4.2: Properties of Determinants:

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $|A| = ad-bc$.

or, $\det(A) = ad-bc$.

Let $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$|A| = a(ei-hf)-b(di-gf)+c(dh-ge)$$

Properties:

1. The determinant of the identity matrix is 1.

$$I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \det(I)=1$$

$$I=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \det(I)=1$$

2. The determinant changes sign when two rows are exchanged.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}, \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$$

3. The determinant depends linearly on the first row.

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} ta & tb \\ tc & td \end{vmatrix} = t^2\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Let A be a matrix of order n. Then

$$\det(tA) = t^n \det(A).$$

4. If two rows of A are equal, then $\det A = 0$.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix} = 0 \quad (\because R_1 = R_3)$$

5. Subtracting a multiple of one row from another row leaves the same determinant.

$$\begin{vmatrix} a-lc & b-ld \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\text{LHS} = \begin{vmatrix} a-lc & b-ld \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} lc & ld \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} c & d \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

6. If A has a row of zeros, then $\det A = 0$.

$$\begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = 0, \quad \begin{vmatrix} a & b & c \\ 0 & 0 & 0 \\ d & e & f \end{vmatrix} = 0.$$

7. If A is triangular, then $\det A$ is the product of the diagonal elements.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \Rightarrow \det A = a_{11} a_{12} a_{13}$$

(upper triangular)

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \det A = a_{11} a_{22} a_{33}$$

(Lower triangular)

If A is a diagonal matrix then $\det A$ is the product of the diagonal elements.

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \Rightarrow |A| = a_{11}\, a_{22}\, a_{33}.$$

All singular matrices have a zero determinant.

8. If A is singular, then $\det A = 0$. If A is invertible then $\det A \neq 0$.

9. $\det(AB) = \det(A).\det(B)$.

$$AA^{-1} = I$$
$$\Rightarrow \det(A\quad) = \det(I)$$
$$\Rightarrow \det(A)\det(A^{-1}) = 1$$
$$\Rightarrow \boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

10. The transpose of A has the same determinant as A itself: $\det A^T = \det A$.

Problem Set 4.2

No. 2. Let A be a 4 by 4 matrix with $\det A = \frac{1}{2}$.

Then $\det(2A) = 2^4 \det(A) = 2^4 \times \frac{1}{2} = 2^3 = 8$.

$$\det(-A) = (-1)^4 \det(A) = \det(A) = \frac{1}{2}$$

$$\det(A^2) = \det(A \times A) = \det(A).\det(A)$$
$$= (\det(A))^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

$$\det(A^{-1}) = \frac{1}{\det(A)} = 2.$$

No. 3. Let A be a 3 by 3 matrix with $\det A = -1$.

Then $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det A = \frac{1}{8} \times (-1) = -\frac{1}{8}$.

$\det(-A) = (-1)^3 \det(A) = 1$

$\det(A^2) = (\det(A))^2 = (-1)^2 = 1$

$\det(A^{-1}) = \frac{1}{\det(A)} = -1$.

No. 4.

Let $A = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix}$.

$= \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix}$ $\quad R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 + R_1$

$= \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix}$ $\quad R_4 \leftarrow R_4 + 2R_2$

$= \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$ $\quad R_4 \leftarrow R_4 + \frac{5}{2}R_3$

Upper triangular form.

$\det(A) = 1(-1)(-2)(10) = 20$.

Let $B = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$.

$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$ $\quad R_2 \leftarrow R_2 + \frac{1}{2}R_1$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \quad R_3 \leftarrow R_3 + \frac{2}{3}R_2$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & -\frac{11}{4} \end{bmatrix} \quad R_4 \leftarrow R_4 + \frac{2}{3}R_3$$

upper triangular form

$$\det(A) = 2 \times \frac{3}{2} \times \frac{4}{3} \times \left(-\frac{11}{4}\right)$$

$$= -11$$

Interchanging the 3rd and 4-th rows of the matrix B, we have

$$C = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix} \quad R_2 \leftarrow R_2 + \frac{1}{2}R_1$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & \frac{4}{3} & -1 \end{bmatrix} \quad R_4 \leftarrow R_4 + \frac{2}{3}R_2$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -\frac{11}{3} \end{bmatrix} \quad R_4 \leftarrow R_4 + \frac{4}{3}R_3$$

$$\det(C) = 2 \times \frac{3}{2} \times (-1) \times \left(-\frac{11}{3}\right) = 11$$

No. 5. (a) $A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}$ (a rank one matrix)

$$= \begin{bmatrix} 2 & -1 & 2 \\ 8 & -4 & 8 \\ 4 & -2 & 4 \end{bmatrix}$$

$\det(A) = 0$ ($\because$ Two columns of A are identical).

