Chapter 2 - Fifth Edition

1. "a" is the condition: "the digits are distinct"
"b" is the condition: "the number is even"

(i) No. of 4-digit numbers made up of 1,2,3,4 or 5's

$$= \underset{\text{1st}}{\langle 5} \,.\, \underset{\text{2nd}}{5} \,.\, \underset{\text{3rd}}{5} \,.\, \underset{\text{4th choice}}{5 \rangle} = (5)^4 = 625$$

(ii) No. of 4-digit numbers made up of 1,2,3,4 or 5's which satisfy condition "a"

$$= \underset{\text{1st}}{\langle 5} \,.\, \underset{\text{2nd}}{4} \,.\, \underset{\text{3rd}}{3} \,.\, \underset{\text{4th choice}}{2 \rangle} = 5.4.3.2 = 120$$

(iii) No. of 4-digit numbers made up of 1,2,3,4 or 5's which satisfy condition "b"

$$= \underset{\text{1st}}{\langle 5} \,.\, \underset{\text{2nd}}{5} \,.\, \underset{\text{3rd}}{5} \,.\, \underset{\text{4th choice}}{2 \rangle} = 5^3.2 = 250$$

(iv) No. of 4-digit numbers made up of 1,2,3,4 or 5's which satisfy both conditions "a" & "b"

$$= \underset{\text{4th}}{\langle 2} \,.\, \underset{\text{3rd}}{3} \,.\, \underset{\text{2nd}}{4} \,.\, \underset{\text{1st choice}}{2 \rangle} = 2.3.4.2 = 48$$

2. First observe that there are 4! ways of arranging the suits in "groups of same-suit cards"

$$\langle \underset{\text{choices}}{4} \,.\, \underset{\text{choices}}{3} \,.\, \underset{\text{choices}}{2} \,.\, \underset{\text{choice}}{1} \rangle$$

Now for any particular arrangement, say ♢ ♡ ♤ ♧, there will be 13! ways of ordering the diamonds, 13! ways for the hearts 13! ways for the spades & 13! ways for the clubs. So there will be $4!(13!)^4$ ways in all.

3 (a) A poker hand is just a set of 5 cards from a deck of 52. Since the cards are dealth one at a time, there will be

$$\langle \underset{\substack{\text{choices for}\\ \text{1st card}}}{52} \cdot \underset{\text{2nd card}}{51} \cdot \underset{\text{3rd card}}{50} \cdot \underset{\text{4th card}}{49} \cdot \underset{\text{5th card}}{48} \rangle$$

$$= 52.51.50.49.48 \cdot \frac{47!}{47!} = \frac{52!}{47!}$$

(b) Number of possible poker hands

= No. of subsets of the 52 cards with 5 elements

$= \binom{52}{5}$ by the formula for 5-combinations

$$= \frac{52!}{47!\,5!}$$

4. (a) A positive divisor of $3^4 . 5^2 . 7^6 . 11$ is any number of the form $3^a . 5^b . 7^c . 11^d$ where $0 \leq a \leq 4$, $0 \leq b \leq 2$, $0 \leq c \leq 6$, $0 \leq d \leq 1$. So there will be

$$\langle \underset{\substack{\text{choices}\\ \text{for } a}}{5} \cdot \underset{\text{for } b}{3} \cdot \underset{\text{for } c}{7} \cdot \underset{\text{for } d}{2} \rangle$$

$= 5.3.7.2 = 210$ positive divisors

(b) $620 = 2^2 . 5 . 31$ $\qquad 2^a . 5^b . 31^c$

So no. of divisors $= \langle \underset{\substack{\text{choices}\\ \text{for } a}}{3} . \underset{\text{for } b}{2} . \underset{\text{for } c}{2} \rangle$

$= 12.$

(c) $10^{10} = 2^{10} . 5^{10}$ $\qquad 2^a . 5^b$, $0 \leq a, b \leq 10$

So no. of divisors $= \langle \underset{\substack{\text{choices}\\ \text{for } a}}{11} . \underset{\substack{\text{choices}\\ \text{for } b}}{11} \rangle = 121$

5. (a) $50! = 50.49.48 \ldots 45 \ldots 40 \cdots 35 \ldots 5. \ldots 2.1$

$$\left[\frac{50}{5}\right] = \text{no. of terms with at least one factor of } 5$$

$$\left[\frac{50}{5^2}\right] = \text{no. of terms with at least two factors of } 5.$$

No. of factors of 5 in 50! will thus be

$$\left[\frac{50}{5}\right] + \left[\frac{50}{5^2}\right] = 10 + 2 = 12.$$

No. of factors of 2 in 50! will similarly be

$$\left[\frac{50}{2}\right] + \left[\frac{50}{2^2}\right] + \left[\frac{50}{2^3}\right] + \left[\frac{50}{2^4}\right] + \left[\frac{50}{2^5}\right] = 25 + 12 + 6 + 3 + 1 = 47$$

