LECTURE-28

Continue from the previous lecture...

Example 4 The eigenvalues of the diagonal are main diagonal element of the matrix.

Solution:

Let

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad has \quad \lambda_1 = 3 \quad and \quad \lambda_2 = 2.(try \quad yourself)$$

Let x_1 be the eigenvector of the corresponding eigenvalue $\lambda_1 = 3$.

$$(A - 3I)x = 0$$

$$i.e. \quad \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvaetor for $\lambda_1 = 3$,

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

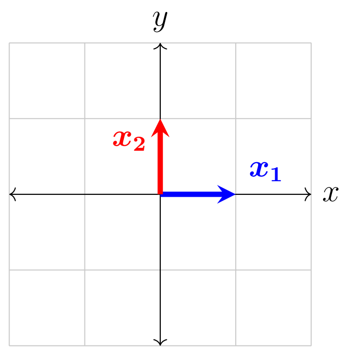
Let x_2 be the eigenvector of the corresponding eigenvalue $\lambda_2 = 2$.

$$(A - 2I)x = 0$$

$$i.e. \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvaetor for $\lambda_2 = 2$,

$$x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Example 5. The eigenvalues of a projection matrix are 1 or 0.

Solution:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad has \quad \lambda_1 = 1 \quad and \quad \lambda_2 = 0.(check \quad yourself)$$

Let x_1 be the eigenvector of the corresponding eigenvalue $\lambda_1 = 1$.

$$(A - I)x = 0$$

$$i.e. \quad \begin{bmatrix} \frac{1}{2} - 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad R_2 \leftarrow R_2 + R_1$$

$$i.e. \quad \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvaetor for $\lambda_1 = 1$,

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let x_2 be the eigenvector of the corresponding eigenvalue $\lambda_2 = 0$.

$$(A - 0 \cdot I)x = 0$$

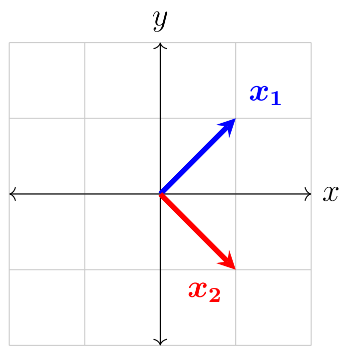
$$i.e. \quad \begin{bmatrix} \frac{1}{2} - 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_2 \leftarrow R_2 - R_1$$

$$i.e. \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvaetor for $\lambda_2 = 0$,

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



Example 6. The eigenvalues are on the main diagonal when A is triangular.

Solution: Let

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & \frac{3}{4} & 6 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

The characteristic equation is

$$det(A - \lambda I) = 0$$

$$i.e. \quad \begin{vmatrix} 1-\lambda & 4 & 5 \\ 0 & \frac{3}{4}-\lambda & 6 \\ 0 & 0 & \frac{1}{2}-\lambda = 0 \end{vmatrix}$$

$$i.e. \quad (1-\lambda)(\frac{3}{4}-\lambda)(\frac{1}{2}-\lambda = 0)$$

$$i.e. \quad \lambda = 1, \quad \lambda = \frac{3}{4} \quad or \quad \frac{1}{2}$$

The above eigenvalues are the main diagonal of the given triangular matrix.

PROPERTIES OF EIGENVALUES

1. The sum of the eigenvalue of a matrix is equal to the trace of the matrix.(Trace means sum of the principal diagonal of the matrix)

e.g. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since A is a 2×2 matirx. Therefore the numbers eigenvalues will be two.

Let λ_1 and λ_2 are the eigenvalue of the matrix A.

$$i.e. \quad \lambda_1 + \lambda_2 = trace of the matrix A = a + d$$

2. The product of the eigenvalue of a matrix A is equal to the determinate of A

$$\lambda_1 \cdot \lambda_2 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

3. Any square matrix A and its transpose A^T have the same eigenvalue.

4. Eigenvalue of the **diagonal matrix or scalar matrix or triangular matrix** are the principal diagonal elements of the matrix.

5. Eigenvalue of the projection matrix are always 1 or 0. If numbers of eigenvalues $\lambda = 1$ repeated r times of the $n \times n$ matrix, then $\lambda = 0$ repeated $n - r$ times.

6. If λ is an eigenvalue of A, then eigenvalue of $A \pm kI$ is $\lambda \pm k$.

7. Eigenvalues of symmetric matrix are always distinct.

8. If λ is an eigenvalue of an **orthogonal matrix** A. Then $\frac{1}{\lambda}$ is also an eigenvalue of the same matrix.

$$Ax = \lambda x \quad (\lambda \quad is \quad an \quad eigenvalue \quad of \quad A)$$

$$i.e. \quad A^T A x = \lambda A^T x$$

$$i.e. \quad Ix = \lambda A^T x$$

$$i.e. \quad \frac{1}{\lambda} x = A^T x$$

$$i.e. \quad A^T x = \frac{1}{\lambda} x$$

$$i.e. \quad \frac{1}{\lambda} is an eigenvalue of A^T$$

we know that A and A^T have same eigenvalue. Therefore λ and $\frac{1}{\lambda}$ is an eigenvalue of A as well as A^T

NOTE: If A is an orthogonal matrix then λ or $\frac{1}{\lambda}$ are an eigenvalue of A as well as A^T.

