1(a) Let a_i = no. of games played by the end of the i-th day
Then $a_{i+1} - a_i \geq 1$ because at least one game was played on the i-th day. Also $a_{77} \leq 12.11 = 132$ because each week at most 12 games are played.
So

$$1 \leq a_1 < a_2 < a_3 < \cdots < a_{77} \leq 132$$

And for each $k = 1, 2, \ldots, 21$

$$1+k \leq a_1 + k < a_2 + k < \cdots < a_{77} + k \leq 132 + k \leq 153$$

Now look at the sequence

$$a_1, a_2, a_3, \ldots, a_{77}, a_1+k, a_2+k, a_3+k, \ldots, a_{77}+k.$$

This sequence has 154 terms each of which is between 1 & 153 (inclusive). So by the P.H.P, two of the terms must be equal. Since $a_1, \ldots, a_{77}$ are all distinct & $a_1+k, \ldots, a_{77}+k$ are all distinct we must have

$$a_i = a_j + k \qquad \text{for some } 1 \leq i, j \leq 77$$

So $a_i - a_j = k$ and hence on the $(j+1)$-th, $(j+2)$-th, $\ldots$, $j+(i-j)$-th days a total of k games were played.

(b) No, not with the proof used in application 4. If $k = 22$, the sequence

$$a_1, a_2, a_3, \ldots, a_{77}, a_1+k, \ldots, a_{77}+k$$

will have 154 terms each of which is between 1 & 154 (inclusive). And we would not be able to conclude that two terms must be equal. By using a more sophisticated proof and the fact that at most 12 games are played each week we can, however, find a succession of days in which 22 games are played.

3. We can generalize Application 5 as follows:

If S is any subset of $\{1,2,3,\ldots,2n\}$ with $n+1$ elements, then we can always find two elements of S such that one divides the other

To prove this let $S = \{a_1, a_2, \ldots, a_{n+1}\}$. Now every positive integer can be uniquely expressed in the form $a = 2^b \cdot c$ where c is odd. The number c is called the <u>odd part</u> of a.

Let $D = \{1, 3, 5, \ldots, 2n-1\}$ and define

$f: S \to D$ by

$f(a_i) =$ odd part of a_i

Since D has only n elements, we must have $f(a_i) = f(a_j)$ for some i & some j because S had $n+1$ elements. So

$$a_i = 2^{b_i} \cdot c \quad \& \quad a_j = 2^{b_j} \cdot c$$

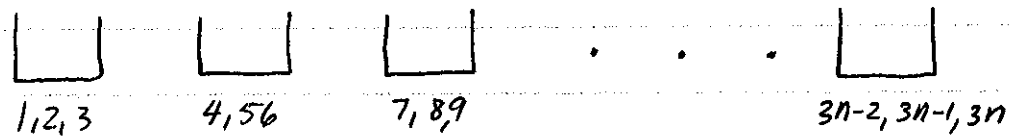
and hence the smaller of a_i & a_j will divide the larger of a_i & a_j.

4. Let S be any subset of $\{1,2,\ldots,2n\}$ with $n+1$ elements. Make n boxes which will accept only the integers shown below the boxes

1,2   3,4   5,6   . . .   2n-1, 2n

Now if we place the $n+1$ integers in S into these n boxes, some box (will get) must get two. So S will always have 2 integers which differ by 1.

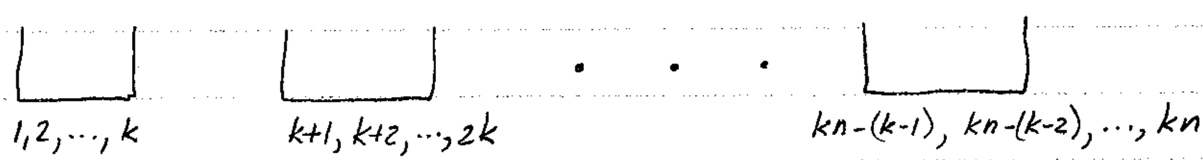
5 Let S be any subset of $\{1,2,3,\ldots,3n\}$ with $n+1$ elements. Make n boxes which will accept only the integers shown below the boxes:



Now if we place the $n+1$ integers in S into these n boxes, some box must get at least two integers from S. So S will always have two integers which differs by at most 2.

6. Let S be any subset of $\{1,2,3,\ldots,kn\}$ with $n+1$ elements. Then S will always have two elements which differ by at most $(k-1)$. This result generalizes Problems #4 & #5.

