Problem Set 2.3

No. 4. Given: $v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, v_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, v_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}, v_4 = \begin{bmatrix}2\\3\\4\end{bmatrix}$

Let $A = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$.

$|A| = 1(1-0) = 1 \neq 0$

So, the vectors v_1, v_2 and v_3 are linearly independent.

Let c_1, c_2, c_3 and c_4 be four scalars.

$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$

$\Rightarrow c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\end{bmatrix} + c_4\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\Rightarrow c_1 + c_2 + c_3 + 2c_4 = 0$

$c_2 + c_3 + 3c_4 = 0$

$c_3 + 4c_4 = 0$

$c_3 + 4c_4 = 0$

$\Rightarrow c_3 = -4c_4$

Let $c_4 = 1$. Then $c_3 = -4$.

$c_2 = -c_3 - 3c_4 = -(-4) - 3 = 4 - 3 = 1$

$c_1 = -c_2 - c_3 - 2c_4 = -1 + 4 - 2 = 1$

So, the four vectors are linearly dependent as ~~at least~~ one of the scalar $c_i \neq 0$, $i = 1, 2, 3, 4$.

No. 10. Let w_1, w_2, w_3 be independent vectors.

Let $v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$ and $v_3 = w_1 + w_2$.

Let c_1, c_2 and c_3 be three scalars.

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$\Rightarrow c_1(w_2 + w_3) + c_2(w_1 + w_3) + c_3(w_1 + w_2) = 0$

$\Rightarrow (c_2+c_3)w_1 + (c_1+c_3)w_2 + (c_1+c_2)w_3 = 0$

$\Rightarrow \quad c_2+c_3=0, \; c_1+c_3=0, \; c_1+c_2=0$

$\Rightarrow \quad c_1=c_2=c_3=0.$

$\Rightarrow \quad v_1, v_2$ and v_3 are independent.

No. 16. Vector space : $\mathbb{R}^3$

(a) Given: vectors $(1,1,-1)$ and $(-1,-1,1)$

Let $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $R_2 \leftarrow R_2 - R_1$, $R_3 \leftarrow R_3 + R_1$, echelon form.

Rank of $A = 1$.

The required subspace of $\mathbb{R}^3$ spanned by the two given vectors is a line passing through origin.

(b) Given: vectors $(0,1,1)$, $(1,1,0)$ and $(0,0,0)$.

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ $R_3 \leftarrow R_3 - R_1$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $R_3 \leftarrow R_3 + R_2$, echelon form

Rank of $A = 2$.

The subspace of $\mathbb{R}^3$ spanned by the three given vectors is a plane passing through origin.

(c) Given: the columns of a 3 by 5 echelon matrix with 2 pivots.

Rank of the matrix = 2.

So, the subspace of $\mathbb{R}^3$ spanned by the columns of the matrix is a plane passing through origin.

(d) Given: all vectors with positive components

Here all vectors with positive components i.e 1st octant contains three linearly independent vectors. So, the subspace of $\mathbb{R}^3$ spanned by all vectors with positive components is the whole space $\mathbb{R}^3$.

No. 19. Plane (P): $x - 2y + 3z = 0$ in $\mathbb{R}^3$.

$$x = 2y - 3z$$

$$y = 0,\ z = 1 \Rightarrow x = -3$$

$$y = 1,\ z = 0 \Rightarrow x = 2$$

So, $\{(-3,0,1), (2,1,0)\}$ is a basis of P.

xy-plane is $z = 0$.

Intersection of the plane P with xy-plane is $x - 2y = 0,\ z = 0$ (a line in $\mathbb{R}^3$).

$$x = 2y$$

A basis for the line is $\{(2,1,0)\}$.

A basis for all vectors perpendicular to the plane is $\{(1,-2,3)\}$.

No.13. Let $v_1=(1,1,0,0)$, $v_2=(1,0,1,0)$, $v_3=(0,0,1,1)$, $v_4=(0,1,0,1)$.

$c_1v_1+c_2v_2+c_3v_3+c_4v_4=0$

$\Rightarrow (c_1,c_1,0,0)+(c_2,0,c_2,0)+(0,0,c_3,c_3)+(0,c_4,0,c_4)$
$=(0,0,0,0)$.

$\Rightarrow$ $c_1+c_2=0$
$c_1+c_4=0$
$c_2+c_3=0$
$c_3+c_4=0$

$\Rightarrow c_1=-1,\ c_2=1,\ c_3=-1,\ c_4=1$

$\Rightarrow$ v_1, v_2, v_3 and v_4 are not linearly independent.

Again,
$c_1v_1+c_2v_2+c_3v_3+c_4v_4=(0,0,0,1)$

$$\Rightarrow \left.\begin{array}{l} c_1+c_2=0 \\ c_1+c_4=0 \\ c_2+c_3=0 \\ c_3+c_4=1 \end{array}\right] \text{impossible}$$

So, v_1, v_2, v_3 and v_4 are not independent.

Hence, they do not span $\mathbb{R}^4$.

No.31 (i) $A=\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$

For $c=0, d=2$:

$$A=\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftarrow R_3-R_2$$

, echelon form.

Since there are two pivot elements, so rank of $A = 2$.

(ii) $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$, $|B| = c^2 - d^2 = 0$

$\Rightarrow c = \pm d$

For $c \neq \pm d$, the matrix B is nonsingular.

Rank of a nonsingular matrix is equal to its order. So, rank of $A = 2$ for $c \neq \pm d$.

No. 40 (a) Vectors $(1,2,0)$ and $(0,1,-1)$.

Let $B = \{(1,2,0),(0,1,-1)\}$.

Since exactly three linearly independent vectors are required for a basis of $\mathbb{R}^3$, so B is not a basis of $\mathbb{R}^3$.

(b). Vectors $(1,1,-1),(2,3,4),(4,1,-1),(0,1,-1)$.

Let $B = \{(1,1,-1),(2,3,4),(4,1,-1),(0,1,-1)\}$

Four or more vectors in $\mathbb{R}^3$ are always linearly dependent. Since B contains four vectors, so B is not a basis.

(c) vectors : $(1,2,2),(-1,2,1),(0,8,0)$.

Let $B = \{(1,2,2),(-1,2,1),(0,8,0)\}$

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 0 \end{bmatrix}$

$|A| = 1(0-8) + 1(0-16) = -24 \neq 0$

So, the three vectors of B are linearly independent. Hence B is a basis of $\mathbb{R}^3$.

(d) Vectors : $(1,2,2), (-1,2,1), (0,8,6)$.

Let $B = \{(1,2,2), (-1,2,1), (0,8,6)\}$

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 6 \end{bmatrix}$

$$|A| = 1(12-8) + 1(12-16) = 4 - 4 = 0$$

So, the vectors of B are linearly dependent.

Hence B is not a basis.