Ex: 3.2

① (a) $a = \begin{bmatrix} \sqrt{y} \\ \sqrt{x} \end{bmatrix}$, $b = \begin{bmatrix} \sqrt{x} \\ \sqrt{y} \end{bmatrix}$

Schwarz inequality,

$$|a^T b| \le ||a|| \, ||b||$$

$$\Rightarrow 2\sqrt{xy} \le \sqrt{x+y} \cdot \sqrt{x+y}$$

$$\Rightarrow 2\sqrt{xy} \le x+y$$

$$\Rightarrow \sqrt{xy} \le \frac{1}{2}(x+y)$$

$$\Rightarrow G.M \le A.M$$

(b) Given, $||x+y|| \le ||x|| + ||y||$

$$\Rightarrow ||x+y||^2 \le (||x|| + ||y||)^2$$

$$\Rightarrow ||x+y||^2 \le ||x||^2 + ||y||^2 + 2\,||x||\,||y||$$

$$\Rightarrow (x+y)^T(x+y) \le x^Tx + y^Ty + 2\,||x||\,||y||$$

$$\Rightarrow \cancel{x^Tx} + y^Tx + x^Ty + \cancel{y^Ty} \le \cancel{x^Tx} + \cancel{y^Ty} + 2\,||x||\,||y||$$

$$\Rightarrow \cancel{2}\, x^Ty \le \cancel{2}\, ||x||\,||y||$$

$\Rightarrow |x^T y| \leq ||x|| \, ||y||$ (Schwarz inequality)

③ By choosing the correct vector b in the Schwarz inequality, prove that

$$(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2)$$

When does equality hold?

Ans:- $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ (Choosen)

We know that,

$|a^T b| \leq ||a|| \, ||b||$ (Schwarz inequality)

$\Rightarrow |a_1 + a_2 + \cdots + a_n| \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \times \sqrt{n}$

(Squaring both sides)

$\Rightarrow (a_1 + a_2 + \cdots + a_n)^2 \leq n \cdot (a_1^2 + a_2^2 + \cdots + a_n^2)$

$\therefore$ Equality holds if

$a_1 = a_2 = \cdots = a_n$ (a is parallel to b)

⑤ (a) Find the projection matrix P_1 onto the line through $a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and also the matrix P_2 that projects onto the line perpendicular to a.

(b) Compute $P_1 + P_2$ and $P_1 P_2$ and explain.

Ans:- (a) $a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Projection Matrix $P_1 = \dfrac{aa^T}{a^T a}$

$$= \frac{\begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix}}{\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}}{10}$$

$$\Rightarrow P_1 = \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}$$

Let's take $b = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ to make the line perpendicular to a.

i.e. $\left[\begin{bmatrix}-3 & 1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = 0\right]$

$P_2 = \frac{bb^T}{b^Tb}$

$= \dfrac{\begin{bmatrix}-3\\1\end{bmatrix}\begin{bmatrix}-3 & 1\end{bmatrix}}{\begin{bmatrix}-3 & 1\end{bmatrix}\begin{bmatrix}-3\\1\end{bmatrix}}$

$= \dfrac{\begin{bmatrix}9 & -3\\-3 & 1\end{bmatrix}}{10}$

$\Rightarrow P_2 = \begin{bmatrix}9/10 & -3/10\\-3/10 & 1/10\end{bmatrix}$

(b) $P_1 + P_2 = \begin{bmatrix}1/10 & 3/10\\3/10 & 9/10\end{bmatrix} + \begin{bmatrix}9/10 & -3/10\\-3/10 & 1/10\end{bmatrix}$

$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$\Rightarrow P_1 + P_2 = I$

$P_1P_2 = \begin{bmatrix}1/10 & 3/10\\3/10 & 9/10\end{bmatrix}\begin{bmatrix}9/10 & -3/10\\-3/10 & 1/10\end{bmatrix}$

$\Rightarrow P_1P_2 = 0$

(8) Prove that the trace of $P = aa^T/a^Ta$, which is the sum of its diagonal entries always equals 1.

