b) $x_n = x_{n-1} + 2x_{n-2}$ for $n \geq 2$ is of order 2

(c) $x_n = x_{n-1} + 0.x_{n-2} + n.x_{n-3}$ for $n \geq 3$ is of order 3.

(d) $x_n = x_{n-1} + x_{n-2} + \cdots + x_1 + x_0$ is of unbounded order.

Note: Although the recurrence equation $x_n = x_{n-1} + \cdots + x_0$ looks pretty "wild," it can be "tamed" by putting

$y_n = x_n + \underbrace{x_{n-1} + \cdots + x_0}$. Then we have

$$y_n = (x_{n-1} + \cdots + x_0) + (x_{n-1} + \cdots + x_0)$$

$$= y_{n-1} + y_{n-1} = 2y_{n-1}$$

So $y_n = 2y_{n-1}$ and this is of order 1

However, not all recurrence equations of unbounded order can be "tamed".

Many counting problems in Combinatorics lead to recurrence equations in which the answer is x_n. We can easily calculate the value of x_n for any given value of n by using the recurrence equations. But this will not tells how fast x_n grows — for that we need to express x_n explicitly in terms of n.

(Fibonacci Rabbit Problem)

Ex. 3 A newly born pair of rabbits of opposite sex is placed in a large enclosure in the middle of January 2014. After a female rabbit is two months old, it gives birth to one pair of rabbits of opposite sex in the middle of all future months. How many pairs of rabbits (of opposite sex) will there be

(a) on Oct. 31st, 2014 (b) n months after Jan. 1st, 2014?

Sol. Let x_n = the number of pairs of rabbits (of opposite sex) n months after Jan 1st, 2014.

Then $x_0 = 0$ and $x_1 = 1$. Also

x_n = no. of pairs after $n-1$ months
  + no. of pairs born in the middle of the n-th month
  $= x_{n-1} + x_{n-2}$

because only the females present after $n-2$ months would be mature enough to produce a pair in the middle of the n-th month.

Thus $x_n = x_{n-1} + x_{n-2}$ for $n \geq 2$ & $x_0 = 0$, $x_1 = 1$.

$\therefore$ $x_2 = x_1 + x_0 = 1+0 = 1$, $x_3 = x_2 + x_1 = 1+1 = 2$

$x_4 = x_3 + x_2 = 2+1 = 3$, $x_5 = x_4 + x_3 = 3+2 = 5$

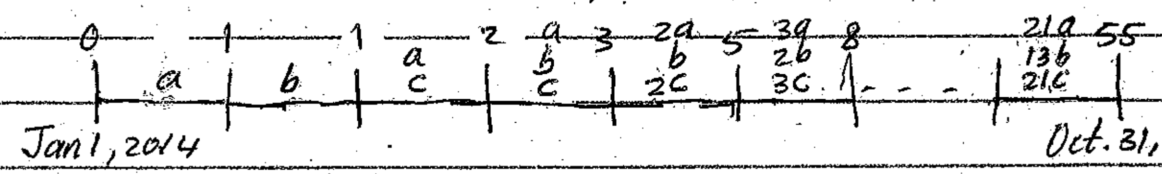
$x_6 = x_5 + x_4 = 5+3 = 8$, $x_7 = x_6 + x_5 = 8+5 = 13$

$x_8 = x_7 + x_6 = 13+8 = 21$ $x_9 = x_8 + x_7 = 21+13 = 34$

$x_{10} = x_9 + x_8 = 34+21 = 55$

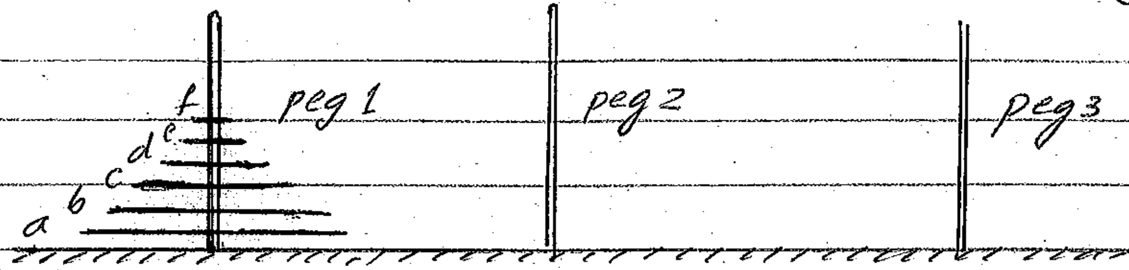
So on Oct. 31st, 2014, there will be 55 pairs.

