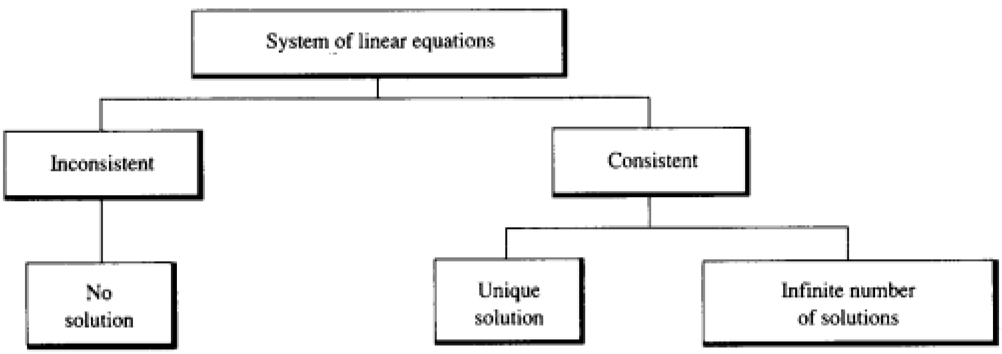
2.2 - Solving Ax=0 And Ax=b

Vocab : Cofficient Mtrix,Augmented Matrix,EchelonForm,Row Reduced Form ,Rank,Pivot Variable,free Variable



Ex 1 - Consider a System of Linear Equation

$$\left.\begin{aligned} &x_1 - 2x_2 + x_3 = 0 \\ &2x_2 - 8x_3 = 8 \\ &-4x_1 + 5x_2 + 9x_3 = -9 \end{aligned}\right\} \tag{1}$$

Solution: The elimination procedure is shown here with and without matrix notation and the results are placed side by side for comparison:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Keep x_1 is the first equation and eliminate it from the other equations. To do so, add 4 times equation 1 to equation 3. After some practice, this type of calculation is usually performed mentally:

$$\begin{array}{rc}
4.[\textit{equation } 1]: & 4x_1 - 8x_2 + 4x_3 = 0 \\
+[\textit{equation } 3]: & -4x_1 + 5x_2 + 9x_3 = -9 \\
\hline
\textit{new equation } 3: & -3x_2 + 13x_3 = -9
\end{array}$$

The result of this calculation is written in place of the original third equation:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Now, multiply equation 2 by 1/2 in order to obtain 1 as the coefficient for x_2. (This calculation will simplify the arithmatic in the next step.)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Use the x_2 in equation 2 to eliminate the $-3x_2$ in equation 3. The "mental" computation is

$$\begin{array}{rc}
3.[\textit{equation } 2]: & 3x_2 - 12x_3 = 12 \\
+[\textit{equation } 3]: & -3x_2 + 13x_3 = -9 \\
\hline
\textit{new equation } 3: & x_3 = 3
\end{array}$$

The new system has a triangular form.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Eventually, you want to eliminate the $-2x_2$ term from equation 1, but it is more efficient to use the x_3 in equation 3 first, to eliminate the $-4x_3$ and $+x_3$ terms in equation 2 and 1. The two "mental" calculations are

$$\begin{array}{rc}
4.[\textit{equation } 3]: & 4x_3 = 12 \\
+[\textit{equation } 2]: & x_2 - 4x_3 = 4 \\
\hline
\textit{new equation } 2: & x_2 = 16
\end{array} \qquad
\begin{array}{rc}
-1.[\textit{equation } 3]: & -x_3 = -3 \\
+[\textit{equation } 1]: & x_1 - 2x_2 + x_3 = 0 \\
\hline
\textit{new equation } 1: & x_1 - 2x_2 = -3
\end{array}$$

It is convenient to combine the results of these two operations:

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Now, having cleaned out the column above the x_3 in equation 3, move back to the x_2 in equation 2 and use it to eliminate the $-2x_2$ above it. Because of the previous work with x_3, there is now no arithmetic involving x_3 terms. Add 2 times equations 2 to equation 1 and obtain the system:

