Linear Algebra

September 29, 2020

1 Introduction

Linear Algebra is a part of Mathematics and applied in every field of life. With the help of Linear Algebra we can convert our real life problems into various models involving, which are easy to deal with. One can visualize the objects in higher dimensions to get more information. It helps in solving Differential Equations, game developing, Machine Learning, Data Mining, Image Processing, Traffic Controlling, Electrical Circuit problems, Genetics, Cryptography, various economic model and several other fields. With the help of linear transformation concept one can study the properties of different entities in different spaces. Also a complicated geometrical problem can be studied by converting it into a simple algebraic problem. Hence Linear Algebra can be considered as a bridge between Geometry and Algebra.

2 System of Linear Equations

Definition : A linear system of equations is formed when two or more linear equations involving two or more unknowns considered together to represent a problem. For example:

$$\begin{gathered}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\
\mathrm{a}_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\
... \; ... \; ... \\
\mathrm{a}_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m
\end{gathered}$$

Which can also be written as $Ax = b$. Where $A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$ is a $m \times n$ matrix having the coefficients of ith unknown as ith column elements and is said to be *coefficient matrix*. $x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$ is a vector of unknowns said to be *solution vector* and

$b = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$ is said to be the *righthand side vector or nonhomogeneous vector.*

The above system of equations can also be represented as

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{m2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}.$$

The values of $x_1, x_2, \cdots, x_n$ for which the given equations are satisfied form a vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$ is a *solution* of this system of equations.

A system of equations is said to *singular* if the corresponding coefficient matrix is singular. A matrix is said to be *singular* iff its rows or columns are linearly related to each other. That is a row (or column) can be obtained from the addition of scalar multiples of other rows(or columns).

The system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

is singular if $\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}}$. Otherwise it is said to *nonsingular*. A nonsingular system of equations has a **unique** solution.

A singular system of equations has either infinitely many solutions or no solutions. Hence for the above system of equations

1. if $\frac{a_{11}}{a_{21}} \neq \frac{a_{12}}{a_{22}}$, then it has **unique** solution.
2. if $\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} \neq \frac{b_1}{b_2}$, then it has **no** solution.
3. if $\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} = \frac{b_1}{b_2}$, then it has **infinitely many** solutions.

The following figures give an pictorial representation of the three cases discussed here.

