ASSIGNMENT-2

Q1) a) Rephrase the definitions for the reflexive, symmetric, transitive, & antisymmetric properties of a relation R (on a set A), using quantifiers.

→ If R is a relation from A to A, then $R \subseteq A \times A$; we say that R is a relation on A.

A relation R on A is said to be:-

* reflexive:- if $(a,a) \in R \ \forall a \in A$,
* symmetric:- if $(a,b) \in R \Rightarrow (b,a) \in R \ \forall a,b \in A$
* transitive:- if $[(a,b) \in R \wedge (b,c) \in R] \Rightarrow (a,c) \in R \ \forall a,b,c \in A$.
* antisymmetric:- if $[(a,b) \in R \wedge (b,a) \in R] \Rightarrow a=b \ \forall a,b \in A$.

b) Use of results of part (a) to specify when a relation R (on a set A) is (i) not reflexive (ii) not symmetric, (iii) not transitive & (iv) not antisymmetric.

→ If R is a relation from A to A, then $R \subseteq A \times A$; we say that R is a relation on A.

A relation R on A is said to be:-

* not reflexive:- if $(a,a) \notin R \ \forall a \in A$
* not symmetric:- if $(a,b) \in R \Rightarrow (b,a) \notin R \ \forall a,b \in A$
* not transitive:- if $[(a,b) \in R \wedge (b,c) \in R] \Rightarrow (a,c) \notin R \ \forall a,b,c \in A$
* not antisymmetric:- if $[(a,b) \in R \wedge (b,a) \in R] \Rightarrow a \neq b \ \forall a,b \in A$.

(2) For each of the following statements about relations on a set A, where $|A| = n$, determine whether the statement is true or false. If it is false, give a counterexample.

(09)

a) if R is a relation on A & $|R| \geq n$, then R is reflexive.

b) if R_1, R_2 are relations on A & $R_2 \supseteq R_1$, then R_1 reflexive, (symmetric, antisymmetric, transitive) $\Rightarrow$ R_2 reflexive (symmetric, antisymmetric, transitive).

c) if R_1, R_2 are relations on A & $R_2 \supseteq R_1$ then R_2 reflexive (symmetric, antisymmetric, transitive) $\Rightarrow$ R_1 reflexive (symmetric, antisymmetric, transitive)

d) if R is an equivalence relation on A then $n \leq |R| \leq n^2$.

Ans:- a) No (False), not everytime R is reflexive $|R| \geq n$

Counterexample:- False: let $A = \{1, 2\}$, & $R = \{(1,2), (2,1)\}$

there R is not reflexive.

b) We have, $R_1, R_2 \rightarrow$ relations

$R_2 \supseteq R_1$

let $A = \{1, 2\}$, $R_1 = \{(1,1)\}$, & $R_2 = \{(1,1), (1,2)\}$.

(i) Reflexive :- True. (because if R_1 is reflexive than it must contain all elements of A, & R_2 is also reflexive as $R_2 \supseteq R_1$ or we can say if R_1 is reflexive

$\Rightarrow R_2$ is reflexive :- as $(R_1 = R_2)$

(ii) Symmetric :- False.

let $A = \{1, 2\}$, $R_1 = \{(1,1)\}$ & $R_2 = \{(1,1), (1,2)\}$

So, R_1 is symmetric but R_2 is not symmetric.

(iii) Antisymmetric :- False

let $A = \{1, 2\}$, $R_1 = \{(1,2)\}$ & $R_2 = \{(1,2), (2,1)\}$

So, R_1 is antisymmetric but R_2 is not antisymmetric.

(IV) Transitive :- False

$A = \{1, 2\}$, $R_1 = \{(1,2)\}$ & $R_2 = \{(1,2), (2,1)\}$

Here R_1 is transitive but R_2 is not transitive.

c) We have $R_1, R_2 \rightarrow$ relations on A.

$R_2 \supseteq R_1$

(i) Reflexive :- False.

Let ($A = \{1, 2\}$), $R_2 = \{(1,1), (2,2)\}$ & $R_1 = \{(1,1)\}$

Here, R_2 is reflexive but R_1 is not reflexive

(II) Symmetric :- False

Let $A = \{1, 2\}$, $R_2 = \{(1,2), (2,1)\}$, $R_1 = \{(1,2)\}$

Here, R_2 is symmetric but R_1 is not symmetric

(III) Antisymmetric :- True.

