1.5 Triangular Factors and Row Exchanges

Course Outcome: Students will have understanding about the triangular factorization like LU and LDU factorization, and permutation matrices that are being used for row exchange purpose.

Triangular Factorization:

Given: $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$ $\quad R_2 \leftarrow R_2 - 2R_1 \ (2)$, $\quad R_3 \leftarrow R_3 + R_1 \ (-1)$

$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad R_3 \leftarrow R_3 + R_2 \ (-1)$

$= U$

The elementary matrices are

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$E_{32} E_{31} E_{21} A = U$

$\Rightarrow MA = U,$

where $M = E_{32} E_{31} E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

$M^{-1} = (E_{32} E_{31} E_{21})^{-1}$

$= E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = L.$

$$MA = U$$

$$\Rightarrow A = M^{-1}U$$

$\Rightarrow \boxed{A = LU}$, which is known as LU factorization of the matrix A.

Ex :- Find the LU and LDU factorization of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Soln :-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow[(3)]{R_2 \leftarrow R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = U$$

LU-factorization:

$$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

$$LU = A$$

LDU-factorization:

$$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$LDU = A$$

Ex :- Find the LU and LDU factorization of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$.

Soln :-

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[(2)]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[(3)]{R_3 \leftarrow R_3 - 3R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}$$

$$\xrightarrow[(2)]{R_3 \leftarrow R_3 - 2R_2}$$

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

LU-factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$LU = A$$

LDU-factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$LDU = A$$

Row Exchanges and Permutation Matrices:

During Gaussian elimination, in case of breakdown problems, zero is appearing in the pivot place. To make that pivot place zero into nonzero, we are taking the help of row exchange. For this row exchange purpose, we will use permutation matrices.

Permutation Matrices:

There are $2! = 2$ permutation matrices of order 2. That are

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

There are $3! = 6$ permutation matrices of order 3. That are

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{21} P_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_{32} P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Points to remember:

1. Elements of permutation matrices are either 0 or 1.
2. Product of two permutation matrices is again a permutation matrix.
3. There are $n!$ permutation matrices of order n.

Ex: $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

The 2nd order permutation matrices are

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = A$$

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$P_{21}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$P_{32}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \xrightarrow[(1)]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 5 & 8 \end{bmatrix} \xrightarrow[(2)]{R_3 \leftarrow R_3 - 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix}$

$\downarrow R_2 \leftrightarrow R_3$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ $\qquad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

Here $LU \neq A$.

$\Rightarrow$ A has no LU-factorization.

But PA has LU-factorization, where the permutation matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

$$PA = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 8 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \ (2) \\ R_3 \leftarrow R_3 - R_1 \ (1) \end{matrix}$$

$$= U$$

LU-factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$LU = PA$$

Problem Set 1.5

No. 2. When is an upper triangular matrix is nonsingular?

Ans: An upper triangular matrix is nonsingular if none of its diagonal elements are zero i.e. it has full set of pivots.

No. 7.

$$Ax = b$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \quad R_3 \leftarrow R_3 - 3R_1$$

$$\Rightarrow Ux = c$$

No. 11. (a)

$$v - w = 2$$
$$u - v = 2$$
$$u - w = 2$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_1$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_2$$

Using back-substitution, we have

$$0 = -2,$$

which is not possible.

So, the system is singular and has no solution.

(b)

$$v - w = 0$$
$$u - v = 0$$
$$u - w = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_1$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_2$$

Using back-substitution, we have

$0 = 0$,

which is an identity.

So, the system is singular and has infinitely many solutions.

ⓒ

$$\begin{aligned} v + w &= 1 \\ u + v \quad &= 1 \\ u \quad + w &= 1 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad R_3 \leftarrow R_3 + R_2$$

Using back-substitution, we have

$$2w = 1 \Rightarrow w = \frac{1}{2}$$

$$v + w = 1 \Rightarrow v = 1 - w = 1 - \frac{1}{2} = \frac{1}{2}$$

$$u + v = 1 \Rightarrow u = 1 - v = 1 - \frac{1}{2} = \frac{1}{2}$$

So, $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the solution and is unique.

No. 28. Tridiagonal matrix: A square matrix is said to be a tridiagonal matrix if all its elements are zero except on the main diagonal and the two adjacent diagonals.

Let us find the LU and LDU factorization of the tridiagonal matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix}$$

(i) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ $R_2 \leftarrow R_2 - R_1 (1)$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $R_3 \leftarrow R_3 - R_2 (1)$

$= U$

LU-factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$LU = A$

LDU-factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$LDU = A$

(ii) $A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix} \xrightarrow[(1)]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & b & b+c \end{bmatrix} \xrightarrow[(1)]{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & 0 & c \end{bmatrix} = U$

LU-factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & 0 & c \end{bmatrix}$$

$LU = A$

LDU-factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$LDU = A$

No. 32. $A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (Since the given matrix is already in upper triangular form, so L is the identity matrix)

$LU = A$

LDU - factorization:

$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$LDU = A$

No. 19. Find the permutation matrix P such that $PA = LDU$ and check it, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

Soln: $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$

$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $R_2 \leftrightarrow R_3$

Since to convert the given matrix into upper triangular form, so the required permutation matrix is $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

$$PA = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow[(1)]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow[(2)]{R_3 \leftarrow R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$$

LDU - factorization:

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$LDU = PA$

LU- factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$LU = PA.$$