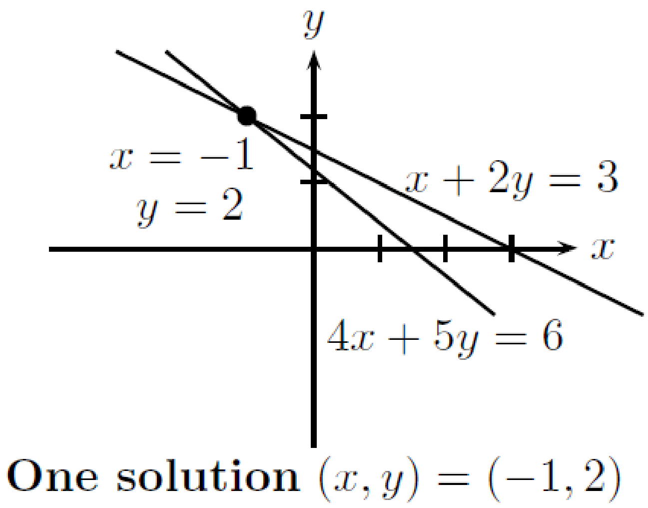


Figure 1: **One Solution**

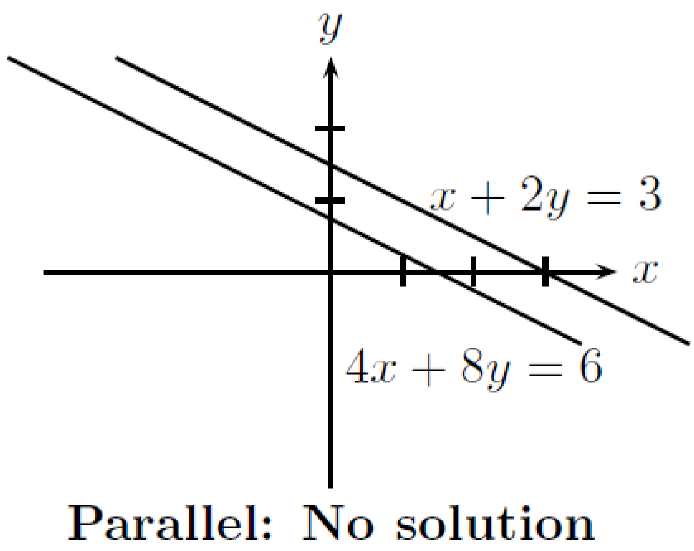


Figure 2: **No Solution**

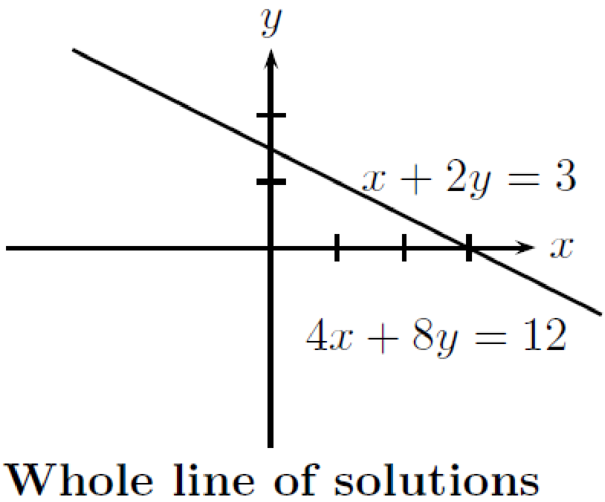


Figure 3: **whole lines of Solution**

THE GEOMETRY OF LINEAR EQUATIONS

To solve a system of equations geometrically two methods are used.

(i) **Row picture method** :

1. Plot the straight lines corresponding to the given equations.
2. Find the points of intersection(s) if exist. The x-coordinate value of the point of intersection represents the value of x and y- coordinate value gives the value of y.
3. Here if the lines are intersecting then unique solution.
4. If they are parallel then no solution.
5. If they represent the same line then every point on the line is a solution of it.

(See the first figure)

(ii) **Column picture method** :

1. Write the given system of equations as a linear combinations of column vectors equal to the rhs vector.

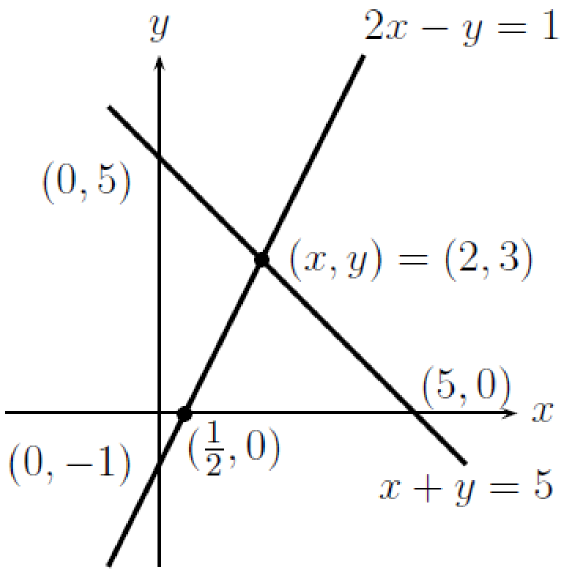
$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} y = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

2. Plot the points $P = (a_{11}, a_{21})$, $Q = (a_{12}, a_{22})$ in xy-plane.
3. Join each point with the origin O. Extend the lines.
4. Plot the point $B = (b_1, b_2)$. Draw a line from B to OP parallel to OQ and get the coordinates of point of intersection (h_1, h_2).
6. Also, draw a line from B to OQ parallel to OP and get the coordinates of point of intersection (k_1, k_2).
7. Find the values of x and y from the equations: $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} y = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$.