(b) $U = \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ (upper triangular matrix)

$\det(U) = 4 \times 1 \times 2 \times 2 = 16$

(c) $U^T = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 8 & 2 & 2 & 0 \\ 8 & 2 & 6 & 2 \end{bmatrix}$ (lower triangular matrix)

$\det(U^T) = 4 \times 1 \times 2 \times 2 = 16.$

(d) $\det(U^{-1}) = \dfrac{1}{\det(U)} = \dfrac{1}{16}.$

(e). $M = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{bmatrix}$ (reverse-triangular)

$$= \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_4$$

$$= \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$\det(M) = (-1)^2\, 4 \times 1 \times 2 \times 2 = 16$

No. 6.

$$\begin{vmatrix} a-mc & b-md \\ c-la & d-lb \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} - \begin{vmatrix} mc & md \\ c-la & d-lb \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ la & lb \end{vmatrix} - \begin{vmatrix} mc & md \\ c & d \end{vmatrix} + \begin{vmatrix} mc & md \\ la & lb \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l\begin{vmatrix} a & b \\ a & b \end{vmatrix} - m\begin{vmatrix} c & d \\ c & d \end{vmatrix} + lm\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - lm\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= (1-lm)\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

No. 13. (i) $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \Rightarrow \det(A) = 12-2 = 10$

(ii) $A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 3/10 & -1/5 \\ -1/10 & 2/5 \end{bmatrix}$

$$\det(A^{-1}) = \frac{6}{50} - \frac{1}{50} = \frac{5}{50} = \frac{1}{10}.$$

(iii) $A - \lambda I = \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$

$$\det(A-\lambda I) = (4-\lambda)(3-\lambda) - 2$$
$$= (\lambda-4)(\lambda-3) - 2$$
$$= \lambda^2 - 7\lambda + 12 - 2$$
$$= \lambda^2 - 7\lambda + 10$$

$$\lambda^2 - 7\lambda + 10 = 0$$
$$\Rightarrow \lambda^2 - 2\lambda - 5\lambda + 10 = 0$$
$$\Rightarrow (\lambda-2)(\lambda-5) = 0.$$
$$\Rightarrow \lambda = 2, 5$$

For $\lambda = 2, 5$, $A - \lambda I$ is singular.

No. 20. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$= \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$$|A^{-1}| = \frac{ad}{(ad-bc)^2} - \frac{bc}{(ad-bc)^2}$$

$$= \frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc}$$

But $|A^{-1}| = \frac{1}{ad-bc}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$

$= \frac{1}{ad-bc}(ad-bc) = 1$ is wrong.

4.4: Application of Determinants

Course Outcomes: Students will have idea how to use determinants to find inverse of a matrix A, to solve $Ax = b$, to find the pivots and the volume of boxes.

The inverse of a matrix A:

Given: Square matrix A

$|A| \neq 0 \Rightarrow A^{-1}$ will exist.

Minor of $A = M$

Cofactor of $A = C$ $\quad \begin{pmatrix} + & - \\ - & + \end{pmatrix}$

Adjoint of $A = C^T$

$$\boxed{A^{-1} = \frac{\text{Adj}\cdot A}{|A|} = \frac{C^T}{|A|}}$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $|A| = 4 - 6 = -2 \neq 0$

$\Rightarrow A^{-1}$ will exist.

Minor of $A = M = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

Cofactor of $A = C = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

$\text{Adj}\cdot A = C^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$$A^{-1} = \frac{C^T}{|A|} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Ex: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| = ad - bc \neq 0$

$\Rightarrow A^{-1}$ will exist.

Minor of $A = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$

$$\text{Cofactor of } A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = C$$

$$\text{Adj. } A = C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ex :- $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$|A| = 1(20-0) - 2(0-0) + 3(0-0) = 20 \neq 0$$

$\Rightarrow A^{-1}$ will exist.

$$\text{Minor of } A = \begin{bmatrix} 20 & 0 & 0 \\ 10 & 5 & 0 \\ -12 & 0 & 4 \end{bmatrix}$$

$$\text{Co-factor of } A = \begin{bmatrix} 20 & 0 & 0 \\ -10 & 5 & 0 \\ -12 & 0 & 4 \end{bmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Adj. A = Transpose of cofactor

$$= \begin{bmatrix} 20 & -10 & -12 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj. } A}{|A|} = \frac{1}{20} \begin{bmatrix} 20 & -10 & -12 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Solution of $Ax = b$: (Cramer's Rule)

Given:
$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}, \quad z = \frac{D_3}{D}$$

If $D \neq 0$, then the system has unique solution.