$\therefore\ 50! = 2^{47}.5^{12}.K$ where K has no factors of 2 or 5

So highest power of 10 dividing 50! will be 12 because $50! = 2^{12}.5^{12}.\ 2^{35}.K$

$$= 10^{12}.\underbrace{2^{35}.K}_{\text{has no factors of } 10}$$

(b) Similarly the highest power of 10 dividing 1000! can be found by just finding how many factors of 5, 1000! has. (There will be at least this many factors of 2 to produce the factors of 10). Now no. of factors of 5 in 1000!

$$= \left[\frac{1000}{5}\right] + \left[\frac{1000}{5^2}\right] + \left[\frac{1000}{5^3}\right] + \left[\frac{1000}{5^4}\right]$$

$$= 200 + 40 + 8 + 1 = 249$$

So the highest power of 10 dividing 1000! will be 249.

6. (a) We want the number of integers ≥ 5400 in which the digits are distinct & neither 2 nor 7 appears.

not 0,2,or 7 ↓   not 2 or 7 or first digit ↙

No. of 8-digit such numbers $= 7.7.6.5.4.3.2.1$

No of 7-digit " $= 7.7.6.5.4.3.2$

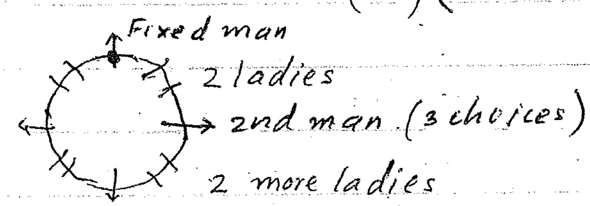
" 6-digit " $= 7.7.6.5.4.3$

" 5-digit " $= 7.7.6.5.4$

No. of 4-digit such numbers $= 3.7.6.5 + 1.4.6.5$

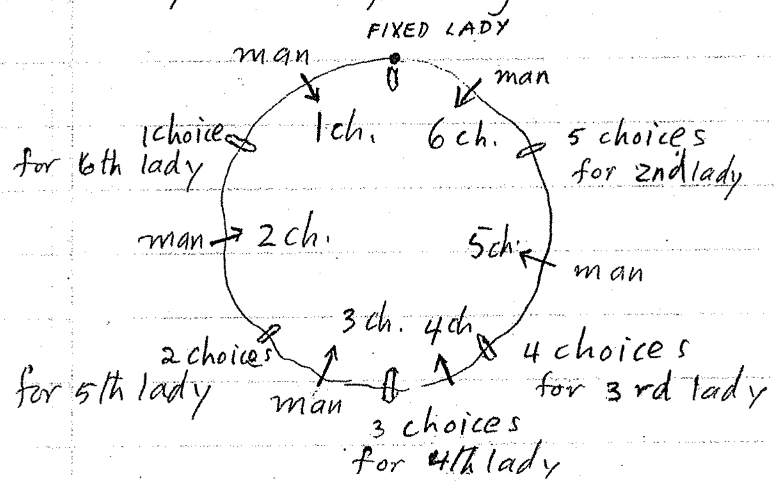
↑ 6,8 or 9   5 (4,6,8 or 9)

So answer $= 7(7!)\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!}\right) + 3.7.6.5 + 4.6.5$

7. 

Ans: $(3!)(8!)$

8 First fix one lady. Then there will be $5!$ ways of placing the other ladies



▯ = lady

↓ = man

After the ladies are seated in the alternate seats as shown — we place the men. There will be $6!$ ways of placing the men for each arrangement of the ladies. So there will be $(6!)(5!)$ ways of seating the men & ladies

9. (a) If we make A & B hold hands and consider them as one person, then there will be 13! circular permutations of these 14 "people". Now A can be on the right of B or on the left of B. So the number of ways in which A & B will be seated next to each other at the round table will be $2.(13!)$
So the number of ways A & B will not be seated next to each other will be $14! - 2.(13!) = 14.(13!) - 2.(13!) = 12.(13!)$

(b) We can now see that the number of ways in which B is not seated on the right of A $= 14! - 13! = 13.(13!)$

10. (a) We first find the different ways that the committee can be constituted:

2 WOMEN + 3 MEN — $\binom{12}{2}\cdot\binom{10}{3}$

3 WOMEN + 2 MEN — $\binom{12}{3}\cdot\binom{10}{2}$

4 WOMEN + 1 MAN — $\binom{12}{4}\cdot\binom{10}{1}$, $\binom{12}{5}\cdot\binom{10}{0}$ ↑ 5 WOMEN + 0 MEN

So answer $= \binom{12}{2}\cdot\binom{10}{3} + \binom{12}{3}\cdot\binom{10}{2} + \binom{12}{4}\cdot\binom{10}{1} + \binom{12}{5}\cdot\binom{10}{0}$

(b) Instead of "two members" of the club refuse to serve together the author should have said that "one woman & one man" refuse to serve together. Now we will count the number of ways this woman & this man can serve together.