(See the second figure)

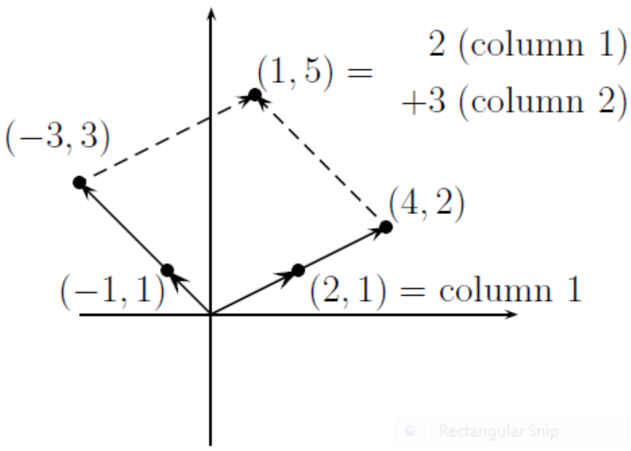
Consider a system of equations

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5. \end{aligned}$$



(a) Lines meet at $x = 2$, $y = 3$

Figure 1: **Row Picture Method**



(b) Columns combine with 2 and 3

Figure 2: **Column Picture Method**

PROBLEM SET 1.2

Q7. Explain why the given system is singular by finding a combination of the three equations that adds up to 0=1. What value should replace the last 0 on the r.h.s. to allow the equations to have solutions and what is one of the solutions?

$$\begin{aligned} u+v+w &= 2 \\ u+2v+3w &= 1 \\ v+2w &= 0. \end{aligned}$$

Ans. Here in the left hand side $Row1 + Row3 = Row2$ but not in right hand side. Hence the system is singular but no solution exists. If the last 0 is replaced by -1 then l.h.s. and r.h.s. both satisfy the condition $Row1 + Row3 = Row2$.

Hence solution exists and $\begin{bmatrix} 3+w \\ -1-2w \\ w \end{bmatrix}$ is a general solution. For every value of w it gives a solution of

$$\begin{aligned} u+v+w &= 2 \\ u+2v+3w &= 1 \\ v+2w &= -1. \end{aligned}$$

In patricular, $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ is a solution of it.

Q 8. Under what condition on y_1, y_2, y_3 do the points $(0, y_1)$, $(1, y_2)$, $(2, y_3)$ lie on a straight line?

Ans. The points $(0, y_1)$, $(1, y_2)$, $(2, y_3)$ lie on a straight line means the slopes of the line joining the points $(0, y_1)$ and $(1, y_2)$ and the points $(1, y_2)$ and $(2, y_3)$ are equal. That is,

$$\frac{y_2 - y_1}{1 - 0} = \frac{y_3 - y_2}{2 - 1}$$

which implies that $y_1 - 2y_2 + y_3 = 0$. Hence for $y_1 - 2y_2 + y_3 = 0$ the points $(0, y_1)$, $(1, y_2)$, $(2, y_3)$ lie on a straight line.

Q 11. The column picture form of a system is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b.$$

Show that the three columns on the left lie in the same plane by expressing the third column as a combination of the first two. What are all the solutions (u, v, w) if b is the zero vector $(0, 0, 0)$?

Ans. Here it can be observed that $2C_2 - C_1 = C_3$ that is, $2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. Hence the system of the equations

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

has infinitely many solutions. For every value of w the vector $(w, -2w, w)$ represents a solution of it. All solutions of it is represented by the set $\{(w, -2w, w) \in R^3 : w \in R\}$.

Q2. Sketch these three lines and decide if the equations are solvable :

$$\begin{aligned} x + 2y &= 2 \\ x - y &= 2 \\ y &= 1. \end{aligned}$$

What happens if all the right-hand sides are zero? Is there any nonzero choice of right hand sides that allows the three lines to intersect at the same point ?

Ans. The first figure represents the straight lines represented in the question.

$$\begin{aligned} x + 2y &= 2 \\ x - y &= 2 \\ y &= 1. \end{aligned}$$

Which shows that there exist no point common to all straight lines. Hence no solution exists.

The second figure gives the straight lines with r.h.s. vector 0, that is

$$\begin{aligned} x + 2y &= 0 \\ x - y &= 0 \\ y &= 0. \end{aligned}$$

Hence $x = 0$ and $y = 0$ is a solution of it.

The third figure gives the graph of

$$\begin{aligned} x + 2y &= 2 \\ x - y &= 2 \\ y &= 0. \end{aligned}$$

With r.h.s. vector $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ and $x = 2$, $y = 0$ satisfies all the equations. Hence it has a solution at $(2, 0)$.