9. If $\lambda_1, \lambda_2, \lambda_3, \ldots.., \lambda_m$ are an eigenvalue of the matrix A then,

(a) The eigenvalue of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \ldots..., \frac{1}{\lambda_m}$

(b) The eigenvalue of A^n are ${\lambda_1}^n, {\lambda_2}^n, {\lambda_3}^n, \ldots.., {\lambda_m}^n$

(c) The eigenvalue of kA are $k\lambda_1, k\lambda_2, k\lambda_3, \ldots.., k\lambda_m$

(d) The eigenvalue of $A \pm kI$ $\lambda_1 \pm k, \lambda_2 \pm k, \lambda_3 \pm k, ..., \lambda_m \pm k$

PROPERTIES OF EIGENVECTOR

1. Eigenvector of A and A^T are always different.

2. Eigenvector of A and $A \pm kI$ are always same.

3. Eigenvector of symmetric matrix for corresponding eigenvalue are orthogonal.

Let A is asymmetric matrix of order 3. Let $\lambda_1, \lambda_2, \lambda_3$ are three eigenvalue of the matrix A. For corresponding eigenvalue $\lambda_1, \lambda_2, \lambda_3$, we have three eigenvector x_1, x_2, x_3 then

$$x_1{}^T \cdot x_2 = 0 \qquad x_2{}^T \cdot x_3 = 0 \qquad x_3{}^T \cdot x_1 = 0$$

i.e. x_1, x_2, x_3 are orthogonal.

Problem Set 5.1

Q.1 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}.$$

Verify that the trace equals the sum of the eigenvalues, and the determinant equals their product.

Solution: The characteristic equation is

$$det.(A - \lambda.I) = 0$$

$$i.e. \quad \begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$i.e. \quad (1-\lambda)(4-\lambda)+2=0$$

$$i.e \quad \lambda^2-5\lambda+6=0$$

$$i.e. \quad \lambda=2 \quad or \quad \lambda=3$$

Let x_1 be the eigenvector of the corresponding eigenvalue $\lambda_1 = 2$.

$$(A-2I)x=0$$

$$i.e. \quad \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad R_2 \leftarrow R_2 + 2R_1$$

Eigenvaetor for $\lambda_1 = 2$,

$$x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let x_2 be the eigenvector of the corresponding eigenvalue $\lambda_2 = 3$.

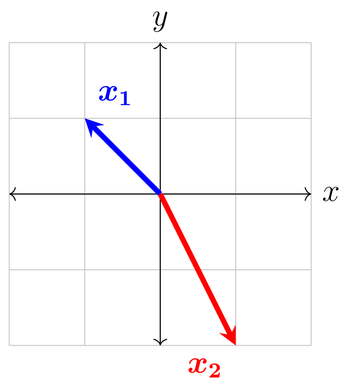
$$(A-3I)x=0$$

$$i.e. \quad \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad R_2 \leftarrow R_2 + R_1$$

Eigenvaetor for $\lambda_2 = 3$,

$$x_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$



Second part

Trace of A=sum of diagonal of the matrix A =$1+4$ =5

and Sum of eigenvalue= $\lambda_1 + \lambda_2$ =$2+3$ =5

Therefore Trace of A=$\lambda_1 + \lambda_2$

Again, Product of the eigenvalue =$\lambda_1 \times \lambda_2$ =$2 \cdot 3$ =6

Determinant of A,

$$\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = 6$$

Therefore Determinant of A=Product of the eigenvalue.

Q.2. With the same matrix, Solve the differential equation $\frac{du}{dt} = Au$,

$u(0) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$, What are the two pure exponential solutions?

Solution: The given coefficient matrix is

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

The pure exponential solution the differential equation $\frac{du}{dt} = Au$ is $u = e^{\lambda t}x$,

and the two special solutions are

$$u(t) = e^{\lambda_1 t}x_1 = e^2 \cdot \begin{bmatrix} -1 \\ \\ 1 \end{bmatrix} \qquad and \qquad u(t) = e^{\lambda_2 t}x_1 = e^3 \cdot \begin{bmatrix} 1 \\ \\ 2 \end{bmatrix}$$

These two special solution gives the complete solution. Therefore the complete solution $\frac{du}{dt} = Au$ is

$$u(t) = c_1 e^{\lambda_1 t}x_1 + c_2 e^{\lambda_2 t}x_2.$$

$$i.e. \quad u(t) = c_1 e^{2t} \cdot \begin{bmatrix} -1 \\ \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \begin{bmatrix} 1 \\ \\ -2 \end{bmatrix}.$$

At initial condition $t = 0$, $u(t) = u(0)$

$$c_1 x_1 + c_2 x_2 = 0$$

$$i.e. \quad \begin{bmatrix} -1 & 1 \\ \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 6 \end{bmatrix} \qquad R_2 \leftarrow R_2 + R_1$$

$$i.e. \quad \begin{bmatrix} -1 & 1 \\ \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 6 \end{bmatrix}$$

The constant are $c_1 = 2$ and $c_2 = 3$ Therefor the solution to the original equation is

$$u(t) = 2e^{2t}x_1 \cdot \begin{bmatrix} -1 \\ \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 \\ \\ 2 \end{bmatrix}$$