10 (b) Let's call the woman x and call the man y.

$x + y + 1$ WOMAN $+ 2$ MEN — $\binom{11}{1}\cdot\binom{9}{2}$ (2 WOMEN)

$x + y + 2$ WOMEN $+ 1$ MAN — $\binom{11}{2}\cdot\binom{9}{1}$ (3 WOMEN)

$x + y + 3$ WOMEN $+ 0$ MEN — $\binom{11}{3}\cdot\binom{9}{0}$ (4 WOMEN)

So number of different committees of 4 with at least 2 women in which x & y do not serve together

$$= \binom{12}{2}\cdot\binom{10}{3} + \binom{12}{3}\cdot\binom{10}{2} + \binom{12}{4}\cdot\binom{10}{1} + \binom{12}{5}\cdot\binom{10}{0} - \binom{11}{1}\cdot\binom{9}{2} - \binom{11}{2}\cdot\binom{9}{1} - \binom{11}{3}\cdot\binom{9}{0}$$

11 Answer $= \binom{20}{3} - 18 - 17.16 - 2.17$

No. of subsets with 3 elements

No. of subsets with 3 consecutive elements
$\{1,2,3\}\ \{2,3,4\}\ \cdots\ \{18,19,20\}$

No. of subsets with exactly 2 consecutive elements which contains neither 1 nor 20.

$\{2,3\}$ + one of $\{5,6,\dots,20\}$ ← 16 choices (avoid 2-subset & 2 neigbr

$\{3,4\}$ + one of $\{1,6,7,\dots,20\}$ ← 16 choices (avoid 2-set & 2 neighbour

$\{4,5\}$ + one of $\{1,2,7,\dots,20\}$ ← 16 choices "

⋮

$\{18,19\}$ + one of $\{1,2,3,\dots,16\}$ ← 16 choices (avoid 2-subset & 2 neighb..

$\{1,2\}$ + one of $\{4,5,6,\dots,20\}$ ← 17 choices

$\{19,20\}$ + one of $\{1,2,3,\dots,17\}$ ← 17 choices

avoid 2-subset and 1 neighbour

(2-subsets that contain 1 or 20)

Answer $= \binom{20}{3} - 1.18 - 18.17 = \binom{20}{3} - 18.18$

$$= \frac{20.19.\cancel{18}^{3}}{\cancel{3}.\cancel{2}} - 18.18 = 1140 - 324 = 816.$$

12. Let's call the two players who can play on the line as well as in the backfield X and Y.

X & Y on the line — $\binom{8}{5}\cdot\binom{5}{4}$

X on line & Y in back + Y on line & X in back — $2\cdot\binom{8}{6}\cdot\binom{5}{3}$

X & Y in backfield — $\binom{8}{7}\cdot\binom{5}{2}$

X on line & Y out + X out & Y on line — $2\cdot\binom{8}{6}\cdot\binom{5}{4}$

X & Y both out — $\binom{8}{7}\cdot\binom{5}{4}$

X in back & Y out + X out & Y in back — $2\cdot\binom{8}{7}\cdot\binom{5}{3}$

$$\text{Answer} = \binom{8}{5}\cdot\binom{5}{4} + 2\cdot\binom{8}{6}\binom{5}{3} + \binom{8}{7}\cdot\binom{5}{2} + 2\cdot\binom{8}{6}\cdot\binom{5}{4} + \binom{8}{7}\cdot\binom{5}{4} + 2\cdot\binom{8}{7}\cdot\binom{5}{3}.$$

13. (a)

$$\underbrace{\binom{100}{25}}_{\text{choices for dorm. A}}\cdot\underbrace{\binom{75}{35}}_{\text{choices for dorm B}}\cdot\underbrace{\binom{40}{40}}_{\text{Choices for dorm C.}} = \frac{100!}{25!\,75!}\cdot\frac{75!}{35!\,40!} = \frac{100!}{25!\,35!\,40!}.$$

(b)

$$\underbrace{\binom{50}{25}}_{\text{25 men to A}}\cdot\underbrace{\binom{25}{25}}_{\text{remaining 25 men To C}}\cdot\underbrace{\binom{50}{35}}_{\text{35 women to dorm B}}\cdot\underbrace{\binom{15}{15}}_{\text{remaining 15 women to dorm C.}} = \frac{50!}{25!\;25!}\cdot\frac{50!}{35!\;15!}$$

