Ch.3 – The Binomial Coefficients & their properties

§1 The Binomial & Multinomial Theorems

A *monomial* in Elementary Algebra is usually taken to be the product of a constant and a finite number of variables. A *multinomial* (or polynomial) is the sum of a finite number of monomials. When we have a sum of two monomials we usually call the expression a binomial – and for three monomials we call it a *trinomial*.

Ex.1 (a) 7, $(-9)x^4$, $5x^2y$, and $(6-\sqrt{2})\cdot xy^2z$ are all monomials

(b) $(x+y)$, (x^2+y^2), $(1+x)$ and $\underbrace{(3-y^3)}_{=(3+(-y^3))}$ are all binomials.

(c) $(x+y+z)$, $(x_1+x_2+x_3x_4)$ and $(1+x^2-yz)$ are all trinomials.

(d) $(1-\sqrt{2})^3$ is a monomial because it is a constant.

In this chapter we will be interested in the expansion of expressions of the form $(x_1+\cdots+x_k)^n$ and in the properties of the coefficients we obtain from these expansions. Recall that we define the expressions $\binom{n}{k}$ & $\binom{n}{n_1\, n_2 \cdots n_k}$ by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \& \quad \binom{n}{n_1,\ldots,n_k} = \begin{cases} \dfrac{n!}{n_1!\, n_2! \cdots n_k!} & \text{if } n = n_1+\cdots+n_k \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\binom{n}{k} = \binom{n}{k,\, n-k}$. We will assume that all the terms in these expressions are non-negative.

Theorem 1 (The Standard Binomial theorem)
Let n be any non-negative integer. Then

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} x^k .$$

Proof: We have $(1+x)^n = \underset{\text{1st term}}{(1+x)}\underset{\text{2nd term}}{(1+x)} \cdots \underset{n\text{-th term}}{(1+x)}$

Now to get the term involving x^k in the expansion we must choose the x from k of the terms and the 1's from the remaining $n-k$ terms. There are $\binom{n}{k}$ ways of choosing k of the x's from n terms and there are $\binom{n-k}{n-k}$ ways of choosing the 1's from the remaining $n-k$ terms. So we have $\binom{n}{k}\cdot\binom{n-k}{n-k} = \binom{n}{k}$ ways of getting x^k. Hence the coefficient of x^k in the expansion will be $\binom{n}{k}$. Since we can only get x^k for $k=0$ all the way to n, it follows that

$$(1+x)^n = \binom{n}{0}\cdot 1 + \binom{n}{1}\cdot x + \binom{n}{2}\cdot x^2 + \dots + \binom{n}{n}\cdot x^n$$
$$= \sum_{k=0}^{n} \binom{n}{k}\cdot x^k .$$

Note: There are $n+1 = \binom{n+2-1}{2-1}$ terms in the expansion of the expression $(1+x)^n$. In particular if $n=0$, there is only one term, namely $\binom{0}{0} = 1$.

We are interested in the expansion of $(x_1 + \dots + x_k)^n$. When $k=1$, this becomes x_1^n and we thus have only $1 = \binom{n+1-1}{1-1} = \binom{n}{0}$ terms. Remember that $0^0 = 1$.

Theorem 2 (The Multinomial Theorem)

Let n be any non-negative integer. Then

$$(x_1+\cdots+x_k)^n = \sum_{\substack{n_1+\cdots+n_k=n \\ \text{and } n_i\in\mathbb{N}}} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

Proof: $(x_1+\cdots+x_k)^n = \underbrace{(x_1+\cdots+x_k)}_{\text{1st term}}\underbrace{(x_1+\cdots+x_k)}_{\text{2nd term}} \cdots \underbrace{(x_1+\cdots+x_k)}_{n\text{-th term}}$.

Now the term $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ is obtained by choosing n_1 x_1's from n_1 of the n terms,

n_2 x_2's from n_2 of the remaining $n-n_1$ terms

n_3 x_3's from n_3 of the remaining $n-n_1-n_2$ terms

⋮

& n_k x_k's from n_k of the remaining $n-n_1-n_2-\cdots-n_{k-1}$ terms.

