Ch. 2 – Combinations & Permutations of Multi-sets

In many situations in Combinatorics we often want to specify a collection containing identical objects. For example, we might want to speify a collection of 3 a's and 2 b's – but if we write $\{a,a,a,b,b\}$, this will just boil down to $\{a,b\}$. And if we use $\langle a,a,a,b,b\rangle$ then we will introduce an order when perhaps none is needed. We shall introduce the notation $[3.a, 2.b]$ and call it a multi-set. Note $[3.a, 2.b] = [a,a,a,b,b] = [b,a,b,a,a]$.

Def. A multi-set is an ordered pair $M = \langle A, f\rangle$ where A is a set and $f: A \to \mathbb{Z}^+ \cup \{\infty\}$ is a function (called the multiplicity function) which tells us the number of times an element of A appears in M.

Ex. 1 $M = [\infty.a, 3.b, \infty.c]$ is a multi-set with an infinite number of a's, 3 b's, and an infinite number of c's. We can write $M = \langle\{a,b,c\}, f\rangle$ where $f(a) = \infty$, $f(b) = 3$, and $f(c) = \infty$ – but this is not as revealing as $[\infty.a, 3.b, \infty.c]$

Ex. 2 Let $M = [3.a, 2.b, 1.c]$ How many 2-combinations of M are there? How many 2-permutations of M are there?

Sol. First observe that a 2-combination of M is a portion (part) of M with 2 elements. So M has the following 2-combinations:

$[a,a]$, $[a,b]$, $[a,c]$, $[b,b]$, and $[b,c]$

Thus M has 5 2-combinations.

Also a 2-permutation of M is a 2-tuple of two elements of M. Two 2-permutations will be the same if they are the same 2-tuple. So M has the following 2-permutations:

$\langle a,a \rangle$, $\langle a,b \rangle$, $\langle a,c \rangle$, $\langle b,a \rangle$,
$\langle b,b \rangle$, $\langle b,c \rangle$, $\langle c,a \rangle$, $\langle c,b \rangle$.

Note that $\langle c,c \rangle$ is not a 2-permutation of M because $[c,c]$ is not a portion of M. We have to take a 2-combination of M and then see how many ways we can order it as a 2-tuple.

$[a,a]$ produces $\langle a,a \rangle$ (only),
$[a,b]$ produces $\langle a,b \rangle$ & $\langle b,a \rangle$,
$[a,c]$ produces $\langle a,c \rangle$ & $\langle c,a \rangle$,
$[b,b]$ produces $\langle b,b \rangle$, and
$[b,c]$ produces $\langle b,c \rangle$ & $\langle c,b \rangle$.

So once again we get 8 2-permutations of M.

Note It does not seem easy to find the number of r-combinations of M & it seems harder to find the no. of r-permutations of M. But there are some special cases in which the answers are easy to obtain. Later on we will give a method of finding the no. of r-combinations of M by using the Inclusion-Exclusion Principle. We can also do this by using the standard Generating functions & we can find the no. of r-permutations of M by using the Exponential Generating functions.

Prop. 5 Let $M = [\infty.a_1, \infty.a_2, \infty.a_3, \ldots, \infty.a_k]$. Then the number of r-permutations of M is given by $P_R(k,r) = (k)^r$.

Proof: An r-permutation of M is an r-tuple obtained from a portion of M with r elements. Now there are k choices for each component of the r-tuple

choices

$$\langle \underset{1st}{\underline{k}}, \underset{2nd}{\underline{k}}, \underset{3rd}{\underline{k}}, \ldots, \underset{r\text{-th}}{\underline{k}} \rangle$$

Since each of these choices produces a different r-tuple we get no. of r-perm. of M

$$= P_R(k,r) = \underbrace{k.k.\ldots k}_{r \text{ times}} = (k)^r$$

Prop. 6 Let $M = [n_1.a_1, n_2.a_2, \ldots, n_k.a_k]$ and put $n_1 + n_2 + \cdots + n_k = n$. Then the number of permutation (i.e., n-permutations) of M is given by

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Proof: An n-permutation of M is an n-tuple of of all the elements of M.

$$\langle \underset{\substack{\Uparrow\\ a_2}}{\_\_}, \underset{\substack{\uparrow\\ a_1}}{\_\_}, \underset{\substack{\uparrow\\ a_1}}{\_\_}, \_\_, \underset{\substack{\Uparrow\\ a_2}}{\_\_}, \cdots, \underset{\substack{\uparrow\\ a_1}}{\_\_} \rangle$$