14. First we have 5 students who can sit in either the front row or the back row. We have to decide how to split these 5 students into the two rows. Then we can see how many ways they can be arranged

3fr + 2bk - 8 IN FRONT + 6 IN BACK — $\binom{5}{3}\binom{2}{2}\binom{8}{0}\cdot 8!\,\binom{8}{2}\cdot 6!$

2fr + 3bk - 7 IN FRONT + 7 IN BACK — $\binom{5}{2}\binom{3}{3}\binom{8}{1}\cdot 7!\,\binom{8}{1}\,7!$

1fr + 4bk - 6 IN FRONT + 8 IN BACK — $\binom{5}{1}\cdot\binom{4}{4}\binom{8}{2}\,6!\,\binom{8}{0}\cdot 8!$

the 5 flexible students.

choose 1 from 5 for FRONT ROW

Choosing 2 empty seats in the front row.

Choosing 0 empty seats in back row

arranging 8 in back row.

arranging 6 in 6 seats in FRONT ROW

15 (a) First we choose 15 men & 15 women to make into couples. Then we find the different ways of making couples. There are

$\binom{15}{15}$ ways of choosing 15 men from 15 men

$\binom{20}{15}$ ways of choosing 15 women from 15 women

Once we pick the 15 men & 15 women we see there are: $\langle \underset{\substack{\uparrow \\ \text{choices of men} \\ \text{for 1st woman}}}{\underline{15}} \cdot \underset{\substack{\uparrow \\ \text{choices of men} \\ \text{for 2nd woman}}}{\underline{14}} \cdot \underline{13} \cdot \underline{\ \ } \cdots \underset{\substack{\uparrow \\ \text{choices of men} \\ \text{for 15th woman}}}{\underline{1}} \rangle$

i.e., 15! ways of making couples.

So answer $= \binom{15}{15} \cdot \binom{20}{15} \cdot 15! = \dfrac{20!}{5!}$

(b) Similarly we get the answer

$$\binom{15}{10} \cdot \binom{20}{10} \cdot 10! = \frac{15!}{10!\,5!} \cdot \frac{20!}{10!\,10!} \cdot 10!$$

$$= \frac{15!\ 20!}{(10!)^2 \cdot 5!}$$

16 $\binom{n}{r}$ = No. of r- subsets of $\{a_1, a_2, \cdots, a_n\}$

$\binom{n}{n-r}$ = No. of $(n-r)$ - subsets of $\{a_1, a_2, \cdots, a_n\}$

We will show that there is a one-to-one correspondence between the collection of all r-subsets & the collection of all $(n-r)$-subsets of $\{a_1, \cdots, a_n\}$

We correspond an r- subset $\{a_{n_1}, a_{n_2}, \cdots, a_{n_r}\}$ with the $(n-r)$-subset $\{a_1, \cdots, a_n\} - \{a_{n_1}, \cdots, a_{n_r}\}$.

It is easy to see that this is a one-to-one correspondence.

So $\binom{n}{r} = \binom{n}{n-r}$.

17. (a) First observe that each row will have to contain exactly one rook. We just get to choose the columns as we go along.

$$\langle \underset{\substack{\uparrow \\ \text{choices of colums} \\ \text{for rook in 1st row}}}{6} \cdot \underset{\substack{\uparrow \\ \text{in 2nd row}}}{5} \cdot \underset{\substack{\uparrow \\ \text{in 3rd row}}}{4} \cdot 3 \cdot 2 \cdot \underset{\substack{\uparrow \\ \text{in 6th row}}}{1} \rangle$$

Hence there are $6!$ ways of placing the rooks

(b) First we find the number of ways we can place the 2 red rooks in the rows. There are $\binom{6}{2}$ ways of choosing 2 rows for the red rooks. The blue rooks will be in the other 4 rows. Then, as above, we see how many choices of columns we have for the rooks in the 1st, 2nd, ..., 6th row. There will be $6!$ ways. So our answer is $\binom{6}{2} \cdot 6!$

18. First we have to pick 2 rows and 2 columns which will not be involved. The 6 rooks will cover the 6×6 board that remains. There are $\binom{8}{2}$ ways of picking 2 rows out of 8 and $\binom{8}{2}$ ways of picking 2 columns out of 8. And because of the analysis in 16(b) we see that our answer will be

$$\binom{8}{2} \cdot \binom{8}{2} \cdot \binom{6}{2} \cdot 6! = \frac{8!}{6!2!} \cdot \frac{8!}{6!2!} \cdot \frac{6!}{4!2!} \quad 6!$$

$$= \frac{(8!)^2}{8 \cdot 4!} = 7 \cdot 6 \cdot 5 \cdot (8!)$$

18. There is a slightly different way to do this problem. First pick 6 of the 8 rows to put the rooks. There are $\binom{8}{6}$ ways to do this. Then pick 2 rows for the red rooks. There are $\binom{6}{2}$ ways of doing this. Then check how many choices of columns you have for each of the 6 rows. $\langle 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \rangle$

