# Chapter-2: Vector Spaces

## 2.1 Vector Spaces and Subspaces:

Course Outcomes:

1. Students will be acquainted with different types of vector spaces and subspaces.
2. Students will be acquainted with the column space and the nullspace of different matrices.

In this article we will discuss about the following important things:

1. Vector Space
2. Subspace
3. The Column Space
4. The Nullspace.

**Vector Space**: A nonempty set V is said to be a vector space if it satisfies the following properties:

1. $x+y = y+x$
2. $x+(y+z) = (x+y)+z$
3. There is a unique vector $0$ (zero vector) such that $x+0=x$ for all $x$.
4. For each $x$ there is a unique vector $-x$ such that $x+(-x)=0$.

Vector addion

5. $1x = x$
6. $(c_1c_2)x = c_1(c_2x)$
7. $c(x+y) = cx+cy$
8. $(c_1+c_2)x = c_1x+c_2x$,

where $x, y, z \in V$ and $c, c_1, c_2 \in \mathbb{R}$.

Scalar multiplication.

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Alternate definition:

A nonempty set $V$ is said to be a vector space if it satisfies the following properties:

1. Vector Addition i.e.

$$x, y \in V \Rightarrow x+y \in V.$$

2. Scalar multiplication i.e.

$$c \in \mathbb{R}, x \in V \Rightarrow cx \in V.$$

Examples of vector spaces : $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots\ldots, \mathbb{R}^n$.

Ex : Show that $\mathbb{R}^2$ is a vector space.

Proof : $\mathbb{R}^2$ contains infinitely many elements and the elements are pairs like $(0,0), (1,2), (-1,2) \ldots$

So, $\mathbb{R}^2$ is nonempty.

1. Vector addition :

Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$.

$$x+y = (x_1+y_1, x_2+y_2) \in \mathbb{R}^2 \text{ as}$$

$$x_1+y_1 \in \mathbb{R} \text{ and } x_2+y_2 \in \mathbb{R}.$$

2. Scalar multiplication :

Let $c \in \mathbb{R}$ and $x = (x_1, x_2) \in \mathbb{R}^2$.

$$cx = (cx_1, cx_2) \in \mathbb{R}^2 \text{ as } cx_1, cy_2 \in \mathbb{R}.$$

So, $\mathbb{R}^2$ is a vector space.

Ex : Verify whether the 1st quadrant of $\mathbb{R}^2$ is a vector space or not.

Proof : Let $V$ be the 1st quadrant.

$$V = \{(x_1, x_2) \in \mathbb{R} : x_1 \geq 0, x_2 \geq 0\}$$
$$= \{(0,0), (1,1), (1,2), (2,3), \ldots\}$$

$V$ is nonempty.

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1. Vector addition:

Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in V$.

$x_1 \geq 0,\ x_2 \geq 0,\ y_1 \geq 0,\ y_2 \geq 0$.

$x + y = (x_1 + y_1,\ x_2 + y_2) \in V$ as $x_1 + y_1 \geq 0$ and $x_2 + y_2 \geq 0$.

2. Scalar multiplication:

Let $c = -2 \in \mathbb{R}$ and $x = (1, 2) \in V$.

$cx = (-2, -4) \notin V$

So, $V$ does not satisfy scalar multiplication property.

Hence, $V$ i.e. 1st quadrant is not a vector space

Ex: Verify whether the 1st and 3rd quardant of $\mathbb{R}^2$ is a vector space or not.

Proof: Let $V$ be 1st and 3rd quadrant.

$V = \{(1,1), (1,2), (-1,-1), (-1,-2), \cdots\}$

So, $V$ is nonempty.

1. Vector addition:

Let $x = (-1, -2)$ and $y = (2, 1) \in V$.

$x + y = (1, -1) \notin V$

So, $V$ does not satisfy vector addition property.

Hence, $V$ is not a vector space.

Subspace: A subspace of a vector space is a nonempty subset that satisfies the requirements for a vector space.

Ex: Show that $y = x$ line is a subspace of the vector space $\mathbb{R}^2$.

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<u>Proof</u>: Let $V$ be $y = x$ line.

$$V = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$$
$$= \{(0,0), (1,1), (2,2), (-1,-1), (-2,-2), \dots\}$$

So, $V$ is a nonempty subset of $\mathbb{R}^2$.

1. <u>Vector addition</u>:

Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in V$.

Then $x_1 = x_2$ and $y_1 = y_2$.

$x + y = (x_1 + y_1, x_2 + y_2) \in V$ as $x_1 + y_1 = x_2 + y_2$.

