Lecture 17

2.4 The Four Fundamental Subspaces

Course Outcomes: Students will have understanding about four fundamental subspaces of matrices and one sided inverse of rectangular matrices. The four fundamental subspaces of matrices are as follows :

- The column space $C(A)$.
- The null space $N(A)$.
- The row space $C(A^T)$.
- The left null space $N(A^T)$.

The Column Space $C(A)$: The column space of A is denoted by C(A). Its dimension is the rank r.

The Null Space $N(A)$: The nullspace of A is denoted by N(A). Its dimension is $n - r$.

The Row Space $C(A^T)$: The row space of A is the column space of A^T. It is $C(A^T)$, and it is spanned by the rows of A. Its dimension is also r.

The Left Null Space $N(A^T)$: The left nullspace of A is the nullspace of A^T. It contains all vectors y such that $A^T y = 0$, and it is written $N(A^T)$. Its dimension is $m - r$.

Notes

- The nullspace N(A) and row space $C(A^T)$ are subspaces of R^n.
- The left nullspace $N(A^T)$ and column space C(A) are subspaces of R^m.

Existence of Inverses:
let A be a matrix of order $m \times n$ with rank r.
(1) Full row rank r = m. Ax = b has at least one solution x for every b if and only if the columns span R^m. Then A has a right-inverse C such that $AC = I_m$ (m by m). This is possible only if $m \leq n$.

(2) Full column rank r = n. Ax = b has at most one solution x for every b if and only if the columns are linearly independent. Then A has an n by m left-inverse B such that $BA = I_n$. This is possible only if $m \geq n$.

Notes: One-sided inverses are $B = (A^T A)^{-1} A^T \ and \ C = A^T (AA^T)^{-1}$

Example. Find a left-inverse and/or a right-inverse (when they exist) for

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

Ans.

$$Let \ A = \begin{bmatrix} \boxed{4} & 0 & 0 \\ 0 & \boxed{5} & 0 \end{bmatrix}$$

The matrix A is already in Echelon form with 2 pivots.
So, rank of A=r=2.
Here m=2 and n=3.
So, m=r=2
$\Rightarrow A$ is a full row rank matrix.
$\Rightarrow$ right inverse C of A will exist and is given by

$$\begin{aligned} C &= A^T (AA^T)^{-1} \\ \Rightarrow C &= \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix} \end{aligned}$$