Now there are $\binom{n}{n_1}$ ways to place the n_1 a_1's in this n-tuple, $\binom{n-n_1}{n_2}$ ways to place the n_2 a_2's, $\binom{n-n_1-n_2}{n_3}$ ways to place the n_3 a_3's and so on. In the end there will be $\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} = \binom{n_k}{n_k}$ ways to place the n_k a_k's. So the number of n-permutations of M will be

$$\binom{n}{n_1}\cdot\binom{n-n_1}{n_2}\cdot\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!\,(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!}\cdot\frac{(n-n_1-n_2)!}{n_3!\,\cdots}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!\,0!}$$

$$= \frac{n!}{n_1!\,n_2!\,\cdots n_k!} = \binom{n}{n_1, n_2, \cdots, n_k}$$ if we define

$$\binom{n}{n_1,\cdots,n_k} \text{ to be } \begin{cases} \dfrac{n!}{n_1!\cdots n_k!} & \text{if } n_1+\cdots+n_k = n \\ 0 & \text{if } n_1+\cdots+n_k \neq n. \end{cases}$$

Ex.3 In how many ways can the letters of MISSISSIPPI be arranged in a row?

Sol. Answer = Number of 11-permutations of $[4.I, 1.M, 2.P, 4.S] = \dfrac{11!}{4!\,1!\,2!\,4!} = 34{,}650.$

Prop. 7 Let $M = [n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k]$ and $n = n_1 + \cdots + n_k$. Then the number of $(n-1)$-permutations of M is the same as the no. of n-permutations of M, i.e.,
$$\binom{n}{n_1, n_2, \ldots, n_k}.$$

Proof: (a) Number of n-permutations of M

$=$ No. of n-perm. of M with 1st component a_1
$+$ No. of n-perm. of M with 1st component a_2
$\vdots$
$+$ No. of n-perm. of M with 1st component a_k

$=$ No. of $(n-1)$-perm. of $M_1 = [(n_1-1)\cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k]$
(because there is a one-to-one correspondence between the no. of n-perm. of M with 1st comp. a_1 and the no. of $(n-1)$-perm. of $M_1 = [(n_1-1)\cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k]$)
$+$ No. of $(n-1)$-perm. of $M_2 = [n_1 \cdot a_1, (n_2-1)\cdot a_2, \ldots, n_k \cdot a_k]$
$\vdots$
$+$ No. of $(n-1)$-perm. of $M_k = [n_1 \cdot a_1, n_2 \cdot a_2, \ldots, (n_k-1)\cdot a_k]$

$=$ No. of $(n-1)$-permutations of M because the set of $(n-1)$ permutations of M_i are all disjoint since they arose by ordering the different multi-sets $M_1, M_2, \ldots, M_k$.

(b) Another way to see this is to observe that no. of $(n-1)$ perm. of M

$$= \frac{(n-1)!}{(n_1-1)!\, n_2! \cdots n_k!} + \frac{(n-1)!}{n_1!(n_2-1)! \cdots n_k!} + \cdots + \frac{(n-1)!}{n_1!\, n_2! \cdots (n_k-1)!}$$

$$= (n-1)! \left[\frac{n_1}{n_1!\, n_2! \cdots n_k!} + \frac{n_2}{n_1!\, n_2! \cdots n_k!} + \cdots + \frac{n_k}{n_1!\, n_2! \cdots n_k!}\right]$$

$$= (n-1)! \frac{(n_1 + n_2 + \cdots + n_k)}{n_1!\, n_2! \cdots n_k!} = \frac{n\cdot(n-1)!}{n_1!\, n_2! \cdots n_k!} = \text{no. of } n\text{-perm. of } M.$$

Prop. 8: Let $M = [\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k]$. Then the number of r-combinations of M is given by
$$C_R(k,r) = \binom{r+k-1}{k-1} = \binom{r+k-1}{r}$$

Proof: Observe that an r-combination of M is just a sub-multiset of M of the form
$$[x_1 \cdot a_1, x_2 \cdot a_2, \ldots, x_k \cdot a_k]$$
with $x_1 + x_2 + \cdots + x_k = r$ & $x_i \in \mathbb{N}$. So the number of r-combinations of M is just the number of non-negative integer solutions of the equation
$$x_1 + x_2 + \cdots + x_k = r$$
Now each solution of this equation corresponds to an arrangement of r 1's and $(k-1)$ +'s in a row. For example,
$$\underbrace{11}_{x_1}+\underbrace{11111}_{x_2}+\underbrace{\ }_{x_3}+\underbrace{111}_{x_4}+\underbrace{\ }_{x_5}$$
corresponds to the solution
$$\underset{x_1}{2} + \underset{x_2}{5} + \underset{x_3}{0} + \underset{x_4}{3} + \underset{x_5}{0} = 10 \qquad (r=10 \ \& \ k=5)$$