2. <u>Scalar multiplication</u>:

Let $c \in \mathbb{R}$ and $x = (x_1, x_2) \in V$.

Then $x_1 = x_2$
$\Rightarrow cx_1 = cx_2$.

$cx = (cx_1, cx_2) \in V$.

So, $V$ i.e. $y = x$ line is subspace of $\mathbb{R}^2$.

<u>Vector Space</u>: $\mathbb{R}^2$

<u>Subspaces</u>:

1. $\mathbb{R}^2$
2. Any line passing through origin.
3. Origin

<u>Vector Space</u>: $\mathbb{R}^3$

<u>Subspaces</u>:

1. $\mathbb{R}^3$
2. Any plane passing through origin.
3. Any line passing through origin
4. Origin.

<u>Points to remember</u>:

1. Every vector space is a subspace of itself.
2. A subspace is a vector space in its own right.
3. Every vector space is the largest subspace of itself and origin is the smallest subspace.

**The Column Space:** Let $A$ be an $m \times n$ matrix. Then the column space contains all the linear combinations of the columns of $A$. It is denoted by $C(A)$. It is a subspace of $\mathbb{R}^m$.

**Ex:** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$Ax = b$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$$

$$\Rightarrow x_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 4 \end{bmatrix} = b$$

$x_1 = 0, x_2 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = 1, x_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$x_1 = 0, x_2 = 1 \Rightarrow b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$x_1 = 1, x_2 = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$\vdots$

$$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \dots \right\}$$

$$= \mathbb{R}^2.$$

**Ex:** $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$Ax = b$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$$

$$\Rightarrow x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 4 \end{bmatrix} = b$$

$x_1 = 0, x_2 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = 1, x_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x_1 = 0, x_2 = 1 \Rightarrow b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$x_1 = 1, x_2 = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$\vdots$

$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \dots \right\} = y = 2x$ line in $\mathbb{R}^2$.

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Ex:- $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$Ax = b$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$

$\Rightarrow x_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = b$

$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ i.e. origin of $\mathbb{R}^2$.

Points to remember:

1. If $A$ is a 2nd order nonsingular matrix, then $C(A) = \mathbb{R}^2$.
2. If $A$ is a 2nd order nonzero singular matrix, then $C(A)$ is a line passing through origin in $\mathbb{R}^2$.
3. If $A$ is a 2nd order zero matrix, then $C(A)$ is the origin of $\mathbb{R}^2$.

The Nullspace: Let $A$ be a matrix of order $m \times n$. The nullspace of the matrix $A$ consists of all vector $x$ such that $Ax = 0$. It is denoted by $N(A)$. It is a subspace of $\mathbb{R}^n$.

Ex:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$Ax = 0$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = 0, x_2 = 0$ (only case)

$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ i.e. origin of $\mathbb{R}^2$.

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Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$Ax = 0$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = 0, x_2 = 0$

$x_1 = 2, x_2 = -1$

$x_1 = 4, x_2 = -2$

$x_1 = 8, x_2 = -4$

$\vdots$

$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \end{bmatrix}, \ldots \right\}$,

which is the line $y = -\frac{x}{2}$ in $\mathbb{R}^2$.

Ex: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$Ax = 0$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Any value of $x_1$ and $x_2$ can satisfy.

So, $N(A) = \mathbb{R}^2$.

Points to remember:

1. If $A$ is a 2nd order nonsingular matrix, then $N(A)$ is the origin of $\mathbb{R}^2$.
2. If $A$ is a 2nd order nonzero singular matrix, then $N(A)$ is a line passing through origin in $\mathbb{R}^2$.
3. If $A$ is a 2nd order zero matrix, then $N(A) = \mathbb{R}^2$.

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4. For the column space of a matrix 'A', we are collecting the 'b' of the system $Ax=b$.

5. For the nullspace of a matrix 'A', we are collecting the 'x' of the system $Ax=0$.

6. If A is a 3rd order nonsingular matrix, then $C(A) = \mathbb{R}^3$ and $N(A)$ = origin of $\mathbb{R}^3$.

7. If A is a 3rd order zero matrix, then $C(A)$ = origin of $\mathbb{R}^3$ and $N(A) = \mathbb{R}^3$.

8. Let A be a 3rd order nonzero singular matrix. Then

(i) $C(A)$ is a line passing through origin and $N(A)$ is a plane passing through origin if rank of $A=1$.

(ii) $C(A)$ is a plane passing through origin and $N(A)$ is a line passing through origin if rank of $A=2$.

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