Ch.5 – The Pigeon-hole Principle & its applications

§1. Two forms of the Pigeon-hole principle

PHP – Simple Form : If $n+1$ pigeons are distributed among n holes, then there exists a hole which contains at least two pigeons.

Proof: Suppose there is no hole with at least 2 pigeons. Then every hole will have at most 1 pigeon. Since there are n holes, there will be at most $1+1+1+\cdots+1$ (n times) $= n$ pigeons, which contradicts the fact that $n+1$ pigeons were distributed into the n holes (or boxes). So there must exist a hole with at least 2 pigeons.

PHP – General Form : If k pigeons are distributed among n holes then there exists a hole which contains at least $\lfloor \frac{k-1}{n} \rfloor + 1$ pigeons.

Proof: Suppose there is no hole with at least $\lfloor \frac{k-1}{n} \rfloor + 1$ pigeons. Then every hole will have at most $(\lfloor (k-1)/n \rfloor)$ pigeons. So the total no. of pigeons in the holes will be $\leq n . \lfloor (k-1)/n \rfloor$

$$\leq n . \frac{k-1}{n} = k-1$$

which contradicts the fact that k pigeons were distributed into the holes. Hence the must exist a hole with at $\lfloor \frac{k-1}{n} \rfloor + 1$ pigeons.

Ex.1 Prove that in any group of 13 people we can always find 2 people whose birthday falls in the same month.

Sol. Let the 12 months of the year be the holes and the 13 people be the pigeons. If we distribute the 13 people among the 12 months according to their birthdays, there must exist a month which has at least 2 people. So two people will have birthdays which fall in the same month.

Ex.2 Suppose we have 10 married couples and we want to select a team of 11 people from the 10 couples. Prove that we will always have at least one married couple in the team.

Sol. Let the 10 married couples be the holes and the team of 11 chosen players be the pigeons. Now if we assign each of the chosen players to the married couple of which they are a part, then there must exist a married couple (hole) which contributed 2 players to the team (otherwise the team will have at most 10 players).

Ex.3 Prove that in any group of 733 people, there are at least 3 people with same birthday.

Sol. Let the holes be the 366 possible birthdays (Feb. 29th being included) and assign the 733 to the holes according to their birthdays. Then by the General form of the PHP, there must be a hole (birth date) in which there are at least $\left\lfloor \frac{733-1}{366} \right\rfloor + 1 = \left\lfloor \frac{732}{366} \right\rfloor + 1 = \lfloor 2 \rfloor + 1 = 3$ people.

PHP: Function Form: If $f: P \to H$ is a function and $|P| > |H|$, then there exists two values $a_1, a_2 \in P$ such that $f(a_1) = f(a_2)$. (This is basically the same as the simple form)

Prop. 1 Let S be any $(n+1)$-subset of $\{1, 2, 3, \ldots, 2n\}$. Then we can find 2 elements of S such that one divides the other.

Proof: Let $H = \{1, 3, 5, \ldots, 2n-1\}$ and defined $f: S \to H$ by $f(a) =$ the unique odd no. c such that $a = 2^b \cdot c$, where $b \in \mathbb{N}$ & $c \in \mathbb{Z}^+$. Since S has $n+1$ elements and H has only n elements, we must have $f(a_1) = f(a_2) = c$ for two elements $a_1, a_2 \in S$. But then this means that $a_1 = 2^{b_1} \cdot c$ and $a_2 = 2^{b_2} \cdot c$. So the $\min\{a_1, a_2\}$ will divide $\max\{a_1, a_2\}$. Thus S will contain 2 elements such that one divides the other.

Note: $S_0 = \{n+1, n+2, n+3, \ldots, 2n\}$ is an n-subset of $\{1, 2, 3, \ldots, 2n\}$ in which no element divides another.

Prop. 2 Let $\underset{\sim}{s} = \langle a_1, \dots, a_n \rangle$ be any seq. of n integers. Then there is a segment of $\underset{\sim}{s}$ in which the consecutive terms add up to a multiple of n.

Proof Let $H = \{0, 1, 2, \dots, n-1\}$ and put
$P = \{0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + \dots + a_n\}$
Then H has n elements and P has $n+1$ elements. Now define $f: P \to H$ by

$$f(k) = k \pmod n.$$

Since $|P| > |H|$, there exists 2 elements $k_1 = a_1 + a_2 + \dots + a_i$ & $k_2 = a_1 + a_2 + \dots + a_j$ with $0 \leq i < j \leq n$ such that $f(k_2) = f(k_1)$.

So for some $0 \leq i < j \leq n$ we have have
$a_1 + a_2 + \dots + a_i + \dots + a_j \equiv a_1 + a_2 + \dots + a_i \pmod n$
Hence $a_{i+1} + a_{i+2} + \dots + a_j \equiv 0 \pmod n$
So the segment $\langle a_{i+1}, a_{i+2}, \dots, a_{j-1}, a_j \rangle$ will add up to a multiple of n.

