Chapter-3: Orthogonality

3.1 Orthogonal Vectors and Subspaces:

Course outcomes: Students will be acquainted with orthogonal vectors, orthonormal vectors, orthogonal subspaces and orthogal complement of subspaces.

Length of a vector:

Let $x = (x_1, x_2)$.

Length of $x = \|x\| = \sqrt{x_1^2 + x_2^2}$

Length square of $x = \|x\|^2 = x_1^2 + x_2^2$

Let $x = (x_1, x_2, x_3)$.

Length of $x = \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

Length square of $x = \|x\|^2 = x_1^2 + x_2^2 + x_3^2$

Let $x = (x_1, x_2, \dots, x_n)$.

Length of $x = \|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

Length square of $x = \|x\|^2 = x_1^2 + x_2^2 + \dots + x_n^2$

Inner product:

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Inner product of x with y is denoted by $x^T y$ and is defined by

$$x^T y = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= x_1 y_1 + x_2 y_2$$

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$.

Then $x^T y = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3$.

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.

Then $x^T y = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$

$x^T x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$= x_1^2 + x_2^2 + \cdots + x_n^2$

$= \|x\|^2$

$\Rightarrow$ Inner product of a vector with itself is equal to the length square of the vector.

$x^T y > 0 \Rightarrow$ angle between x and y is less than 90°.

$x^T y < 0 \Rightarrow$ angle between x and y is greater than 90°.

$x^T y = 0 \Rightarrow$ angle between x and y is 90°

$\Rightarrow x$ is orthogonal to y.

i.e. $x \perp y$.

Notes: 1. Zero is the only vector with length zero.

2. Zero is the only vector orthogonal to itself.

Definition: Two vectors x and y are said to be orthogonal to each other iff $x^T y = 0$.

Ex: Let $x = (2, 2, -1)$ and $y = (-1, 2, 2)$.

$\|x\| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{4+4+1} = 3$

$\|y\| = \sqrt{(-1)^2 + 2^2 + 2^2} = \sqrt{1+4+4} = 3$

Inner product of x with y is

$$x^T y = \begin{bmatrix} 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = -2+4-2=0$$

$\Rightarrow x \perp y$

$\Rightarrow x$ is orthogonal to y.

Orthogonal vectors in $\mathbb{R}^2$:

Let $v_1 = (\cos\theta, \sin\theta)$ and $v_2 = (-\sin\theta, \cos\theta)$.

$$\|v_1\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\|v_2\| = \sqrt{(-\sin\theta)^2 + \cos^2\theta} = 1$$

$$v_1^T v_2 = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$= -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$$

$\Rightarrow v_1 \perp v_2$

$\Rightarrow v_1$ is orthogonal to v_2.

Here assigning different values to θ, we can generate somany orthogonal vectors in $\mathbb{R}^2$.

Orthogonal Subspaces:

Two subspaces V and W of the same space $\mathbb{R}^n$ are orthogonal if every vector of V is orthogonal to every vector of W.

Ex:- 1. x-axis and y-axis are subspaces of $\mathbb{R}^2$ and every vector of x-axis is orthogonal to every vector of y-axis. So, x-axis $\perp$ y-axis in $\mathbb{R}^2$.

2. $y=x$ line $\perp$ $y=-x$ line in $\mathbb{R}^2$.

3. All the three axes in $\mathbb{R}^3$ are orthogonal to each other.

Notes:

1. The subspace $\{0\}$ is orthogonal to all subspaces.
2. A line is orthogonal to another line or it can be orthogonal to a plane but a plane can not be orthogonal to a plane.

The Fundamental Theorem of Orthogonality:

The row space is orthogonal to the nullspace in $\mathbb{R}^n$ and the column space is orthogonal to the left nullspace in $\mathbb{R}^m$ for a matrix of order $m \times n$.

In symbol, $C(A) \perp N(A^T)$ in $\mathbb{R}^m$

and $C(A^T) \perp N(A)$ in $\mathbb{R}^n$

for the matrix A of order $m \times n$.

Ex:- $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$C(A)$ is xy-plane

$C(A^T)$ is yz-plane

$N(A)$ is x-axis of $\mathbb{R}^3$

$N(A^T)$ is z-axis of $\mathbb{R}^3$

xy-plane $\perp$ z-axis in $\mathbb{R}^3$

$\Rightarrow C(A) \perp N(A^T)$ in $\mathbb{R}^3$

yz-plane $\perp$ x-axis in $\mathbb{R}^3$

$\Rightarrow C(A^T) \perp N(A)$ in $\mathbb{R}^3$.