If $D = 0$ and at least one of D_1, D_2 and D_3 is nonzero, then the system has no solution.

If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the system has infinite number of solution.

Ex:- $x_1 + 3x_2 = 0$

$2x_1 + 4x_2 = 6$

$$D = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}, \quad D_1 = \begin{vmatrix} 0 & 3 \\ 6 & 4 \end{vmatrix}, \quad D_2 = \begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix}$$

$= 4 - 6 = -2$, $\quad = 0 - 18 = -18$, $\quad = 6 - 0 = 6$

$$x_1 = \frac{D_1}{D} = \frac{-18}{-2} = 9$$

$$x_2 = \frac{D_2}{D} = \frac{6}{-2} = -3$$

Ex:- $x + 4y - z = 1$

$x + y + z = 0$

$2x \quad + 3z = 0$

$$D = \begin{vmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 1(3-0) - 4(3-2) - 1(0-2)$$

$$= 3 - 4 + 2 = 1$$

$$D_1 = \begin{vmatrix} 1 & 4 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 1(3-0) = 3$$

$$D_2 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 1(0-0) - 1(3-2) - 1(0-0)$$
$$= -1$$

$$D_3 = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 1(0-0) - 4(0-0) + 1(0-2)$$
$$= -2.$$

$$x = \frac{D_1}{D} = \frac{3}{1} = 3, \quad y = \frac{D_2}{D} = \frac{-1}{1} = -1, \quad z = \frac{D_3}{D} = \frac{-2}{1} = -2.$$

To find the pivots :

Let A be a matrix of order n with fullset of pivots. Let $A_1, A_2, \ldots, A_n$ be the leading submatrices of A. Let $d_1, d_2, \ldots, d_n$ be the pivots of A. Then

$$\boxed{d_k = \frac{\det(A_k)}{\det(A_{k-1})},}$$

where $k = 1, 2, \ldots, n$ and $\det(A_0) = 1$

Ex : $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$

The leading submatrices of A are

$$A_1 = [2], \quad A_2 = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}, \quad A_3 = A$$

$$|A_1| = 2, \quad |A_2| = -16, \quad |A_3| = |A| = 2(-12) - 1(8) + 1(28-12)$$
$$= -24 - 8 + 16$$
$$= -16$$

The pivots of A are

$$d_1 = \frac{|A_1|}{|A_0|} = \frac{2}{1} = 2$$

$$d_2 = \frac{|A_2|}{|A_1|} = \frac{-16}{2} = -8$$

$$d_3 = \frac{|A_3|}{|A_2|} = \frac{-16}{-16} = 1$$

The volume of a box:

Let l_1, l_2 and l_3 be three adjacent sides of a box. Then the volume of the box is

$$V = l_1 l_2 l_3$$

Let l_1, l_2, l_3 and l_4 be four adjacent sides of a box. Then the volume of the box is

$$V = l_1 l_2 l_3 l_4$$

Similarly, if $l_1, l_2, \ldots, l_n$ be n adjacent edges of a box, then volume of the box is

$$V = l_1 l_2 \ldots l_n.$$

Let A be a matrix whose rows are the edges $l_1, l_2, \ldots, l_n$. Then

$$A = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix}, \quad A^T = \begin{bmatrix} l_1 & l_2 & \ldots & l_n \end{bmatrix}$$

$$AA^T = \begin{bmatrix} l_1^2 & l_1 l_2 & \cdots & l_1 l_n \\ l_2 l_1 & l_2^2 & \cdots & l_2 l_n \\ \vdots & & & \vdots \\ l_n l_1 & l_n l_2 & \cdots & l_n^2 \end{bmatrix}$$

$$= \begin{bmatrix} l_1^2 & 0 & \cdots & 0 \\ 0 & l_2^2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & l_n^2 \end{bmatrix} \quad (\because l_i \perp l_j \text{ for } i \neq j)$$

$$\Rightarrow \det(AA^T) = l_1^2 l_2^2 \ldots l_n^2$$

$\Rightarrow \det(A) \cdot \det(A^T) = l_1^2 l_2^2 \cdots l_n^2$

$\Rightarrow (\det(A))^2 = l_1^2 l_2^2 \cdots\cdots l_n^2 \quad (\because \det(A) = \det(A^T))$

$\Rightarrow \det \cdot (A) = l_1 l_2 \cdots\cdots l_n$

$\Rightarrow \det \cdot (A) =$ volume of the box.

