Ex: 4.2

(2) If a 4×4 matrix has $\det A = \frac{1}{2}$, find $\det(2A)$, $\det(-A)$; $\det(A^2)$ and $\det(A^{-1})$

Ans: $\det A = \frac{1}{2}$

so, $\det(2A) = 2^4 \det(A) = 16 \times \frac{1}{2} = 8$

$\det(-A) = (-1)^4 \det(A) = \frac{1}{2}$

$\det(A^2) = \det(AA) = (\det(A))^2 = \frac{1}{4}$

$\det(A^{-1}) = \frac{1}{\det(A)} = 2$

(4) By applying row operations to produce an upper triangular U, compute

$$\det\begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} \text{ and } \det\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

Exchange rows 3 and 4 of the second matrix and recompute the pivots and determinant.

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} \quad \begin{pmatrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + R_1 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} \quad (R_4 \to R_4 + 2R_2)$$

$$= \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \quad (R_4 \to R_4 \times 2R_3)$$

$= U$

$\therefore \det A = \det U = 1(-1)\cdot(-2)(10)$

$= 20$ (Ans)

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \quad (R_2 \to R_2 - (-0.5)R_1)$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \quad (R_3 \to R_3 - (-0.6)R_2)$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & -11/4 \end{bmatrix} \quad (R_4 \to R_4 - (-0.75)R_3)$$

$$= U$$

$$\therefore \det A = \det U = \cancel{2} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{4}}{\cancel{3}} \cdot \frac{-11}{\cancel{4}}$$

$$= -11$$

Exchange row 3 to row 4

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix} \quad (R_2 \to R_2 - (-0.5)R_1)$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 4/3 & -1 \end{bmatrix} \quad (R_4 \to R_4 - (-0.6)R_2)$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -11/3 \end{bmatrix} \quad (R_4 \to R_4 - (-0.75)R_3)$$

$$\therefore \det A = \det U = \cancel{2} \cdot \cancel{3}/\cancel{2} \cdot (-1)(-11/\cancel{3})$$

$$= 11 \text{ (Ans)}$$

(5) Find the determinants of:

(a) a rank one matrix

$$A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}$$

Ans: $A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 8 & -4 & 8 \\ 4 & -2 & 4 \end{bmatrix}$

$\det A = 0$ [$\because$ A is singular]

(b) the upper triangular matrix

Ans: $U = \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$\Rightarrow \det U = 4 \cdot 1 \cdot 2 \cdot 2 = 16$

(c) the lower triangular matrix U^T

Ans: $\det(U^T) = \det(U) = 16$

(d) the inverse matrix U^{-1}

Ans: $\det(U^{-1}) = \frac{1}{\det(U)} = \frac{1}{16}$

(e) The "reverse-triangular" matrix that results from row exchanges.

Ans:
$$M = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{bmatrix}$$

$$\det(M) = (-1)^2 \det(U) = 16$$

(6) Suppose you do two row operation at once, going from

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ to } \begin{bmatrix} a-mc & b-md \\ c-la & d-lb \end{bmatrix}$$

Find the determinant of the new matrix, by rule 3 or by direct calculation.

Ans:
$$|A| = \begin{vmatrix} a-mc & b-md \\ c-la & d-lb \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} + \begin{vmatrix} -mc & -md \\ c-la & d-lb \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} - m \begin{vmatrix} c & d \\ c-la & d-lb \end{vmatrix}$$

$$= - \begin{vmatrix} c-la & d-lb \\ a & b \end{vmatrix} + m \begin{vmatrix} c-la & d-lb \\ c & d \end{vmatrix}$$

$$= - \begin{vmatrix} c & d \\ a & b \end{vmatrix} + m \left[\begin{vmatrix} c & d \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ c & d \end{vmatrix} \right]$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + m \left[0 - l \begin{vmatrix} a & b \\ c & d \end{vmatrix} \right]$$

$$= (1-lm) \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (1-ml)(ad-bc)$$

(13) Find the determinants of

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}, \quad A^{-1} = \frac{1}{10} \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$$

For which values of λ is $A - \lambda I$ a singular matrix?

Ans:
$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\det A = 12 - 2 = 10$$

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$$

$$\det A^{-1} = \frac{1}{\det A} = \frac{1}{10}$$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (4-\lambda)(3-\lambda) - 2$$

$$= \lambda^2 - 7\lambda + 10$$

To make $A - \lambda I$ singular,

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 10 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 5) = 0$$

$$\Rightarrow \lambda = 2 \text{ or } \lambda = 5 \text{ (Ans)}$$