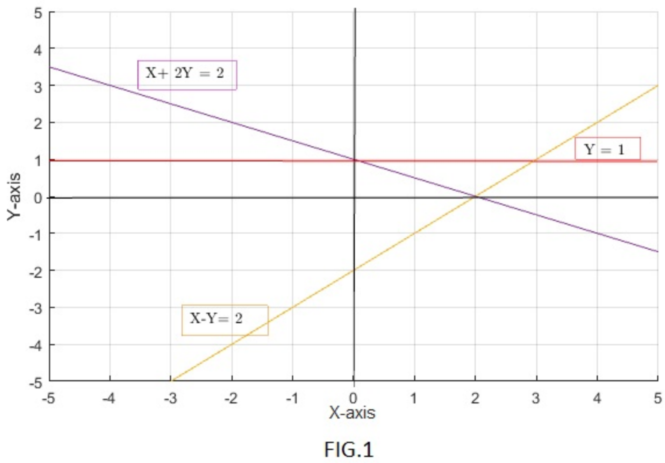


Figure 1: **Solution does not exist**

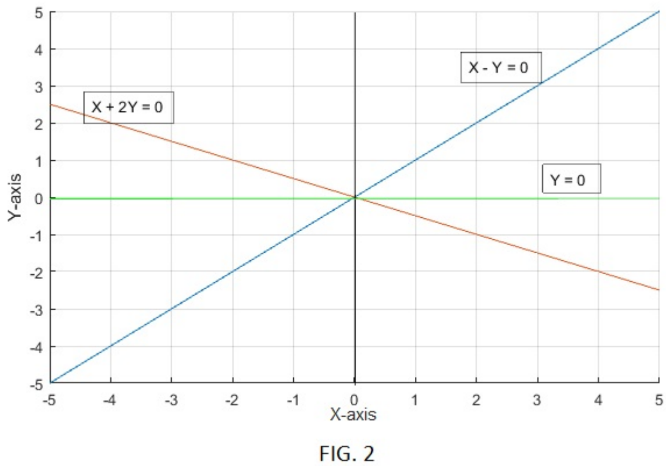


Figure 2: **Solution exist and is the zero solution**

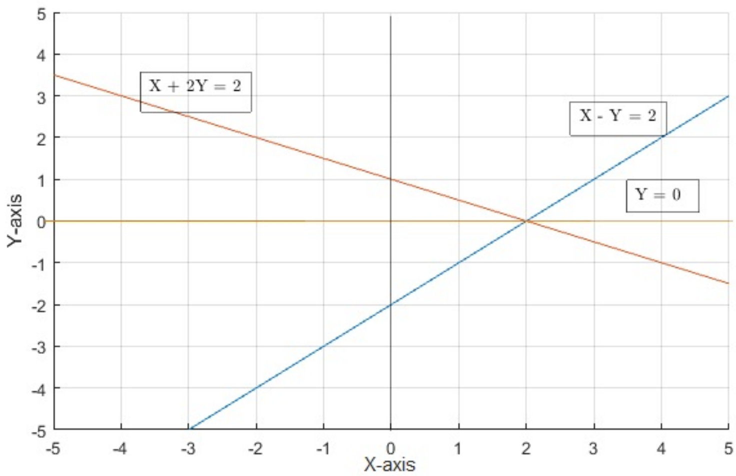


FIG.3

Figure 3: **Nonzero solution exists**

Gaussian Elimination

Consider a system of linear equations.

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ .... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{gathered}$$

Which can also be written as $Ax = b$. Where $A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$ is a $m \times n$ matrix having the coefficients of ith unknown as ith column elements and is said to be **coefficient matrix**. $x = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ . \\ . \\ . \\ \mathrm{x}_n \end{bmatrix}$ is a vector of unknowns said to be **solution vector** and $b = \begin{bmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \\ . \\ . \\ . \\ \mathrm{b}_m \end{bmatrix}$ is said to be the **righthand side vector or nonhomogeneous vector.** The values of $x_1, x_2, \cdots, x_n$ for which the given equations are satisfied form a vector $x = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ . \\ . \\ . \\ \mathrm{x}_n \end{bmatrix}$ is a **solution** of this system of equations.

Let us try to understand elimination method by an example :

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \quad &= -2 \\ -2u + 7v + 2w &= 9. \end{aligned}$$

Here u, v, w are the unknowns.

To eliminate a variable, means making its coefficient 0.
(a) Let us subtract 2 times the first equation from the second
(b) Let us subtract -1 times the first equation from the third to get

$$\begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ 8v + 3w &= 14. \end{aligned}$$

Here the first variable u is eliminated.