Prop. 3 Let $\underset{\sim}{s} = \langle a_1, a_2, a_3, \dots, a_{n^2+1} \rangle$ be a sequence of n^2+1 distinct (different) real numbers. Then there exists a strictly increasing subsequence of $\underset{\sim}{s}$ of length $n+1$ or there exists a strictly decreasing subsequence of $\underset{\sim}{s}$ of length $n+1$.

Proof: Suppose there is no strictly increasing subsequence of $\underset{\sim}{s}$ of length $n+1$. We will show that there must exist a strictly decreasing subsequence of $\underset{\sim}{s}$ of length $n+1$. Let $f(a_i)$ = the length of the longest str. incr. subseq. of $\underset{\sim}{s}$ starting at a_i. Then for each i, $1 \leq f(a_i) \leq n$, because $\underset{\sim}{s}$ has no str. incr. subseq. of length $\geq n+1$.

Now assign the n^2+1 terms of $\underset{\sim}{s}$ to the n boxes (holes) $1, 2, 3, \ldots, n$ according to the value of $f(a_i)$. Then we must have some box with at least $\left\lfloor \frac{(n^2+1)-1}{n} \right\rfloor + 1 = \left\lfloor \frac{n^2}{n} \right\rfloor + 1 = n+1$ terms by the General form of the PHP. So we can find a subsequence $\langle a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}} \rangle$ of $n+1$ terms of $\underset{\sim}{s}$ such that

$$f(a_{i_1}) = f(a_{i_2}) = \cdots = f(a_{i_{n+1}}).$$

Now consider a_{i_1} and a_{i_2}. We cannot have $a_{i_1} < a_{i_2}$ otherwise we would get $f(a_{i_1}) > f(a_{i_2})$. So $a_{i_1} > a_{i_2}$ because all the a_i's are different. Similarly if $a_{i_2} < a_{i_3}$, then $f(a_{i_2}) > f(a_{i_3})$, so we must have $a_{i_2} > a_{i_3}$. The same argument tells us that we must also have $a_{i_3} > a_{i_4}$, $a_{i_5} > a_{i_6}$, $\ldots$, and $a_{i_n} > a_{i_{n+1}}$

So $a_{i_1} > a_{i_2} > a_{i_3} > \cdots > a_{i_n} > a_{i_{n+1}}$. Thus $\langle a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}} \rangle$ is a strictly decreasing subsequence of $\underset{\sim}{s}$ of length $n+1$.

Hence $\underset{\sim}{s}$ always has a strictly incr. subseq. of length $n+1$ or a strictly decr. subseq. of length $n+1$.

Ex. 4. Consider the sequence $\langle 5,3,9,2,7,0,8,4,6,1\rangle$ of $3^2+1=10$ terms. Let us show the longest increasing subsequences beginning at a_i.

	$\langle 5,$	$3,$	$9,$	$2,$	$7,$	$0,$	$8,$	$4,$	$6,$	$1\rangle$
	7	7		4	8	4		6		
	8	8		6		6				
$f(a_i)=$	3	3	1	3	2	3	1	2	1	1

Now if we look at the terms corresponding to the box (hole) with the four 3's, we see that we get the terms $\langle 5,3,2,0\rangle$ which is indeed a decr. subsequence of length $3+1$.

Also if we look the terms corresp. to the box with the four 1's, we see that we get the terms $\langle 9,8,6,1\rangle$ which is another decr. subseq. of length $3+1$.

Ex. 5 If we look at the sequence of 4^2 terms
$\langle 4,3,2,1,\ 8,7,6,5,\ 12,11,10,9,\ 16,15,14,13\rangle$
we can easily check that there is no increasing subseq. of length $4+1$, nor any decreasing subsequence of length $4+1$.

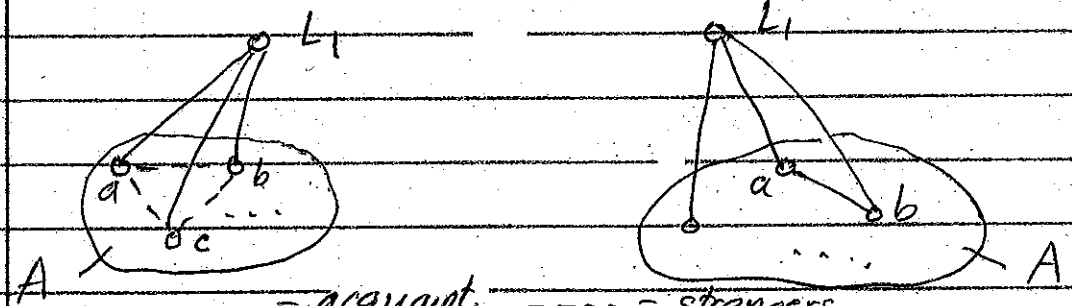
So Prop. 3 is the best possible result. If s was a sequence of only n^2 terms, we could not guarantee an incr. subseq. of length $n+1$ or a decr. subseq. of length $n+1$.