We can do all this in

$$\binom{n}{n_1}\cdot\binom{n-n_1}{n_2}\cdot\binom{n-n_1-n_2}{n_3} \cdots \binom{n-n_1-n_2-\cdots n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdot \frac{(n-n_1-n_2)!}{n_3! \cdots} \cdots \frac{(n-n_1-\cdots-n_{k-1})!}{n_k!(n-n_1-\cdots-n_k)!}$$

$$= \frac{n!}{n_1!\, n_2! \cdots n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}.$$ Hence

$$(x_1+\cdots+x_k)^n = \sum_{\substack{n_1+\cdots+n_k=n \\ \& \ n_i\in\mathbb{N}}} \binom{n}{n_1, \ldots, n_k} \cdot x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

Note: If $n=2$, the Multi-nomial Theorem becomes

$$(x_1+x_2)^n = \sum_{n_1+n_2=n} \binom{n}{n_1, n_2} x_1^{n_1} x_2^{n_2} = \sum_{k=0}^{n} \binom{n}{k} \cdot x_1^k \cdot x_2^{n-k}$$

It is often written as $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Prop 3 There are $\binom{n+k-1}{k-1}$ terms in the expansion of the expression $(x_1+\cdots+x_k)^n$.

Proof Number of terms in the expansion of $(x_1+\cdots+x_n)^n$

$=$ no. of non-negative integer solutions of the equation $(n_1+n_2+\cdots+n_k=n)$

$=$ no. of permutations of the multiset $[n.\text{"}1\text{"}, (k-1).\text{"}+\text{"}]$

$$= \frac{(n+k-1)!}{n!\,(k-1)!} = \binom{n+k-1}{k-1} \left[= \binom{n+k-1}{n} \text{ also} \right]$$

Ex. 2 $(x+y+z)^3$ will have $\binom{3+3-1}{3-1} = \binom{5}{2} = 10$ terms in its expansion.

Ex. 3 Find the coefficients of x^2y^3z and xy^2z^2 in the expansion of $(x-2y+3z)^6 \longrightarrow (x-2y+3z)^6$

Sol. (a) Coefficent of $x^2(-2y)^3(3z)^1$ in the expansion of $(x-2y+3z)^6$

$$= \binom{6}{2,3,1} = \frac{6!}{2!\,3!\,1!} = \frac{6.5.4}{2!} = 60.$$

So the coefficient of $x^2y^3z^1$ in the expansion will be $60.(1)^2(-2)^3.(3)^1 = -1440.$

(b) Coefficient of xy^2z^2 in the expansion of $(x-2y+3z)^6$ will be zero because $\binom{6}{1,2,2}=0$ since $1+2+2 \neq 6$.

Note: There are many reasons why $0^0=1$. Here is one. We expect that $(1+x)^0 = 1$ for any x - this is not unreasonable. So $0^0 = [1+(-1)]^0 = (1+x)^0 = 1$ with $x=-1$.

Also $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$. Putting $x=0$, gives $(1-0)^{-1} = x^0$

$\therefore 1 = 0^0$

§2 Properties of the Binomial Coefficients.

There are quite a large number of identities which involve the Binomial Coefficients. Most of these (actually, perhaps all of these) identities can be proved in an analytic way and also in a combinatorial way. The analytic way uses algebra (and analysis sometimes) – but we are never quite sure about the reason why the identity is true. This is in some ways like Mathematical Induction – it justifies the result and makes us certain that it is true – but we are still left wondering what was the "real reason" the result is true. The combinatorial way gives us an idea why the result is true – but sometimes it can be harder to understand than the analytic way. Below we will give several examples in why we prove the results both ways.

Prop. 4 $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$, for any $n \in \mathbb{N}$.

Proof: (Analytic way): From the Binomial theorem we have

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \dots + \binom{n}{n} \cdot x^n, \text{ for any } n \in \mathbb{N}.$$

Putting $x=1$, we get

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} \cdot 1 + \binom{n}{2} \cdot 1^2 + \dots + \binom{n}{n} \cdot 1^n.$$

$$\therefore \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n.$$

Proof (Combinatorial Way): $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}$

= Number of subsets of $\{1,2,\ldots,n\}$ with 0 elements
+ No. of subsets of " " 1 element
+ No. of subsets of " " 2 elements
⋮
+ No. of subsets of $\{1,2,\ldots,n\}$ with n elements
= Total no. of subsets of $\{1,2,\ldots,n\}$ = 2^n.

