Lecture 20
3.1 Orthogonal Vectors and Subspaces

Problem Set-3.1

1. Which pairs are orthogonal among the vectors v_1, v_2, v_3, v_4?

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Ans.

$$\begin{aligned} Here \quad v_1^T v_2 &\neq 0 \\ v_1^T v_3 &= 0 \\ v_1^T v_4 &\neq 0 \\ v_2^T v_3 &= 0 \\ v_2^T v_4 &\neq 0 \\ v_3^T v_4 &\neq 0 \end{aligned}$$

So $v_1 \& v_3$ *and* $v_2 \& v_3$ are orthogonal pairs.

7. Find the lengths and the inner product of $x = (1, 4, 0, 2)$ and $y = (2, -2, 1, 3)$.

Ans.

$$\begin{aligned} x &= (1,4,0,2) \\ \Rightarrow ||x|| &= \sqrt{21} \\ y &= (2,-2,1,3) \\ \Rightarrow ||y|| &= \sqrt{18} \\ x^T y &= 0 \\ \Rightarrow x &\perp y \end{aligned}$$

9. Find a basis for the orthogonal complement of the row space of A:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$$

Split $x = (3,3,3)$ into a row space component x_r and a nullspace component x_n.

Ans. Basis for the orthogonal complement of the row space of A is same as basis for nullspace. i.e. $C(A^T)^{\perp} = N(A)$.

The Null Space $N(A)$:

$$\begin{aligned} Ax &= 0 \\ \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow u + 2w = 0, u + v + 4w = 0$$

$$\Rightarrow u = -2, v = -2, w = 1$$

$$\begin{aligned} Hence\ x &= \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \end{aligned}$$

$$Basis\ for\ N(A) = \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$So, Basis\ for\ C(A^T)^{\perp} = \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$\begin{aligned} x &= x_r + x_n \\ \Rightarrow \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} &= x_r + \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \\ \Rightarrow x_r &= \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix} \end{aligned}$$

12. Show that $x - y$ is orthogonal to $x + y$ if and only if $||x|| = ||y||$
Proof.

$$x - y\ is\ orthogonal\ to\ x + y$$

$$\begin{aligned} \Leftrightarrow (x - y) &\perp (x + y) \\ \Leftrightarrow (x - y)^T (x + y) &= 0 \\ \Leftrightarrow (x^T - y^T)(x + y) &= 0 \\ \Leftrightarrow x^T x + x^T y - y^x - y^T y &= 0 \\ \Leftrightarrow ||x||^2 - ||y||^2 &= 0 \quad since\ x^T y = y^T x \\ \Leftrightarrow ||x|| &= ||y|| \end{aligned}$$