We will later find an explicit formula for x_n.



a = newborn pair, b = 1 month old, c = at least 2 months.

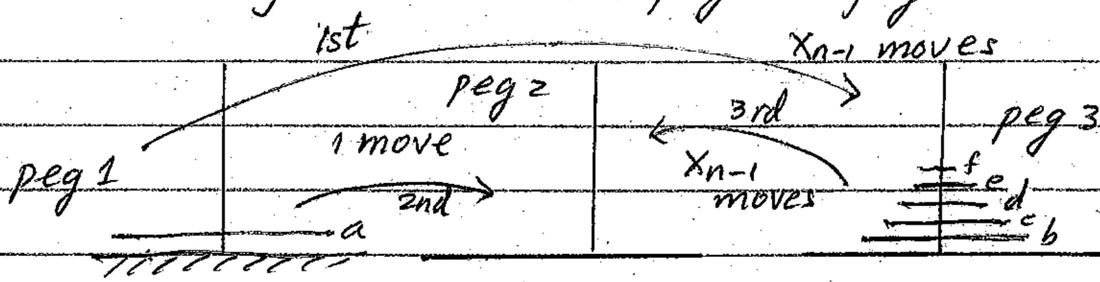
Ex. 4 (Tower of Brahma Problem). We have n disks of decreasing sizes that are stacked as shown on peg #1. What is the least number of moves needed to transfer the n disks from peg #1 to peg #2, if we must always store the disks on one of the three pegs & if we cannot place a larger disk on a smaller one?



Sol. Let X_n = least no. of moves needed transfer the n disks from peg 1 to peg 2.

Then X_n = least no. of moves need to transfer the n disks from any peg to any other of the 3 pegs.

Now at some point in the process of transferring the n disks from peg 1 to peg 2 we must have the top n-1 disks on peg 3 and be just about to move the largest disk from peg 1 to peg 2.



So $\quad X_n = X_{n-1} + 1 + X_{n-1}$

$\therefore \quad X_n = 2X_{n-1} + 1. \qquad$ Also $a_0 = 0$.

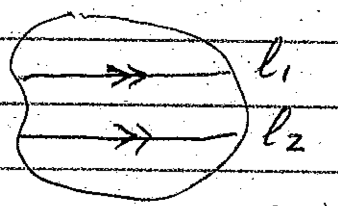
Thus
$$\begin{aligned}
X_1 &= 2(X_0)+1 = 2(0)+1 = 1 = 2^1-1\\
X_2 &= 2(X_1)+1 = 2(1)+1 = 3 = 2^2-1\\
X_3 &= 2(X_2)+1 = 2(3)+1 = 7 = 2^3-1\\
X_4 &= 2(X_3)+1 = 2(7)+1 = 15 = 2^4-1\\
X_5 &= 2(X_4)+1 = 2(15)+1 = 31 = 2^5-1
\end{aligned}$$

From this it is easy to conjecture that $X_n = 2^n-1$ and we can verify this by using induction. We will also show how to find X_n, without guessing, later on.

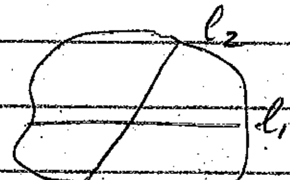
Ex. 5 (The Plane partitioning problem) Let $S_n = \{l_1, \ldots, l_n\}$ be a set of n straight lines in the plane. What is the largest number of regions into which S_n can partition the plane?

Sol. Let x_n = largest number of regions into which S_n can partition the plane.

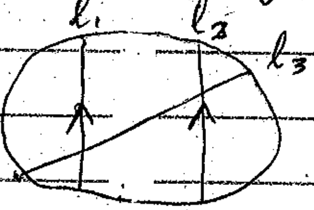
Then $x_0 = 1$ and $x_1 = 2$. Now to get the maximum no. of regions we should ensure that l_n intersect all the lines $l_1, \ldots, l_{n-1}$ & that no three lines intersect in the same point.



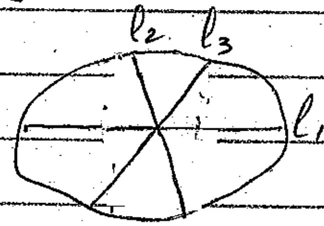
3 regions



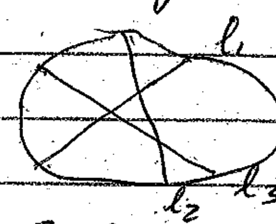
4 regions



5 regions



6 regions



7 regions

So $x_2 = 4$ and $x_3 = 7$. If we add a new line l_n to $\{l_1, \ldots, l_{n-1}\} = S_{n-1}$ and ensure that l_n intersect $l_1, \ldots, l_{n-1}$ in different places then l_n will be split into n segments and each of these segments will divide an old region into 2 new ones. So $x_n = x_{n-1} + n$ & $x_0 = 1$

It is easy to see the pattern for x_n but it it is not so easy to come up with a formula for x_n.