Let $A = \{1, 2\}$, $R_2 = \{(1,2), (1,1)\}$, $R_1 = \{(1,2)\}$

Hence both R_1 & $R_2 \rightarrow$ Antisymmetric

(IV) Transitive :- False.

$A = \{1, 2\}$, $R_2 = \{(1,2), (2,1), (1,1)\}$ $R_1 = \{(2,1)\}$

Hence, R_2 is transitive but R_1 is not transitive

d) True. (As, if $\mathbb{R}$ is an equivalence rel^n
then n no. of elements are min. required for th
$\mathbb{R}$ to be Reflexive also, n^2 no. of max eleme
required for the $\mathbb{R}^n$ to be reflexive, symmetri
& transitive.)

(1)

(3) If $A = \{w, x, y, z\}$, determine the number of relations on A that are (a) reflexive $\rightarrow$ $N=4$ so, $2^{4^2-4} = 2^{12}$ (ans)

(b) symmetric $\rightarrow$ $N=4$ so, $2^{4^2+4/2} = 2^{10}$ (ans)

(c) reflexive & symmetric

$\hookrightarrow N=4$ so, $2^{\frac{4^2-4}{2}} = 64$ (ans)

(d) reflexive & contain (x,y).

$\rightarrow$ Any rel^n that is reflexive & contains (x,y) certainly contains 5 pairs (all of the form (a,a) & (x,y)).
All of the other 11 pairs can either be included or not, so we choose b/w 2 options eleven times

giving us $2^{11} = 2048 \rightarrow$ (ans)

(e) symmetric & contain (x,y).

$\rightarrow$ All the symmetric rel^n can be divided into 2 categories: those who don't contain $\{(x,y), (y,x)\}$ & those who do & there is a one to one correspondence b/w them, hence the required answer is

$\frac{1}{2} \cdot 2^{10} = 2^9 \rightarrow$ (ans)

(f) Antisymmetric : $N=4$, $2^4 \cdot 3^{\frac{4^2-4}{2}} = 2^4 \cdot 3^6$.

(g) Antisymmetric & contain (x,y):

$\rightarrow$ If a pair (a,b) is in such a rel^n, since it is antisymmetric containing (x,y) guarantees non-containment (y,x).
Thus, we can divide all the antisymmetric relations into 3 categories, containing exactly (x,y), but not (y,x); containing (y,x), but not (x,y); & containing none.
There is a one-to-one correspondence b/w each of these, So, each category contains exactly one-third of the

$\{1, 2\}$
$(1,2)$ $(2,1)$
(1) $(1,2)$
or $(2,1)$
$\{(1,1), (2,2), (1,2)\}$
$\frac{2^2-2}{2} = 2$
$\{(1,1)\ (1,2)\}$ $2+\frac{2^2-2}{2}$
$2+1=3$

antisymmetric relations, i.e.

$$\frac{1}{3} \cdot 2^4 \cdot 3^6 = 2^4 \cdot 3^5 \rightarrow \text{(ans)}$$

(b) symmetric & antisymmetric :-

→ If R is a relation on A which is both symmetric & antisymm-
etric, then the set $S = \{(a,b) : a, b \in A,\ a \neq b\} \subseteq R$
is empty, hence there are 2 options for each of the pairs
$\{(a,a),\ a \in A\}$, hence, the required answer is 2^4.

(c) Reflexive, symmetric & antisymmetric.

→ In h, such relations can only contain pair of form (a, a)
for some $a \in A$. However, reflexivity also means that they must
contain all such pairs. Thus, only one among the 2^4 such rel^n.
fits the bill, hence the answer is 1, particularly :-

$$\{(w, w),\ (x, x),\ (y, y),\ (z, z))\}$$

(4) Let A be a set with $|A| = n$, let $\mathbb{R}$ be a relation on A that
is antisymmetric. What is the maximum value for $|R|$?
How many antisymmetric rel^n can have this size ?

