Problem Set 3.2

No. 1. (a) Let $a = (\sqrt{y}, \sqrt{x})$ and $b = (\sqrt{x}, \sqrt{y})$ where x and y are positive numbers.

Schwarz inequality is

$$|a^T b| \le \|a\| \|b\|$$

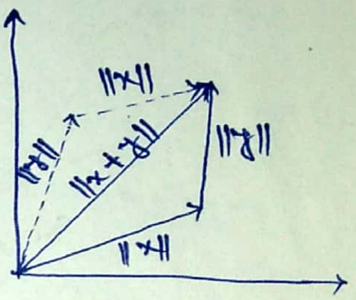
$$\Rightarrow \sqrt{xy} + \sqrt{xy} \le \sqrt{y+x}\,\sqrt{x+y}$$

$$\Rightarrow 2\sqrt{xy} \le (x+y)$$

$$\Rightarrow \sqrt{xy} \le \frac{x+y}{2}.$$

$$\Rightarrow GM \le AM.$$

(b) Triangle inequality is



$$\|x+y\| \le \|x\| + \|y\|$$

$$\Rightarrow \|x+y\|^2 \le (\|x\| + \|y\|)^2$$

$$\Rightarrow (x+y)^T(x+y) \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$\Rightarrow x^Tx + x^Ty + y^Tx + y^Ty \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$\Rightarrow \|x\|^2 + 2x^Ty + \|y\|^2 \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$\Rightarrow 2x^Ty \le 2\|x\|\|y\|$$

$$\Rightarrow x^Ty \le \|x\|\|y\|$$

$$\Rightarrow |x^Ty| \le \|x\|\|y\|,$$

which is the Schwarz inequality.

No. 3. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (1, 1, \ldots, 1) \in \mathbb{R}^n$.

The Schwarz Inequality is

$$|a^T b| \le \|a\| \|b\|$$

$$\Rightarrow (a_1 + a_2 + \cdots + a_n) \le (a_1^2 + a_2^2 + \cdots + a_n^2)^{\frac{1}{2}} \cdot (1 + 1 + \overset{n\text{-times}}{\cdots\cdots} + 1)^{\frac{1}{2}}$$

$$\Rightarrow (a_1 + a_2 + \cdots + a_n)^2 \le n(a_1^2 + a_2^2 + \cdots + a_n^2)$$

Equality will hold when

$$a_1 = a_2 = \cdots = a_n.$$

No. 5. Projection matrix P_1 onto the line through $a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is

$$P_1 = \frac{aa^T}{a^Ta} = \frac{aa^T}{\|a\|^2} = \frac{1}{10}\begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{9}{10} \end{bmatrix}$$

The matrix P_2 that projects onto the line perpendicular to a is

$$P_2 = I - P_1$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{9}{10} \end{bmatrix} = \begin{bmatrix} \frac{9}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{10} \end{bmatrix}$$

So, $P_1 + P_2 = I$

and $P_1 P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

No. 8. Let $a = (a_1, a_2, \ldots, a_n) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

$$P = \frac{aa^T}{a^Ta} = \frac{1}{a^Ta}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

$$= \frac{1}{a^Ta}\begin{bmatrix} a_1^2 & a_1a_2 & \cdots & a_1a_n \\ a_{21} & a_2^2 & \cdots & a_2a_n \\ \vdots & & & \vdots \\ a_na_1 & a_na_2 & \cdots & a_n^2 \end{bmatrix}$$

$$\text{Trace} = \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{a^Ta} = \frac{a^Ta}{a^Ta} = 1.$$

No. 9. Line: $x + 2y = 0$

It passes through the point $(-2, 1)$.

The matrix that projects onto the line through $(-2, 1)$ is

$$P = \frac{aa^T}{a^Ta} = \frac{1}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}.$$

11. $a = (1,1,1)$

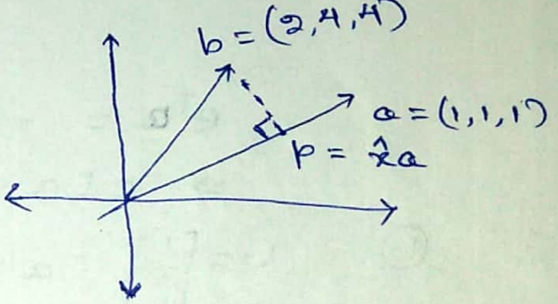
Let $p = \hat{x}a$ be closest to the point $b = (2,4,4)$.

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{a^T b}{\|a\|^2}$$

$$= \frac{1}{3}\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}2\\4\\4\end{bmatrix} = \frac{10}{3}$$

$$\therefore p = \hat{x}a = \left(\frac{10}{3}, \frac{10}{3}, \frac{10}{3}\right).$$

The point closest to a on the line through b is



$$p = \frac{b^T a}{b^T b} b$$

$$= \frac{1}{36}\begin{bmatrix}2 & 4 & 4\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} b$$

$$= \frac{10}{36} b = \frac{10}{36}[2,4,4] = \frac{5}{18}(2,4,4) = \left(\frac{5}{9}, \frac{10}{9}, \frac{10}{9}\right).$$

No. 17 (a) $b = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ and $a = \begin{bmatrix}1\\0\end{bmatrix}$

$$\hat{x} = \frac{a^T b}{a^T a} = a^T b \quad (\because a^T a = 1)$$

$$= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} = \cos\theta$$

Projection of b onto a is

$$p = \hat{x}a = \cos\theta\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\0\end{bmatrix} = (\cos\theta, 0)$$

(b). $b = \begin{bmatrix}1\\1\end{bmatrix}$ and $a = \begin{bmatrix}1\\-1\end{bmatrix}$

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{1}{2}a^T b = \frac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}\times 0 = 0$$

The projection of b onto a is

$$p = \hat{x}a = 0\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = (0,0).$$

No.19 (a) $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{5}{3}$$

Projection of the vector b onto the line through a is

$$p = \hat{x} a$$
$$= \frac{5}{3}(1,1,1) = \left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)$$

$$e = b - p = \left(1-\frac{5}{3}, 2-\frac{5}{3}, 2-\frac{5}{3}\right)$$
$$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

$$e^T a = -\frac{2}{3} + \frac{1}{3} + \frac{1}{3} = 0$$

$\Rightarrow e \perp a$.

(b) $b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $a = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{1}{11} \begin{bmatrix} -1 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \frac{-11}{11} = -1$$

Projection of the vector b onto the line through a is $p = \hat{x} a$

$$= -1(-1,-3,-1) = (1, 3, 1)$$

$$e = b - p = (0, 0, 0)$$

$e^T a = 0 \Rightarrow e \perp a$.

No. 21 $a_1 = (-1, 2, 2)$, $a_2 = (2, 2, -1)$

$$P_1 = \frac{a_1 a_1^T}{a_1^T a_1} = \frac{1}{9} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$$

$$P_2 = \frac{a_2 a_2^T}{a_2^T a_2} = \frac{1}{9} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$P_1 P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = zero matrix since $a_1 \perp a_2$.