The proof of this result is similar to that of #4 & #5. Let S be any subset of $\{1,2,3,\ldots,kn\}$ with $n+1$ elements. Make n boxes which will accept only the integers shown below the boxes:



Now if we place the $n+1$ integers in S into these n boxes, then some box must get at least 2 integers from S. These two integers will differ by at most $(k-1)$. So S will always have at least two integers which differ by at most $(k-1)$.

7. Let $S = \{a_1, a_2, a_3, \cdots, a_{52}\}$ be the set of the 52 given integers. Classify these integers into two two types according to $a_i \pmod{100}$.

Type I: those with $0 \leq a_i \pmod{100} \leq 50$

Type II: those with $51 \leq a_i \pmod{100} \leq 99$

Now define a function $f: S \to \{0, 1, 2, 3, \cdots, 50\}$ by

$$f(a_i) = \begin{cases} a_i \pmod{100} & \text{if } a_i \text{ is type I} \\ 100 - a_i \pmod{100} & \text{if } a_i \text{ is type II} \end{cases}$$

Since S has 52 integers and $\{0, 1, 2, \cdots, 50\}$ has only 51 elements, we must have

$$f(a_i) = f(a_j) \text{ for some } 1 \leq i < j \leq 52.$$

Now if a_i & a_j were both of type I or both of type II, then $a_i - a_j = 0 \pmod{100}$ and so $a_i - a_j$ will be divisible by 100.

And if a_i & a_j were of different types, then

$a_i + a_j = 100 = 0 \pmod{100}$ and so $a_i + a_j$ will be divisible by 100.

So we will always be able to find 2 integers in S whose sum or difference is divisible by 100.

8 Consider a rational number such as 12/7 and list the quotient & remainders at each stage obtaining its decimal expansion

$\frac{12}{7} =$ 1 . 7 1 4 2 8 5 7 1 4 ... quotient

5 , 1 , 3 , 2 , 6 , 4 , 5 , 1 , 3 , 2 remainders

8. Now the remainders must be an integer between 1 and 6. So after a string of 6 remainders we are guaranteed that the 7th will repeat one of the previous 6 and this will mean that the remainders will repeat from that point. In our example as soon as we get 5 for a second time, we know that 1, 3, 2 ... will follow. Because the remainders are the same, the quotients will be the same (because we will just be adding 0's and dividing by 7 after a point). So we will get a repeating decimal after some point.

For an arbitry rational number m/n, there are two cases: $n = 2^a 5^b$ for some a & b

$n \neq 2^a 5^b$ for any a & any b.

In the first case, the decimal expansion eventually repeats with 0's (i.e., it terminates) In the second case the decimal will repeat after some point. The length of the portion that repeats is always a divisor of $n-1$

$\frac{1}{7} = 0.142857\ 148257\ 148257\ ...$ period = 6

$\frac{3}{11} = 0.27\ 27\ 27\ 27\ ...$ period = 2

$\frac{1}{13} = 0.076923\ 076923\ 076923\ ...$ period = 6

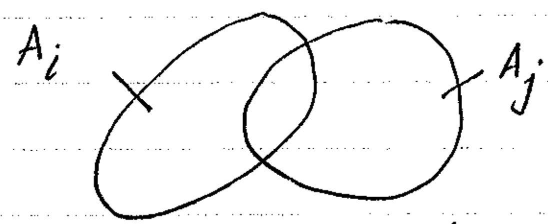
From these examples you can see that the period is not always $n-1$. It is, however, always a divisor of $n-1$.

9. Let $S = \{P_1, \dots, P_{10}\}$ be the set of the 10 people. There are $2^{10}-1 = 1023$ non-empty subsets of S. Let us write these as $\{A_1, \dots, A_{1023}\}$. Since the max. age in 60 and the min age is 1, the sum of the ages in any A_i is between 1 and 600. Since there are 1023 subsets and only 600 possible sums, we must have

sum of ages in A_i = sum of ages in A_j

for some $1 \le i < j \le 1023$.

