1-3 An Example of Gaussian Elimination

Outcome of the study: Students will be acquainted with the Guassian elimation method, pivot elements and breakdown ~~of~~ elimination.

Ex-1 ...

$$
\begin{aligned}
2u + v + w &= 5 \\
4u - 6v \quad\quad &= -2 \\
-2u + 7v + 2w &= 9
\end{aligned}
$$

The augmented matrix is

$$
\left[\begin{array}{ccc|c}
2 & 1 & 1 & 5 \\
4 & -6 & 0 & -2 \\
-2 & 7 & 2 & 9
\end{array}\right]
$$

$$
\approx \left[\begin{array}{ccc|c}
2 & 1 & 1 & 5 \\
0 & -8 & -2 & -12 \\
0 & 8 & 3 & 14
\end{array}\right]
\qquad
\begin{array}{l}
R_2 \leftarrow R_2 - 2R_1 \\
R_3 \leftarrow R_3 + R_1
\end{array}
$$

$$
\approx \left[\begin{array}{ccc|c}
2 & 1 & 1 & 5 \\
0 & -8 & -2 & -12 \\
0 & 0 & 1 & 2
\end{array}\right]
\qquad
R_3 \leftarrow R_3 + R_2
$$

Using backsubstitution, we have

$$w = 2$$

$$-8v - 2w = -12 \Rightarrow v = 1$$

$$2u + v + w = 5$$

$$\Rightarrow 2u + 1 + 2 = 5$$

$$\Rightarrow u = 1$$

So, the solution is $(1, 1, 2)$.

Ex-2. $u+v+w=1$

$2u+2v+5w=2$

$4u+6v+8w=6$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 5 & 2 \\ 4 & 6 & 8 & 6 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 4 & 2 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 4R_1 \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 3 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

Using backsubstitution, we have

$3w=0$

$\Rightarrow w=0$

$2v+4w=2$

$\Rightarrow v=1$

$u+v+w=1$

$\Rightarrow u=0$

So, $(0,1,0)$ is the solution.

Ex-3 $u+v+w=1$

$2u+2v+5w=2$

$4u+4v+8w=8$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 5 & 2 \\ 4 & 4 & 8 & 8 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 4R_1 \end{array}$$

Using backsubstitution, we have

$$\left.\begin{array}{l} 4w = 4 \Rightarrow w = 1 \\ 3w = 0 \Rightarrow w = 0 \end{array}\right] \text{ is not possible.}$$

So, the system has no solution.

<u>Ex-4</u>

$$u + v + w = 1$$

$$2u + 2v + 5w = 8$$

$$4u + 4v + 8w = 12$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 5 & 8 \\ 4 & 4 & 8 & 12 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 4 & 8 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 4R_1 \end{array}$$

Using backsubstitution, we have

$$4w = 8 \Rightarrow w = 2$$

$$3w = 6 \Rightarrow w = 2.$$

$$u + v + w = 1$$

$$\Rightarrow u + v = -1$$

$$\Rightarrow v = -1 - u$$

Let $u = 1$. Then $v = -2$.

So, $(1, -2, 2)$ is a solution and the system has infinite number of solutions.

The Pivot elements:

In example-1, the pivot elements are 2, −8 and 1.

No. of pivots = 3 = order of the coefficient matrix.

So, the system has full set of pivots and is nonsingular and has unique solution.

In example-2, the pivot elements are 1, 2 and 3.

No. of pivots = 3 = order of the coefficient matrix.

So, the system has full set of pivots and is nonsingular and has unique solution.

In example-3, the pivot elements are 1 and 3.

No. of pivots = 2 ≠ order of the coefficient matrix.

So, the system has not full set of pivots and is singular and has no solution.

In example-4, the pivot elements are 1 and 3.

No. of pivots = 2 ≠ order of the coefficient matrix.

So, the system has not full set of pivots and is singular and has infinite no. of solutions.

The Breakdown of Elimination:

If during Gaussian elimination a zero appears in pivot place, then there is a breakdown of the elimination. The breakdown may occur at the initial stage or at an intermediate stage during elimination. There are two types of breakdown. That are

1. Temporary breakdown.
2. Permanent breakdown.

Temporary Breakdown:

In case of a breakdown, the elimination algorithm needs repair. If we can repair i.e. the pivot place zero we can make nonzero and will be able to get full set of pivots, then the breakdown is temporary.

Permanent Breakdown:

In case of a breakdown, if the elimination algorithm can not be repairable i.e. to the pivot place zero we can not make nonzero and will be not able to get full set of pivots then the breakdown is permanent.

In example-2, there is a breakdown and it is a temporary breakdown.

In examples-3 and 4, there are breakdowns and they are permanent breakdown.

Temporary breakdown $\Rightarrow$ The system is nonsingular and the solution is unique.

Permanent breakdown $\Rightarrow$ The system is singular and the system has either no solution or infinite number of solutions.

Problem Set 1.3

No. 1

$3x + 2y = 10$

$6x + 4y = $ —.

Let the right hand side of the 2nd equation be α. Then

$$\left[\begin{array}{cc|c} 3 & 2 & 10 \\ 6 & 4 & \alpha \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 3 & 2 & 10 \\ 0 & 0 & \alpha-20 \end{array}\right] \quad R_2 \leftarrow R_2 - 2R_1$$

For no solution, $\alpha - 20 \neq 0$

$\Rightarrow \boxed{\alpha \neq 20}$

For infinitely many solutions,

$\alpha - 20 = 0$.

