Existence of One Sided Inverse :

Let A be a matrix of order $m \times n$ with rank r.

(i) If $r = m$, then A is a full row rank matrix.

Right inverse C of A will exist and

$$\boxed{C = A^T (AA^T)^{-1}}$$

$$AC = I$$

(ii) If $r = n$, then A is a full column rank matrix.

Left inverse B of A will exist and

$$\boxed{B = (A^T A)^{-1} A^T}$$

$$BA = I$$

<u>Ex</u> :- $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}_{2\times 3}$

The matrix A is already in echelon form with two pivots. So, rank of $A = r = 2$.

Here $m = 2$ and $n = 3$.

$m = r = 2$.

$\Rightarrow$ A is a full row rank matrix.

$\Rightarrow$ Right inverse C of A will exist.

$$C = A^T (AA^T)^{-1}$$

$$AA^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 25 \end{bmatrix}$$

$$(AA^T)^{-1} = \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{25} \end{bmatrix}$$

$$C = A^T(AA^T)^{-1}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}_{3\times 2}$

The matrix A is already in echelon form with two pivots. So, rank of $A = r = 2$.

Here $m = 3$ and $n = 2$.

$r = n = 2$

$\Rightarrow$ A is a full column rank matrix.

$\Rightarrow$ Left inverse B of A will exist.

$$B = (A^TA)^{-1}A^T$$

$$A^TA = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 25 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{25} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Matrices of rank 1:

Every matrix of rank 1 has the simple form

$A = uv^T$ = column times row.

Ex: $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix}$

Every row of A is a multiple of the 1st row.

$\Rightarrow$ dim. $C(A^T) = 1$

$\Rightarrow$ Rank of $A = 1$.

$A = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} = uv^T$,

where $u = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Problem Set 2.4

No. 1. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

The column space of A is

$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ldots \right\}$,

which is $z = 0$ i.e. xy-plane

The row space of A is

$C(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ldots \right\}$,

which is $x = 0$ i.e. yz-plane.

The nullspace of A is

$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \ldots \right\}$,

which is x-axis of $\mathbb{R}^3$.

The left nullspace of A is

$N(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \ldots \right\}$,

which is z-axis of $\mathbb{R}^3$.

No. 2. $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}_{3\times 4}$

$= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $R_3 \leftarrow R_3 - R_1$, echelon form

Rank of $A = r = 2$.

Here $m = 3$ and $n = 4$.

Dim. $C(A) = r = 2$

Dim. $C(A^T) = r = 2$

Dim. $N(A) = n - r = 4 - 2 = 2$

Dim. $N(A^T) = m - r = 3 - 2 = 1$.

Basis for the $C(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$

Basis of $C(A^T)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

$Ax = 0$

$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + 2x_2 + x_4 = 0$

$x_2 + x_3 = 0$

$\Rightarrow x_2 = -x_3$

$x_1 = -2x_2 - x_4 = 2x_3 - x_4$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Basis of $N(A)$ is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$A^T y = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y_1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow y_1 = 1, \; y_2 = 0, \; y_3 = -1.$

Basis of $N(A^T)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

No. 3. $A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}_{2\times 4}$ $(m = 2, n = 4)$

$= \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $R_2 \leftarrow R_2 - 2R_1$, echelon form

Rank of $A = r = 1$.

$\dim. C(A) = r = 1$, $\quad \dim. N(A) = n - r = 4 - 1 = 3$

$\dim. C(A^T) = r = 1$, $\quad \dim N(A^T) = m - r = 2 - 1 = 1$

Basis of $C(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Basis of $C(A^T)$ is $\{ (0, 1, 4, 0) \}$

$Ax = 0$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Basis of $N(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$A^T y = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 4 & 8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y_1\begin{bmatrix}0\\1\\4\\0\end{bmatrix}+y_2\begin{bmatrix}0\\2\\8\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow y_1 = 2,\ y_2 = -1$

Basis of $N(A^T)$ is $\left\{\begin{bmatrix}2\\-1\end{bmatrix}\right\}$.

No. 6 (i) $A = \begin{bmatrix}1 & 0 & 0 & 3\\0 & 0 & 0 & 0\\2 & 0 & 0 & 6\end{bmatrix}$

Every row of A is a multiple of the 1st row.

$\Rightarrow$ dim. $C(A^T) = 1$

$\Rightarrow$ Rank of $A = 1$

$A = \begin{bmatrix}1\\0\\2\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 3\end{bmatrix} = uv^T$ = column times row,

where $u = \begin{bmatrix}1\\0\\2\end{bmatrix}$ and $v = \begin{bmatrix}1\\0\\0\\3\end{bmatrix}$.

(ii) $A = \begin{bmatrix}2 & -2\\6 & -6\end{bmatrix}$

Second row of A is a multiple of 1st row.

$\Rightarrow$ dim. $C(A^T) = 1$

$\Rightarrow$ rank of $A = 1$

$A = \begin{bmatrix}2\\6\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} = uv^T$ = column times row,

where $u = \begin{bmatrix}2\\6\end{bmatrix}$ and $v = \begin{bmatrix}1\\-1\end{bmatrix}$.

No. 9. (i) $A = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\end{bmatrix}_{2\times 3}$

The matrix A is already in echelon form with two pivots. So, rank of $A = r = 2$.

Here $m = 2$ and $n = 3$.

$r = m = 2$

$\Rightarrow$ A is a full row rank matrix.

$\Rightarrow$ Right inverse C of A will exist.

$$C = A^T (AA^T)^{-1}$$

$$AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(AA^T)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$C = A^T (AA^T)^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

(ii) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}_{3 \times 2}$, Here $m = 3$ and $n = 2$.

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad R_2 \leftarrow R_2 - R_1$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_2$$

, echelon form

Rank of $A = r = 2$.

$r = n = 2$

$\Rightarrow$ A is a full column rank matrix.

$\Rightarrow$ Left inverse B of A will exist.

$$B = (A^T A)^{-1} A^T$$

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$