Problem Set 2.1

If $|A| \neq 0$, then $C(A) =$ the whole space and

$N(A) = \{\text{zero vector}\}$.

If A is a null matrix then $C(A) = \{\text{zero vector}\}$

and $N(A) =$ the whole space.

No. 1. (a) Let $V = \{(u, v) :$ u and v are the ratios $\frac{p}{q}$ of integers, where $q \neq 0\}$

$x, y \in V \Rightarrow x+y \in V$ and $x-y \in V$.

So, V is closed under vector addition and subtraction.

But $\alpha = \sqrt{2} \in \mathbb{R}$ and $x = \left(\frac{1}{2}, \frac{3}{2}\right) \in V$

$\alpha x = \sqrt{2}\left(\frac{1}{2}, \frac{3}{2}\right) = \left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) \notin V$.

So, V is not closed under scalar multiplication.

(b) Let $V = \{(u, v) :$ where $u = 0$ or $v = 0\}$

$\alpha \in \mathbb{R}$ and $x \in V \Rightarrow \alpha x \in V$

So, V is closed under scalar multiplication.

But $x = (2, 0) \in V$ and $y = (0, 3) \in V$

$x + y = (2, 0) + (0, 3) = (2, 3) \notin V$

So, V is not closed under vector addition.

No. 2.

(a) $V = \{$The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0\}$

(i) Let $b = (b_1, b_2, b_3) \in V$ and $c = (c_1, c_2, c_3) \in V$.

Then $b_1 = 0$ and $c_1 = 0$.

$b+c = (b_1+c_1, b_2+c_2, b_3+c_3)$

$b_1 = 0$ and $c_1 = 0 \Rightarrow b_1+c_1 = 0$.

$\Rightarrow b+c \in V$

$\Rightarrow V$ is closed under vector addition.

(ii) Let $\alpha \in \mathbb{R}$ and $b = (b_1, b_2, b_3) \in V$.

Then $b_1 = 0$.

$\Rightarrow \alpha b_1 = 0$.

Now, $\alpha b = (\alpha b_1, \alpha b_2, \alpha b_3) \in V$

$\Rightarrow V$ is closed under scalar multiplication.

$\therefore V$ is a subspace of $\mathbb{R}^3$.

(b) $V = \{$The plane of vectors $b = (b_1, b_2, b_3)$ with $b_1 = 1\}$

(i) Let $b = (b_1, b_2, b_3) \in V$ and $c = (c_1, c_2, c_3) \in V$

Then $b_1 = 1$ and $c_1 = 1$.

$\Rightarrow b_1 + c_1 = 2$.

Now, $b+c = (b_1+c_1, b_2+c_2, b_3+c_3) \notin V$

So, V is not closed under vector addition.

$\therefore V$ is not a subspace of $\mathbb{R}^3$.

(e) $V = \{$The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0\}$

(i) Let $b = (b_1, b_2, b_3) \in V$ and $c = (c_1, c_2, c_3) \in V$.

Then $b_3 - b_2 + 3b_1 = 0$ and $c_3 - c_2 + 3c_1 = 0$.

$b+c = (b_1+c_1, b_2+c_2, b_3+c_3)$

Now, $b_3+c_3 - (b_2+c_2) + 3(b_1+c_1)$

$= (b_3 - b_2 + 3b_1) + (c_3 - c_2 + 3c_1)$

$= 0 + 0$

$= 0$

$\Rightarrow b+c \in V$. So, V is closed under vector addition.

Scanned with CamScanner

(ii) Let $\alpha \in \mathbb{R}$ and $b = (b_1, b_2, b_3) \in V$.

Then $b_3 - b_2 + 3b_1 = 0$.

$\alpha b = (\alpha b_1, \alpha b_2, \alpha b_3)$

Now, $\alpha b_3 - \alpha b_2 + 3\alpha b_1 = \alpha(b_3 - b_2 + 3b_1) = \alpha \cdot 0 = 0$

$\Rightarrow \alpha b \in V$

So, V is closed under scalar multiplication

$\therefore$ V is a subspace of $\mathbb{R}^3$.

No. 5. (i) $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$Ax = b$

$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = b$

$\Rightarrow u \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ 0 \end{bmatrix} = b$

$u = 0, v = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$u = 1, v = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$u = 0, v = 1 \Rightarrow b = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$Ax = 0$

$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow u \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow u = 1, v = 1$

N(A) is the line through (1,1).

i.e. y = x line.

$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \ldots \right\}$ which is x-axis of $\mathbb{R}^2$.