Prop. 5 For any $n, k \in \mathbb{N}$; $\displaystyle\sum_{n_1+\cdots+n_k=n \,\&\, n_i\in\mathbb{N}} \binom{n}{n_1,\ldots,n_k} = (k)^n$.

Proof: (Analytic way): From the Multi-nomial Theorem we have

$$(x_1+\cdots+x_k)^n = \sum_{n_1+\cdots+n_k=n \,\&\, n_i\in\mathbb{N}} \binom{n}{n_1,\ldots,n_k} x_1^{n_1} x_2^{n_2}\cdots x_k^{n_k}$$

Putting $x_1 = x_2 = \cdots = x_k = 1$, we get

$$\sum_{n_1+\cdots+n_k=n \,\&\, n_i\in\mathbb{N}} \binom{n}{n_1,\ldots,n_k} = (\underbrace{1+1+\cdots+1}_{k \text{ times}})^n = (k)^n.$$

(Combinatorial way): $\displaystyle\sum_{n_1+\cdots+n_k=n \,\&\, n_i\in\mathbb{N}} \binom{n}{n_1,\ldots,n_k}$

$= \displaystyle\sum_{n_1+\cdots+n_k=n \,\&\, n_i\in\mathbb{N}} \{\text{Number of } n\text{-permutations of } [n_1\cdot a_1,\ldots,n_k\cdot a_k]\}$

= Total number of n-permutations of $[n\cdot a_1, n\cdot a_2,\ldots,n\cdot a_k]$

= Total no. of n-perm. of $[\infty\cdot a_1,\ldots,\infty\cdot a_k] = (k)^n$.

Prop 6 (Pascal's identity): $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
for any $n \geq 1$ & $k \geq 1$.

Proof: (Analytic way): $(1+x)^n = (1+x).(1+x)^{n-1}$. So

Proof $\binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n$

$= (1+x)\cdot\left\{\binom{n-1}{0} + \cdots + \binom{n-1}{k-1}x^{k-1} + \binom{n-1}{k}x^k + \cdots + \binom{n-1}{n-1}x^{n-1}\right\}$

$= \binom{n-1}{0} + \cdots + \left\{\binom{n-1}{k-1} + \binom{n-1}{k}\right\}x^k + \cdots + \binom{n-1}{n-1}x^k$

Since the coefficents of x^k in the two expansions must be the same, we get $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

(Algebraic way): $\binom{n-1}{k-1} + \binom{n-1}{k}$

$= \dfrac{(n-1)!}{(k-1)!\,[(n-1)-(k-1)]!} + \dfrac{(n-1)!}{k!\,[(n-1)-k]!}$

$= \dfrac{(n-1)!}{k!\,(n-k)!}\left[\dfrac{k}{1} + \dfrac{(n-k)}{1}\right] = \dfrac{n(n-1)!}{k!\,(n-k)!} = \binom{n}{k}$.

(Combinatorial way): $\binom{n}{k}$ = no. of k-subsets of $\{1,2,\ldots,n\}$

= no. of k-subsets of $\{1,2,\ldots,n\}$ containing 1
+ no. of k-subsets of $\{1,2,\ldots,n\}$ not containing 1

$= \underbrace{\binom{n-1}{k-1}}_{\text{choosing } k-1 \text{ elements of } \{2,3,\ldots,n\} \text{ to get a } k\text{-subset of } \{1,2,\ldots,n\} \text{ containing } 1.} + \binom{n-1}{k}$, and we are done.

choosing k elements of $\{2,\ldots,n\}$ to get a k-subset of $\{1,\ldots,n\}$ not containing 1.

Prop. 7 For any $n \in \mathbb{N}$, $\sum_{k=0}^{n}\left\{\binom{n}{k}^2\right\} = \binom{2n}{n}$

Proof: (Analytic way):

$\binom{2n}{n}$ = coefficient of x^n in the expansion of $(1+x)^{2n}$.

