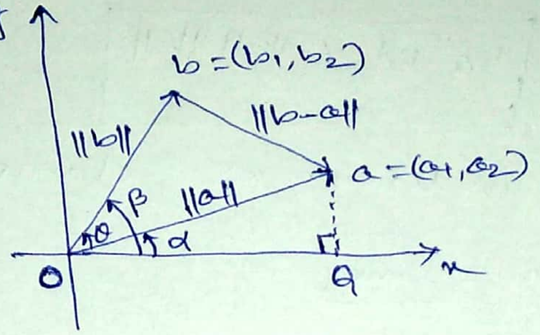
3.2: Cosines and Projections onto Lines

Course Outcomes: Students will have idea how to project vectors onto lines, how to construct projection matrix and about Schwarz inequality.

Inner Products and Cosines:



Let the vectors a and b make angles α and β with x-axis respectively. The angle between a and b is θ. From the figure, it is clear that $\theta = \beta - \alpha$.

$$\sin\alpha = \frac{a_2}{\|a\|}, \quad \cos\alpha = \frac{a_1}{\|a\|}$$

$$\sin\beta = \frac{b_2}{\|b\|}, \quad \cos\beta = \frac{b_1}{\|b\|}$$

Now, $\theta = \beta - \alpha$

$$\Rightarrow \cos\theta = \cos(\beta-\alpha)$$
$$= \cos\beta.\cos\alpha + \sin\beta\sin\alpha$$
$$= \frac{a_1b_1 + a_2b_2}{\|a\|\,\|b\|}$$
$$= \frac{a^Tb}{\|a\|\,\|b\|}$$

$$\Rightarrow \boxed{\cos\theta = \frac{a^Tb}{\|a\|\,\|b\|}}$$

We know that

$$|\cos\theta| \le 1$$

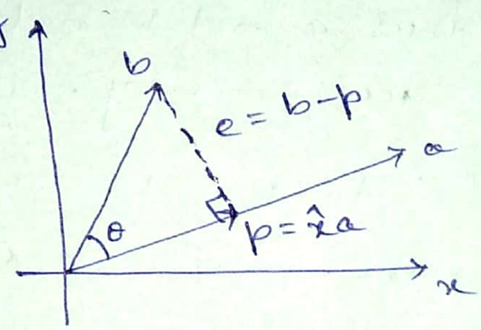
$$\Rightarrow \left| \frac{a^T b}{\|a\| \, \|b\|} \right| \le 1$$

$$\Rightarrow \frac{|a^T b|}{\|a\| \, \|b\|} \le 1$$

$$\Rightarrow \boxed{|a^T b| \le \|a\| \, \|b\|}$$

This is known as Schwarz Inequality.

<u>Projection onto a line:</u>



From the figure, it is clear that

$$e \perp a$$

$$\Rightarrow (b - p) \perp a$$

$$\Rightarrow (b - \hat{x}a) \perp a$$

$$\Rightarrow a^T(b - \hat{x}a) = 0$$

$$\Rightarrow a^T b - \hat{x}\, a^T a = 0$$

$$\Rightarrow \boxed{\hat{x} = \frac{a^T b}{a^T a}}$$

The projection of the vector b onto the line in the direction of a is

$$p = \hat{x}a$$

$$\Rightarrow \boxed{p = \frac{a^T b}{a^T a} a}$$

Note: Equality holds in Schwarz inequality $|a^T b| \le \|a\| \, \|b\|$ if and only if b is a multiple of a. The angle $\theta = 0^\circ$ or 180° and $\cos\theta = 1$ or -1. In this case b is identical with its projection p, and the distance between b and the line is zero.

Ex: Find the projection of $b = (1,2,3)$ onto the line through $a = (1,1,1)$ and verify Schwarz inequality.

Soln: Given: $a = (1,1,1)$, $b = (1,2,3)$

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{1+2+3}{3} = \frac{6}{3} = 2.$$

The projection is $p = \hat{x} a = (2,2,2)$.

$$\|a\| = \sqrt{3}, \quad \|b\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

Schwarz Inequality is

$$|a^T b| \le \|a\| \, \|b\|$$

$$\Rightarrow |6| \le \sqrt{3} \cdot \sqrt{14}$$

$$\Rightarrow 6 \le \sqrt{42}$$

$$\Rightarrow \sqrt{36} \le \sqrt{42}$$, which is true.

Projection Matrix of Rank 1:

$$p = \hat{x} a$$

$$= \frac{a^T b}{a^T a} a$$

$$= a \frac{a^T b}{a^T a}$$

$$= \frac{a a^T}{a^T a} b$$

$$= Pb,$$

where $\boxed{P = \frac{a a^T}{a^T a}}$ is the projection matrix.

Notes: 1. P is a symmetric matrix

2. $P^2 = P$.

Ex: Find the projection matrix that projects any vector onto the line through $a = (1,1,1)$.

Soln: Given: $a = (1,1,1)$

$$a^T a = \|a\|^2 = 3$$

$$aa^T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The required projection matrix is

$$P = \frac{aa^T}{a^T a} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Ex: Find the projection matrix that projects any vector onto the line through $a = (\cos\theta, \sin\theta)$.

Soln: Given: $a = (\cos\theta, \sin\theta)$.

$$a^T a = \|a\|^2 = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$aa^T = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

The required projection matrix is

$$P = \frac{aa^T}{a^T a} = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}.$$