$\langle x_n \rangle = 1, 2, 4, 7, 11, 16, 22, \ldots \qquad x_n = \frac{1}{2}(n^2 + n + 2)$

§2. Linear constant-coefficient recurrence equations of order k

Prop. 1 Any recurrence equation $x_n = g(n, x_{n-1}, \dots, x_{n-k})$ (*) of order k with initial conditions $x_0 = a_0, \dots, x_{k-1} = a_{k-1}$ has a unique solution.

Proof: First observe that we know $x_0 = a_0, \dots, x_{k-1} = a_{k-1}$. Now by repeatedly using (*) for $n = k, k+1, \dots$ we will get the value of $x_k, x_{k+1}, \dots$. So this will give us a solution of (*) which satisfy the initial conditions. Now any other solution, x_n' say, is forced to have the same first k values $a_0, \dots, a_{k-1}$ and because of (*), x_n' will be forced to have all the other values the same as the first solution. Hence we have a unique solution.

Def. A linear recurrence equation of order k is one which can be written in the form

$$x_n + p_1(n) x_{n-1} + p_2(n) x_{n-2} + \dots + p_k(n) x_{n-k} = q(n)$$

where

$p_1(n), \dots, p_k(n)$ & $q(n)$ are functions of n only.

If we replace n by $n+k$, we can write this linear recurrence equation in the alternative form

$$x_{n+k} + p_{k-1}'(n) x_{n+k-1} + p_{k-2}'(n) x_{n+k-2} + \dots + p_0'(n) x_n = q'(n).$$

We will use both forms of the lin. rec. eq. in whatever follows.

Def A linear <u>constant-coefficient</u> recurrence equation is one in which all of the functions $p_1(n), \ldots, p_k(n)$ are constants. [The function $q(n)$ does not have to be a constant.]

A linear recurrence equation is said to be <u>homogeneous</u> if $q(n) \equiv 0$ (i.e., $q(n) = 0$ for all n).

We will shortly see that the linear recurrence equations can be written in a special form by using the <u>next-term</u> operator E.

Def. We define the operator E on a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ by $E(\langle x_n \rangle_{n \in \mathbb{N}}) = \langle x_{n+1} \rangle_{n \in \mathbb{N}}$. Note that

$$E^2(\langle x_n \rangle_{n \in \mathbb{N}}) = E(\langle x_{n+1} \rangle_{n \in \mathbb{N}}) = \langle x_{n+2} \rangle_{n \in \mathbb{N}}$$

and $E^0 = I$ = <u>the identity operator</u> which is defined by $I(\langle x_n \rangle_{n \in \mathbb{N}}) = \langle x_n \rangle_{n \in \mathbb{N}}$.

Ex. 1 Consider the recurrence eq. $x_n - 3x_{n-1} = 0$ for $n \geq 1$. We can also write this as $x_{n+1} - 3x_n = 0$ for $n \geq 0$. We can further rewrite this as

$$(E - 3I)(\langle x_n \rangle) = O$$

where $O = \langle 0 \rangle$ is the sequence of 0's. We will usually just write this as $(E - 3I)(x_n) = 0$.

Ex. 2 The linear rec. eq. $x_{n+2} - 2n \cdot x_{n+1} + n^2 x_n = 3^n$ can be written in the form $\mathcal{L}(E) x_n = 3^n$ where

$$\mathcal{L}(E) = E^2 - 2nE + n^2 E^0$$
$$= E^2 - 2nE + n^2 I$$

Prop. 2 Let $\langle a_n \rangle$ & $\langle b_n \rangle$ be solutions of the linear recurrence equation $\mathcal{L}(E)(X_n) = 0$. Then

(a) $\langle a_n + b_n \rangle$ is also a solution of $\mathcal{L}(E)(X_n) = 0$ and

(b) $\langle A a_n \rangle$ is also a solution of $\mathcal{L}(E)(X_n) = 0$ for any arbitrary constant $A \in \mathbb{C}$.