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Ex 2 - A system of linear equations is a list of linear equations with the same unknowns. In particular, a system of 2 linear equations in 2 unknowns x_1, x_2can be put in the standard form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{aligned}\right\} \tag{2}$$

where a_{ij}, b_i are constant.and we can rewrite system (2) as :

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{bmatrix} \tag{3}$$

again we can rewrite system (3) to as

$$\left[\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right] \tag{4}$$

again we can rewrite system (4)(without using unknown,for the simplicity) to as

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right] \tag{5}$$

$$A = \left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right], C = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where A is called **Augmented Matrix** and C is called **Cofficient Matrix** of the system

$$\begin{aligned} &\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{a_{11}} R_1} \left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ a_{21} & a_{22} & b_2 \end{array}\right] \\ &\xrightarrow{R_2 \to R_2 - a_{21} R_1} \left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & a_{22} - \frac{a_{21}a_{12}}{a_{11}} & b_2 - \frac{a_{21}b_1}{a_{11}} \end{array}\right] \end{aligned} \tag{6}$$

(6) can be written as

$$\left.\begin{aligned} &x_1 + \tfrac{a_{12}}{a_{11}} x_2 = \tfrac{b_1}{a_{11}} \\ &0x_1 + (a_{22} - \tfrac{a_{21}a_{12}}{a_{11}})x_2 = b_2 - \tfrac{a_{21}b_1}{a_{11}} \end{aligned}\right\} \tag{7}$$

Case 1 : if $a_{22} - \frac{a_{21}a_{12}}{a_{11}} \neq 0$ then $x_2 = \frac{b_2 - \frac{a_{21}b_1}{a_{11}}}{a_{22} - \frac{a_{21}a_{12}}{a_{11}}}$, x_1 ca be calculated from $x_1 + \frac{a_{12}}{a_{11}}x_2 = \frac{b_1}{a_{11}}$
Conclusion- unique solution of (2)

Case 2 : if $a_{22} - \frac{a_{21}a_{12}}{a_{11}} = 0$ and $b_2 - \frac{a_{21}b_1}{a_{11}} = 0$ then second equation of (7) becomes $0x_1 + 0x_2 = 0$
Conclution - infinite solution of (2)

Case 3 : if $a_{22} - \frac{a_{21}a_{12}}{a_{11}} = 0$ and $b_2 - \frac{a_{21}b_1}{a_{11}} \neq 0$ then second equation of (6)becomes $0x_1 + 0x_2 = b_2 - \frac{a_{21}b_1}{a_{11}}$ i.e. $0 = b_2 - \frac{a_{21}b_1}{a_{11}}$ which is not true
Conclution - solution does not exit of (2).

Elementary Row Operations

Supppose A is a matrix with rows R_1, R_2,..., R_m. The following operations on A are called elementary row operations.

[E_1] (Row Interchange): Interchange rows R_i and R_j. This may be written as

"Interchange R_i and R_j" or "$R_i \leftrightarrow R_j$"

[E_2] (Row Scaling): Replace row R_i by a nonzero multiple kR_i of itself. This may be written as

" Replace R_i by $kR_i (k \neq 0)$ or "$kR_i \rightarrow R_i$"

[E_3] (Row Addition): Replace row R_j by the sum of a multiple kR_i of a row R_i and itself. This may be written as

" Replace R_j by $kR_i + R_j$" or "$kR_i + R_j \rightarrow R_j$.

The arrow $\rightarrow$ in E_2 and E_3 may be read as "replaces".

Sometimes (say to avoid fractions when all the given scalars are integers) we may apply [E_2] and [E_3] in one step; that is, we may apply the following operation:

[E] Replace R_j by the sum of a multiple kR_i of a row R_i and a nonzero multiple $k'R_j$ of itself. This may ge written as

"Replace R_j by $kR_i + k'R_j$ ($k' \neq 0$)" or "$kR_i + k'R_j \rightarrow R_j$"

We emphasize that in operations [E_3] and [E] only row R_j is changed.

Echelon Matrices (or in echelon form) U and Row Reduced Form R

Echelon Matrices U

A Matrix U is called an echelon matrix or is said to be in echelon form , if the following two conditions hold :

(1) All zero rows,if any, are at the bottem of the matrix.

