LECTURE-29

continue from the previous lecture...

Q.6. Shoe that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into

$$det(A - \lambda t) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)....(\lambda_n - \lambda)$$

and making a cleaver choice of λ.

Solution:

If $\lambda = 0$ then,

$$det(A - 0) = (\lambda_1 - 0)(\lambda_2 - 0)....(\lambda_n - 0)$$

$$detA = \lambda_1\lambda_2\lambda_3....\lambda_n$$

Q.7. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 0 \end{bmatrix}$$

Check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace and $\lambda_1\lambda_2\lambda_3$ equals the determinant.

Solution:

Since the A is triangular matrix. Therefore the eigenvalues of the matrix A are $\lambda_1 = 3, \quad \lambda_2 = 1 \quad and \quad \lambda_3 = 0$

Sum of the eigenvalue

$$\lambda_1 + \lambda_2 + \lambda_3 = 3 + 1 + 0 = 4$$

Trace of of the matrix A=$3+1+0$=4

Therefore Sum of the eigenvalue is equal to the trace of of the matrix A.

Product of the eigenvalue

$$\lambda_1\lambda_2\lambda_3 = 3 \cdot 1 \cdot 0 = 0$$

and determinant of the matrix A is

$$\begin{vmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 3(0-0) - 4(0-0) + 2(0-0) = 0$$

Therefore Product of the eigenvalue is equal to the determinant of the matrix A.

Second part

Let Let x_1 be the eigenvector of the corresponding eigenvalue $\lambda_1 = 3$.

$$(A - 3I)x = 0$$

$$i.e. \quad \begin{bmatrix} 3-3 & 4 & 2 \\ 0 & 1-3 & 2 \\ 0 & 0 & 0-3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \leftarrow R_2 + \frac{1}{2}R_1$$

$$i.e. \quad \begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \leftarrow R_3 + R_2$$

$$i.e. \quad \begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here x is free variable since x's pivot is missing.

Eigenvaetor for $\lambda_1 = 3$, is

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let x_2 be the eigenvector of the corresponding eigenvalue $\lambda_2 = 3$.

$$(A - I)x = 0$$

$$i.e. \quad \begin{bmatrix} 3-1 & 4 & 2 \\ 0 & 1-1 & 2 \\ 0 & 0 & 0-1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_3 \leftarrow R_3 + \frac{1}{2}R_2$$

$$i.e. \quad \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here y is free variable since y's pivot is missing. Here $z = 0$ and $x = 2y$

Eigenvaetor for $\lambda_2 = 1$, is

$$x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Let x_3 be the eigenvector of the corresponding eigenvalue $\lambda_3 = 0$.

$$(A - 0)x = 0$$

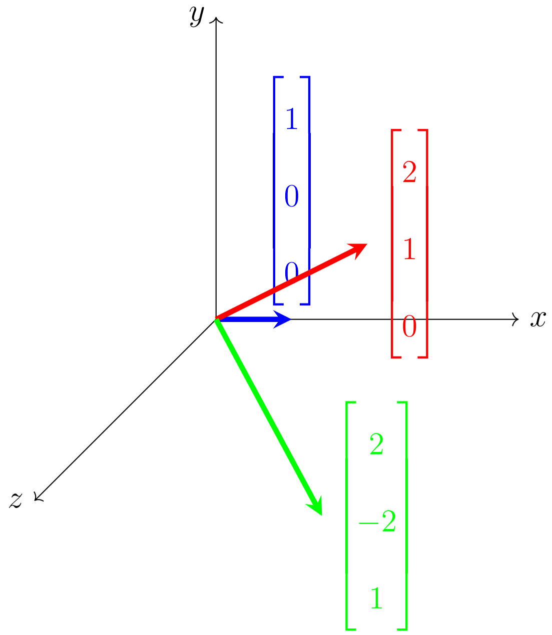
$$i.e. \quad \begin{bmatrix} 3-0 & 4 & 2 \\ 0 & 1-0 & 2 \\ 0 & 0 & 0-0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here z is the free variable since z's pivot is missing. Here $y = -2z$ and $x = 2z$

Eigenvaetor for $\lambda_3 = 0$, is

$$x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$



Q. 15. Find the eigenvalue and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \qquad and \quad B = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

Solution:

We know that, sum of the eigenvalue of the matrix is equal to trace of the matrix.