$\underset{\text{column choices for 1st row among the 6 rows}}{8} \quad \underset{\text{2nd}}{7} \quad \ldots \quad \underset{\text{6th}}{3}$

So answer $= \binom{8}{2} \cdot \binom{6}{2} \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$

$$= \frac{8!}{6!\,2!} \cdot \frac{6!}{2!\,4!} \cdot \frac{8!}{2!} = 7 \cdot 6 \cdot 5 \cdot (8!)$$

19. (a) Pick 5 out of the 8 rows for red rooks. Then you'll get $\binom{8}{5} \cdot 8!$ as your answer for the problem.

(b) Pick 4 rows & 4 columns which will not be involved – this will leave an 8 by 8 board. Pick 5 out of the 8 rows for the red rooks. As above you'll get the answer

$$\binom{12}{4} \cdot \binom{12}{4} \cdot \binom{8}{5} \cdot 8!$$

20. Fix 0 as the anchor position. If you place 9 opposite 0, there will be $8!$ ways of placing the other digits. Since there are $9!$ total circular permutations of the 10 digits, our answer is $= 9! - 8! = 8 \cdot 8!$

21. (a) $9!/(1!\,2!\,1!\,3!\,2!)$ (b) $8!\left(\frac{1}{2!\,3!\,2!} + \frac{1}{3!\,2!} + \frac{1}{2!\,3!\,2!} + \frac{1}{2!\,2!\,2!} + \frac{1}{2!\,3!}\right) = (a)$

27 First we have to pick 3 rows & 3 columns which will not be involved. Since we can't pick the 1st row or first column, we will have $\binom{7}{3}\cdot\binom{7}{3}$ ways to do this. Now we are left with a 5 by 5 board and there are $5!$ ways of placing rooks (none attacking another) on such a board. Hence our answer will be $\binom{7}{3}\cdot\binom{7}{3}\cdot 5!$.

Note: $\binom{7}{4}^2\cdot 4! + 7^2\cdot\binom{6}{3}^2\cdot 3! = \binom{7}{3}^2\cdot 4! + \binom{7}{4}^2\cdot 4^2\cdot\binom{6}{3}^2\cdot 3! = \binom{7}{3}^2\cdot(4! + 4\cdot 4!)$
$= \binom{7}{3}^2\cdot 5!$

28(a) The secretary has to walk 17 blocks, E or N. If we know the blocks which he walked E on, then he'll automatically have to walk N on the other blocks. For example, he can walk east on his 1st, 2nd, 4th, 5th, 6th, 10th, 12th, 13th, 14th,

So number of routes

$=$ no. of 9-subsets of $\{1,2,3,\dots,17\}$

$= \binom{17}{9}$

(b) First find the no. of ways he can walk along the flooded block and then subtract from the answer in (a).

In a manner similar to that in (a) we get

$\binom{7}{4}\cdot\binom{1}{1}\cdot\binom{9}{4}$ ways he can reach using the flooded block.

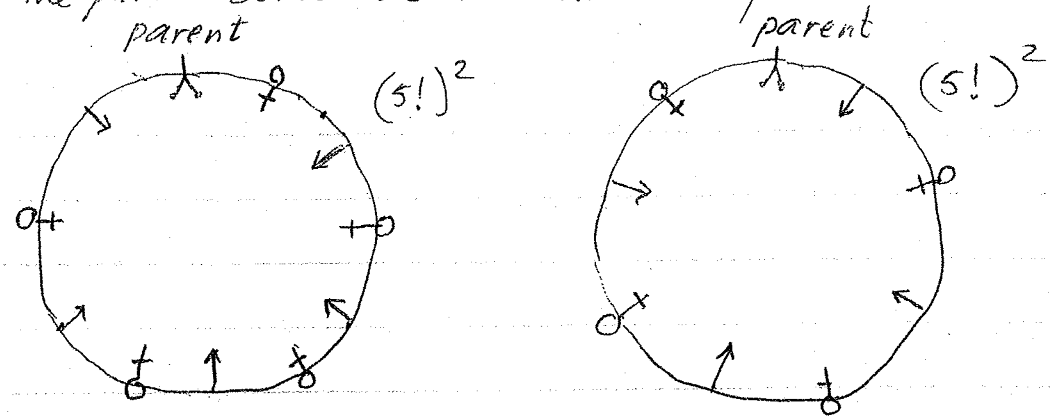
reaching the point 4 blocks east, 3 north

using the flooded block

going from end of flooded block to work place.