(2) Each leading nonzero entry in a row is to the right of the leading nonzero entry in the preceding row.

Row Reduced Form R

A Matrix is said to be in row reduced form R if it is an echelon matrix and if satisfies the following additional two properties:

(3)Each pivot(leading nonzero entry) is equal to 1.

(4) Each pivot is the only nonzero entry in its column.

EX 3 The following is an echelon matrix whose pivots have been circled

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 & 5 & 9 & 0 & 7 \\ 0 & 0 & 0 & 3 & 4 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 5 & 7 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

NOTE 1 - The major difference between an echelon matrix in row reduced form is that in an echelon matrix there must be zeros below the pivots [properties(1)and (2)] but in a matrix in row reduced form , each pivot must also equal 1 [property (3)] and there must also be zeros above the pivots [properties(4)].

Ex-4 The following are echelon matrices whose pivots have been circled

$$\begin{bmatrix} 2 & 3 & 2 & 0 & 4 & 5 & -6 \\ 0 & 0 & 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The Third matrix is also an example of a matrix in row reduced form. the second matrix is not in row reduced form ,since it does not satisfy property(4),taht is,there is a nonzero entry above the second pivot in the third column.The first matrix is not in row reduced form, because it satisfies neither property (3) nor property (4); that is, some pivots are not equal to 1 and there are nonzero entries above the pivots.

Ex-5 The entries of a 5 by 8 echelon matrix U and its reduced form R

$$U = \begin{bmatrix} \bullet & * & * & * & * & * & * & * \\ 0 & \bullet & * & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} \mathbf{1} & \mathbf{0} & * & \mathbf{0} & * & * & * & \mathbf{0} \\ 0 & \mathbf{1} & * & \mathbf{0} & * & * & * & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{1} & * & * & * & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot Variable and Free Variable

Pivot Variable: Pivot Variable are those variable that correspond to columns with pivots.
Free Variable : Free Variable are those variable that correspond to columns without pivots.

Note : If $Ax = 0$ has more unknowns than equations $(n > m)$, it has at least one special solution: There are more solutions than the trivial $x = 0$.
Note : $x_{complete} = x_{particlar} + x_{nullspace}$
Note : if there are n column in a matrix A and there are r pivots then there are r pivot variables and $n - r$ free variable.and this important number r is called **Rank** of a Matrix.

Rank of a Matrix = The rank of a matrix A, written rank(A), is equal to the maximum number of linearly independent columns of A
= number of pivot column in the echelon form of a matrix A
=maximum number of linearly independent rows of A
= dimension of the column space of A
= dimension of the row space of A.

Note : Let A be an n-square matrix. then A is invertible if and only if $rank(A) = n$

Ex 6: Find Rank of A

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Sol. Since Echelon form of A is itself A.and 1st and 3rd column are pivot column.
So Rank of A is 2.

Method for solving System of linear equation

Method-1

Ex 7 - Consider a System of linear equation

$$\left.\begin{aligned} 1x_1+2x_2+3x_3+5x_4&=b_1\\ 2x_1+4x_2+8x_312x_4&=b_2\\ 3x_1+6x_2+7x_3+13x_4&=b_3 \end{aligned}\right\} \tag{8}$$

Sol.

Step 1: Reduce $Ax=b$ to $Ux=c$

i.e. Reduce Augmented Matrix [A b] to Augmented Matrix [U c]

$$\begin{aligned} \begin{bmatrix} A & b \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1\\ 2 & 4 & 8 & 12 & b_2\\ 3 & 6 & 7 & 13 & b_3 \end{array}\right] &\xrightarrow{R_2\to R_2-2R_1, R_3\to R_3-3R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1\\ 0 & 0 & 2 & 2 & b_2-2b_1\\ 0 & 0 & -2 & -2 & b_3-3b1 \end{array}\right]\\ &\xrightarrow{R_3\to R_3+R_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1\\ 0 & 0 & 2 & 2 & b_2-2b_1\\ 0 & 0 & 0 & 0 & b_3+b_2-5b_1 \end{array}\right]\\ &= \begin{bmatrix} U & c \end{bmatrix} \end{aligned} \tag{9}$$

