# Lecture 18
# 2.4 The Four Fundamental Subspaces

### Problem Set-2.4

**2.** Find the dimension and a basis for the four fundamental subspaces for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

**Ans.**

$$\begin{aligned} Let\ A &= \begin{array}{cl} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} & \begin{array}{l} R_1 \\ R_2 \\ R_3 \leftarrow R_3 - R_1 \end{array} \end{array} \\ \Rightarrow A &= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (echelon form) \end{aligned}$$

Rank of $A = r = 2$
Here $m = 3,\ n = 3$
$dim C(A) = r = 2$
$dim C(A^T) = r = 2$
$dim N(A) = n - r = 2$
$dim N(A^T) = m - r = 2$

The Column Space $C(A)$:

$$Basis\ for\ C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

The Null Space $N(A)$:

$$\begin{aligned} Ax &= 0 \\ \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow x_1 + 2x_2 + x_4 = 0, x_2 + x_3 = 0$$

$$\begin{aligned} Hence\ x &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ &= \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} \\ &= x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$Basis\ for\ N(A) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The Row Space $C(A^T)$:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$
\begin{aligned}
Let\ A^T &= \begin{bmatrix} \boxed{1} & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \leftarrow R_2 - 2R_1 \\ R_3 \\ R_4 \leftarrow R_4 - R_1 \end{matrix} \\
\Rightarrow A^T &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \leftarrow R_3 - R_2 \\ R_4 \end{matrix} \\
\Rightarrow A^T &= \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (echelon form)
\end{aligned}
$$

$$
Basis\ for\ C(A^T) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}
$$

The Leftnull Space $N(A^T)$:

$$
\begin{aligned}
A^T y &= 0 \\
\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\Rightarrow y_1 + y_3 = 0, y_2 = 0
$$

$$\begin{aligned}
Hence\ y &= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\
&= \begin{bmatrix} -y_3 \\ 0 \\ y_3 \end{bmatrix} \\
&= y_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

$$Basis\ for\ N(A^T) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

**13.** Find a basis for each of the four subspaces of

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

**Ans.**

$$\begin{aligned}
Let\ A &= \begin{bmatrix} 0 & \boxed{1} & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \leftarrow R_2 - R_1 \\ R_3 \end{matrix} \\
\Rightarrow A &= \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \\
\Rightarrow A &= \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
Basis\ for\ C(A) &= \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\} \\
Basis\ for\ C(A^T) &= \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 6 \end{bmatrix} \right\} \\
Basis\ for\ N(A) &= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} \\
Basis\ for\ N(A^T) &= \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}
\end{aligned}
$$

**18.** Find a 1 by 3 matrix whose nullspace consists of all vectors in $R^3$ such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace.
**Ans.** A 1 by 3 matrix whose nullspace consists of all vectors in $R^3$ such that $x_1 + 2x_2 + 4x_3 = 0$ is $A = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$.
Since $x_1 + 2x_2 + 4x_3 = 0$ in the matrix form can be written as

$$
\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}
$$

A 3 by 3 matrix with that same nullspace is $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 12 \end{bmatrix}$

**24.** Construct a matrix with the required property, or explain why you can't.

**(a)** Column space contains $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, row space contains $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

**(b)** Column space has basis $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, nullspace has basis $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

**(c)** Dimension of nullspace = 1+ dimension of left nullspace.
**(d)** Left nullspace contains $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, row space contains $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
**(e)** Row space = column space, nullspace $\neq$ left nullspace.
**Ans.**
**(a)** A matrix with the required property is given by

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**(b)** Impossible: dimensions $1 + 1 \neq 3$.
**(c)** A matrix with the required property is given by

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

**(d)** A matrix with the required property is given by

$$A = \begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$$

**(e)** Impossible: Row space = column space requires m = n.
Then $m - r = n - r$ .

**29.** Without elimination, find dimensions and bases for the four subspaces for

$$A = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \textit{ and } B = \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 5 & 5 \end{bmatrix}$$

**Ans.**