→ Each element in A can be related to itself, so there are n
such pairs. For the remaining pairs, if (a, b) is in R, then (b, a)
cannot be in R due to the antisymmetry. So, for each pair of
distinct elements, we can include atmost one of (a, b) & (b, a)
in R. There are $n(n-1)/2$ such pairs of distinct elements in
A. Therefore, the max. size of R is $n + n(n-1)/2 = n(n+1)/2$.
The no. of antisymmetric relations that can have this size
is $2^{\frac{n^2-n}{2}}$ → (ans)

(13)

(5) With $A = \{1, 2, 3, 4\}$, let $R = \{(1,1), (1,2), (2,3), (3,3), (3,4), (4,4)\}$

be a relation on A. Find 2 relations S & T on A where $S \neq T$. but $R \circ S = R \circ T = \{(1,1), (1,2), (1,4)\}$.

ans:- We have:- $R = \{(1,1), (1,2), (2,3), (3,3), (3,4), (4,4)\}$.

$S = \{(2,1), (2,2), (1,4)\}$.

& $T = \{(1,1), (1,2), (1,4)\}$.

(6) Let A be a set with $|A| = n$ & let R be an equivalence relation on A with $|R| = r$. Why is $r-n$ always even ?

ans:- In R, n no. of rel^n. are there which are reflexive

So, $r-n \Rightarrow$ it counts the elements in R of the form (x, y), where $x \neq y$.

So, R is an equivalence rel^n :-

So, R is also symmetric, (if it contains (a,b) then (b,a) also there)

hence, $r-n$ is even.

(7) Determine how many integer solutions, there are to

$x_1 + x_2 + x_3 + x_4 = 19$, if

a) $0 \leq x_i$ for all $1 \leq i \leq 4$

b) $0 \leq x_i < 8 \ \forall \ 1 \leq i \leq 4$

c) $0 \leq x_1 < 5$, $0 \leq x_2 \leq 6$, $3 \leq x_3 \leq 7$, $3 \leq x_4 \leq 8$.

s:- a) Given $x_1 + x_2 + x_3 + x_4 = 19$

$0 \leq x_i$, $1 \leq i \leq 4 \Rightarrow \binom{19+4-1}{19} = \binom{22}{19} = \binom{22}{3}$

$= \frac{\cancel{22}^{11} \times \cancel{21}^{7} \times 20}{\cancel{3} \times \cancel{2} \times 1} = 77 \times 20 = 1540 \rightarrow$ (ans)

(b) for $1 \le i \le 4$:

$N = 1540$,

Non accepting state ; $x_i \ge 8 \Rightarrow y_i = x_i - 8 \Rightarrow \boxed{x_i = y_i + 8}$

$N(C_1) = x_1 + x_2 + x_3 + x_4 = 19$

$\Rightarrow y_1 + 8 + x_2 + x_3 + x_4 = 19$

$\Rightarrow \boxed{y_1 + x_2 + x_3 + x_4 = 11}$

So, $\binom{11+4-1}{11} = \binom{14}{11} = \binom{14}{3} = N(C_2) = N(C_3) = N(C_4)$.

So, $N(C_1) = N(C_2) = N(C_3) = N(C_4) = \frac{14 \times 13 \times \cancel{12}^{2}}{\cancel{3 \times 2 \times 1}} = 364$.

$N(C_1C_2) \Rightarrow x_1 + x_2 + x_3 + x_4 = 19$

$\rightarrow y_1 + 8 + y_2 + 8 + x_3 + x_4 = 19$

$\Rightarrow y_1 + y_2 + x_3 + x_4 = 3$

So, $N(C_1C_2) = \binom{3+4-1}{3} = \binom{6}{3} = N(C_1C_3) = N(C_1C_4) = N(C_2C_3) = N(C_2C_4) = N(C_3C_4) = 20$.

$\therefore N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4}) = N - (N(C_1) + N(C_2) + N(C_3) + N(C_4))$
$+ (N(C_1C_2) + N(C_1C_3) + N(C_2C_3))$
$\cdots N(C_3C_4))$
$= 1540 - 4 \times 364 + 120 (6 \times 20) = 204$ (ans)

(c) We have:- $x_1 + x_2 + x_3 + x_4 = 19$

$0 \le x_1 \le 5 \longrightarrow x_1 \ge 6$ (Unaccepted state)

$0 \le x_2 \le 6 \longrightarrow x_2 \ge 7$

$3 \le x_3 \le 7 \Rightarrow 0 \le x_3 - 3 \le 4 \Rightarrow 0 \le y_3 \le 4 \rightarrow y_3 \ge 5$

