Ex:- 3.3

(1) Solve $Ax = b$ by least squares, and find $p = A\hat{x}$ if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Verify that the error $b - p$ is perpendicular to the columns of A.

Ans:- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

We know that, $A^T A \hat{x} = A^T b$ ——— (1)

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

from equ^n (1),

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \hat{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \hat{x} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \hat{x} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

$p = A\hat{x}$

$\Rightarrow p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$

$= \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix}$

$e = b - p$

$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}$

$e^T(\text{col } 1) = \begin{bmatrix} 2/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$

$e^T(\text{col } 2) = \begin{bmatrix} 2/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$

$\therefore (b-p)$ is $\perp$r to the columns of A (verified)

(2) Write out $E^2 = \|Ax - b\|^2$ and set to zero its derivatives with respect to u and v, if

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $x = \begin{bmatrix} u \\ v \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

Compare the resulting equations with $A^TA\hat{x} = A^Tb$. Find the solution $\hat{x}$ and the projection $p = A\hat{x}$. Why is $p = b$?

Ans:- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $x = \begin{bmatrix} u \\ v \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

$Ax - b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} u-1 \\ v-3 \\ u+v-4 \end{bmatrix}$

$E^2 = \|Ax - b\|^2$

$\Rightarrow E^2 = (u-1)^2 + (v-3)^2 + (u+v-4)^2$

To minimize E^2,

$\frac{\partial E^2}{\partial u} = 0$

$\Rightarrow 2(u-1) + 2(u+v-4) = 0$

$\Rightarrow 4u + 2v = 10$ — (1)

and $\frac{\partial E^2}{\partial v} = 0$

$\Rightarrow 2(v-3) + 2(u+v-4) = 0$

$\Rightarrow 2u + 4v = 14$ — (2)

From (1) & (2) $\Rightarrow$ we get

$4u + 2v = 10$	$2u + 4v = 14$
$\Rightarrow 2u + v = 5$	$\Rightarrow u + 2v = 7$

$\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ — (3)

$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$A^Tb = \begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1\\3\\4\end{bmatrix} = \begin{bmatrix}5\\7\end{bmatrix}$

So, $A^TA\hat{x} = A^Tb$

$\Rightarrow \begin{bmatrix}2&1\\1&2\end{bmatrix}\hat{x} = \begin{bmatrix}5\\7\end{bmatrix}$ —— (4)

So, (3) and (4) are same

$\therefore \hat{x} = \begin{bmatrix}2&1\\1&2\end{bmatrix}^{-1}\begin{bmatrix}5\\7\end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix}3\\9\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix} = \hat{x}$ (Ans)

And $p = A\hat{x}$

$= \begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}$

$\Rightarrow p = \begin{bmatrix}1\\3\\4\end{bmatrix} = b$ (Ans)

Here $p = b$ becoz $b \in C(A)$

$\begin{bmatrix}1\\3\\4\end{bmatrix} = 1\begin{bmatrix}1\\0\\1\end{bmatrix} + 3\begin{bmatrix}0\\1\\1\end{bmatrix}$

(4) The following system has no solu^n:

$Ax = \begin{bmatrix}1&-1\\1&0\\1&1\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}4\\5\\9\end{bmatrix} = b$

Sketch and solve a straight line fit that leads to the minimization of the quadratic $(C-D-4)^2 + (C-5)^2 + (C+D-9)^2$? What is the projection of b onto the column space of A?

Ans: $Ax = \begin{bmatrix}1&-1\\1&0\\1&1\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}4\\5\\9\end{bmatrix} = b$

$\Rightarrow C - D = 4$

$C = 5$

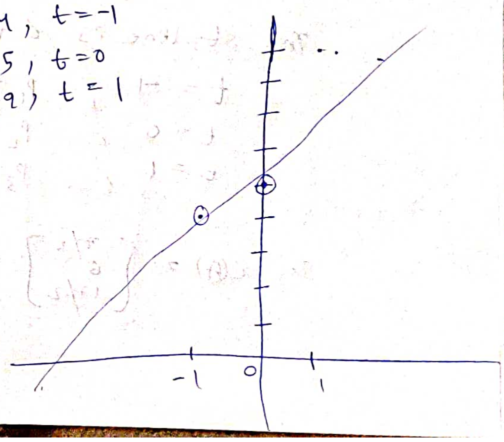
$C + D = 9$

Let the st. line be $C + Dt$

So, $b = 4,\ t = -1$

$b = 5,\ t = 0$

$b = 9,\ t = 1$



Minimize the quadratic,

$$E^2 = (C-D-4)^2 + (C-5)^2 + (C+D-9)^2$$

$$\frac{dE^2}{dC} = 0$$

$$\Rightarrow 2(C-D-4) + 2(C-5) + 2(C+D-9) = 0$$

$$\Rightarrow 6C = 36$$

$$\Rightarrow \boxed{C=6}$$

and $$\frac{dE^2}{dD} = 0$$

$$\Rightarrow -2(C-D-4) + 2(C+D-9) = 0$$

$$\Rightarrow 4D = 10$$

$$\Rightarrow \boxed{D = 5/2}$$

The st. line is $6 + 5/2\, t$

$t = -1$, $p_1 = 7/2$

$t = 0$, $p_2 = 6$

$t = 1$, $p_3 = \frac{17}{2}$

So, $C(A) = \begin{bmatrix} 7/2 \\ 6 \\ 17/2 \end{bmatrix}$

(6) Find the projection of b onto the column space of A.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

Split b into $p+q$, with p in the column space & q $\perp$r to that space. Which of the four subspaces contain q?