So our answer is $\binom{17}{9} - \binom{7}{4}\cdot\binom{1}{1}\cdot\binom{9}{4}$

30. (a) Let the parent serve as the anchor position.



↑ = boy , ♀ = girl , ⅄ = parent

There are two cases. We can have a girl to the left of the parent or we can have a boy there. Each case gives us $(5!)^2$ ways. So our answer is $2 \cdot (5!)^2$

(b) Split into cases depending where the parents are.

32. $[2 a, 4b, 5c]$ — produces $\frac{11!}{2!\,4!\,5!}$ 11-perm

$[3a, 3b, 5c]$ — produces $\frac{11!}{3!\,3!\,5!}$ 11-perm.

$[3a, 4b, 4c]$ — produces $\frac{11!}{3!\,4!\,4!}$ 11-perm.

Answer is $\frac{11!}{2!\,4!\,5!} + \frac{11!}{3!\,3!\,5!} + \frac{11!}{3!\,4!\,4!}$

$= \frac{11!}{3!\,4!\,5!}(3+4+5) = \frac{12!}{3!\,4!\,5!}$

33.

$[a, 4b, 5c] \to 10!/(1!\,4!\,5!) = 6 \cdot 10!/(3!\,4!\,5!)$

$[2a, 3b, 5c] \to 10!/(2!\,3!\,5!) = 12 \cdot$ "

$[2a, 4b, 4c] \to 10!/(2!\,4!\,4!) = 15 \cdot$ "

$[3a, 2b, 5c] \to 10!/(3!\,2!\,5!) = 12 \cdot$ "

$[3a, 3b, 4c] \to 10!/(3!\,3!\,4!) = 20 \cdot$ "

$[3a, 4b, 3c] \to 10!/(3!\,4!\,3!) = 20 \cdot$ "

Ans: $85 \cdot 10!/(3!\,4!\,5!)$

34. $11!\left(\frac{1}{2!\,3!\,3!\,3!} + \frac{1}{3!\,2!\,3!\,3!} + \frac{1}{3!\,3!\,2!\,3!} + \frac{1}{3!\,3!\,3!\,2!}\right) = \frac{12!}{(3!)^4}$

35 (a) $[a,a,b]$ $[a,a,c]$ $[a,b,c]$ $[a,c,c]$ $[b,c,c]$ $[c,c,c]$

(b) $[a,a,b,c]$ $[a,a,c,c]$ $[a,b,c,c]$ $[a,c,c,c]$ $[b,c,c,c]$.

36 A combination (of any size) of the multi-set $M = [n_1 \cdot a_1, \ldots, n_k \cdot a_k]$ is just a multi-set of the form $[x_1 \cdot a_1,\ x_2 \cdot a_2, \ldots, x_k \cdot a_k]$ where $0 \le x_i \le n_i$

So there (n_1+1) choices for x_1

(n_2+1) choices for x_2

$\vdots \qquad \vdots$

(n_k+1) choices for x_k

Hence the total number of combinations is

$(n_1+1) \cdot (n_2+1) \cdot \cdots \cdot (n_k+1)$

37. (a) No. of different dozens of pastry

= No. of solutions of "$x_1 + x_2 + \cdots + x_6 = 12$" in non-negative integers

$= \binom{12+6-1}{6-1} = \binom{17}{5}$

(b) No. of different dozens of pastry

= No. of solutions of $x_1 + \cdots + x_6 = 12$ in non-neg. integers with $x_i \ge 1$

= No. of solutions of $y_1 + \cdots + y_6 = 6$ in non-neg integers

$= \binom{6+6-1}{6-1} = \binom{11}{5}$.

38 No. of integral solutions of $x_1+x_2+x_3+x_4=30$ satisfying $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$, and $x_4 \geq 8$

= No of solutions of $(y_1+2)+(y_2)+(y_3-5)+(y_4+8)=30$ in non-negative integers

= No. of solutions of $y_1+y_2+y_3+y_4 = 25$ in non negative integers

$$= \binom{25+4-1}{4-1} = \binom{28}{3}.$$

39. (a) No. of ways of picking 6 sticks out of the 20

= No. of ways of arranging 14 1's and 6 + signs in a row

(the + signs represent the sticks we picked)

= No. of solutions of $x_1+x_2+\cdots+x_7 = 14$ in non-neg. integers

$$= \binom{14+7-1}{7-1} = \binom{20}{6}$$

<u>Aside</u>: Of course we can just say that there are $\binom{20}{6}$ ways of picking 6 sticks out of 20 by the theorem on combinations – but we did the problem as above to show how to do (b) and (c).