= coefficient of x^n in the expansion of $(1+x)^n\cdot(1+x)^n$

$= \text{coefficient of } x^n \text{ in } \left\{\sum_{k=0}^{n} \binom{n}{k} x^k\right\} \cdot \left\{\sum_{k=0}^{n} \binom{n}{k} \cdot x^k\right\}$

$= \binom{n}{0}\cdot\binom{n}{n} + \binom{n}{1}\cdot\binom{n}{n-1} + \cdots + \binom{n}{k}\cdot\binom{n}{n-k} + \cdots + \binom{n}{n}\cdot\binom{n}{0}$

$= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \left\{\binom{n}{k}^2\right\}$ bec. $\binom{n}{n-k} = \binom{n}{k}$.

(Combinatorial way):

$\binom{2n}{n} =$ No. of n-subsets of $\{1, 2, \ldots, 2n\}$

$= \sum_{k=0}^{n} \begin{pmatrix}\text{No. of } k\text{-subsets} \\ \text{of } \{1,2,\ldots,n\}\end{pmatrix} \cdot \begin{pmatrix}\text{No. of } (n-k)\text{-subsets} \\ \text{of } \{n+1, n+2, \ldots, 2n\}\end{pmatrix}$

The union of the k-subset of $\{1,2,\ldots,n\}$ and the $(n-k)$-subset of $\{n+1,\ldots,2n\}$ will produce an n-subset of $\{1,\ldots,2n\}$

$= \sum_{k=0}^{n} \binom{n}{k}\cdot\binom{n}{n-k} = \sum_{k=0}^{n} \left\{\binom{n}{k}^2\right\}$.

Prop. 8 For any $n, k \geq 1$; $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$.

Proof: (Algebraic way): $k \cdot \binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!}$

$= \frac{k}{k} \cdot \frac{n \cdot (n-1)!}{(k-1)!\,(n-k)!} = n \cdot \frac{(n-1)!}{(k-1)!\,[(n-1)-(k-1)]!} = n \cdot \binom{n-1}{k-1}$.

(Combinatorial way): We shall calculate the number of ways of picking a team of k players out of n individuals and designating a captain who must be in the team.

$k \cdot \binom{n}{k} = \binom{n}{k} \cdot \binom{k}{1}$ Pick the k players first & then choose the captain.

$=$ no. of ways of picking the team & the captain

$= \binom{n}{1} \cdot \binom{n-1}{k-1}$ Pick the captain first & then choose $k-1$ more players from $n-1$ to get the team.

$= n \cdot \binom{n-1}{k-1}$.

Ex. 1 Prove that $\binom{n}{k} = \binom{n}{n-k}$ combinatorially. Put $A = \{1, 2, 3, \ldots, n\}$ and $B =$ coll. of k-subsets of A & $C =$ coll. of $(n-k)$ subsets

Define $f: B \to C$ by $f(S) = A - S$. Then f is a bijection. of A.
So $\binom{n}{k} = |B| = |C| = \binom{n}{n-k}$.

§3. Pascal's triangle (or Pascal's infinite array)

By repeatedly using Pascal's identity we can build an infinite array with the Binomial Coefficients. Some patterns immediately "jump out" at us while others take a little more time to appear. So let us take a peek at the $\binom{n}{k}$ array for $n, k \in \mathbb{N}$.

$n \downarrow$ \ $k \rightarrow$	0	1	2	3	4	5	6	7	8	$\sum_{k=0}^{n} \binom{n}{k} =$
8	1	8	28	56	70	56	28	$\binom{n}{k}$	1	256
7	1	7	21	35	35	21	$\binom{n-1}{k-1}$	$\binom{n-1}{k}$	0	128
6	1	6	15	20	15	6	1	0	0	64
5	1	5	10	10	5	1	0	0	0	32
4	1	4	6	4	1	0	0	0	0	16
3	1	3	3	1	0	0	0	0	0	8
2	1	2	1	0	0	0	0	0	0	4
1	1	1	0	0	0	0	0	0	0	2
0	1	0	0	0	0	0	0	0	0	1

The first thing we notice is that the non-zero coefficients form a right-triangle — that is why the array is called Pascal's triangle (but over the years this triangle has been drawn in different ways). Next we notice that the sum of the entries in the n-th row is exactly 2^n. This is because $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

Observe also that the non-zero entries are enclosed by the $x=y$ main diagonal with 1's & column 0 with 1's.

If we look at the sum of the terms in any column up to and including the $\binom{n}{k}$ term, we immediately see from the square-cornered rectangles that there is a pattern there

$1+6=7$, $1+5+15=21$, $1+4+10+20=35$.