Now to eliminate v, let us subtract (-1) times of the second equation from the third to get,

$$\begin{aligned}\mathbf{2}u + v + w &= 5\\ \mathbf{-8}v - 2w &= -12\\ \mathbf{1}w &= 2.\end{aligned}$$

These values 2, -8, 1 are called **pivots**. The cofficient of u in the first equation and the coefficient of v in the second equation and the coefficient of w in the third equation in the triangular form are called **1st, 2nd, 3rd pivots** respectively.

In matrix form

$$\begin{bmatrix} 2 & 1 & 1 & 5\\ 4 & -6 & 0 & -12\\ -2 & 7 & 2 & 9\end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 5\\ 0 & -8 & -2 & -12\\ 0 & 8 & 3 & 14\end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 5\\ 0 & -8 & -2 & -12\\ 0 & 0 & 1 & 2\end{bmatrix}$$

Now back substitution yields the complete solution in the opposite order, beginning with the last unknown. The last equation $1w = 2$ gives $w = 2$. Then the second equation $-8v - 2w = -12$ gives $v = 1$. Finally, the first equation $2u + v + w = 5$ gives $u = 1$. Hence $\begin{bmatrix} u\\ v\\ w\end{bmatrix} = \begin{bmatrix} 1\\ 1\\ 2\end{bmatrix}$ is the required solution.

The Breakdown of Elimination :

If a zero appears in a pivot position, elimination has to stop either temporarily or permanently.

If the zero pivot can be replaced by a nonzero value by row exchange process then the **breakdown of elemination process is temporary or else it is permanent.**
Consider an example of Nonsingular case :

$$\begin{aligned} u + v + w &= -\\ 2u + 2v + 5w &= -\\ 4u + 6v + 8w &= -\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} u + v + w &= -\\ 3w &= -\\ 2v + 4w &= -\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} u + v + w &= -\\ 2v + 4w &= -\\ 3w &= -\end{aligned}$$

The breakdown is **temporary**.

Consider an example of singular case :

$$\begin{aligned} u + v + w &= - \\ 2u + 2v + 5w &= - \\ 4u + 4v + 8w &= - \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} u + v + w &= - \\ 3w &= - \\ 4w &= -. \end{aligned}$$

In this case, there is no exchange of equations that can avoid zero in the second pivot position. Hence the breakdown id **permanent.**

Singular system of equations: A system of linear equations is said to be singular if and only if the corresponding coefficient matrix is singular.

A matrix is **singular** if its one row (column) can be written as a linear combination of other rows (columns).
The breakdown is temporary for a nonsingular system of equations (Having full set of pivots). The breakdown is permanent if the system of linear equations is singular.

Problem set 1.3

Q 1. Choose a r.h.s. which gives no solution and another r.h.s. which gives infinitely many solutions. What are two of those solutions?

$$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= ?. \end{aligned}$$

Ans. Here

$$\frac{3}{6} = \frac{2}{4} = \frac{10}{?} \Longrightarrow ? = 20.$$

Hence, if the r.h.s. $? \neq 20$ then no solution exists. For r.h.s. $? = 20$ the system has infinitely many solutions. Every point on the straight line $3x+2y = 10$ is a solution. In particular, $x = 2$, $y = 2$ and $x = 1$, $y = 3.5$ are two solutions.

Q 3. Choose a coefficient b that makes this system singular.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g \end{aligned}$$

Then choose a r.h.s. g that makes it solvable. Find two solutions in that singular case.

Ans. The system is singular $\iff \frac{2}{4} = \frac{b}{8} \implies b = 4$. The system is singular and solvable $\iff \frac{2}{4} = \frac{b}{8} = \frac{16}{g} \implies b = 4 \;\; \& \;\; g = 32$. In this case, the system has infinitely many solutions. Every point on the straight line $2x + 4y = 16$ is a solution. In particular, $x = 2$, $y = 3$ and $x = 6$, $y = 1$ are two solutions.

Q 6. What multiple l of equation 1 should be subtracted from equation 2.

$$\begin{aligned} 2x + 3y &= 1 \\ 10x + 9y &= 1. \end{aligned}$$

After this elimination step, write down the upper triangular system and darken the two pivots.

Ans. Here $\frac{10}{2} = 5 = l$. Hence 5 Multiple of equation 1 is subtracted from equation 2 to get the coefficient matrix $\begin{bmatrix} 2 & 3 \\ 10 & 9 \end{bmatrix} \sim \begin{bmatrix} \mathbf{2} & 3 \\ 0 & \mathbf{-6} \end{bmatrix}$. The pivots are 2 and -6.