(Sketch of)

Proof: Suppose $\langle a_n \rangle$ & $\langle b_n \rangle$ are solutions of $\mathcal{L}(E)(X_n) = 0$. Then $\mathcal{L}(E)(a_n) = 0$ & $\mathcal{L}(E)(b_n) = 0$. Since $\mathcal{L}(E)$ is a linear operator $\mathcal{L}(E)(a_n + b_n) = \mathcal{L}(E)(a_n) + \mathcal{L}(E)(b_n) = 0 + 0 = 0$. So $\langle a_n + b_n \rangle$ is a solution of $\mathcal{L}(E)(X_n) = 0$. Also $\mathcal{L}(E)(A a_n) = A\,\mathcal{L}(E)(a_n) = A(0) = 0$. So $\langle A a_n \rangle$ is a solution of $\mathcal{L}(E)(X_n) = 0$

Ex. 3 Find the solution of the recurrence equation

$$X_{n+1} - 3X_n = 0 \text{ for } n \geq 0 \text{ with } X_0 = 4.$$

Sol. Suppose we have a solution of the form $X_n = \alpha^n$. Then $X_{n+1} = \alpha^{n+1}$. So $X_{n+1} - 3X_n = 0$ becomes

$$\alpha^{n+1} - 3\alpha^n = 0 \quad \text{for } n \geq 0.$$

$$\therefore \quad \alpha^n(\alpha - 3) = 0 \quad \text{for } n \geq 0$$

Since $\alpha^0 \neq 0$, it follows that $\alpha - 3 = 0$. So $\alpha = 3$. Hence $X_n = 3^n$. Let us check it.

We have $X_{n+1} - 3X_n = 3^{n+1} - 3.3^n = 0$, for all $n \geq 0$. Also $X_0 = 3^0 = 1 \neq 4$. So we have not satisfied the initial conditions. But this is easily remedied because $X_n = A.3^n$ is a solution for each $A \in \mathbb{C}$, by Prop. 2. So $4 = A.3^0 = A \Rightarrow A = 4$. $\therefore X_n = 4.3^n$ is the solution.

Note: The equation $x_{n+1} - 3x_n = 0$ can be written in the form $(E - 3I)\, x_n = 0$. And from Ex.3 we know that $x_n = A.(3)^n$ gives us all the possible solutions. So $x_n = A.3^n$ is called the <u>general solution</u> of $(E-3I)x_n = 0$ and it is understood that no initial conditions are imposed as yet.

Ex.3 Find the solution of the recurrence equation $x_{n+2} + x_{n+1} - 6x_n = 0$ for $n \geq 0$ with $x_0 = 7$ & $x_1 = -6$.

Sol. The equation can be written as

$$(E^2 + E - 6I)(x_n) = 0$$

So $(E+3I)\underbrace{(E-2I)(x_n)} = 0$ & $(E-2I)\underbrace{(E+3I)(x_n)} = 0$.

$\therefore$ $(E-2I)(x_n) = 0$ or $(E+3I)(x_n) = 0$

$\therefore$ $x_n = A.(2)^n$ or $x_n = B.(-3)^n$. So by Prop 2 the general solution of the equation is

$$x_n = A.(2)^n + B.(-3)^n, \quad \text{where } A \,\&\, B \text{ are arb.}$$

$\therefore$ $7 = x_0 = A.(2)^0 + B(-3)^0 \Rightarrow A + B = 7$

$-6 = x_1 = A.(2)^1 + B(-3)^1 \qquad 2A - 3B = -6$

$\therefore$ $B = 7 - A$. So $2A - 3(7-A) = -6$

$\therefore$ $5A = -6 + 21 = 15 \Rightarrow A = 3$. So $B = 7 - A = 4$.

$\therefore$ $x_n = A.(2)^n + B.(-3)^n = 3.(2)^n + 4.(-3)^n$.

Note: Let $\mathcal{L}(E)(x_n) = 0$ be a linear constant coeff. recurrence equation of order k. Then we must have $\mathcal{L}(E) = E^k + c_1 E^{k-1} + \dots + c_1 E^1 + c_0 I$ where each $c_i \in \mathbb{C}$, Theorem 3: Let $\mathcal{L}(E)(X_n) = 0$ be a lin. constant coeff. recurrence eq. If $\mathcal{L}(E) = (E-\alpha_1 I)(E-\alpha_2 I)\cdots(E-\alpha_k I)$ and $\alpha_1, \ldots, \alpha_k$ are all distinct, then the general solution of $\mathcal{L}(E)(X_n) = 0$ is given by $X_n = A_1.(\alpha_1)^n + A_2(\alpha_2)^n + \cdots + A_k.(\alpha_k)^n$ where the $A_1, \ldots, A_k$ are arbitrary constants.