This comes from the following result.

<u>Prop. 9</u> For any $n \geq k$, we have

$$\binom{0}{k}+\binom{1}{k}+\binom{2}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}$$

Actually, since $\binom{n}{k}=0$ for $0 \leq n<k$, we can write this as $\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}$.

<u>Proof:</u> We shall prove the result by parametric induction on n (k will be the temporarily fixed parameter)

<u>Basis</u>: For $n=k$, we have $\binom{k}{k}=1=\binom{k+1}{k+1}=\binom{n+1}{k+1}$.

So the result is true for $n=k$.

<u>Ind. Step</u>: Suppose the result is true n (where $n \geq k$).

Then $\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}$. So

$$\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k}+\binom{n+1}{k}$$

$$=\binom{n+1}{k+1}+\binom{n+1}{k}=\binom{n+1}{k}+\binom{n+1}{k+1} \text{ by the Induc. Hyp.}$$

$$=\binom{n+2}{k+1}=\binom{(n+1)+1}{k+1} \text{ by Pascal's Identity.}$$

So, if the result is true for n, it will be true for $n+1$.

<u>Conclusion</u>: By the Principle of Parametric Math Induction the result is true for all n, k.

Also if we look at the terms in a diagonal parallel to the $x=y$ diagonal, then the sum of the terms in the rounded-rectangles show us another pattern.

$1+6+21=28$, $1+5+15+35=56$, $1+4+10+20+35=70$.

This comes from the following result.

Prop. 10 For any $n,k \in \mathbb{N}$ we have
$$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+k}{k}=\binom{n+k+1}{k}$$

Proof. This time we shall prove the result by parametric induction on k (n will be the temporarily fixed parameter.

Basis For $k=0$, we have $\binom{n}{0}=1=\binom{n+1}{0}=\binom{n+k+1}{k}$.
So the result is true for $k=0$.

Ind. Step. Suppose the result is true for k. Then
$$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+k}{k}=\binom{n+k+1}{k}.$$
We shall prove the result for $k+1$. Now
$$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+k}{k}+\binom{n+(k+1)}{k+1}$$
$$=\binom{n+k+1}{k}+\binom{n+k+1}{k+1} \quad \text{by the Induc. Hyp.}$$
$$=\binom{n+(k+2)}{k+1}=\binom{n+(k+1)+1}{k+1} \quad \text{by Pascal's Identity}$$

So if the result is true for true for k, it will be true for $k+1$. By the Principle of Parametric Math Induction, it follows that the result is true for all k,n.

Finally if we look at terms in the diagonals that parallel to the $x+y=0$ diagonal, then we see the following pattern from the terms enclosed in the squiggly rectangles . .

$$1=1, \quad 1+0=1, \quad 1+1=2, \quad 1+2=3, \quad 1+3+1=5.$$

A moment's thought would indicate that is the Fibonacci sequence a_{n+1} which is defined by recursion as follows.

$$a_0=0, \; a_1=1; \text{ and } a_n=a_{n-1}+a_{n-2} \text{ for } n\geq 2.$$

This pattern comes from the following result

Prop. 11 For any $n\in\mathbb{N}$, we have

$$\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\cdots+\binom{0}{n}=a_{n+1}$$

Proof. Let $b_n=\binom{n-1}{0}+\binom{n-2}{1}+\binom{n-3}{2}+\cdots+\binom{0}{n-1}$. Then for $n\geq 2$

$$b_{n-1}+b_{n-2}=\binom{n-2}{0}+\binom{n-3}{1}+\binom{n-4}{2}+\binom{n-5}{3}+\cdots+\binom{0}{n-2}$$

$$\binom{n-3}{0}+\binom{n-4}{1}+\binom{n-5}{2}+\cdots+\binom{0}{n-3}$$

$$=\binom{n-2}{0}+\binom{n-2}{1}+\binom{n-3}{2}+\binom{n-4}{3}+\cdots+\binom{1}{n-2} \quad \text{Pascal's identity}$$

$$=\binom{n-1}{0}+\binom{n-2}{1}+\binom{n-3}{2}+\binom{n-4}{3}+\cdots+\binom{1}{n-2}+\underbrace{\binom{0}{n-1}}_{=0}=b_n$$

Also $b_0=$ emptysum $=0$ & $b_1=\binom{0}{0}=1$. So $\langle a_n\rangle$ & $\langle b_n\rangle$ satisfy the same recurrence equation and the same initial conditions. Hence $a_n=b_n$ for each $n\in\mathbb{N}$. So

$$\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\cdots+\binom{0}{n}=b_{n+1}=a_{n+1}$$

and we are done.