$\Rightarrow \boxed{\alpha = 20}$

$3x + 2y = 10$

$\Rightarrow y = \dfrac{10-3x}{2}$

$x = 0 \Rightarrow y = 5$

$x = 2 \Rightarrow y = 2$

So, $(0,5)$ and $(2,2)$ are two solutions.

No. 3. $2x + by = 16$

$4x + 8y = g$

$$\left[\begin{array}{cc|c} 2 & b & 16 \\ 4 & 8 & g \end{array}\right]$$

$$\approx \left[\begin{array}{cc|c} 2 & b & 16 \\ 0 & 8-2b & g-32 \end{array}\right] \quad R_2 \leftarrow R_2 - 2R_1$$

The system is singular if

$8 - 2b = 0$

$\Rightarrow 2b = 8$

$\Rightarrow \boxed{b = 4}$

The system is solvable if

$g - 32 = 0$

$\Rightarrow \boxed{g = 32}$

$2x + 4y = 16$

$\Rightarrow x + 2y = 8$

$\Rightarrow y = \frac{8-x}{2}$

$x = 0 \Rightarrow y = 4$

$x = 2 \Rightarrow y = 3$

So, $(0,4)$ and $(2,3)$ are two solutions for the singular case.

No. 4. $2x + 3y = 1$

$10x + 9y = 11$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 10 & 9 & 11 \end{array}\right]$$

$$\approx \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -6 & 6 \end{array}\right] \quad R_2 \leftarrow R_2 - 5R_1$$

The numbers 1 and 11 have no influence on those pivots.

5 multiple of equation 1 should be subtracted from eqn 2.

No. 7.

$$3x - 2y = b_1$$
$$6x - 4y = b_2$$

$$\left[\begin{array}{cc|c} 3 & -2 & b_1 \\ 6 & -4 & b_2 \end{array}\right]$$

$$\approx \left[\begin{array}{cc|c} 3 & -2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \end{array}\right] \quad R_2 \leftarrow R_2 - 2R_1$$

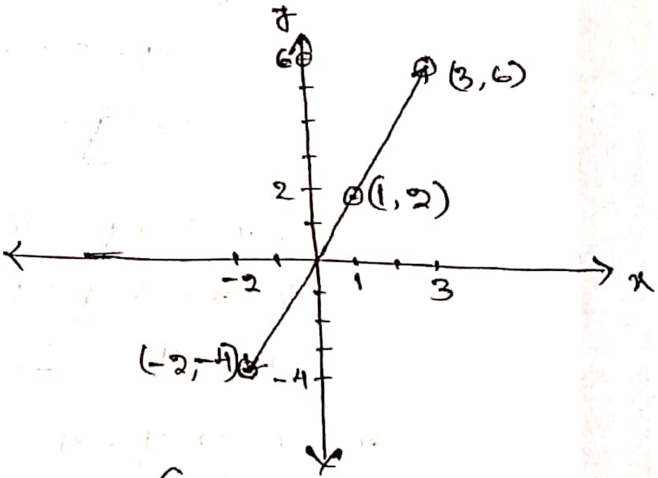
$b_2 - 2b_1 = 0$ allows the system for solutions i.e. for $b_2 - 2b_1 = 0$ the system is solvable and the system has infinitely many solutions.

$$b_2 = 2b_1$$
$$b_1 = 1 \Rightarrow b_2 = 2.$$

$$3x - 2y = 1$$
$$6x - 4y = 2$$

$$\Rightarrow x\begin{bmatrix} 3 \\ 6 \end{bmatrix} + y\begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$1\begin{bmatrix} 3 \\ 6 \end{bmatrix} + 1\begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



(column picture)

No. 8.

$$ax + 3y = -3$$
$$4x + 6y = 6$$

$$\left[\begin{array}{cc|c} a & 3 & -3 \\ 4 & 6 & 6 \end{array}\right]$$

$$\approx \left[\begin{array}{cc|c} a & 3 & -3 \\ 0 & 6 - \frac{12}{a} & 6 + \frac{12}{a} \end{array}\right] \quad R_2 \leftarrow R_2 - \frac{4}{a}R_1$$

$$6 - \frac{12}{a} = 0$$
$$\Rightarrow a = 2$$

For $a = 2$, the elimination breaks down permanently.

For $a=0$, the elimination breaks down temporarily.

No. 14.

$$x+4y-2z=1$$
$$x+7y-6z=6$$
$$3y+qz=t$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 1 & 7 & -6 & 6 \\ 0 & 3 & q & t \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 3 & q & t \end{array}\right] \quad R_2 \leftarrow R_2 - R_1$$

$$\approx \left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & q+4 & t-5 \end{array}\right] \quad R_3 \leftarrow R_3 - R_2$$

The system is singular if

$$q+4=0$$
$$\Rightarrow \boxed{q=-4}$$

The system has infinitely many solutions if

$$t-5=0$$
$$\Rightarrow \boxed{t=5}$$

For the singular case,

$$3y-4z=5$$
$$\Rightarrow y=\frac{5+4z}{3}$$

Let $z=1$. Then $y=3$.

$$x+4y-2z=1$$
$$\Rightarrow x=-9$$

So, the solution is $(-9,3,1)$.