§2. Ramsey theory and Ramsey numbers.

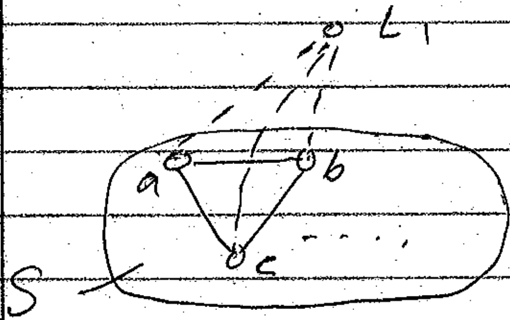
Prop. 4 In any group G of 6 ladies, there always exist either 3 mutual acquaintances or 3 mutual strangers (non-acquaintances)

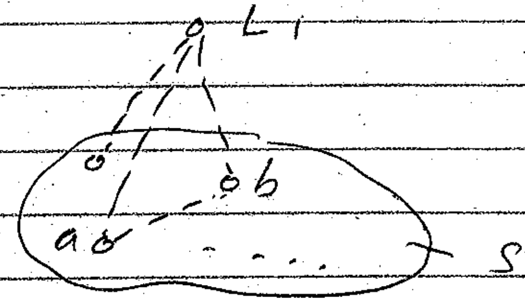
Proof: Choose one out the 6 ladies and call her L_1. Then let A = set of all acquaintances of L_1 in G and S = set of all strangers to L_1 in G. Since A & S are disjoint sets and $|A \cup S| = 5$, we must have $|A| \geq 3$ or $|S| \geq 3$.

Case (i): $|A| \geq 3$. In this case we know that either all the ladies in A are mutual strangers or A contains 2 mutual acquaintances. Now if the ladies in A are mutual strangers, then we know three mutual strangers exists in G because $|A| \geq 3$. And if A contains 2 mutual acquaintances, then by adding L_1 we will get 3 mutual acquaintances in G.



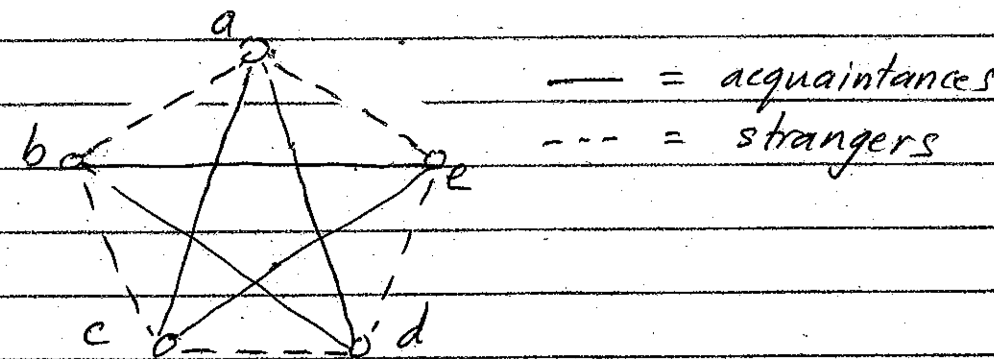
Case (ii) $|S| \geq 3$. In this we know that either all the ladies in S are mutual acquaintances or there S contains 2 mutual strangers. Now if the ladies in S are all mutual acquaintances, then we know that 3 mutual acquaintances exists in G because $|S| \geq 3$. And if S contains 2 mutual strangers, then by adding L_1, we will get 3 mutual strangers in G.





So in all possible scenarios we found either 3 mutual acquaintances in G or 3 mutual strangers in G. This completes the proof.

Ex.1 If we have a group G of 5 ladies, then we cannot guarantee that there will always exist 3 mutual acquaintances or 3 mutual strangers. The scenario below gives one example in which we have neither 3 mutual acquaintances nor 3 mutual strangers



There is no solid triangle or broken-line triangle.

Def. We define the Ramsey number $R(a,b)$ to be the smallest number k such that in any set of k ladies, we can find a mutual acquaintances or b mutual strangers.

Ex.2 From Ex.1 and Prop. 4 we can deduce that $R(3,3) = 6$. It has also been shown that
$$R(3,4) = R(4,3) = 9,$$
$$R(3,5) = R(5,3) = 14,$$
$$R(4,4) = 18, \quad R(4,5) = R(5,4) = 25,$$
and that $43 \leq R(5,5) \leq 48$. These results all take a lot of effort to prove.