(b) No. of ways of picking 6 sticks out of the 20 so that no two are consecutive

= No. of solutions of $x_1+x_2+\cdots+x_7=14$ with $x_1 \geq 0$, $x_2 \geq 1$, $x_3 \geq 1$, $x_4 \geq 1$, $x_5 \geq 1$, $x_6 \geq 1$, $x_7 \geq 0$

39 $= $ No. of solutions of $y_1+(y_2+1)+(y_3+1)+\cdots+(y_6+1)+y_7=14$ in non-negative integers

$= $ No. of solutions of $y_1+y_2+\cdots+y_6+y_7=9$ in non-negative integers

$$= \binom{9+7-1}{7-1} = \binom{15}{6}$$

(c) No. of ways of picking 6 sticks out of the 20 so that there are at least 2 sticks between each pair of chosen sticks

$= $ No. of solutions of $x_1+x_2+\cdots+x_6+x_7=14$ with $x_1 \geq 0,\ x_2 \geq 2,\ x_3 \geq 2, \ldots, x_6 \geq 2,\ x_7 \geq 0$

$= $ No. of solutions of $y_1+(y_2+2)+(y_3+2)+\cdots+(y_6+2)+y_7=14$ in non-neg. integers

$= $ No. of solutions of $y_1+y_2+\cdots+y_6+y_7=4$

$$= \binom{4+7-1}{7-1} = \binom{10}{6}.$$

40. Using the same method as in # 32 we get the following answers

(a) $\binom{(n-k)+(k+1)-1}{(k+1)-1} = \binom{n}{k}$

(b) $\binom{n-k-(k-1)+(k+1)-1}{(k+1)-1} = \binom{n+1-k}{k}$

40 (c) $\binom{n-k-\ell.(k-1)+(k+1)-1}{(k+1)-1} = \binom{n+\ell-\ell.k}{k}$

41 Let x_i = no. of apples that the i-th child gets
If child #1 gets the orange, then the number of ways of distributing the 12 apples

= No. of integer solutions of $x_1+x_2+x_3=12$ with $x_1 \geq 0$, $x_2 \geq 1$, and $x_3 \geq 1$.

= No. of solutions of $y_1+y_2+y_3=10$ in non-neg. integers

$= \binom{10+3-1}{3-1} = \binom{12}{2}$

Similarly if child #2 gets the orange, number of ways of distributing the 12 apples

= No. of integer solutions of $x_1+x_2+x_3=12$ with $x_1 \geq 1$, $x_2 \geq 0$, and $x_3 \geq 1$

$= \cdots = \binom{12}{2}$

And finally if child #3 gets the orange, number of ways of distributing the 12 apples

= No. of integer solutions of $x_1+x_2+x_3=12$ with $x_1 \geq 1$, $x_2 \geq 1$, and $x_3 \geq 0$

$= \cdots = \binom{12}{2}$

So the total no. of ways of distributing the 12 apples & the orange so that each child gets at least one piece $= 3.\binom{12}{2} = \frac{3.12.11}{2} = 198.$

42 First we find the number of ways of distributing the 1 lemon drink & the 1 lime drink to different students. Let the students be #1, 2, 3 & 4.

$\langle 4 \cdot 3 \rangle$

4 choices for lemon drink

3 choices for lime drink (same person can't get both)

Now let's say #1 gets lemon & #3 gets the lime drink
Then no. of ways of distributing the 10 orange drinks
= No. of integer solutions of $x_1 + \cdots + x_4 = 10$ with $x_1 \geq 0$, $x_2 \geq 1$, $x_3 \geq 0$, and $x_4 \geq 1$
= No. of solutions of $y_1 + y_2 + y_3 + y_4 = 8$ in non-neg. integers

$$= \binom{8+4-1}{4-1} = \binom{11}{3}.$$

Since the same thing will happen in all the 11 other cases, the total number of ways of distributing the drinks $= 4 \cdot 3 \cdot \binom{11}{3} = 12 \cdot \binom{11}{3}$.

43. No. of r-comb. of $[1 \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k]$

= No. of r-combin. that contains a_1
+ No. of r-comb. that does not contain a_1

= No. of (r−1)-comb. of $[\infty \cdot a_2, \infty \cdot a_3, \ldots, \infty \cdot a_k]$
+ No. of r-comb. of $[\infty \cdot a_2, \infty \cdot a_3, \ldots, \infty \cdot a_k]$

$$= \binom{(r-1)+(k-1)-1}{(k-1)-1} + \binom{r+(k-1)-1}{(k-1)-1} = \binom{r+k-3}{k-2} + \binom{r+k-2}{k-2}.$$

44. There are k choices of children to give the 1st object, and k choices of children to give the 2nd object, and ... and k choices of children to give the n-th object. So there will be $k.k\ldots k$ (n times) $= k^n$ ways of distributing the n objects to the k children.