Now $A_i \not\subset A_j$ otherwise sum of age in A_i will be strictly less than the sum of ages in A_j. Similarly $A_j \not\subset A_i$. Let $A_i' = A_i - A_j$ and $A_j' = A_j - A_i$. Then A_i' and A_j' are non-empty subsets of S



and sum of ages in A_i'

= sum of ages in A_i − sum of ages in $A_i \cap A_j$

= sum of ages in A_j − sum of ages in $A_i \cap A_j$

= sum of ages in A_j'

So we have found two disjoint non-empty subsets of S such that the sum of the ages in each subset are the same.

10. Let a_i = number of hours of TV that the child watches by the end of the i-th day.

Then, as in problem #1,

$$1 \le a_1 < a_2 < a_3 < \cdots < a_{49} \le 7 \cdot 11 = 77$$

10. So $1+20 \leq a_1+20 < a_2+20 < \cdots < a_{49}+20 \leq 77+20 = 97$
Now look at the sequence

$$a_1, a_2, a_3, \ldots, a_{49}, a_1+20, a_2+20, \ldots, a_{49}+20$$

This sequence has 98 terms and each term is between 1 & 97 (inclusive). So by the P.H.P two of these terms must be the same. But $a_1, \ldots, a_{49}$ are all distinct because this is an increasing sequence. Also $a_1+20, \ldots, a_{49}+20$ are all distinct. So we must have

$$a_i = a_j + 20$$

for some $1 \leq i, j \leq 49$. So $a_i - a_j = 20$ and hence on the

$$(j+1)\text{-th}, (j+2)\text{-th}, \ldots, \& \ j+(i-j)\text{-th}$$

days the child would have watched 20 hours of television.

11. Let b_i = no. of hours that the student studies by the end of the i-th day.
Then as in problem #1

$$1 \leq b_1 < b_2 < b_3 < \cdots < b_{37} \leq 60$$

So

$$1+13 \leq b_1+13 < b_2+13 < \cdots < b_{37}+13 \leq 60+13 = 73.$$

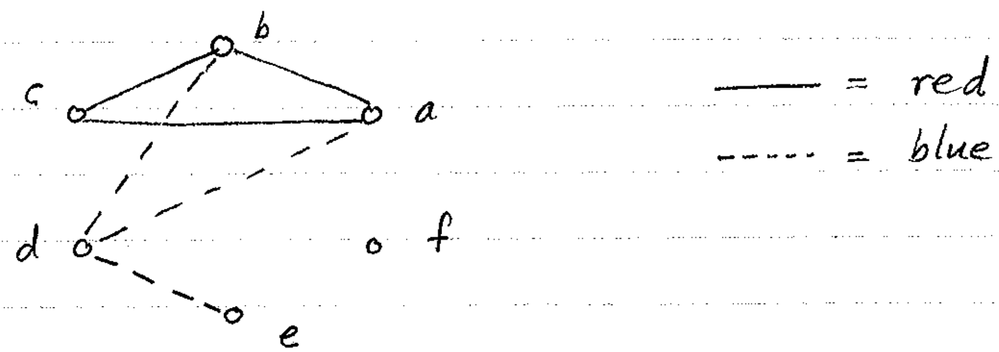
Now look at the seq.

$$b_1, b_2, b_3, \ldots, b_{37}, b_1+13, b_2+13, \ldots, b_{37}+13.$$

This seq. has 74 terms each of which is between 1 & 73 (inclusive). So by the P.H.P. two terms must be equal. So, as in problem #10, we must have $a_i = a_j + 13$ for some i & j. Hence the student will study 13 hours from the beginning of the $j+1$-th day to the end of the $j+(i-j)$-th day.

13 First observe that if we paint the 15 edges (between the 6 vertices) red or blue, then we will get a blue triangle or a red triangle because of formula 2.1 on page 36. There are two cases:

<u>Case (i)</u>: We get a red triangle
Let a, b, c, d, e, f be the six vertices of the problem and let a, b, c be the vertices of the red triangle. Suppose there is no more red triangles. We will show that there must be a blue triangle. (So we will get either two red triangles, or one red and one blue triangle).



