CH-3

Ex:- 3.1

① Which pairs are orthogonal among the vectors v_1, v_2, v_3, v_4?

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Ans:-

$$v_1^T v_2 = \begin{bmatrix} 1 & 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix} = -4 \neq 0$$

$$v_1^T v_3 = \begin{bmatrix} 1 & 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \cancel{1} - \cancel{2} + \cancel{2} - \cancel{1} = 0$$

$$v_1^T v_4 = \begin{bmatrix} 1 & 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1 + \cancel{2} - \cancel{2} + 1 = 2$$

$$v_2^T v_3 = \begin{bmatrix} 4 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \cancel{4} + 0 - \cancel{4} + 0 = 0$$

$$v_2^T v_4 = \begin{bmatrix} 4 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 4 + 0 + 4 + 0 = 8$$

$$v_3^T v_4 = \begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1 - 1 - 1 - 1 = -2$$

$\therefore$ (v_1, v_3) and (v_2, v_3) are orthogonal (Ans)

② Find a vector x orthogonal to the row space of A, and a vector y orthogonal to the column space, and a vector z orthogonal to the nullspace:-

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$$

Ans:- $N(A)$ is orthogonal to $C(A^T)$

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \in N(A) \quad \text{~~is orthogonal to $C(A^T)$~~}$$

$\therefore$ $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is orthogonal to the rowspace of A.

$N(A^T)$ is orthogonal to $C(A)$

$$y = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \in N(A^T) \quad [\text{solving } A^T y = 0]$$

$\therefore$ $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is orthogonal to the column space of A.

$C(A^T)$ is orthogonal to the $N(A)$

$$z = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \in C(A^T) \quad [\text{1st row of } A]$$

$\therefore$ $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is orthogonal to the nullspace of A.

7. Find the lengths and the inner product of $x = (1,4,0,2)$ and $y = (2,-2,1,3)$

Ans: Length of $x = ||x|| = \sqrt{x^T x}$

$$= \sqrt{\begin{bmatrix}1 & 4 & 0 & 2\end{bmatrix}\begin{bmatrix}1\\4\\0\\2\end{bmatrix}}$$

$$= \sqrt{1^2+4^2+0+2^2}$$

$$= \sqrt{21}$$

Length of $y = ||y|| = \sqrt{y^T y}$

$$= \sqrt{\begin{bmatrix}2 & -2 & 1 & 3\end{bmatrix}\begin{bmatrix}2\\-2\\1\\3\end{bmatrix}}$$

$$= \sqrt{2^2+(-2)^2+(1)^2+(3)^2}$$

$$= \sqrt{18} = 3\sqrt{2}$$

Inner product of x and $y = x^T y$

$$= \begin{bmatrix}1 & 4 & 0 & 2\end{bmatrix}\begin{bmatrix}2\\-2\\1\\3\end{bmatrix}$$

$$= 2-8+0+6$$

$$= 0$$

9. Find a basis for the orthogonal complement of row space of A:

$$A = \begin{bmatrix}1 & 0 & 2\\1 & 1 & 4\end{bmatrix}$$

Split $x = (3,3,3)$ into a row space component x_r and a nullspace component x_n.

Ans: $$A = \begin{bmatrix}1 & 0 & 2\\1 & 1 & 4\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 2\\0 & 1 & 2\end{bmatrix} \quad (R_2 \to R_2 - R_1)$$

So, $$\begin{bmatrix}1 & 0 & 2\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Let's take $z = 1$ (from mind)

$$\begin{bmatrix}1 & 0 & 2\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$\Rightarrow x + 2 = 0 \Rightarrow \boxed{x = -2}$ | $y + 2 = 0 \Rightarrow \boxed{y = -2}$

$\therefore$ Basis $= \left\{\begin{bmatrix}-2\\-2\\1\end{bmatrix}\right\}$

Now, $a\begin{bmatrix}1\\0\\2\end{bmatrix} + b\begin{bmatrix}1\\1\\4\end{bmatrix} + c\begin{bmatrix}-2\\-2\\1\end{bmatrix} = \begin{bmatrix}3\\3\\3\end{bmatrix}$