Ex.5 Find the solution of the recurrence equation
$$X_{n+2} - X_{n+1} - X_n = 0 \quad \text{with } X_0 = 0 \;\&\; X_1 = 1.$$

Sol. We have $(E^2 - E - I)(X_n) = 0$. So
$$E = \left[-(-1) \pm \sqrt{1-4(-1)}\right]/2 = (1 \pm \sqrt{5})/2$$
$\therefore \quad X_n = A.\left(\frac{1+\sqrt{5}}{2}\right)^n + B.\left(\frac{1-\sqrt{5}}{2}\right)^n$. So

$$X_0 = 0 = A + B \implies B = -A.$$
$$X_1 = 1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$$
$$\implies 1 = A\left(\frac{1+\sqrt{5}}{2}\right) - A\left(\frac{1-\sqrt{5}}{2}\right) \Rightarrow A\sqrt{5} = 1$$
$\therefore \quad A = 1/\sqrt{5}$ and $B = -A = -1/\sqrt{5}$
$$\therefore \quad X_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^n$$
and now we have the solution to Fibonacci Rabbit Problem.

Ex 6 Find the solution of $X_{n+2} - 2X_{n+1} + 2X_n = 0$ with $X_0 = 0 \;\&\; X_1 = 2$.

Sol. We have $(E^2 - 2E + 2I)(X_n) = 0$. $\therefore E = 1 \pm i$. So $X_n = A.(1-i)^n + B(1+i)^n$ is the general solution.
$X_0 = 0 = A + B$, $X_1 = 2 = A(1-i) + B(1+i)$. $\therefore B = -A$. So
$2 = -2iA \Rightarrow A = i$, $\therefore B = -i$. So $X_n = i(1-i)^n - i(i+1)^n$.

Ex.7 Find the solution of $x_{n+2}-6x_{n+1}+9x_n=0$ with $x_0=-1$ and $x_1=9$.

Sol. We have $(E^2-6E+9I)(x_n)=0$. So $(E-3I)(E-3I)x_n=0$.

$\therefore$ $x_n=A.(3)^n$ is a solution for each $A\in\mathbb{C}$.

Now if we try to impose the initial conditions we get

$$-1=x_0=A.(3)^0 \Rightarrow A=-1$$

$$9=x_1=A.(3)^1 \Rightarrow 3A=9 \Rightarrow A=3.$$

Thus we have a contradiction. So we need another solution. Let $y_n=(E-3I)x_n$. Then Since $(E-3I)\underbrace{(E-3I)(x_n)}=0$, we get $(E-3I)y_n=0$.

Thus $y_n=B.3^n$. So

$$(E-3I)x_n=B.3^n.$$

$$\therefore\ x_{n+1}-3x_n=B.3^n. \qquad (1)$$

So

$$3^1(x_n-3x_{n-1})=3^1.B.3^{n-1}=B.3^n \qquad (2)$$

$$3^2(x_{n-1}-3x_{n-2})=3^2.B.3^{n-2}=B.3^n \qquad (3)$$

$$\vdots \qquad\qquad \vdots$$

$$3^n(x_1-3x_0)=3^n.B.3^0=B.3^n \qquad (n+1)$$

Adding equations (1) ... (n+1) we get

$$x_{n+1}-3^{n+1}x_0=(n+1).B.3^n=(n+1)\frac{B}{3}.3^{n+1}$$

So $x_{n+1}=3^{n+1}x_0+C.(n+1)3^{n+1}$ where $C=B/3$.

$\therefore$ $x_n=x_0.3^n+C.n.3^n$. Thus we now have the general solution. $x_0=-1$ & $x_1=9\Rightarrow$

$9=(-1).3^1+C.1.3^1 \Rightarrow 3C-3=9 \Rightarrow C=4$

$\therefore$ $x_n=(Cn+x_0).3^n=(4n-1).3^n$.

Theorem 4: The general solution of $(E-\alpha I)^m(x_n)=0$ is given by $x_n=(A_0+nA_1+n^2A_2+\dots+n^{m-1}A_{m-1}).(\alpha)^n$, where $A_0,A_1,\dots,A_{m-1}$ are arbitrary constants.

§3. Non-homogeneous constant-coefficients linear rec. eq.

Theorem 5: The general solution of the non-homogeneous linear rec. eq. of order k, $\mathcal{L}(E)(X_n) = q(n)$ is given by

$$X_n = X_n^c + X_n^p$$

where X_n^c is the general solution of the homog. eq. $\mathcal{L}(E)(X_n) = 0$, & X_n^p is a particular solution of $\mathcal{L}(E)(X_n) = q(n)$.