Ans: $$A^TA = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{44} \begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$$A(A^TA)^{-1} = \frac{1}{22} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

$$= \frac{1}{22} \begin{bmatrix} 13 & 17 \\ 5 & 1 \\ -2 & 4 \end{bmatrix}$$

$$p = \frac{1}{22} \begin{bmatrix} 13 & 7 \\ 5 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -11 \\ 27 \end{bmatrix} \quad (p = A\hat{x} = A(A^TA)^{-1}A^Tb)$$

$$= \frac{1}{22} \begin{bmatrix} 46 \\ -28 \\ 130 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix}23\\-14\\65\end{bmatrix} = \begin{bmatrix}23/11\\-14/11\\65/11\end{bmatrix} \quad \text{(Ans)}$$

We know that,

$$e = b - p$$

i.e $e = q = b - p = \begin{bmatrix}1\\2\\7\end{bmatrix} - \begin{bmatrix}23/11\\-14/11\\65/11\end{bmatrix} = \begin{bmatrix}-12/11\\36/11\\12/11\end{bmatrix}$

i.e $b = p + q$

So, q is $\perp$r to the $C(A)$

So, $q^T A = \begin{bmatrix}-12/11 & 36/11 & 12/11\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\\-2 & 4\end{bmatrix} = \begin{bmatrix}0 & 0\end{bmatrix} = 0$

So, $q \in N(A^T)$ (becoz $N(A^T) \perp C(A)$)

(9) Find the best st. line fit to the measurements

$b = 4$ at $t = -2$, $b = 3$ at $t = -1$

$b = 1$ at $t = 0$, $b = 0$ at $t = 2$

then find the projectⁿ of $b = (4, 3, 1, 0)$ onto the column space of

$$A = \begin{bmatrix}1 & -2\\1 & -1\\1 & 0\\1 & 2\end{bmatrix}$$

Ans: The best straight-line fit is $b = C + Dt$

So, $b = 4$ at $t = -2 \Rightarrow C - 2D = 4$

$b = 3$ at $t = -1 \Rightarrow C - D = 3$

$b = 1$ at $t = 0 \Rightarrow C = 1$

$b = 0$ at $t = 2 \Rightarrow C + 2D = 0$

Minimize, $E^2 = ||Ax - b||^2$

$$= (C - 2D - 4)^2 + (C - D - 3)^2 + (C-1)^2 + (C+2D)^2$$

$$\frac{dE^2}{dC} = 2(C - 2D - 4) + 2(C - D - 3) + 2(C - 1) + 2(C + 2D) = 0$$

$\Rightarrow 8C - 2D = 16$

$\Rightarrow 4C - D = 8$ —— (1)

$$\frac{dE^2}{dD} = 2(C - 2D - 4)(-2) + 2(C - D - 3)(-1) + 2(C + 2D)\cdot 2 = 0$$

$\Rightarrow -2C + 18D = -22$

$\Rightarrow C - 9D = 11$ —— (2)

From (1) and (2) $\Rightarrow$ we get

$$\begin{bmatrix}4 & -1\\1 & -9\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}8\\11\end{bmatrix}$$

$\Rightarrow A^T A \hat{x} = A^T b$

$\Rightarrow \hat{x} = (A^T A)^{-1} A^T b$

$\Rightarrow \hat{x} = \frac{1}{-35}\begin{bmatrix}-9 & 1\\ -1 & 4\end{bmatrix}\begin{bmatrix}8\\11\end{bmatrix}$

$\Rightarrow \hat{x} = \begin{bmatrix}\frac{61}{35}\\ \frac{-36}{35}\end{bmatrix}$

So, $C = \frac{61}{35}$ and $D = \frac{-36}{35}$

Best st. line is $\frac{61}{35} - \frac{36}{35}t$ — (1)

$p = A\hat{x}$

$\Rightarrow p = \begin{bmatrix}1 & -2\\ 1 & -1\\ 1 & 0\\ 1 & 2\end{bmatrix}\begin{bmatrix}61/35\\ -36/35\end{bmatrix} = \begin{bmatrix}133/35\\ 97/35\\ 61/35\\ -11/35\end{bmatrix}$ (Ans)

From equⁿ (1) we get:

$t = -2, \; p_1 = \frac{133}{35}$

$t = -1, \; p_2 = 97/35$

$t = 0, \; p_3 = 61/35$

$t = 2, \; p_4 = -11/35$

(12) Find the projection matrix P onto the space spanned by $a_1 = (1,0,1)$ and $a_2 = (1,1,-1)$

Ans: $a_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \; a_2 = \begin{bmatrix}1\\1\\-1\end{bmatrix}$

So, $A = \begin{bmatrix}1 & 1\\ 0 & 1\\ 1 & -1\end{bmatrix}$

$P = A(A^TA)^{-1}A^T$

$A^TA = \begin{bmatrix}1 & 0 & 1\\ 1 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$

$(A^TA)^{-1} = \frac{1}{6}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$

$A(A^TA)^{-1} = \frac{1}{6}\begin{bmatrix}1 & 1\\ 0 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix} = \frac{1}{6}\begin{bmatrix}3 & 2\\ 0 & 2\\ 3 & -2\end{bmatrix}$

So, $P = A(A^TA)^{-1}A^T$

$= \frac{1}{6}\begin{bmatrix}3 & 2\\ 0 & 2\\ 3 & -2\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 1 & 1 & -1\end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix}5 & 2 & 1\\ 2 & 2 & -2\\ 1 & -2 & 5\end{bmatrix}$

$\Rightarrow P = \begin{bmatrix}5/6 & 1/3 & 1/6\\ 1/3 & 1/3 & -1/3\\ 1/6 & -1/3 & 5/6\end{bmatrix}$