$\Rightarrow \; a + b - 2c = 3$ —— ①

$b - 2c = 3$ —— ②

$2a + 4b + c = 3$ —— ③

So, $a + \cancel{b} - \cancel{2c} = 3$

$\cancel{b} - \cancel{2c} = 3$

$(-) \; (+) \qquad (-)$

$\boxed{a = 0}$

Again, $b - 2\cancel{c} = 3$

Equⁿ②×2 $\Rightarrow 8b + 2\cancel{c} = 6$

$9b = 9$

$\Rightarrow \boxed{b = 1}$

So, $b - 2c = 3$

$\Rightarrow -2c = 3 - 1$

$\Rightarrow -2c = 2$

$\Rightarrow \boxed{c = -1}$

$\therefore x = \begin{bmatrix}3\\3\\3\end{bmatrix} = \begin{bmatrix}1\\1\\4\end{bmatrix} + \begin{bmatrix}2\\2\\-1\end{bmatrix} = x_p + x_n$

(10) Let P be the plane in R^2 with equⁿ $x+2y-z=0$. Find a vector perpendicular to P. What matrix has the plane P as its nullspace, and what matrix has P as its row space?

Ans: $x + 2y - z = 0$

$\Rightarrow \begin{bmatrix}1 & 2 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = 0$

So, $\begin{bmatrix}1\\2\\-1\end{bmatrix}$ is the vector perpendicular to P

Verification: Let $(1, 2, 5)$ be a point on P

So, $\begin{bmatrix}1 & 2 & -1\end{bmatrix}\begin{bmatrix}1\\2\\5\end{bmatrix} = 1 + 4 - 5$
$= 0$

Since P in R^2, then if $N(A) = P$, so A should be of order 2×3

$x + 2y - z = 0$

$\Rightarrow z = x + 2y$

Let $x = 1, y = 1 \Rightarrow z = 3$

and $x = 0, y = 1, \Rightarrow z = 2$

$\therefore A = \begin{bmatrix}1 & 1 & 3\\0 & 1 & 2\end{bmatrix}$ & $N(A) = P$

(11) Find all the vectors that are perpendicular to $(1, 4, 4, 1)$ and $(2, 9, 8, 2)$.

Ans: $\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix} = 0$

$\Rightarrow a + 4b + 4c + d = 0$

and $\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 2 \\ 9 \\ 8 \\ 2 \end{bmatrix} = 0$

$\Rightarrow 2a + 9b + 8c + 2d = 0$

So, $\begin{bmatrix} 1 & 4 & 4 & 1 \\ 2 & 9 & 8 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow Ax = 0$

So Basis of $N(A) = \left\{ \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(12) Show that $x - y$ is orthogonal to $x + y$ if and only if $||x|| = ||y||$

Ans: $x - y \perp x + y$

$\Rightarrow (x - y)^T (x + y) = 0$

$\Rightarrow (x^T - y^T)(x + y) = 0$

$\Rightarrow x^T x + \cancel{x^T y} - \cancel{y^T x} - y^T y = 0$

$\Rightarrow ||x||^2 - ||y||^2 = 0$

$\Rightarrow ||x|| = ||y||$

(18) If V and W are orthogonal subspaces, show that the only vector they have in common is the zero vector :- $V \cap W = \{0\}$

Ans: Since, V and W are orthogonal subspaces

$\therefore v^T w = 0 \;\; \forall \; v \in V$ and $w \in W$

So, V and W are subspaces, so, $0 \in V$ and $0 \in W$

So, they intersect at 0.

$\therefore V \cap W = \{0\}$

(33) Suppose A is a symmetric matrix ($A^T = A$).

(a) Why is its column space perpendicular to its nullspace?

(b) If $Ax = 0$ and $Az = 5z$, which subspaces contain these "eigenvectors" x and z?

Ans: (a) A is symmetric matrix ($A^T = A$)

$\therefore$ $C(A) = C(A^T)$ = row space of A

We know that $C(A^T) \perp N(A)$

so, $C(A) \perp N(A)$

(b) $x \in N(A)$, $z \in C(A^T) = C(A)$