$$\left\{\begin{array}{l} (9) means\\ 1x_1+2x_2+3x_3+5x_4=b_1\\ 0x_1+0x_2+2x_3+2x_4=b_2-2b_1\\ 0x_1+0x_2+0x_3+0x_4=b_3+b_2-5b_1\\ \textbf{third equation hold only if } b_3+b_2-5b_1=0\\ \text{it means if } b_3+b_2-5b_1=0 \text{ then system of equation has infinite solution.}\\ \text{if } b_3+b_2-5b_1\neq 0 \text{ then system of equation has no solution.} \end{array}\right\}$$

Here

$$U=\begin{bmatrix} 1 & 2 & 3 & 5\\ 0 & 0 & 2 & 2\\ 0 & 0 & 0 & 0 \end{bmatrix}, C=\begin{bmatrix} b_1\\ b_2-2b_1\\ b_3+b_2-5b_1 \end{bmatrix}$$

Step 2 :

Find Special Solution : $Ux=0$

Take particularly $b_1 = 0$,$b_2 = 6$,$b_3 = -6$

$$\begin{bmatrix} U & 0 \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{10}$$

Here x_2 **and** x_4 **are free variables**

Let $x_2 = a$,$x_4 = b$ where a,b belongs to Set of Real Number

Now we can rewrite (10) as

$1x_1 + 2x_2 + 3x_3 + 5x_4 = 0$—–(*)

$0x_1 + 0x_2 + 2x_3 + 2x_4 = 0$—–(**)

now put the value of x_2 in (**)

$2x_3 + 2b = 0$

i.e. $x_3 = -b$

now put the value of x_3 in (*)

$x_1 + 2a - 3b + 5b = 0$

i.e. $x_1 + 2a + 2b = 0$

i.e. $x_1 = -2a - 2b$

Special Solution $x_n = \begin{bmatrix} -2a - 2b \\ a \\ -b \\ b \end{bmatrix}$

$x_n = a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ where a, b belongs to set of real number

Step 3 :

Find Particular Solution x_p, $Ux_p = c$ and put all free variables$= 0$

So put $x_2 = a = 0, x_4 = b = 0$

$$\begin{bmatrix} U & c \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{11}$$

(11) Can be rewritten as

$1x_1 + 2x_2 + 3x_3 + 5x_4 = 0$—–(*)

$0x_1 + 0x_2 + 2x_3 + 2x_4 = 6$—–(**)

Now put $b = 0$ in (**)

$2x_3 + 0 = 6$

$x_3 = 3$
Now $a = 0$,$x_3 = 3$,$b = 0$ in (*)
$x_1 0 + 9 + 0 = 0$
$x_1 = -9$

$$x_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Step 4 :

Complete Solution $x = x_n + x_p = a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2a - 2b - 9 \\ a \\ -b + 3 \\ b \end{bmatrix}$

$$= \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2a \\ a \\ 0a \\ 0a \end{bmatrix} + \begin{bmatrix} -2b \\ 0b \\ b \\ -b \end{bmatrix}$$

$$= \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

where a, b belongs to Set of real numbers

Method-2

Ex - Consider a System of linear equation

$$\left.\begin{aligned} 1x_1 + 2x_2 + 3x_3 + 5x_4 &= 0 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= 6 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 &= -6 \end{aligned}\right\} \tag{12}$$

Step 1: Reduce $Ax = b$ to $Ux = c$
i.e. Reduce Augmented Matrix [A b] to Augmented Matrix [U c]

$$\begin{aligned} \begin{bmatrix} A & b \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 2 & 4 & 8 & 12 & 6 \\ 3 & 6 & 7 & 13 & -6 \end{array}\right] &\xrightarrow{R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & -2 & -2 & -6 \end{array}\right] \\ &\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ &= \begin{bmatrix} U & c \end{bmatrix} \end{aligned} \tag{13}$$

$$\left\{\begin{array}{l}(13) means \\ 1x_1 + 2x_2 + 3x_3 + 5x_4 = 0 \\ 0x_1 + 0x_2 + 2x_3 + 2x_4 = 6 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 = 0\end{array}\right\}$$

