2.3: Linear Independence, Basis and Dimension ①

Course Outcomes: Students will have understanding about linear independence, dependence, spanning a subspace, basis and dimension of vector spaces.

The aim of this section is to explain and use four ideas:

1. Linear Independence or dependence.
2. Spanning a subspace.
3. Basis for a subspace.
4. Dimension of a subspace.

Linear Independence or Dependence of Vectors:

Definition: A set of vectors $v_1, v_2, \ldots, v_n$ are said to be linearly independent if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

$\Rightarrow$ all scalars $\alpha_i = 0$ for $i = 1, 2, \ldots, n$.

Definition: A set of vectors $v_1, v_2, \ldots, v_n$ are said to be linearly dependent if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

$\Rightarrow$ at least one $\alpha_i \neq 0$, for $i = 1, 2, \ldots, n$.

Ex: Let $v_1 = (1, 2)$ and $v_2 = (3, 4)$.

Let α_1 and α_2 be two scalars.

$$\alpha_1 v_1 + \alpha_2 v_2 = 0$$

$$\Rightarrow \alpha_1 (1, 2) + \alpha_2 (3, 4) = 0$$

$\Rightarrow (\alpha_1, 2\alpha_1) + (3\alpha_2, 4\alpha_2) = 0$

$\Rightarrow \alpha_1 + 3\alpha_2 = 0$

$2\alpha_1 + 4\alpha_2 = 0$

$\Rightarrow \alpha_1 = 0,\ \alpha_2 = 0.$

So, the two given vectors are linearly independent.

Ex: Let $v_1 = (1, 2)$ and $v_2 = (2, 4)$.

Let α_1 and α_2 be two scalars.

$\alpha_1 v_1 + \alpha_2 v_2 = 0$

$\Rightarrow \alpha_1(1, 2) + \alpha_2(2, 4) = 0$

$\Rightarrow \alpha_1 + 2\alpha_2 = 0$

$2\alpha_1 + 4\alpha_2 = 0$

$\Rightarrow \alpha_1 = 2,\ \alpha_2 = -1.$

So, the given two vectors are linearly dependent.

Alternate methods:

Determinant method:

Ex: Let $v_1 = (1, 2)$ and $v_2 = (3, 4)$.

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$|A| = 4 - 6 = -2 \neq 0$

So, the two given vectors are linearly independent.

Ex: Let $v_1 = (1, 2)$ and $v_2 = (2, 4)$.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$|A| = 4 - 4 = 0$

So, the given vectors are linearly dependent.

Note: 1. This determinant method is applicable when A is a square matrix.

2. This determinant method is applicable when 2 vectors in $\mathbb{R}^2$, 3 vectors in $\mathbb{R}^3$ and so on.

Rank method:

Ex: Let $v_1 = (1, 2)$ and $v_2 = (3, 4)$.

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$, $R_2 \leftarrow R_2 - 2R_1$, echelon form

No. of pivots = 2 = no. of columns

$\Rightarrow$ The two given vectors are linearly independent.

Ex: Let $v_1 = (1, 2)$ and $v_2 = (2, 4)$.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $R_2 \leftarrow R_2 - 2R_1$, echelon form

No. of pivots = 1 $\neq$ no. of columns

$\Rightarrow$ The given two vectors are linearly dependent.

Ex: $v_1 = (1, 1)$, $v_2 = (2, 3)$, $v_3 = (1, 2)$.

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$.

$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $R_2 \leftarrow R_2 - R_1$, echelon form

No. of pivots = 2 $\neq$ no. of columns.

$\Rightarrow$ The three given vectors are linearly dependent.

Notes:

1. Three or more vectors in $\mathbb{R}^2$ are always linearly dependent.

2. Four or more vectors in $\mathbb{R}^3$ are always linearly dependent.

3. The set of n vectors in $\mathbb{R}^m$ must be linearly dependent if $n > m$.

Ex: Decide the dependence or independence of the vectors $(1,0,0)$, $(0,1,0)$ and $(-2,0,0)$.

Soln: Let $v_1 = (1,0,0)$, $v_2 = (0,1,0)$ and $v_3 = (-2,0,0)$.

Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The matrix A is already in echelon form.

No. of pivots $= 2 \neq$ no. of columns

$\Rightarrow$ The three given vectors are linearly dependent.

Alternate method:

Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$|A| = 1(0-0) - 2(0-0) = 0$.

So, the three given vectors are linearly dependent.

Ex: Let $v_1 = (1,2,0)$ and $v_2 = (0,1,-1)$.

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ (The two vectors are the two rows of the matrix)

The matrix A is already in echelon form.

No. of pivots $= 2 =$ no. of rows.

$\Rightarrow$ The two vectors v_1 and v_2 are linearly independent.

Ex: Given: $v_1 = (1,2,2)$, $v_2 = (-1,2,1)$, $v_3 = (0,8,0)$.

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 0 \end{bmatrix}$.

$$|A| = 1(0-8) + 1(0-16) = -8-16 = -24 \neq 0$$

$\Rightarrow$ The three vectors are linearly independent.

Ex: Given: $v_1 = (1,1,-1)$, $v_2 = (2,3,4)$, $v_3 = (4,1,-1)$ and $v_4 = (0,1,-1)$.

The given vectors are in $\mathbb{R}^3$. We know that four or more vectors in $\mathbb{R}^3$ are always linearly dependent.

So, the given four vectors are linearly dependent.

Points to remember:

1. Two vectors are dependent if they lie on the same line.
2. Three vectors are dependent if they lie on the same plane.
3. If the nullspace of a matrix is the zero vector only, then the columns of A are linearly indepenent.
4. The rows and the columns in which pivot elements present in the echelon form U and in the ~~reduced row~~ echelon form R of a matrix are linearly independent. The corresponding rows and columns of the given matrix are also linearly independent.

5. In $\mathbb{R}^2$, maximum two linearly independent vectors are present.

6. In $\mathbb{R}^3$, maximum three linearly independent vector are present.

7. In $\mathbb{R}^n$, maximum n linearly independent vectors are present.

Ex: Choose three linearly independent columns of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}.$$

Soln:

$$A = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -5 \end{bmatrix} \quad R_3 \leftarrow R_3 - 2R_2$$

, echelon form

Since the pivots are present in the 1st, 3rd and 4-th column of the echelon form, so the 1st, 3rd and 4-th column of the given matrix are linearly independent.

The 2nd, 3rd and 4-th columns of the given matrix are also linearly independent.

The given matrix has three linearly independent columns.