Chapter - 2 Subgroups

2.1 Def^n (subgroup):

A non-empty subset H of a group G is called a subgroup of G if

(i) H is closed under the composition defined in G,
(i.e. $a \in H, b \in H \Rightarrow ab \in H$)

(ii) H itself is a group for the composition induced by that of G.

Proper and Improper (or Trivial) subgroups:

Every group G of order greater than 1 has atleast two subgroups, which are:

(i) G (itself)
(ii) $\{e\}$, i.e. the group of the identity alone.

These two groups (subgroups) are known as improper or trivial subgroups.

A group other than these two is known as a proper subgroups.

Examples of subgroup:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

(i) Additive Group:

Ex 1. $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Q}, +)$
2: $(\mathbb{Q}, +)$ is a subgroup of $(\mathbb{R}, +)$

(ii) Multiplicative Group:

1: $(\mathbb{Q}^*, \times)$ is a subgroup of $(\mathbb{R}^*, \times)$

2: $\{1,-1\}$, $\{1, \omega, \omega^2\}$, $\{1,-1,i,-i\}$ are subgroups of $(\mathbb{C}^*, \times)$

Proposition 1 (The subgroup criterion):

A subset H of a group G is a subgroup iff

(i) $H \neq \phi$ and

(ii) $\forall\ x, y \in H$, $xy^{-1} \in H$.

Proof: (i) let H be a subgroup of G and $y \in H$

then $y \in H \Rightarrow y^{-1} \in H$ [by existence of inverse in G]

$\therefore\ x \in H,\ y \in H \Rightarrow x \in H,\ y^{-1} \in H$

$\Rightarrow xy^{-1} \in H$

$\therefore$ if H is a subgroup of G, the defined condition is necessary.

Conversely: suppose given condition is true in H, then we will prove that H is a subgroup.

$\because\ H \neq \phi$, let $x \in H$

Therfore identity exists in H.

By given condition $e \in H,\ x^{-1} \in H \Rightarrow ex^{-1} = x^{-1} \in H$

Thus inverse of every element exist in H.

Finally $x \in H,\ y \in H \Rightarrow x \in H,\ y^{-1} \in H$

$\Rightarrow x(y^{-1})^{-1} = xy \in H$

H is closed for the operation G.

$\therefore$ H is a subgroup of G which proves that the given condition is sufficient for H to be a subgroup.

Art 2.1

Exercise.

Dummit & Foote

Que 1: Prove that subset is a subgroup of the given group:

(a) the set of comple numbers of the form $a + ai$, $a \in \mathbb{R}$ (under addition).

Proof: Suppose $G = \{ a + ai ; a \in \mathbb{R} \}$, $G \neq \phi$

Now for any $a + ai$, $b + bi \in G$

$(a + ai) - (b + bi) = (a - b) + (a - b)i \in G$.

Hence G is a subgroup of $\mathbb{C}$.

(b) set of complex number of absolute value 1, i.e. the unit circle in the complex plane (under multiplication).

Proof: let $G = \{ Z = a + ib : |Z| = 1 \}$

let $\bar{z}$ be the conjugate of z.

$\because G \neq \phi$ and for any $z, w \in G$, we have

$|zw^{-1}| = |z||w^{-1}| = \dfrac{|z| \cdot |\bar{w}|}{|w|^2} = 1$

$\therefore zw^{-1} \in G$. Therefore G is subgroup of $\mathbb{C}^*$.

Que 3: Show that the following subsets of the dihedral group D_8 are actually subgroups: $D_{2n} = \{ rs : r^n = s^2 = 1, rs = sr^{-1} \}$

(a) $\{ 1, r^2, s, sr^2 \}$

Proof: Since this is a finite subset so it suffices to show that it is closed under the group operation of composition. We have

$r^2(r^2) = 1$, $r^2(s) = sr^2$, $r^2(sr^2) = s$

$s(r^2) = sr^2$, $s(s) = s^2 = 1$, $s(sr^2) = s^2r^2 = r^2$

$sr^2(r^2) = s$, $sr^2(s) = r^2$, $sr^2(sr^2) = 1$

$\therefore$ This subset is a subgroup of D_8

(b) $\{1, r^2, sr, sr^3\}$.

Proof: We find that

$r^2(r^2) = sr(sr) = (sr^3)(sr^3) = 1$

$r^2(sr) = sr(r^2) = sr^3$

$r^2(sr3) = sr3(r^2) = sr$.

$sr(sr3) = sr3(sr) = r^2$.

Hence this subset is a subgroup.