Proof: First of all $\mathcal{L}(E)(X_n^c + X_n^p) = \mathcal{L}(E)(X_n^c) + \mathcal{L}(E)(X_n^p) = 0 + q(n) = q(n)$. So $X_n^c + X_n^p$ is always a solution of $\mathcal{L}(E)(X_n) = q(n)$. Since X_n^c will have k linearly independent solutions, the solution $X_n^c + X_n^p$ will have k arbitrary constants and we will always be able to satisfy the k independent initial conditions $X_0 = a_0, \ldots, X_{k-1} = a_{k-1}$.

Ex.1 Find the general solution of the equation

$$X_{n+1} - 3X_n = 4.(5)^n.$$

Sol. We have $(E - 3I)(X_n) = 4.(5)^n$. So

$$(E-5I)(E-3I)(X_n) = (E-5I)\,4(5)^n$$
$$= 4.5^{n+1} - 4.5.5^n = 0$$

So $X_n = A.(3)^n + B.(5)^n$. But $X_{n+1} - 3X_n = 4.5^n$

So $(A.3^{n+1} + B.5^{n+1}) - 3.(A.3^n + B.5^n) = 4.5^n$

$\therefore\ (3A - 3A).3^n + (5B - 3B).5^n = 4.5^n$

$\therefore\ 2B.5^n = 4.5^n \Rightarrow 2B = 4 \Rightarrow B = 2.$

Hence the general solution is $X_n = A.(3)^n + 2.(5)^n$.

Notice $X_n^c = A.(3)^n$ & $X_n^p = 2.(5)^n$.

Ex. 2 Find the general solution of the equation

$$X_{n+1} - 3X_n = 5n.(2)^n$$

Sol. We have $(E-3I)(X_n) = n.2^n$. So

$$(E-2I)^2(E-3I)(X_n) = (E-2)^2(5n.2^n)$$
$$= (E^2-4E+4I)(5n.2^n)$$
$$= 5(n+2).2^{n+2} - 4.5.(n+1).2^{n+1} + 4.5.n.2^n$$
$$= (20n+40-40n-40+20.n).2^n = 0.$$

$\therefore \quad (E-2I)^2(E-3I)(X_n) = 0$

$\therefore \quad X_n = A.(3)^n + (Bn+C).(2)^n$

But $X_{n+1} - 3X_n = 5n.2^n$. So

$$[A.3^{n+1} + (B(n+1)+C).2^{n+1}] - 3[A.3^n + (Bn+C).2^n] = 5n.2^n$$

$\therefore \{[2B(n+1)+2C] - 3[Bn+C]\}.2^n = 5n.2^n$

$\therefore \quad (2B-3B)n + [2B+2C-3C] = 5n$

$\therefore \quad -Bn + 2B - C = 5n + 0$

$$\therefore \left.\begin{array}{l} -B = 5 \\ 2B - C = 0 \end{array}\right\} \begin{array}{l} \Rightarrow B = -5 \\ \Rightarrow C = 2B = -10 \end{array}$$

$\therefore \quad X_n = A.3^n + (Bn+C).2^n$

$\quad\quad = A.3^n - (5n+10).2^n.$

Ex. 3 Find the general solution of the equation

$$X_{n+1} - 2X_n = 6.2^n$$

Sol. We have $(E-2I)X_n = 6.2^n$. So

$$(E-2I)(E-2I)(X_n) = (E-2I)(6.2^n)$$
$$= 6.2^{n+1} - 2.6.2^n = 0$$

$\therefore \quad (E-2I)^2 X_n = 0. \quad \therefore \quad X_n = (A+Bn).2^n.$

But $X_{n+1} - 2X_n = 6.2^n$. So

$$\{A + B(n+1)\} \cdot 2^{n+1} - 2[A + Bn] \cdot 2^n = 6 \cdot 2^n$$

$$\therefore \quad (2A + 2Bn + 2B - 2A - 2Bn) \cdot 2^n = 6 \cdot 2^n$$

$$\therefore \quad (2B) \cdot 2^n = 6 \cdot 2^n \Rightarrow 2B = 6 \Rightarrow B = 3$$

$\therefore \quad x_n = (A + Bn) \cdot 2^n = (A + 3n) \cdot (2)^n$ is the gen. sol.

From these examples we can easily see why the following theorems will be true.