Ex. 1 Let $n \in \mathbb{Z}^+$. Put $D =$ set of all subsets of $\{1, \ldots, n\}$ with an odd number of elements and $E =$ set of all subsets of $\{1, 2, \ldots, n\}$ with an even no. of elements. Prove that $|D| = |E|$.

Sol. We know that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, from the Biomial Theorem.

Putting $x = -1$, we get

$$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k = \sum_{\substack{k \in \{0, \ldots, n\} \\ \& \ k \text{ is even}}} (-1)^k \cdot \binom{n}{k} + \sum_{\substack{k \in \{0,1, \ldots, n\} \\ \& \ k \text{ is odd}}} (-1)^k \binom{n}{k}$$

$$\therefore \sum_{\substack{k \in \{0,1, \ldots, n\} \\ \& \ k \text{ is even}}} \binom{n}{k} + \sum_{\substack{k \in \{0,1, \ldots, n\} \\ \& k \text{ is odd}}} (-1) \cdot \binom{n}{k} = 0$$

$$\therefore \sum_{\substack{k \in \{0,1, \ldots, n\} \\ \& \ k \text{ is even}}} \binom{n}{k} = \sum_{\substack{k \in \{0,1, \ldots, n\} \\ \& \ k \text{ is odd}}} \binom{n}{k}. \quad \therefore |D| = |E|$$

because $\binom{n}{k} =$ number of subsets of $\{1, \ldots, n\}$ with k elements.

Ex. 2 Prove that $\frac{1}{1}\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1} - 1}{n+1}$

Proof. We know that $(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n$

Integrating both sides w.r.t. x from 0 to 1, we get

$$\int_0^1 (1+x)^n \, dx = \int_0^1 \left\{ \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} \cdot x^2 + \cdots + \binom{n}{n} \cdot x^n \right\} dx$$

$$\therefore \left[\frac{(1+x)^{n+1}}{n+1} \right]_0^1 = \left[\binom{n}{0} + \binom{n}{1} \cdot \frac{x^2}{2} + \binom{n}{2} \frac{x^3}{3} + \cdots + \binom{n}{n} \frac{x^{n+1}}{n+1} \right]_0^1$$

$$\therefore \frac{2^{n+1} - 1}{n+1} = \frac{1}{1}\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n}$$

Ex. 3 Prove that $\frac{1}{1}\binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{(-1)^n}{n+1}\binom{n}{n} = \frac{1}{n+1}$

Sol. Do for H.W. (Hint: Integrate $(1+x)^n$ from -1 to 0.)

Ex. 1 (again): Define $f: D \to E$ by $f(B) = \begin{cases} B - \{1\} & \text{if } 1 \in B \\ B \cup \{1\} & \text{if } 1 \notin B. \end{cases}$

Then f is a bijection. So $|D| = |E|$. Here $B \subseteq \{1, 2, \ldots, n\}$.

§4. Newton's Binomial Theorem

Theorem 12 (Newton's Binomial Theorem)

Let α be any real number and $|x|<1$. Then $\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$ converges & $\operatorname{val}\left(\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k\right) = (1+x)^\alpha$.

Here $\binom{\alpha}{k} = \begin{cases} \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(k-1))}{k!} & \text{for } k \in \mathbb{Z}^+ \\ 1 \text{ (if } k=0\text{), and } 0 \text{ (if } k \in \mathbb{Z}^-\text{)}. \end{cases}$

Proof The proof is beyond our reach in this course because it needs a background in real analysis. But we will illustrate it for special values of α.