Ans: Let $a = \begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}$

$\therefore P = \frac{aa^T}{a^Ta}$

$\Rightarrow P = \frac{1}{a^Ta}\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}\begin{bmatrix}a_1 & a_2 & \cdots & a_n\end{bmatrix}$

$\Rightarrow P = \frac{1}{a^Ta}\begin{bmatrix}a_1^2 & a_1a_2 & \cdots & a_1a_n\\ \vdots & & & \\ a_na_1 & a_na_2 & \cdots & a_n^2\end{bmatrix}$

Traces of $P = \frac{a_1^2}{a^Ta} + \frac{a_2^2}{a^Ta} + \cdots + \frac{a_n^2}{a^Ta}$

$= \frac{a_1a_1 + a_2a_2 + \cdots + a_na_n}{a^Ta}$

$= \frac{a^Ta}{a^Ta} = 1$ (proved)

(9) Find the matrix that projects every point in the place onto the line $x+2y=0$.

Ans:- Given $x+2y=0$

$\Rightarrow x=-2y$

Let $a_1=(2,-1)$, $a_2=(-2,1)$, $a_3(-4,2)$ are the three points on the line $x+2y=0$.

Let P_1, P_2 and P_3 are the projection matrix for a_1, a_2 and a_3 respectively.

So, $P_1 = \frac{a_1a_1^T}{a_1^Ta_1} = \frac{\begin{bmatrix}2\\-1\end{bmatrix}\begin{bmatrix}2 & -1\end{bmatrix}}{\begin{bmatrix}2 & -1\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}}$

$= \frac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$

$P_2 = \frac{a_2a_2^T}{a_2^Ta_2} = \frac{\begin{bmatrix}-2\\1\end{bmatrix}\begin{bmatrix}-2 & 1\end{bmatrix}}{\begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}}$

$= \frac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$

$P_3 = \frac{a_3a_3^T}{a_3^Ta_3} = \frac{\begin{bmatrix}-4\\2\end{bmatrix}\begin{bmatrix}-4 & 2\end{bmatrix}}{\begin{bmatrix}-4 & 2\end{bmatrix}\begin{bmatrix}-4\\2\end{bmatrix}}$

$= \frac{1}{20}\begin{bmatrix}16 & -8\\-8 & 4\end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$ (Took '4' common)

$\therefore P_1 = P_2 = P_3$

So, $P = \begin{bmatrix}4/5 & -2/5\\-2/5 & 1/5\end{bmatrix}$ is the matrix that projects every point in the place onto the line $x+2y=0$

(11) What multiple of $a=(1,1,1)$ is closest to the point $b=(2,4,4)$? Find also the point closest to a on the line through b.

Ans:- Given, $a=\begin{bmatrix}1\\1\\1\end{bmatrix}$, $b=\begin{bmatrix}2\\4\\4\end{bmatrix}$

$\hat{x} = \frac{a^Tb}{a^Ta} = \frac{\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}2\\4\\4\end{bmatrix}}{\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}} = 10/3$

So, $10/3$ of $a=(1,1,1)$ is closest to the point $b=(2,4,4)$

Now, closest to a on the line through b.

$$\Rightarrow p' = \frac{b^T a}{b^T b} b$$

$$= \frac{\begin{bmatrix} 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

$$= \frac{10}{36}\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5/9 \\ 10/9 \\ 10/9 \end{bmatrix}$$

(17) Draw the projection of b onto a and also compute it from $p = \hat{x} a$:-

(a) $b = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(b) $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Ans:- (a) $\hat{x} = \frac{a^T b}{a^T a} = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$$= \frac{\cos\theta}{1} = \cos\theta$$

$$\therefore p = \hat{x} a = \cos\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}.$$

$$P = \frac{aa^T}{a^T a} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) $\hat{x} = \frac{a^T b}{a^T a} = \frac{\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = 0.$

$$\therefore p = \hat{x} a = 0 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$$P = \frac{aa^T}{a^T a} = \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix}}{\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

(19) Project the vector b onto the line through a. Check that e is perpendicular to a:-

(a) $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) $b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $a = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$

Ans: (a) $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (given)

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = 5/3$$

$$\therefore p = \hat{x} a = 5/3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix}$$

$$\therefore e = b - p = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\therefore e^T a = \begin{bmatrix} -2/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \Rightarrow e \perp a$$

(checked)

(b) (Same process)

$$\left.\begin{aligned} \hat{x} &= \frac{a^T b}{a^T a} = -1 \\ p &= \hat{x} a = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \\ e &= b - p = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ e^T a &= 0 \Rightarrow e \perp a \end{aligned}\right\} \text{(only answers)}$$