47. Let x_i = no. of books on shelf i $(i = 1,2,3)$. Then no. of ways of distributing the identical books so that no shelf has more than the other two combined =

no. of non-neg. integer solutions of $x_1 + x_2 + x_3 = 2n+1$, $0 \leq x_i \leq n$, which is equal to

No. of non.neg. integer solutions of $x_1 + x_2 + x_3 = 2n+1$

− No. of solutions of $x_1 + x_2 + x_3 = 2n+1$ with $x_1 \geq n+1$ or $x_2 \geq n+1$ or $x_3 \geq n+1$.

$$= \binom{2n+1+3-1}{3-1} - 3.\binom{n+(3-1)}{3-1} = \binom{2n+3}{2} - 3.\binom{n+2}{2} = \binom{n+1}{2}.$$

48. No. of perm. of m. A's & at most n B's =

No. of perm. of $[m.A, 0.B]$ + No. of perm. of $[m.A, 1.B]$ + ... + No. of perm. of $[m.A, n.B]$

$$= \binom{m}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+n}{n} = \binom{m+n+1}{n}$$

by formula 5.18 on p. 138. $= \binom{m+n+1}{m+1}$

49. No. of perm. of at most m A's & at most n B's

No. of perm. of of 0 A's & at most n B's

+ ... + No. of perm. of mA's & at most n B's

$$= \binom{0+n+1}{0+1} + \binom{1+n+1}{1+1} + \binom{2+n+1}{2+1} + \cdots + \binom{m+n+1}{m+1}$$

$$= \binom{m+n+1+1}{m+1} - \binom{n}{0} = \binom{m+n+2}{m+1} - 1$$ by formula 5.18 on p. 138,

because $\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \binom{n+3}{3} + \cdots + \binom{m+n+1}{m+1} = \binom{m+n+2}{m+1}$ ...(5.18)

Chapter 4

1. Remember the algorithm is just a nice, programming-friendly way of listing the permutations as on page 86. So things will appear from 3124 in the following order

$$\begin{array}{l} 3\ 1\ 2\ 4\ 5 \\ 3\ 1\ 2\ 5\ 4 \\ 3\ 1\ 5\ 2\ 4 \\ 3\ 5\ 1\ 2\ 4 \\ 5\ 3\ 1\ 2\ 4 \end{array}$$

because 3124 was <u>ninth</u> on the list given on page 90. So we have to insert the "5" beginning on the right.
∴ 31524 follows 31524 & 31254 comes before 31524

2. The mobile integers are 3, 7, and 8.

4. In the Algorithm, the directions of all integers p with $p >$ max. mobile integer is changed in step 3. Since 1 is the smallest integer, it cannot point to a smaller integer – so 1 is never mobile. So max. mobile integer is always ≥ 2. So only integers > 2 can possibly change their directions. So the directions of 1 and 2 never change.

6. (a) $\langle 2,4,0,4,0,0,1,0 \rangle$
(b) $\langle 6,5,1,1,3,2,1,0 \rangle$

b_i = no. of integers bigger than i that are in front of i

7. (a)

8
8 7
8 6 7
8 6 5 7
4 8 6 5 7
4 8 6 5 7 3
4 8 6 5 7 2 3
4 8 1 6 5 7 2 3

(b)

8
7 8
7 6 8
7 6 5 8
7 6 5 8 4
7 3 6 5 8 4
7 3 6 5 8 4 2
7 3 6 5 8 4 1 2

8. No. of inversions = sum of the terms in the inversion sequence.

(a) There is $\binom{6}{0} = 1$ perm. with 15 inversions. It is 6 5 4 3 2 1. Its inversion seq. is $\langle 5,4,3,2,1,0 \rangle$. [0 out of the 6 terms has to be reduced]

(b) There are $\binom{5}{1}$ perm. with 14 inversions. The inversion sequences of these permut. are

$\langle 5,4,3,2,0,0 \rangle$
$\langle 5,4,3,1,1,0 \rangle$ ← [Pick 1 out of the first 5 terms and reduce it by 1.]
$\langle 5,4,2,2,1,0 \rangle$
$\langle 5,3,3,2,1,0 \rangle$
$\langle 4,4,3,2,1,0 \rangle$

[Pick 2 out of the first 5 terms and reduce them by 1 each]
[or Pick 1 out of the first 4 terms and reduce that term by 2.]

(c) There are $\binom{5}{2} + \binom{4}{1}$ perm. with 13 inversions.

<u>Hint</u>: Look at the inversion sequence in (a)

$$\langle 5, 4, 3, 2, 1, 0 \rangle$$

and subtract 1 from 2 of the first 5 terms or subtract 2 from one of the first 4 terms.