Here

$$U = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, c = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

Step 2:

Here x_2 **and** x_4 **are free variables**

Let $x_2 = a$,$x_4 = b$ where a,b belongs to Set of Real Number

Now we can rewrite (13) as

$1x_1 + 2x_2 + 3x_3 + 5x_4 = 0$——(*)

$0x_1 + 0x_2 + 2x_3 + 2x_4 = 6$——(**)

now put the value of x_2 in (**)

$2x_3 + 2b = 6$

i.e. $x_3 = 3 - b$

now put the value of x_3 in (*)

$x_1 + 2a + 3(3 - b) + 5b = 0$

i.e. $x_1 + 2a + 9 + 2b = 0$

i.e. $x_1 = -9 - 2a - 2b$

Complete Solution $x = x_n + x_p = a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} -2a - 2b - 9 \\ a \\ -b + 3 \\ b \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2a \\ a \\ 0a \\ 0a \end{bmatrix} + \begin{bmatrix} -2b \\ 0b \\ b \\ -b \end{bmatrix}$$

$$= \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

where a, b belongs to Set of real numbers

Exercise 2.2.1 : find the value of c that makes it possible to solve $Ax = b$, and solve it:

$$u + v + 2w = 2$$

$$2u + 3v - w = 5$$

$$3u + 4u + w = c$$

Solution Aug matrix $= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 3 & -1 & 5 \\ 3 & 4 & 1 & c \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 1 & -5 & c \end{array}\right] \xrightarrow{R_2 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & c-7 \end{array}\right]$
Solution Exit only if $c - 7 = 0$ so assume $w = k \in R$

$$v - 5w = 1$$

$$v - 5k = 1$$

$$v = 1 + 5k$$

$$u + v + 2w = 2$$

$$u + (1 + 5k) + 2k = 2$$

$$u = 1 - 7k$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1-7k \\ 1+5k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + k \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix} \text{ where } k \in R$$

Exercise 2.2.4 Write the complete solution $x = x_p + x_n$ to these systems ,(as in equation (4))

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix},$$

Solution (1) Aug matrix $= \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 4 & 5 & 4 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$
Since v is free variable so take $v = k$, $k \in R$

$$w = 2$$

$$u + 2v + 2w = 1$$

$$u + 2k + 4 = 1$$

$$u = -3 - 2k$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -3-2k \\ k \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + k \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \text{ where } k \in R$$

(2) Aug matrix $= \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 4 & 4 & 4 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$
i.e.

$$u + 2v + 2w = 1$$

$$0u + 0v + 0w = 2$$

i.e. $0 = 2$ which is not true.
So there is no solution.

Exercise 2.2.5 Reduce A and B to echelon form, to find their ranks, which variables are free ?
$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ find the special solutions to $Ax = 0$ and $Bx = 0$. find all solutions.

Solution:(1) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$. Since first two column in U are L.I. So $\rho(A) = 2$.
Now for solving $Ax = 0$.
Aug. matrix $= [A|0] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$
Since x_3 and x_4 are free variable; so assume $x_3 = k_1$, $x_4 = k_2$, where $k_1, k_2 \in R$
$x_2 + x_3 = 0 \Rightarrow x_2 + k = 0 \Rightarrow x_2 = -k$

$$x_1 + 2x_2 + x_4 = 0$$

$$x_1 - 2k_1 + k_2 = 0$$

$$x_1 = 2k_1 - k_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2k_1 - k_2 \\ -k_1 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $k_1, k_2 \in R$.

This is general solution.

Hence special solutions are $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(2) $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} = U$

Since U has two pivot columns, so $\rho(B) = 2$.
for solving $Bx = 0$

Aug. matrix $= [B|0] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Since x_3 is free variable.
So $x_3 = k, k \in R$.

$$-3x_2 - 6x_3 = 0$$

$$x_2 = -2k$$

$$x_1 + 2x_2 + 3x_3 = 0$$

$$x_1 - 4k + 3k = 0$$

$$x_1 - k = 0 \Rightarrow x_1 = k.$$

General solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, k \in R$. Special solution is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.