Problem Set-6.2

1) $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

A is symmetric.

$A_1 = [2] \Rightarrow |A_1| = 2 > 0$

$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \Rightarrow |A_2| = 4 - 1 = 3 > 0$

$A_3 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \Rightarrow |A_3| = 2(4-1) + 1(-2-1) - 1(1+2)$

$= 6 - 3 - 3$

$= 0$

$\therefore$ A is not positive definite. (Ans:-)

$B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & +1 \\ -1 & 1 & 2 \end{bmatrix}$

B is symmetric

$B_1 = [2] \Rightarrow |B_1| = 2 > 0$

$B_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \Rightarrow |B_2| = 3 > 0$

$B_3 = B \Rightarrow |B_3| = 2(4-1) + 1(-2+1) - 1(-1+2)$

$= 6 - 1 - 1 = 4 > 0$

$\therefore$ B is positive definite. (Ans:-)

$C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$

C is symmetric.

$C_1 = [5] \Rightarrow |C_1| = 5 > 0$

$C_2 = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \Rightarrow |C_2| = 10 - 4 = 6 > 0$

$C_3 = C \Rightarrow |C_3| = 5(10-4) - 2(10-2) + 1(4-2)$

$= 30 - 16 + 2$

$= 16 > 0$

$\therefore$ C is positive definite. (Ans:-)

3) Given,

A & B are positive definite.

$\therefore x^T A x > 0$ & $x^T B x > 0$ for non-zero real vec

Now, $x^T (A+B) x$

$= x^T (Ax + Bx)$

$= x^T A x + x^T B x$

> 0

$\therefore$ A+B is positive definite. (Proved)

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11) $|x^T A y|^2 = |x^T R^T R y|^2$ [$\because A = R^T R$ given]

$\equiv |(Rx)^T (Ry)|^2$

$\leq \|Rx\|^2 \|Ry\|^2$ (Using Schwarz inequality)

$= (Rx)^T (Rx) (Ry)^T (Ry)$

$= (x^T R^T R x)(y^T R^T R y)$

$= (x^T A x)(y^T A y)$

$\therefore |x^T A y|^2 \leq (x^T A x)(y^T A y)$ (Proved)

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25) a) $f = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 = x^T A x$

$\Rightarrow f = x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0$

$\therefore A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \rightarrow$ symmetric

$A_1 = [2] \Rightarrow |A_1| = 2 > 0$

$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \Rightarrow |A_2| = 4 - 1 = 3 > 0$

$A_3 = A \Rightarrow |A_3| = 2(4-1) + 1(-2) + 0$

$= 6 - 2 = 4 > 0$

$\therefore$ A is positive definite.

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b) $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1)$

$\Rightarrow f = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1$

$\Rightarrow f = (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 = x^T A x \geq 0$

$\therefore A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \longrightarrow$ symmetric

$A_1 = 2 \Rightarrow |A_1| = 2 > 0$

$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \Rightarrow |A_2| = 4 - 1 = 3 > 0$

$A_3 = A \quad \Rightarrow \quad |A_3| = 2(4-1) + 1(-2-1) - 1(1+2)$

$= 6 - 3 - 3$

$= 0$

$\therefore$ A is not positive definite.

34> $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$\therefore x^T A x = [x_1 \ x_2 \ x_3] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= x_1^2 + x_2^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$

$= (x_1 + x_2)^2 + 2x_3(x_1 + x_2)$

≥ 0 [$\because x \neq 0$ if $x_1 = -x_2$ & $\cancel{x_3 = 0}$]

$\therefore$ A is +'ve ~~semidefinite~~

$\therefore |A_1| = 1 > 0$

$|A_2| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

$|A_3| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1(-1) - 1(-1) + 1(0)$

$= -1 + 1 = 0$

~~$\therefore$ A is +'ve semidefinite.~~

34) Given,
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Test No. 1)

$$x^T A x = x_1^2 + x_2^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$$
$$= (x_1+x_2)^2 + 2x_3(x_1+x_2)$$
$$= (x_1+x_2)^2 + (x_2+x_3)^2 + (x_3+x_1)^2 - (x_1^2+x_2^2+2x_3^2)$$

If $x_1 = 1,\ x_2 = 0,\ x_3 = 1$
$$x^T A x = 1+1+4-3$$
$$= 3 > 0$$

If $x_1 = 1,\ x_2 = -2,\ x_3 = 1$
~~$x^T A x = 0+0+4$~~
$$x^T A x = 1+1+4-7$$
$$= -1 < 0$$

Test No. 2)

$$A_1 = [1] \Rightarrow |A_1| = 1 > 0$$
$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow |A_2| = 0$$
$$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow |A_3| = 1(-1) - 1(-1) + 1(1-1)$$
$$= -1 + 1 + 0$$
$$= 0$$

$\therefore$ It is not positive definite.

Test No. 3)

$$|A - \lambda I| = 0$$
$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$
$$\Rightarrow (1-\lambda)[\lambda(\lambda-1)-1] - 1[-\lambda-1] + 1[\cancel{1}-\cancel{1}+\lambda] = 0$$
$$\Rightarrow (1-\lambda)(\lambda^2-\lambda-1) + (\lambda+1) + \lambda = 0$$
$$\Rightarrow -\lambda^3 + 2\lambda^2 - \cancel{1} + 2\lambda + \cancel{1} = 0$$
$$\Rightarrow \lambda(\lambda^2 - 2\lambda - 2) = 0$$

$\therefore \lambda_1 = 0$ | $\lambda^2 - 2\lambda - 2 = 0$

$$\therefore \lambda = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm 2\sqrt{3}}{2}$$

$$\therefore \lambda_2 = 1+\sqrt{3} > 0 \quad | \quad \lambda_3 = 1-\sqrt{3} < 0$$

A is indefinite.

Given,

$$B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Test No. 1>

$$x^T B x = 2x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_2 + 4x_1x_3 + 2x_2x_3$$

$$= (x_1 + x_2)^2 + (x_1 + 2x_3)^2 + (x_2 + x_3)^2 - (3x_3^2 + x_2^2)$$

Test No. 2>

$$|B_1| = |2| = 2 > 0$$

$$|B_2| = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 > 0$$

$$|B_3| = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = 2(2-1) - 1(2-2) + 2(1-2)$$

$$= 2 - 0 - 2$$

$$= 0$$

Test No. 3>

$$|B - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 2 \\ 1 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[\lambda^2 - 3\lambda + 2 - 1] - 1[\cancel{2} - \lambda - \cancel{2}] + 2(1 - 2 + 2\lambda) = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2 - 3\lambda + 1) + \lambda - 2 + 4\lambda = 0$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 - 7\lambda + \cancel{2} + 5\lambda - \cancel{2} = 0$$

$$\Rightarrow \lambda(\lambda^2 - 5\lambda + 2) = 0$$

$\lambda_1 = 0$ | $\lambda^2 - 5\lambda + 2 = 0$

$$\Rightarrow \lambda = \frac{5 \pm \sqrt{25 - 8}}{2}$$

$$\therefore \lambda_2 = \frac{5 + \sqrt{17}}{2} > 0 \quad \Big| \quad \lambda_3 = \frac{5 - \sqrt{17}}{2} > 0$$

B is positive semi-definite.