PROBLEM SET 2.2 CONTINUED...

Exercise 2.2.13: (a) find the special solutions to $Ux = 0$, Reduce U to R and repeat

$$Ux = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) If the Right hand side is changed from (0,0,0) to (a,b,0) what is solution?

Solution: (a) Aug matrix $= [U|0] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Since x_2, x_4 are free variables. So assume $x_2 = a$, $x_4 = b$, $a, b \in R$.

$$x_3 + 2x_4 = 0$$

$$x_3 = -2b$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

$$x_1 - 2a - 6b + 4b = 0$$

$$x_1 - 2a - 2b = 0$$

$$x_1 = 2a + 2b.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2a + 2b \\ a \\ -2b \\ b \end{bmatrix} = a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}, a, b \in R.$$

Since $U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$

(b) $Ux = b$

Aug matrix $= [U|b] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & a \\ 0 & 0 & 1 & 2 & b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Since x_2, x_4 are free variables. So assume $x_2 = c$, $x_4 = d$, $c, d \in R$.

$$x_2 + 2x_4 = b$$

$$x_3 + 2d = b \Rightarrow x_3 = b - 2d$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = a$$

$$x_1 + 2c + 3(b - 2d) + 4d = a$$

$$x_1 + 2c + 3b - 6d + 4d = a$$

$$x_1 + 2c + 3b - 2d = a$$

$$x_1 = a - 3b - 2c + 2d$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a - 3b - 2c + 2d \\ c \\ b - 2d \\ d \end{bmatrix} = \begin{bmatrix} a - 3b \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}, c, d \in R.$$

Exercise 2.2.34: What conditions on b_1, b_2, b_3, b_4 make each system solvable? solve for x,

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Solutions:(a) Aug. matrix $= \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{array}\right] \xrightarrow[R_4 \to R_4 - 3R_1]{R_2 \to R_2 - 2R_1, R_3 \to R_3 - 2R_1} \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 3 & b_4 - 3b_1 \end{array}\right]$

$$\xrightarrow{R_4 \to R_4 - 3R_3} \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 0 & b_4 - 3b_3 + 3b_1 \end{array}\right]$$

solution exist only if $\begin{array}{cc} b_2 - 2b_1 = 0, & b_4 - 3b_3 + 3b_1 = 0 \\ b_2 = 2b_1, & b_4 = b_1 + b_3 \end{array}$

Since
$$x_2 = b_3 - 2b_1$$

$$x_1 + 2x_2 = b_1$$

$$x_1 + 2(b_3 - 2b_1) = b_1$$

$$x_1 + 2b_3 - 4b_1 = b_1$$

$$x_1 = 5b_1 - 2b_3$$

General solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix}$

(b)Aug. matrix $= \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{array}\right] \xrightarrow[R_4 \to R_4 - 3R_1]{R_2 \to R_2 - 2R_1,\, R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \end{array}\right] \xrightarrow{R_4 \to R_4 - 3R_3}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 0 & 0 & b_4 - 3b_3 + 3b_1 \end{array}\right]$$

solution exist only if $b_2 - 2b_1 = 0$ and $b_4 - 3b_3 + 3b_1 = 0$ for the general solution.
Since x_3 is free variable assume $x_3 = k, k \in R$.

$$x_2 + k = b_3 - 2b_1$$

$$x_2 = b_3 - 2b_1 - k$$

$$x_1 + 2x_2 + 3x_3 = b_1$$

$$x_1 + 2(b_3 - 2b_1 - k) + 3k = b_1$$

$$x_1 + 2b_3 - 4b_1 - 2k + 3k = b_1$$

$$x_1 = 5b_1 - 2b_3 - k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 - k \\ b_3 - 2b_1 - k \\ k \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + k \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$