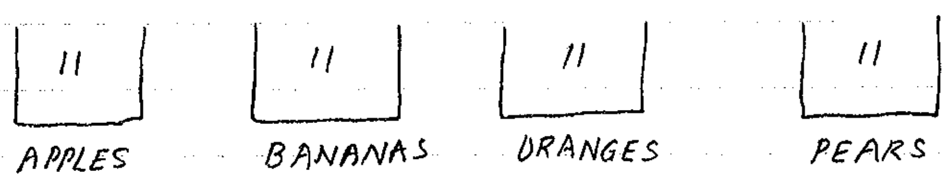
Since we supposed that there is no more red triangles, at least one of the edges of the triangle def must be blue. Let's say de is blue. Now if two or more of the edges da, db and dc were red, then we would get another red triangle by using the appropriate edge from abc. So two of the edges da, db, and dc must be blue. Let's say da & db are blue. Now look at the edges eb & ea. If both eb & ea are red, then we would get another red triangle -

13 and since we supposed that there is no more red triangles, one of the edges eb & ea must be blue. But if eb is blue, then ebd will be a blue triangle and if ea is blue, then ead will be a blue triangle. So if there is only one red triangle then there will be a blue triangle. Hence there must be two red triangles, or one red triangle and one blue triangle.

<u>Case (ii)</u> : We get a blue triangle
A similar proof to that of case (i) will show that if there is no more blue triangles, then there will be a red triangle. Hence we must have two blue triangles, or one blue and one red triangle.

Hence in either case we get two mono-chromatic triangles (i.e., two red triangles, two blue triangles, or one red and one blue triangle).

14. We can pick up to 44 fruits and still not get a dozen of any kind.



But if we pick 45 fruits, we are guaranteed to get a dozen of some kind. It will take 45 min. to pick 45 fruits.

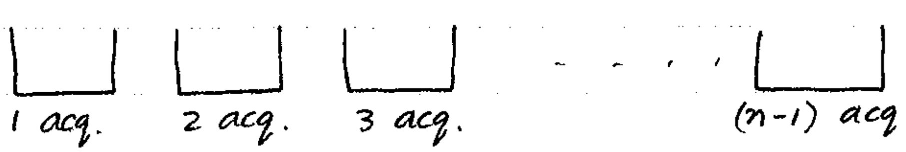
15. Let $S = \{a_1, \ldots, a_{n+1}\}$ and $D = \{0, 1, 2, \ldots, n-1\}$
Define $f: S \to D$ by
$$f(a_i) = a_i \pmod n$$
Since S has $n+1$ elements and D has only n elements, we must have
$$f(a_i) = f(a_j) \quad \text{for some } 1 \le i < j \le n+1.$$
So $a_i = a_j \pmod n$

$\therefore \; a_i - a_j = 0 \pmod n$

Hence $a_i - a_j$ is divisible by n. So we we can always find two integers a_i & a_j in S with $i \neq j$ and $a_i - a_j$ divisible by n.

16. We define a person to be a <u>loner</u> if they have no acquaintances. We will prove that in any group of n people there are two people with the same number of acquaintances. There are two cases:

<u>Case (i)</u>: The group has no loners.
In this case the possible number of acquaintances a person can have are
$$1, 2, 3, \ldots, n-1$$
since no one is allowed to be acquainted with themselves. Since we have n people and only $(n-1)$ possibilities, by the P.H.P., two people must have the same no. of acquaintances.



16 <u>Case (ii)</u> The group has at least one loner.
In this case the possible number of acquaintances a person can have are

$$0, 1, 2, \ldots, n-2$$

because a person cannot be acquainted with the loner or themselves. Since we have n people & $n-1$ possibilities, two people must have the same number of acquaintances.

So in either case two people will have the same number of acquaintances.

17 We will prove that there are three people at the party with the same number of acquaintances by splitting the problem into three cases.

<u>Case (i)</u>: The party has at least 2 loners
In this case the possible number of acquaintances a person can have are

$$0, 2, 4, 6, \ldots, 94, 96$$

because the person has to exclude the 2 loners and themselves and also posses an even no. of acquaintances. Since we have 100 people and only 49 possibilities the must be 3 people with the same no. of acquaintances.

<u>Case (ii)</u>: The party has exactly one loner
In this case the possible number of acquaintances of the

17. other 99 people are

$2, 4, 6, \ldots, 94, 96, 98$

Since we have 99 people and only 49 possibilities, we must have 3 people with the same number of acquaintances.