Since A is 2×2 matrix, therefore Matrix A has two eigenvalue. Let $\lambda_1 \quad and \quad \lambda_2$.

$$\lambda_1 + \lambda_2 = 3 - 3 = 0 \tag{1}$$

$$\lambda_1 \times \lambda_2 = \begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix} = -25 \tag{2}$$

Solving Eq. (1) and (2), $\quad \lambda_1 = 5 \quad or \quad \lambda_2 = -5$ Let x_1 be the eigenvector of the corresponding eigenvalue $\lambda_1 = 5$.

$$(A - 5I)x = 0$$

$$i.e. \quad \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_2 \leftarrow R_2 + 2R_1$$

$$i.e. \quad \begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvaetor for $\lambda_1 = 5$,

$$x_1 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Let x_2 be the eigenvector of the corresponding eigenvalue $\lambda_2 = -5$.

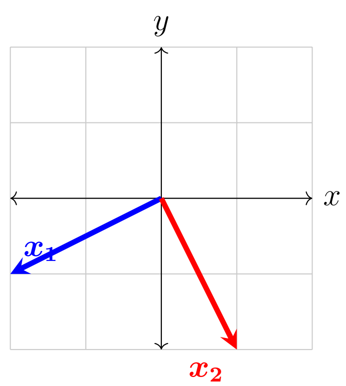
$$(A + 5I)x = 0$$

$$i.e. \quad \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} 8 & 4 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_2 \leftarrow R_2 - \frac{1}{2}R_1$$

Eigenvaetor for $\lambda_2 = -5$,

$$x_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$



Q.17. The Powers A^k of this matrix A approaches a limit as $k \to \infty$

$$A = \begin{vmatrix} .8 & .3 \\ .2 & .7 \end{vmatrix} \quad and \quad A^2 = \begin{vmatrix} .70 & .45 \\ .30 & .55 \end{vmatrix} \quad and \quad A^\infty = \begin{vmatrix} .6 & .6 \\ .4 & .4 \end{vmatrix}$$

The matrix A^2 is halfway between A and A^∞. Explain why $A^2 = \frac{1}{2}(A + A^\infty)$ from the eigenvalues and eigenvectors of these three matrices.

Solution: Given

$$A = \begin{vmatrix} .8 & .3 \\ .2 & .7 \end{vmatrix}$$

The characteristic equation for the matrix A is

$$det(A - \lambda I) = \begin{vmatrix} .8 - \lambda & .3 \\ .2 & .7 - \lambda \end{vmatrix}$$

$$= (.8 - \lambda)(.7 - \lambda) - 0.03 \cdot 0.2 = 0$$

$$= \lambda^2 - 1.5k + 0.5$$

$$= \lambda_1 = 1 \quad or \quad \lambda_2 = 0.5$$

Therefore the eigenvalue of A^2 are $\mu_1 = 1$ and $\mu_2 = 0.25$

Similarly, the eigenvalue of A^∞ are $\nu_1 = 1$ and $\nu_2 = 0$

Again,

$$\mu_i = \frac{1}{2} \cdot (\lambda_i + \nu_i)$$
$$\mu_i x_i = \frac{1}{2} \cdot (\lambda_i m u_i + \nu_i) x_i$$
$$A^2 x_i = \frac{1}{2} \cdot (A + A^\infty) x_i$$
$$A^2 x_1 + A^2 x_2 = \frac{1}{2} \cdot (A + A^\infty) x_1 + \frac{1}{2} \cdot (A + A^\infty) x_i$$
$$A^2 (x_1 + X_2) = \frac{1}{2} (A + A^\infty)(x_1 + X_2)$$
$$A^2 = \frac{1}{2} (A + A^\infty)$$

Q.3. If you shift to $A - 7I$, what are the eigenvalue and eigenvectors and how are they related to those of A?

Q.9 The eigenvalue of A equal the eigenvalue of A^T. This is because $det(A - \lambda I)$ equals $det(A^T - \lambda I)$. That is true because .... Show by an example that the eigenvectors of A and AT are not the same.

Q.39. When $a + b = c + d$ show that $(1, 1)$ is an eigenvector and find both eigenvalues

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Solution: Show that $(1, 1)$ is a vector of the matrix A for some eigenvalue λ. So, we have to find the eigenvalue for the given eigenvector.

Let $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$Ax = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \end{bmatrix} = (a+b) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad since \quad a+b = c+d$$

1.e. $\quad Ax = \lambda x, \quad where \quad \lambda = a+b$

$\therefore (1,1)$ is the eigenvector for the corresponding eigenvalue $\lambda = a+b$.