In the next few results we will just show that $R(4,3) \leq 10$, $R(3,4) \leq 10$, & $R(4,4) \leq 20$.

Prop.5 (a) In any group G of 10 ladies, there always exist 4 mutual acquaintances or 3 mutual strangers.

(b) In any group H of 10 ladies there always exist 3 mutual acquaintances or 4 mutual strangers.

Proof: (a) Choose one lady out of the 10 & call her L_1.
Let A = set of all acquaintances of L_1 in G
& S = set of all strangers to L_1 in G.
Since A & S are disjoint and $|A \cup S| = 9$, we must have $|A| \geq 6$ or $|S| \geq 4$.
(If $|A| < 6$ & $|S| < 4$, then $|A| \leq 5$ & $|S| \leq 3$ and so $|A \cup S| = |A| + |S| \leq 5 + 3 \leq 8$ which contradicts the fact that $|A \cup S| = 9$.)

Case(i): $|A| \geq 6$. In this case we know from Prop. 4, that A contains 3 mutual acquaintances or A contains 3 mutual strangers. Now if A contains 3 mutual strangers, then we get 3 mutual strangers in G. And if A contains 3 mutual acquaintances, then we can add L_1 to get 4 mutual acquaintances in G.

Case(ii) $|S| \geq 4$. In this case we know that all of the ladies in S are mutual acquaintances or there are 2 mutual strangers in S. Now if all the ladies in S are mutual acquaintances, then we will get 4 mutual acquaintances in G because $|S| \geq 4$. And if S contained 2 mutual strangers, then we can add L_1 to get 3 mutual strangers in G.

So in all the scenarios we got either 4 mutual acquaintances in G or 3 mutual strangers in G. This proves part (a).

(b) Choose one lady out of the 10 in H and call her L_1. Let A = set of all acquaintances of L_1 in H and S = set of all strangers to L_1 in H. Since A & S are disjoint and $|A \cup S| = 9$, we must have $|A| \geq 4$ or $|S| \geq 6$.
(If $|A| < 4$ & $|S| < 6$, then $|A| \leq 3$ & $|S| \leq 5$ & we would get $|A \cup S| = |A| + |S| \leq 3 + 5 = 8$ – a contradiction.)

Case (i): $|A| \geq 4$. In this case we know that all the ladies in A are mutual strangers or A contains 2 mutual acquaintances. Now if all the ladies in A are mutual strangers, then we will get 4 mutual strangers in H, because $|A| \geq 4$. And if A contains 2 mutual acquaintances, then we can add L_1 to get 3 mutual acquaintances in H.

Case (ii) $|S| \geq 6$. In this case we know from Prop. 4 that S contains 3 mutual acquaintances or 3 mutual strangers. Now if S contains 3 mutual acquaintances, then we get 3 mutual acquaintances in H. And if S contains 3 mutual strangers, then we can add L_1 to get 4 mutual strangers in H.

So in all the scenarios, we got either 3 mutual acquaintances in H or 4 mutual strangers in H. This proves part (b)

Prop. 6 : In any group J of 20 ladies, there exists 4 mutual acquaintances or 4 mutual strangers.

Proof Choose one lady out of the 20 ladies in J and call her L_1. Let A = set of all acquaintances of L_1 in J and S = set of all strangers to L_1 in J. Since A & S are disjoint & $|A \cup S| = 19$, $|A| \geq 10$ or $|S| \geq 10$.

Case (i): $|A| \geq 10$. In this case, we know from Prop 5(b) that A contains 3 mutual acquaintances or A contains 4 mutual strangers. Now if A contains 3 mutual acquaintances, then we can add L_1 to get 4 mutual acquaintances in J. And if A contains 4 mutual strangers, then J will also contain 4 mutual strangers.

Case (ii) $|S| \geq 10$. In this case, we know from Prop 5(a) that S contains 4 mutual acquaintances or 3 mutual strangers. Now if S contains 4 mutual acquaintances, then J will contain 4 mutual acquaintances. And if S contains 3 mutual strangers, then we can add L_1 to get 4 mutual strangers in J.

So in all the scenarios we either got 4 mutual acquaintances in J or 4 mutual strangers in J. Hence this completes the proof of Prop 6.

Note 1. It can be shown that $R(4,3) = 9 =$ [redacted] $R(3,4)$. From this we can get $R(4,4) \leq 9 + 9 = 18$, just as in Prop 6.

2. We can show by this same technique that $R(p,q) \leq R(p,q-1) + R(p-1,q)$.

3. We can also show $R(p,q) \leq \dfrac{(p+q-2)!}{(p-1)!\,(q-1)!}$ and $R(p,q) = R(q,p)$.

END