Prop. 13 For any $n \in \mathbb{Z}^+$ and any $k \in \mathbb{N}$, we have

$$\binom{-n}{k} = (-1)^k \cdot \binom{n+k-1}{k}.$$

Proof:

$$\binom{-n}{k} = \frac{(-n)(-n-1)(-n-2)\cdots(-n-(k-1))}{k!}$$

$$= (-1)^k \cdot \frac{(n-1)!}{(n-1)!} \cdot \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!}$$

$$= (-1)^k \frac{[(n-1)+k]!}{(n-1)!\cdot k!} = (-1)^k \cdot \binom{n+k-1}{k}.$$

Ex. 4

$$(1+x)^{-n} = \frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} (-1)^k \cdot \binom{n+k-1}{k} \cdot x^k, \quad |x|<1$$

Replacing x by $(-x)$, we get that

$$(1-x)^{-n} = \frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \quad \text{for } |x|<1.$$

Ex. 5

$$\binom{-1/2}{k} = \left[(-\tfrac12)(-\tfrac12-1)(-\tfrac12-2)\cdots(-\tfrac12-(k-1))\right]/k!$$

$$= \frac{(-1)}{2}\frac{(-3)}{2}\frac{(-5)}{2}\frac{(-7)}{2}\cdots\frac{[-(2k-1)]}{2} \cdot \frac{1}{k!}$$

$$= \frac{(-1)^k}{2^k} \cdot \frac{(1)(2)(3)(4)\cdots(2k-1)(2k)}{2\cdot 4\cdots(2k-2)\cdot(2k)} \cdot \frac{1}{k!}$$

$$= \frac{(-1)^k}{2^k} \cdot \frac{1}{2^k} \cdot \frac{(2k)!}{k!} \cdot \frac{1}{k!} = \frac{(-1)^k}{1} \binom{2k}{k} \cdot \frac{1}{2^k}$$

$$= (-1)^k \cdot \frac{1}{2^{2k}} \cdot \binom{2k}{k} \checkmark$$

Ex.5 So $(1+x)^{-1/2} = \sum_{k=0}^{\infty} \binom{-1/2}{k} \cdot x^k = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^{2k}} \binom{2k}{k} \cdot x^k$.

Ex.6 $\binom{1/2}{k} = (1/2)(1/2-1)(1/2-2)\cdots[1/2-(k-1)]/k!$

$$= \frac{(1)}{2}\frac{(-1)}{2}\frac{(-3)}{2}\frac{(-5)}{2}\cdots\frac{[-(2k-3)]}{2} \cdot \frac{1}{k!}$$

$$= \frac{(-1)^{k-1}}{2^k \cdot k!} \cdot \frac{(1)(2)\cdot(3)\cdot(4)\cdot(5)\cdots(2k-3)(2k-2)(2k-1)(2k)}{2 \cdot 4 \cdots (2k-4)(2k-2)(2k)}$$

$$= \frac{(-1)^{k-1}}{2^{2k}} \cdot \frac{(2k)!}{k!\,k!} \cdot \frac{1}{2k-1} = \frac{(-1)^{k-1}}{(2k-1)\cdot 2^{2k}} \binom{2k}{k}.$$

$$\therefore (1+x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} \cdot x^k = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{(2k-1)\cdot 2^{2k}} \binom{2k}{k} \cdot x^k.$$

The extended Pascal's infinite array is obtained from $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$ for $n \in \mathbb{Z}^+$ & $k \in \mathbb{N}$ and $\binom{n}{k} = 0$ when $k \in \mathbb{Z}^-$.

k=0

n	..	0	0	1	7	21	35	35	21	7	1
6	..	0	0	1	6	15	20	15	6	1	0
5	..	0	0	1	5	10	10	5	1	0	0
4	..	0	0	1	4	6	4	1	0	0	0
3	..	0	0	1	3	3	1	0	0	0	0
2	..	0	0	1	2	1	0	0	0	0	0
1	..	0	0	1	1	0	0	0	0	0	0
0	..	0	0	1	0	0	0	0	0	0	0
-1	..	0	0	1	-1	1	-1	1	-1	1	-1
-2	..	0	0	1	-2	3	$\boxed{-4}$	5	-6	7	-8
-3	..	0	0	1	-3	6	-10	15	-21	28	-36
-4	..	0	0	1	-4	10	-20	35	-56	84	-120
-5	..	0	0	1	-5	15	-35	70	-126	210	-330
-6	..	0	0	1	-6	21	-56	126	-252	462	-792
n											
	k:	-2	-1	0	1	2	3	4	5	6	7 ← k

old array

n=0