Theorem 5: If $q(n) = (b_0 + b_1 n + \cdots + b_r n^r) \cdot (\alpha)^n$ and α is not a root of the auxiliary equation $\mathcal{L}(E) = 0$, then the recurrence equation

$$\mathcal{L}(E)(x_n) = q(n)$$

has a particular solution of the minimal form

$$x_n^p = (B_0 + B_1 n + \cdots + B_r n^r)(\alpha)^n.$$

Theorem 6: If $q(n) = (b_0 + b_1 n + \cdots + b_r n^r) \cdot (\alpha)^n$ and α is a root of multiplicity m of the auxiliary equation $\mathcal{L}(E) = 0$, then the recurrence eq.

$$\mathcal{L}(E)(x_n) = q(n)$$

has a particular solution of the minimal form

$$x_n^p = (B_0 + B_1 n + \cdots + B_r n^r) \cdot n^m \cdot (\alpha)^n.$$

Ex. 4 Find the complimentary solution x_n^c of each of the following recurrence equation and als give the minimal form of a particular solution of each equation.

(a) $(E - 3I)^2 (E + I)(x_n) = 6n^2$

(b) $(E^2 + 9I)(E + 2I)^2 (x_n) = 5 \cdot n \cdot (-2)^n$

(c) $(E + I)^2 (E - I)^3 (x_n) = 8n$

(d) $(E^2 + 4I)^2 (E - 2I)^3 = 10 \cdot n \cdot 2^n$

(a) The auxiliary equation is $(E-3)^2(E+1)=0$.
So $E=3$ (twice) or $E=-1$. Hence
$$X_n^c = (A_1 + nA_2).(3)^n + A_3.(-1)^n$$
The $(\alpha)^n$ associated with the RHS $6n^2$ is $(1)^n$.
and since $6n^2$ is a polynomial of degree 2 and 1 is not a root of the auxiliary equation, the minimal form of a particular solution is
$$X_n^p = (B_0 + n.B_1 + n^2B_2).(1)^n = B_0 + nB_1 + n^2B_2.$$

(b) The auxiliary equation is $(E^2+9)(E+2)^2=0$.
So $(E+3i)(E-3i)(E+2)^2=0$. Hence
$E = -3i,\ 3i,$ or -2 (twice).
$$\therefore\ X_n^c = A_1.(-3i)^n + A_2.(3i)^n + (A_3 + n.A_4).(-2)^n.$$
Since $5n$ is a polynomial of degree 1 and -2 is a root of multiplicity 2 of the auxiliary equation, the minimal form of X_n^p is
$$X_n^p = (B_0 + nB_1).n^2.(-2)^n$$

(c) The auxiliary equation is $(E+1)^2(E-1)^3=0$.
So $E=-1$ (twice) or 1 (three times).
$$\therefore\ X_n^c = (A_1 + nA_2).(-1)^n + (A_3 + nA_4 + n^2A_5).(1)^n$$
$$= (A_1 + nA_2).(-1)^n + (A_3 + nA_4 + n^2A_5).$$
Since $8n$ is a polynomial of degree 1 and 1 is a root of multiplicity 3 of the auxiliary equation, the minimal form of X_n^p is
$$X_n^p = (B_0 + nB_1).n^3.(1)^n = (B_0 + nB_1).n^3.$$

(d) The auxiliary equation is $(E^2+4)^2(E-2)^3$. So
$[(E-2i)(E+2i)]^2(E-2)^3=0$. $\therefore\ (E-2i)^2(E+2i)^2(E-2)^3=0.$

(d) So $E = 2i$ (twice), $-2i$ (twice), or 2 (thrice).

$\therefore X_n^c = (A_1 + nA_2)(2i)^n + (A_3 + nA_4).(-2i)^n + (A_5 + nA_6 + n^2 A_7) 2^n$.

Since $10n$ is a polynomial of degree 1 and 2 is a root of multiplicity 3 of the auxiliary equation, the minimal form of X_n^p is

$$X_n^p = (B_0 + nB_1).n^3.(2)^n.$$

Ex.5 (The Tower of Brahma again). Find the solution of the recurrence equation $X_n = 2X_{n-1} + 1$ for $n \geq 1$ with the initial condition $X_0 = 0$.

Sol. Replacing n by $n+1$ we get $X_{n+1} = 2X_n + 1$ for $n \geq 0$. So $X_{n+1} - 2X_n = 1$. $\therefore (E - 2I) X_n = 1$.

$\therefore X_n^c = A.(2)^n$.

Try $X_n^p = B$. Then $X_{n+1}^p = B$. So $X_{n+1} - 2X_n = 1$ becomes $B - 2B = 1$.

$\therefore -B = 1$. So $B = -1$. Thus the general solution is $X_n = A.(2)^n - 1$

Since $X_0 = 0$, it follows that

$$0 = X_0 = A.(2)^0 - 1 = A - 1$$

$\therefore A - 1 = 0 \Rightarrow A = 1$.

$\therefore X_n = A.2^n - 1 = 2^n - 1$.