But the number of ways of placing r 1's and $(k-1)$ +'s in a row is just the number of permutations of the multi-set $[r \cdot \text{"1"}, (k-1) \cdot \text{"+"}]$ and by Prop. 6, this is
$$\frac{[r+(k-1)]!}{r!\,(k-1)!} = \binom{r+k-1}{k-1} = \binom{r+k-1}{r} \text{ also.}$$
So if we let $C_R(k,r)$ = no. of r-combinations of M, then $C_R(k,r) = \binom{r+k-1}{k-1} = \binom{r+k-1}{r}$.

Note: If $M = [n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k]$ and for i $n_i \geq r$, then the number of r-combinations of M is also $\binom{r+k-1}{k-1} = \binom{r+k-1}{r}$ because any non-negative integer solution of the equation $x_1 + x_2 + \cdots + x_k = r$ will also satisfy $0 \leq x_i \leq r$.

Ex. 4 In how many ways can we purchase a bag of 10 sodas if the store has large numbers of 4 different kinds of sodas only.

Sol. Answer = No. of r-comb. of $[\infty \cdot s_1, \infty \cdot s_2, \infty \cdot s_3, \infty \cdot s_4]$
$= \binom{10+4-1}{4-1} = \binom{13}{3} = 286$

Prop 9 Let $M = [n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k]$ and $n = n_1 + \cdots + n_k$. Prove that the total number of r-combinations of M with r taking any value between 0 & n is $(n_1+1)(n_2+1)\cdots(n_k+1)$.

Proof: An r-combination of M is just a portion of M with r elements (some of which may be identical). Now we have (n_1+1) choices for deciding how many a_1's will in the r-comb., (n_2+1) choices for how many a_2's will be in the r-comb., ..., and (n_k+1) choices for how many n_k's will be in the r-comb. So the total no. of all the r-comb. for any value of r with $0 \leq r \leq n$ will be $(n_1+1)(n_2+1)\cdots(n_k+1)$.

Ex.5 Find the number of integer solutions of the linear equation $x_1 + x_2 + x_3 = 10 \quad \cdots (*)$ with $x_1 \geq 2$, $x_2 \geq 5$, and $x_3 \geq -4$.

Sol. Let $x_1 = y_1 + 2$, $x_2 = y_2 + 5$, and $x_3 = y_3 + (-4)$.
Then $x_1 + x_2 + x_3 = 10$ & $x_1 \geq 2, x_2 \geq 5$ & $x_3 \geq -4$ becomes $(y_1+2) + (y_2+5) + (y_3-4) = 10$ and $y_1 + 2 \geq 2$, $y_2 + 5 \geq 5$, & $y_3 - 4 \geq -4$. So we get $y_1 + y_2 + y_3 = 7$ and $y_1 \geq 0$, $y_2 \geq 0$, & $y_3 \geq 0$.
So our answer will be no. of non-negative integer solutions of $y_1 + y_2 + y_3 = 7$ and this is the number of permutations of the multi-set $[7 \cdot \text{"1"}, (3-1) \cdot \text{"+"}]$ which is

$$\binom{7+3-1}{3-1} = \binom{9}{2} = \frac{9 \cdot 8}{2 \cdot 1} = 36.$$

Ex.6 Find the no. of 15-combinations of the multi-set $M = [\infty \cdot a, \infty \cdot b, \infty \cdot c]$ with at least 2 a's, at least 4 b's and at least 1 c.

Sol. Answer = no. of integer solution of $x_1 + x_2 + x_3 = 15$ with $x_1 \geq 2$, $x_2 \geq 4$, and $x_3 \geq 1$.
Now let $x_1 = y_1 + 2$, $x_2 = y_2 + 4$ and $x_3 = y_3 + 1$.
Then answer $= \left\{ \begin{array}{l} \text{no. of non-negative integer} \\ \text{integer solution of the equation} \\ (y_1+2) + (y_2+4) + (y_3+1) = 15 \end{array} \right.$
= no. of non-negative integer solution of the equation $y_1 + y_2 + y_3 = 8$, which is $\binom{8+3-1}{3-1} = \binom{10}{2} = 45$.

Ex.7 Find the no. of divisors of $180 = 2^2 \cdot 3^2 \cdot 5^1$ & their sum.

Sol. (a) $(2+1)(2+1)(1+1) = 18$; (b) $[1 + 2^1 + 2^2] \cdot [1 + 3^1 + 3^2][1 + 5^1] = 546$.