$(x_3 - 3)$

$(3 \le x_4 \le 8) \Rightarrow 0 \le x_4 - 3 \le 5$

$\Rightarrow 0 \le y_4 \le 5$

$(y_4 = x_4 - 3)$

$(y_4 \ge 6)$

So, new, eq^n:- $x_1 + x_2 + (y_3 + 3) + (y_4 + 3) = 19$

$$\Rightarrow \boxed{x_1 + x_2 + y_3 + y_4 = 13}$$

$$\text{So, } N = \binom{13+4-1}{13} = \binom{16}{13} = \binom{16}{3} = \frac{\cancel{16}^{8} \times \cancel{15}^{5} \times 14}{\cancel{3} \times \cancel{2} \times 1} = \underline{\underline{560}}$$

Now, $N(C_1) = x_1 + x_2 + y_3 + y_4 = 13$

$\Rightarrow (y_1 + 6) + x_2 + y_3 + y_4 = 13$

$\Rightarrow y_1 + x_2 + y_3 + y_4 = 7$.

$$= \binom{7+4-1}{7} = \binom{10}{7} = \binom{10}{3}$$

$= 120$.

$N(C_2) \Rightarrow x_1 + x_2 + y_3 + y_4 = 13$

$\Rightarrow x_1 + (y_2 + 7) + y_3 + y_4 = 13$

$\Rightarrow x_1 + y_2 + y_3 + y_4 = 6$

$$= \binom{6+4-1}{6} = \binom{9}{6} = \binom{9}{3} = 84.$$

$N(C_3) = x_1 + x_2 + y_3 + y_4 = 13$

$\Rightarrow x_1 + x_2 + (z_3+5) + y_4 = 13$

$\Rightarrow x_1 + x_2 + z_3 + y_4 = 8$

$N(C_3): \binom{8+4-1}{8} = \binom{11}{8} = \binom{11}{3}$

$= \frac{11 \times \cancel{10}^{5} \times \cancel{9}^{3}}{\cancel{3} \times \cancel{2} \times 1} = 165$

Now taking combination of 2 cases :-

$N(C_1 C_2) = 1$

$N(C_1 C_3) \Rightarrow x_1 + x_2 + y_3 + y_4 = 13$

$(y_1+6) + x_2 + (z_3+5) + y_4 = 13$

$\Rightarrow y_1 + x_2 + z_3 + y_4 = 2$.

$N(C_1 C_3): \binom{2+4-1}{2} = \binom{5}{2} = \frac{5 \times 4}{2} = 10$.

$N(C_1 C_4) = \binom{4}{1} = 4$

$N(C_2 C_3) = \binom{4}{1} = 4$

$N(C_2 C_4) = 1$ & $N(C_3 C_4) = \binom{5}{2} = 10$.

Now, $N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4}) = N - (120+120+84+165) - (1+1+4+4+10+10)$

$= 560 - 489 + 30$

$= 101 \longrightarrow$ (ans)

(8) For the directed graph $G = (V, E)$ in fig 7.12 classify each of the following statements as true or false.

(17)

a) Vertex c is the origin of two edges in G

→ True

b) Vertex g is adjacent to vertex h

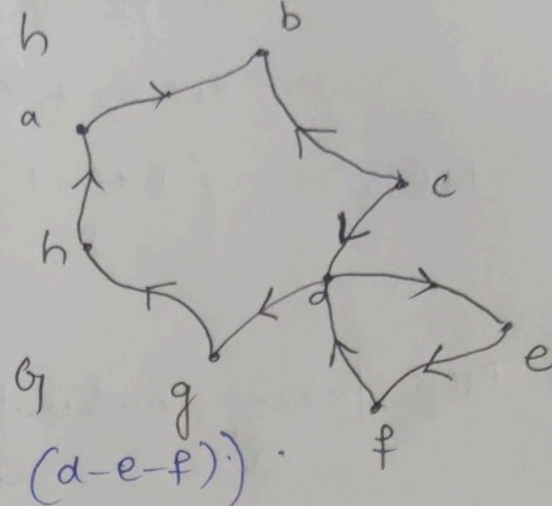
→ True

c) There is a directed path in G from d to b

→ True

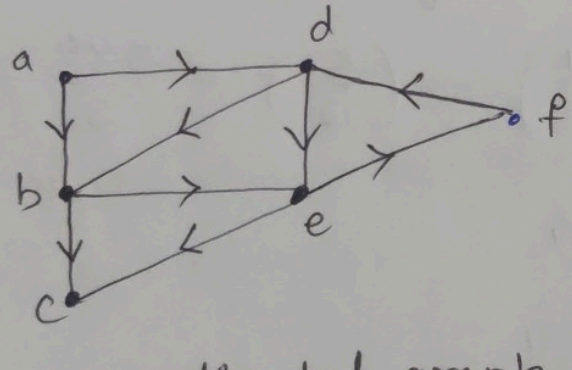
d) There are 2 directed cycles in G

→ False (only 1 directed cycle (d-e-f)).