Ex.6 (The Plane partition problem again). Find the solution of the recurrence equation $X_n = X_{n-1} + n$ for $n \geq 1$ with the initial condition $X_0 = 1$.

Sol. Replacing n by $n+1$, we get $X_{n+1} = X_n + n + 1$ for $n \geq 0$. So $X_{n+1} - X_n = n+1$. $\therefore (E - I) X_n = n + 1$.

Ex6 $\therefore\ x_n^c = A$. Since 1 is a root of multiplicity 1 of the auxiliary equation, we should try

$$x_n^p = (B + Cn).\,n.(1)^n = Bn + Cn^2.$$

$\therefore\ x_{n+1}^p = B(n+1) + C(n+1)^2$. So $x_{n+1} - x_n = n+1$ becomes $B(n+1) + C(n+1)^2 - Bn - Cn^2 = n+1$

$$\therefore\ (C-C)n^2 + (2C + B - B)n + (B+C) = n+1$$

$$\therefore\ \left.\begin{array}{l} 2C = 1 \\ B + C = 1 \end{array}\right\} \begin{array}{l} \Rightarrow C = 1/2 \\ \Rightarrow B = 1 - C = 1 - 1/2 = 1/2. \end{array}$$

$\therefore\ x_n^p = Bn + Cn^2 = \frac{1}{2}n + \frac{1}{2}n^2$. So

$$x_n = x_n^c + x_n^p = A + \tfrac{1}{2}n + \tfrac{1}{2}n^2.$$

But $x_0 = 1$. So $1 = A + \frac{1}{2}(0) + \frac{1}{2}(0)^2 \Rightarrow A = 1$

$$\therefore\ x_n = 1 + \tfrac{1}{2}n + \tfrac{1}{2}n^2 = \tfrac{1}{2}(n^2 + n + 2)$$

The method we used to solve the constant coefficient linear recurrence eq. is called the E-method. Unfortunately, the E-method cannot easily handle variable coefficients or complicated RHS such as 2^{n^2}.

Ex.7 Find the solution of $x_n - n.x_{n-1} = 3(n!)$ with $x_0 = 2$

Sol. Let $y_n = x_n/(n!)$. Then $x_n = (n!).y_n$ & $y_0 = \frac{x_0}{0!} = 2$

So $(n!)y_n - n.(n-1)!\,y_{n-1} = 3(n!)$ Thus

$$(y_n - y_{n-1})\,n! = 3(n!) \qquad \therefore\ y_n - y_{n-1} = 3$$

So $y_n^c = A.(1)^n = A$. Also try $y_n^p = Bn$. Then $y_n^p - y_{n-1}^p = 3$ becomes $Bn - B(n-1) = 3$

$\therefore\ B = 3$ $\therefore\ y_n = y_n^c + y_n^p = A + 3n$. Since $y_0 = 2$, we get $2 = y_0 = A + 3(0) \Rightarrow A = 2$.

$\therefore\ y_n = 2 + 3n$. Hence $x_n = (n!)y_n = (2+3n).(n!)$

Ex. 8 Find the solution of $x_n - \frac{x_{n-1}}{n} = \frac{6}{(n-1)!}$ for $n \geq 1$ with $x_0 = 2$.

Sol. Let $y_n = n!\,x_n$. Then $x_n = y_n/n!$ & $y_0 = 0!\,x_0 = 2$.

So $\frac{y_n}{n!} - \frac{y_{n-1}}{(n-1)!}\cdot\frac{1}{n} = \frac{6}{(n-1)!} = \frac{6n}{n!}$

Thus $y_n - y_{n-1} = 6n$

$\therefore \quad (E-I)\,y_{n-1} = 6n = 6n.(1)^n$

$\therefore \quad y_n^c = A$.

Try $y_n^p = Bn + Cn^2$ (since 1 is a root of the aux. eq.)

Then $y_{n-1}^p = B(n-1) + C(n-1)^2$, So $y_n^p - y_{n-1}^p = 6n$

$\therefore \quad Bn + Cn^2 - B(n-1) - C(n-1)^2 = 6n$

$(C-C)n^2 + (B-B+2C)n + (B-C) = 6n$

$\therefore \quad 2C = 6$ & $B - C = 0$

$\therefore \quad C = 3$ & $B = 3$.

$\therefore \quad y_n = y_n^c + y_n^p = A + Bn + Cn^2 = A + 3n + 3n^2$

Since $y_0 = 2$, $\quad 2 = y_0 = A + 0 + 0 \Rightarrow A = 2$.

$\therefore \quad y_n = 2 + 3n + 3n^2$.

$\therefore \quad x_n = \frac{y_n}{n!} = \frac{2+3n+3n^2}{n!}$ END