Lecture 16

2.3 Linear Independence, Basis and Dimension

Course Outcomes: Students will have understanding about linear independence,dependence, spanning a subspace, basis and dimension of a vector space.
The aim of this section is to explain and use four ideas:

- Linear Independence or dependence.
- spanning a subspace.
- basis for a subspace.
- dimension of a subspace.

Spanning a Subspace: If $S = \{w_1, w_2, ..., w_l\}$ is a set of vectors in a vector space V, then the **span of S** is the set of all linear combinations of the vectors in S.

$$Span(S) = \{c_1w_1 + c_2w_2 + ... + c_lw_l \mid for\ all\ c_i \in R\}$$

If every vector in a given vector space can be written as the linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.

Notes:

- The column space is spanned by its columns.
- The row space is spanned by its rows.

Basis for a Vector Space: A basis for a vector space V is a subset with a sequence of vectors having two properties at once:

- The vectors are linearly independent.(not too many vectors)
- They span the space V.(not too few vectors)

Notes:

- A basis of a vector space is the maximal independent set.
- A basis of a vector space is also a minimal spanning set.
- Spanning involves the column space and independence involves the null space.
- No elements of a basis will be wasted.

Example: Check whether the following sets are the basis of R^3 or not?

$$
\begin{aligned}
(a) B_1 &= \{(1,2,2),(-1,2,1),(0,8,0)\} \\
(b) B_2 &= \{(1,2,2),(-1,2,1),(0,8,6)\} \\
(c) B_3 &= \{(1,2,2),(-1,2,1)\} \\
(d) B_4 &= \{(1,0,0),(0,1,0),(0,0,1)\} \\
(e) B_5 &= \{(1,1,-1),(2,3,4),(4,1,-1),(0,1,-1)\}
\end{aligned}
$$

Solution:
(a)

$$
\begin{aligned}
Let\ A &= \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 0 \end{bmatrix} \\
\Rightarrow |A| &= -24 \\
\Rightarrow |A| &\neq 0
\end{aligned}
$$

The vectors $(1,2,2),(-1,2,1)$ and $(0,8,0)$ are LI.
So, B_1 is a basis of R^3.

Dimension of Vector Spaces: Dimension of a vector space is the maximum number of LI vectors of the vector space.

OR. The no. of elements present in the basis of a vector space is known as the dim. of vector space.

$$dimR = 1, dimR^2 = 2, dimR^3 = 3, ...dimR^n = n$$

Problem Set-2.3

16. Describe the subspace of R^3 (is it a line or a plane or R^3?) spanned by
(a) the two vectors $(1, 1, -1)$ *and* $(-1, -1, 1)$.
(b) the three vectors $(0, 1, 1)$ *and* $(1, 1, 0)$ *and* $(0, 0, 0)$.
(c) the columns of a 3 by 5 echelon matrix with 2 pivots.
(d) all vectors with positive components.
Ans.(a)

$$\begin{aligned} Let\ A &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 + R_1 \end{array} \\ \Rightarrow A &= \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (echelonform) \end{aligned}$$

Here rank of A=1.
$\therefore$ The subspace of R^3 spanned by the two vectors $(1, 1, -1)$ *and* $(-1, -1, 1)$ is a line passing through the origin of R^3.
(b)

$$Let\ A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Interchanging R_1 by R_2 we have,

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \leftarrow R_3 - R_1 \end{array} \\ \Rightarrow A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \leftarrow R_3 + R_2 \end{array} \\ \Rightarrow A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (echelonform) \end{aligned}$$

Here rank of A=2.
$\therefore$ The subspace of R^3 spanned by the three vectors $(0,1,1)$ *and* $(1,1,0)$ *and* $(0,0,0)$ is a plane of R^3.
(c)

$$Let\ A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here rank of A=2.
$\therefore$ The subspace of R^3 spanned by the the columns of a 3 by 5 echelon matrix with 2 pivots is a plane.

19. Find a basis for the plane $x - 2y + 3z = 0$ in R^3. Then find a basis for the intersection of that plane with the xy-plane. Then find a basis for all vectors perpendicular to the plane.
Ans. The basis for the plane $x - 2y + 3z = 0$ in R^3 is the nullspace of the matrix $A = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$.
The plane $x - 2y + 3z = 0$ in the matrix form can be written as

$$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Here x is the pivot variable and y, z are the free variables.
So, $x = 2y - 3z$

$\therefore x = \begin{bmatrix} 2y - 3z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Basis for the plane is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

A basis for the intersection of the plane $x - 2y + 3z = 0$ with the xy-plane i.e z=0 is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

A basis for all vectors perpendicular to the plane is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}$

23 Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspace.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

Ans.

$$\begin{aligned} Let\ A &= \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \leftarrow R_3 - R_1 \end{matrix} \\ \Rightarrow A &= \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (echelon form) \end{aligned}$$

$$Basis\ for\ C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$Basis\ for\ C(A^T) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

To find basis for nullspace we have,

$$\begin{aligned} Ax &= 0 \\ \Rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow x + 3y + 2z = 0, y + z = 0$$

$$Hence\ x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$Basis\ for\ N(A) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$