Lecture-10

2.1 Vector Spaces and Subspaces

Course Outcomes: Students will have understanding about vector spaces and subspaces. Also, will be acquainted with the column space and the nullspace of different matrices.

In this article we will discuss about the followings:

(i) Vector Space

(ii) Subspaces

(iii) The Column Space

(iv) The Nullspace

Vector Space: A nonempty set V is said to be a vector space if it satisfies the following properties:

1. $x + y = y + x$ (Commutative law of addition)

2. $x + (y + z) = (x + y) + z$ (Associative law of addition)

3. There is a unique vector '0' (zero vector) such that $x + 0 = x$ for all x. (Additive identity property)

4. For each x there is a unique vector $-x$ such that $x + (-x) = 0$.(Additive inverse property)

5.$1x = x$

6. $(c_1c_2)x = c_1(c_2x)$

7. $c(x + y) = cx + cy$

8. $(c_1 + c_2)x = c_1x + c_2x$,

where $x, y, z \in$ V and $c,\ c_1,\ c_2\ \in\ R$.

Since out of the above eight properties first four are coming under vector addition and last four are coming under scalar multiplication, so the following is an alternate definition of vector space.

A nonempty set V is said to be a vector space if it satisfies the following properties:

1. Vector Addition

i.e. $x \in V, \ y \in V \implies x + y \in V$

2. Scalar Multiplication

i.e. $c \in R, x \in V \implies cx \in V$

Examples of vector spaces: R, R^2, R^3,........., R^n.

Example.

Show that R^2 is a vector space.

Proof. R^2 contains infinitely many elements and the elements are pairs like $(0, 0), (1, 2), (-1, 2), \ldots\ldots\ldots$

So, R^2 is nonempty.

1. Vector addition:

Let $x = (x_1, x_2)$ *and* $y = (y_1, y_2) \in R^2$.

$x + y = (x_1 + y_1, x_2 + y_2) \in R^2$ *as* $x_1 + y_1 \in R$ *and* $x_2 + y_2 \in R$.

2. Scalar multiplication:

Let $c \in R$ and $(x_1, x_2) \in R^2$.

$cx = (cx_1, cx_2) \in R^2$ *as* $cx_1 \in R$ *and* $cx_2 \in R$.

So, R^2 is a vector space.

Example. Verify whether the 1st quadrant of R^2 is a vector space or not.

Proof: Let the set V be the 1st quadrant of R^2.

Then $V = \{(x_1, x_2) \in R^2 : \quad x_1 \geq 0, \, x_1 \geq 0\}$

$= \{(0, 0), (1, 1), (1, 2), (2, 3), \ldots\ldots\ldots\}$

So, V is nonempty.

1. Vector addition:

Let $x = (x_1, x_2)$ *and* $y = (y_1, y_2) \in V$.

Then $\quad x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0$.

$x + y = (x_1 + y_1, x_2 + y_2) \in V \ as \ x_1 + y_1 \geq 0 \ and \ x_2 + y_2 \geq 0$.

2. Scalar multiplication:

Let $c = -2 \in R$ and $x = (1, 2) \in V$.

$cx = (-2, -4) \notin V$

So, V does not satisfy scalar multiplication property.

Hence, V i.e. 1st quadrant is not a vector space.

Example. Verify whether combinedly the 1st and 3rd quadrant of R^2 is a vector space or not.

Proof: Let the set V be the 1st and 3rd quadrant of R^2.

Then $V = \{(1, 1), (1, 2), (-1, -1), (-1, -2), \dots\dots\dots\}$.

So, V is nonempty.

1. Vector addition:

Let $x = (-1, -2)$ and $y = (2, 1) \in$ V.

$x + y = (1, -1) \notin$ V.

So, V does not satisfy vector addition property.

Hence, V i.e. 1st and 3rd quadrant combinedly is not a vector space.

Subspace: A subspace of a vector space is a nonempty subset that satisfies the requirements for a vector space.

Example. Show that $y = x$ line is a subspace of the vector space R^2.

Proof. Let the set V be $y = x$ line.

Then $V = \{(x_1, x_2) \in R^2 : x_1 = x_2\}$

$= \{(0,0),(1,1),(2,2),(-1,-1),(-2,-2),\ldots\ldots\ldots\}$.

So, V is a nonempty subset of R^2.

1. Vector addition:

Let $x = (x_1, x_2)\ and\ y = (y_1, y_2) \in V.$

Then $x_1 = x_2\ and\ y_1 = y_2.$

$x + y = (x_1 + y_1, x_2 + y_2) \in V\ as\ x_1 + y_1 = x_2 + y_2.$

2. Scalar multiplication:

Let $c \in R\ and\ (x_1, x_2) \in V.$

$cx = (cx_1, cx_2) \in V\ as\ cx_1 = cx_2.$

So, V i.e. $y = x$ line is a subspace space R^2 .

Vector Space: R^2

Subspaces:

1. R^2

2. Any line passing through origin.

3. Origin i.e. {(0,0)}.

Vector Space: R^3

Subspaces:

1. R^3

2. Any plane passing through origin.

3. Any line passing through origin.

3. Origin i.e. {(0,0,0)}.

Points to remember:

1. Every vector space is a subspace of itself.

2. A subspace is a vector space in its own right.

3. Every vector space is the largest subspace of itself and origin is the smallest

subspace.

No.2 Which of the following subsets of R^3 are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.

(b) The plane of vectors b with $b_1 = 1$.

Sol. (a).

Let $V = \{The\ plane\ of\ vectors\ (b_1, b_2, b_3)\ with\ first\ component\ b_1 = 0\}$.

$= \{(0, 0, 0), (0, 1, 0), (0, 1, 2), \dots\dots\dots\}$

So, V is a nonempty subset of R^3.

1. Vector addition:

Let $b = (b_1, b_2, b_3)\ and\ c = (c_1, c_2, c_3) \in V.\ Then\ b_1 = 0\ and\ c_1 = 0.$

$b + c = (b_1 + c_1, b_2 + c_2, b_3 + c_3) \in V\ as\ b_1 + c_1 = 0.$

2. Scalar multiplication:

Let $\alpha \in R\ and\ b = (b_1, b_2, b_3) \in V.\ Then\ b_1 = 0 \implies \alpha b_1 = 0$

$\alpha b = (\alpha b_1, \alpha b_2, \alpha b_3) \in V\ as\ \ \alpha b_1 = 0$

So, V is a subspace of R^3.

Sol. (b).

Let $V = \{The\ plane\ of\ vectors\ b = (b_1, b_2, b_3)\ with\ first\ component\ b_1 = 1\}$.

$= \{(1, 0, 0), (1, 1, 0), (1, 1, 2), \dots\dots\dots\}$

So, V is a nonempty subset of R^3.

1. Vector addition:

Let $b = (b_1, b_2, b_3)\ and\ c = (c_1, c_2, c_3) \in V.\ Then\ b_1 = 1\ and\ c_1 = 1 \implies$

$b_1 + c_1 = 2.$

$b + c = (b_1 + c_1, b_2 + c_2, b_3 + c_3) \notin V\ as\ b_1 + c_1 \neq 1.$

So, V does not satisfy vector addition property.

Hence, V is not a subspace of R^3.