Lecture 19
Chapter 3: Orthogonality

3.1 Orthogonal Vectors and Subspaces

Course Outcomes: Students will be acquainted with orthogonal vectors, orthonormal vectors, orthonormal subspaces, and orthogonal compliment of subspaces.

Length of a vector: It is denoted by $||x||$.
Let $x = (x_1, x_2)$.
Length in 2D = $||x|| = \sqrt{x_1^2 + x_2^2}$.
Length squared=$||x||^2 = x_1^2 + x_2^2$.
Let $x = (x_1, x_2, x_3)$.
Length in 3D = $||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.
Length squared=$||x||^2 = x_1^2 + x_2^2 + x_3^2$
Let $x = (x_1, x_2, ..., x_n)$.
Length in R^n = $||x|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$.
Length squared=$||x||^2 = x_1^2 + x_2^2 + ... + x_n^2$

Inner Product:
Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then the inner product of two vectors x and y is denoted by $x^T y$ and defined as $x^T y = x_1 y_1 + x_2 y_2$
Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then $x^T y = x_1 y_1 + x_2 y_2 + x_3 y_3$
Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$.
Then $x^T y = x_1 y_1 + x_2 y_2 + ... + x_n y_n$

Note:

$$x^T x = \begin{pmatrix} x_1 & x_2 & ......x_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ ...... \\ x_n \end{pmatrix}$$

$$= x_1^2 + x_2^2 + ... + x_n^2 = ||x^2||$$

Hence inner product of a vector with itself is equal to the length square of the vector.

- The inner product $x^T y$ is zero if and only if x and y are orthogonal vectors.
- If $x^T y > 0$, their angle is less than $\deg 90$. If $x^T y < 0$, their angle is greater than $\deg 90$.
- The only vector with length zero and the only vector orthogonal to itself is the zero vector.

Orthogonal Vectors:
Two vectors x and y are said to be orthogonal iff $x^T y = 0$

Orthogonal Subspaces:
Two subspaces V and W of the same space R^n are orthogonal if every vector v in V is orthogonal to every vector w in W i.e $v^T w = 0$ for all v and w.

Examples

- x-axis and y-axis are subspaces of R^2 and every vector of x-axis is orthogonal to every vector in y-axis. So, x-axis $\perp$ y-axis in R^2
- $y = x$ line $\perp$ $y = -x$ line in R^2.
- All the three axes in R^3 are orthogonal to each other.

Notes

- The subspace $\{0\}$ is orthogonal to all subspaces.
- A line can be orthogonal to another line, or it can be orthogonal to a plane, but a plane cannot be orthogonal to a plane.

Fundamental theorem of orthogonality: The row space is orthogonal to the nullspace (in R^n). The column space is orthogonal to the left nullspace (in R^m).

Orthogonal Complement Given a subspace V of R^n, the space of all vectors orthogonal to V is called the orthogonal complement of V. It is denoted by $V^\perp =$ "V perp."

Examples

- x-axis is the orthogonal complement of y-axis in R^2.
- $y = x$ line is the orthogonal complement of $y = -x$ line in R^2.
- x-axis is the orthogonal complement of yz-plane in R^3.

Fundamental Theorem of Linear Algebra: The nullspace is the orthogonal complement of the row space in R^n. The left nullspace is the orthogonal complement of the column space in R^m.