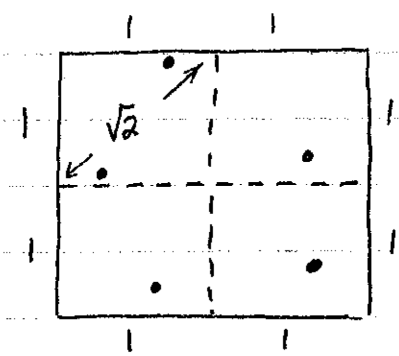
<u>Case (iii)</u>: The party has no loners

<u>In</u> this case the possible number of acquaintances a person can have are

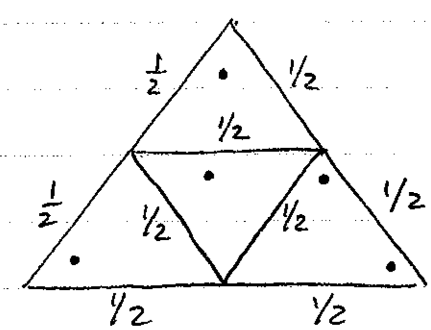
$2, 4, 6, 8, \ldots, 96, 98$

Since we have 100 people and only 49 possibilities, we must have 3 people with the same number of acquaintances.

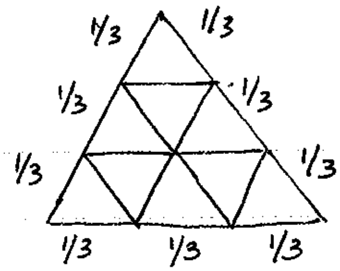
18. Divide the square into four smaller squares as shown on the right. If we choose any 5 points in the 2×2 square, at least two will fall in one of the smaller squares. The distance between these two points will be at most $\sqrt{2}$.



19. (a) Hint: Split the equi-lateral triangle into 4 smaller equilateral triangles as shown on the right.



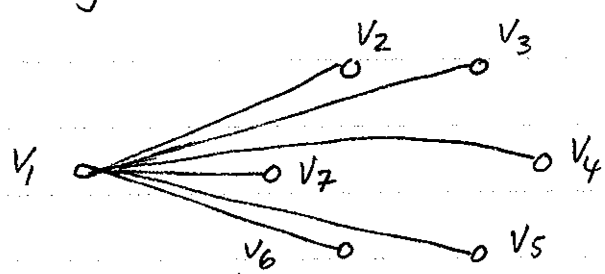
19 (b) Hint: Divide the triangle into 10 equilateral triangles as shown on the right



(c) $m_n = (1 + 3 + 5 + 7 + \cdots + 2n-1) + 1$
$= n^2 + 1.$

20. To show that $r(3,3,3) \leq 17$ is equivalent to showing that if we color the $\binom{17}{2}$ edges between the 17 vertices of K_{17} with red, blue, or green, then we can always find a red triangle or a blue triangle or a green triangle. We will prove this below.

Let v_1 be of the 17 vertices and consider the 16 edges that come out of v_1. If at most 5 edges out of v_1 have the same color, then v_1 will have at most 15 edges coming out of it. ∴ So v_1 must have at least 6 red edges or 6 blue or 6 green edges coming out of it.



—— = red

Let's say that at least 6 red edges comes out of v_1 (the cases with 6 blue or 6 green are entirely similar). If one of the edges between $v_2, v_3, \ldots,$ and v_7 is red then we get a red

20. triangle instantly and we are done. And if none of the edges between $v_2, v_3, \ldots, v_7$ are red – then all the edges between these 6 vertices are either blue or green, and by formula 2.1 p.36 we know that there must either be a green triangle or a blue triangle. So with 17 vertices we will always be able to get a blue triangle, or a green triangle, or a red triangle. Hence $r(3,3,3) \leq 17$

23. Let $v_1, v_2, \ldots, v_{10}$ be the 10 ten vertices and consider the 9 edges coming out of v_{10}. If at most 5 of these edges were blue and at 3 edges were red, then only 8 edges will come out of v_{10}. So at least 6 edges coming out of v_{10} must be blue or at least 4 edges coming out of v_{10} must be red.

Now if 4 of these edges are red, look at the other four endpoints v_1, v_2, v_3, v_4, say. If there is a red edge between two of these vertices, we will get a red triangle by adding v_{10}; and if there are no red edges between these 4 vertices, then v_1, v_2, v_3 & v_4 will form a blue K_4 (i.e., all six edges will be blue).

Also if 6 of these are blue, look at the other six endpoints a, b, c, d, e, f say. Now the edges between these 6 vertices are red or blue. So by a previous theorem they contain a blue triangle or a red triangle. If we add v_{10} to the blue triangle we will get a blue K_4.

So in all the cases we will get either a blue K_4 or a red triangle. So we are done.