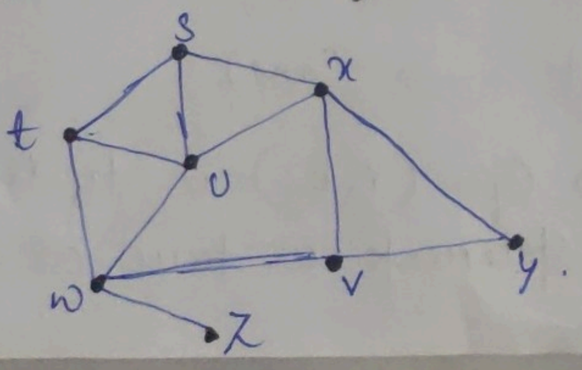
(9) a) Draw the digraph $G_1 = (V_1, E_1)$ where $V_1 = \{a, b, c, d, e, f\}$ & $E_1 = \{(a,b), (a,d), (b,c), (b,e), (d,b), (d,e), (e,c), (e,f), (f,d)\}$.

Ans:- a)

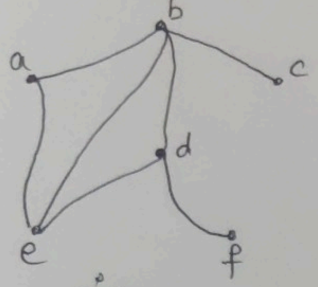


b) Draw the undirected graph $G_2 = (V_2, E_2)$ where $V_2 = \{s, t, u, v, w, x, y, z\}$ & $E_2 = \{\{s,t\}, \{s,u\}, \{s,x\}, \{t,u\}, \{t,w\}, \{u,w\}, \{u,x\}, \{v,w\}, \{v,x\}, \{v,y\}, \{w,z\}, \{x,y\}\}$.

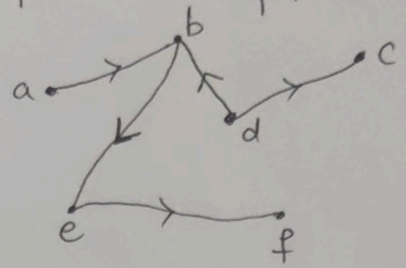
Ans:- b)



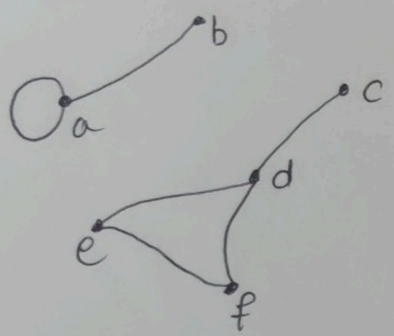
(10) For $A = \{a, b, c, d, e, f\}$, each graph or digraph Fig 7.13 represents a relⁿ R on A. Determine the relations.



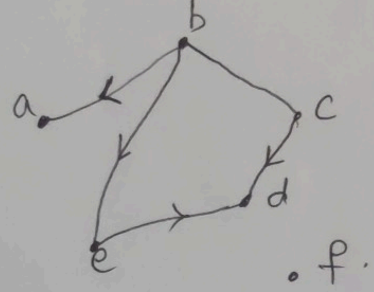
(i)



(ii)



(iii)



(iv)

Ans:- (i) $R_1 = \{(a,b), (b,a), (a,e), (e,a), (b,c), (c,b), (b,d), (d,b), (b,e), (e,b), (d,e), (e,d), (d,f), (f,d)\}$.

(ii) $R_2 = \{(a,b), (b,e), (d,b), (d,e), (e,f)\}$

(iii) $R_3 = \{(a,a), (a,b), (b,a), (c,d), (d,c), (d,e), (e,d), (d,f), (f,d), (e,f), (f,e)\}$.

(iv) $R_4 = \{(b,a), (b,c), (c,b), (b,e), (c,d), (e,d)\}$.

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