<u>Ex</u>:- Find the volume of a box whose three adjacent edges are $(-1,2,2)$, $(2,-1,2)$ and $(2,2,-1)$.

<u>Solⁿ</u>:- The edge vectors are

$l_1 = (-1,2,2)$, $l_2 = (2,-1,2)$ and $l_3 = (2,2,-1)$.

Volume of the box is

$$V = \begin{vmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{vmatrix}$$

$$= -1(1-4) - 2(-2-4) + 2(4+2)$$

$$= 3 + 12 + 12$$

$= 27$ cube unit.

<u>Ex</u>:- Find area of the triangle with vertices $(2,2)$, $(-1,3)$ and $(0,0)$.

<u>Solⁿ</u>:- Given: vertices $(2,2)$, $(-1,3)$, $(0,0)$.

Area of the triangle is

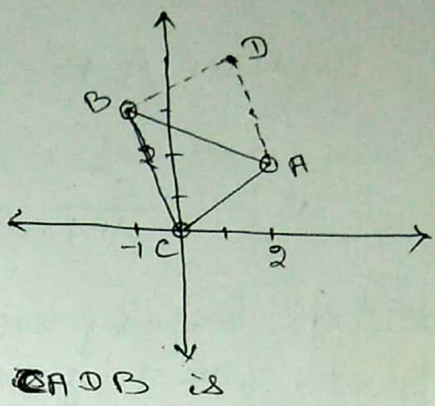
$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times 2(6+2) = 8$$

Problem Set 4.4

<u>No. 2</u> (a). $A(2,2)$, $B(-1,3)$, $C(0,0)$



1st edge of CADB is

$$l_1 = CA = (2,2)$$

2nd edge of CADB is

$$l_2 = CB = (-1,3).$$

Area of the parallelogram CADB is

$$\det\begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix} = \begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix} = 6+2 = 8$$

Area of the triangle ABC is

$$\frac{1}{2}\det.\begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix} = \frac{1}{2}\times 8 = 4.$$

(b) $A = (2,2)$, $B = (-1,3)$, $C = (1,-4)$.

$$\text{Area of } ABC = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

$$= \frac{1}{2}\left|\begin{vmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -4 & 1 \end{vmatrix}\right| = \frac{1}{2}\left[2(3+4) - 2(-1-1) + 1(4-3)\right]$$

$$= \frac{1}{2}(14+4+1)$$

$$= \frac{19}{2}.$$

<u>No. 5</u> (a) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$|A| = 2(4-1) + 1(-2-0) = 6-2 = 4 \neq 0.$$

$$\text{Minor of } A = M = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\text{Cofactor of } A = C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{C^T}{|A|} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

(b). $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$$|B| = 1(6-4) - 1(3-2) + 1(2-2) = 2-1 = 1 \neq 0.$$

$$\text{Minor of } B = M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\text{Cofactor of } B = C = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{C^T}{|B|} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

No. 7 (i) $ax + by = 1$

$cx + dy = 0$

$\Rightarrow AX = b,$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$D_1 = \begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix} = d$, $D_2 = \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix} = c$

$x = \frac{D_1}{D} = \frac{d}{ad-bc}$, $y = \frac{D_2}{D} = \frac{c}{ad-bc}$

(ii) $x + 4y - z = 1$

$x + y + z = 0$

$2x + 3z = 0$

$\Rightarrow AX = b,$

where $A = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$D = \begin{vmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 1(3-0) - 4(3-2) - 1(0-2) = 3 - 4 + 2 = 1$

$D_1 = \begin{vmatrix} 1 & 4 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 3$

$D_2 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{bmatrix} = 1(0-0) - 1(3-2) - 1(0-0) = -1$

$D_3 = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} = 1(0-0) - 4(0) + 1(0-2) = -2.$

$x = \frac{D_1}{D} = 3$, $y = \frac{D_2}{D} = -1$, $z = \frac{D_3}{D} = -2.$

No. 27. Given: parallelogram with sides (2,1) and (2,3).

Area of the parallelogram $= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 6-2=4$

Another parallelogram with sides (2,2) and (1,3).

Area $= \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 6-2=4.$

Area of both the parallelograms are equal.

No. 29. 1st edge of the box $= l_1 = (3,1,1)-(0,0,0) = (3,1,1)$

2nd edge of the box $= l_2 = (1,3,1)-(0,0,0) = (1,3,1)$

3rd edge of the box $= l_3 = (1,1,3)-(0,0,0) = (1,1,3).$

Volume of the box $= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix}$

$= 3(9-1)-1(3-1)+1(1-3)$

$= 24-2-2$

$= 20.$