(ii) $B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

$Bx = b$

$\Rightarrow \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = b$

$\Rightarrow u \begin{bmatrix} 0 \\ 1 \end{bmatrix} + v \begin{bmatrix} 0 \\ 2 \end{bmatrix} + w \begin{bmatrix} 3 \\ 3 \end{bmatrix} = b$

$u = 0, v = 0, w = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$u = 1, v = 0, w = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$u = 0, v = 1, w = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$u = 0, v = 0, w = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$$C(B) = \left\{ \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}, \begin{bmatrix}3\\3\end{bmatrix}, \cdots\cdots \right\}$$

$C(B) = \mathbb{R}^2$.

$Bx = 0$

$$\Rightarrow \begin{bmatrix}0 & 0 & 3\\1 & 2 & 3\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow u\begin{bmatrix}0\\1\end{bmatrix} + v\begin{bmatrix}0\\2\end{bmatrix} + w\begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$\Rightarrow u = -2, v = 1, w = 0$

$N(B)$ is the line through $(-2, 1, 0)$.

(iii) $C = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

Since C is a null matrix, so $C(C) = \{\text{zero vector}\}$ $= \{(0,0)\} = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ and $N(C)$ = the whole space $\mathbb{R}^3$.

No. 24 (a)

$$\begin{bmatrix}1 & 4 & 2\\2 & 8 & 4\\-1 & -4 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & 4 & 2\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_2-2b_1\\b_3+b_1\end{bmatrix} \quad \begin{matrix}R_2 - 2R_1\\R_3 + R_1\end{matrix}$$

condition for solvability are

$$\boxed{b_2 - 2b_1 = 0 \text{ and } b_3 + b_1 = 0}$$

$b = \begin{bmatrix}1\\2\\-1\end{bmatrix}$

b_1 is any finite real value

In general, $b = (c, 2c, -c)$.

(b)

$$\begin{bmatrix}1 & 4\\2 & 9\\-1 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & 4\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}b_1\\b_2-2b_1\\b_3+b_1\end{bmatrix} \quad \begin{matrix}R_2 - 2R_1\\R_3 + R_1\end{matrix}$$

condition for solvability are

$$\boxed{b_3 + b_1 = 0 \text{ and } b_2 - 2b_1 \text{ is any finite real value}}$$

No. 8

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 + x_3 = 0$$
$$x_1 \quad + 2x_3 = 0$$

$$\Rightarrow x_1 = -2x_3$$

Let $x_3 = 1$. Then $x_1 = -2$ and $x_2 = -x_1 - x_3 = 2-1 = 1$.

$(-2, 1, 1)$

$N(A)$ = line of vectors $(-2x, x, x)$.

(b) a line

(d) a subspace

(e) the nullspace of A.

No. 26.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$Ax = b$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$$

$$\Rightarrow x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = b$$

$x_1 = 0,\ x_2 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 1,\ x_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 0,\ x_2 = 1 \Rightarrow b = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 1,\ x_2 = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

$$C(A) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \ldots\ldots \right\}$$

i.e. x-axis of $\mathbb{R}^3$.

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$Bx = b$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$$

$$\Rightarrow x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = b$$

$x_1 = 0,\ x_2 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 1,\ x_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 0,\ x_2 = 1 \Rightarrow b = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$x_1 = 1,\ x_2 = 1 \Rightarrow b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\vdots$

$$C(B) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \ldots\ldots \right\}$$

i.e. $z = 0$ i.e. xy-plane.

$$C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$Cx = b$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$

$\Rightarrow x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = b$

$x_1 = 0,\ x_2 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 1,\ x_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$x_1 = 1,\ x_2 = 1 \Rightarrow b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$x_1 = 2,\ x_2 = 1 \Rightarrow b = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$

$\vdots$

$$C(C) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \ldots\ldots \right\}$$

i.e. line of vectors $(x, 2x, 0)$.

No. 7. Given : Plane (P) : $x+2y+z=6$.

Equation of the plane P_0 through origin parallel to P is

$P_0: x+2y+z=0$

P_0 is a subspace of $\mathbb{R}^3$ but P is not a subspace of $\mathbb{R}^3$.

Q. Show that all combinations of two given vectors (1,1,0) and (2,0,1) is a subspace of $\mathbb{R}^3$.

Proof : Given : (1,1,0) and (2,0,1) two vectors.

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}_{3\times 2}$.

All combinations of the two vectors (1,1,0) and (2,0,1) means the column space C(A) of the matrix A.

From the definition of column space, we know that if A is a matrix of order $m\times n$, then the column space C(A) is a subspace of $\mathbb{R}^m$.

Here A is a matrix of order 3×2. So, column space of the given matrix A is a subspace of $\mathbb{R}^3$.

$\Rightarrow$ All combinations of the two given vectors is a subspace of $\mathbb{R}^3$.