Q7. What test on b_1 and b_2 decides where these two equations allow a solution? How many solutions will they have? Draw the column pictures.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2. \end{aligned}$$

Ans. The given system of equations can be written as $x \begin{bmatrix} 3 \\ 6 \end{bmatrix} + y \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Note $\frac{3}{6} = \frac{-2}{-4} \implies 2b_1 = b_2$. Hence the two equations allow a solution and they have infinitely many solutions. If (b_1, b_2) point lies on the straight line joining $(-2, -4)$, $(0, 0)$ and $(3, 6)$. Then the system has infinite number of solutions.

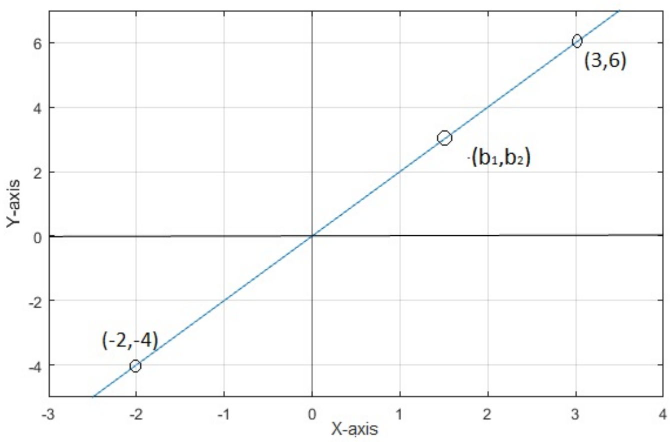


FIG.6

Figure 1: **Column Picture**

Problem set 1.3 continued...

Q 8. For which numbers a does elimination breakdown (a) Permanently (b) Temporarily ?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6. \end{aligned}$$

Ans. Here

$$\frac{a}{4} = \frac{3}{6} \implies a = 2.$$

$\implies$ The system is singular. Hence there is a permanent breakdown of elimination process. $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$. The second pivot is missing and cannot be replaced by any nonzero value by any row exchange process. Hence there exist a breakdown and the breakdown is permanent.

For $a = 0$, $\frac{a}{4} \neq \frac{3}{6}$, the system is nonsingular. Hence the breakdown is temporary. $\begin{bmatrix} 0 & 3 \\ 4 & 6 \end{bmatrix}$ $\implies$ The first pivot is missing, hence the elimination process breaksdown. By interchanging the 1st and 2nd rows, we have $\begin{bmatrix} 4 & 6 \\ 0 & 3 \end{bmatrix}$. Both the pivots 4, 3 are nonzero, hence the breakdown is temporary.

Q 14. Which number q makes this system singular and which right hand side t gives it infinitely many soluions? Find the solution that has $z = 1$

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t. \end{aligned}$$

Ans. The augment matrix of the given system of equations is :

$$\begin{bmatrix} 1 & 4 & -2 & 1 \\ 1 & 7 & -6 & 6 \\ 0 & 3 & q & t \end{bmatrix} R_2 - R_1 \to R_2 \sim \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 3 & q & t \end{bmatrix} R_3 - R_2 \to R_3 \sim \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & q+4 & t-5 \end{bmatrix}.$$

Here $q + 4 = 0$ makes the 3rd pivot missing and hence the system is singular. For $q + 4 = 0$, $t - 5 = 0$ we get infinitely many solutions since

$$\begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & q+4 & t-5 \end{bmatrix}. = \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence $3y - 4z = 5$, $\quad z = 1 \implies y = 3$ and $x + 4y - 2z = 1 \implies x = -9$.

The solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \\ 1 \end{bmatrix}$.

Q 16. If rows 1 and 2 are the same, how far can you get with elimination? Which pivot is missing?

$$\begin{aligned} 2x - y + z &= 0 \\ 2x - y + z &= 0 \\ 4x + y + z &= 2 \end{aligned}$$

Ans. The augment matrix of the given system of equations is :

$$\begin{bmatrix} 2 & -1 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 4 & 1 & 1 & 2 \end{bmatrix} R_2 - R_1 \to R_2 \sim \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 2 \end{bmatrix} R_3 - 2R_1 \to R_3 \sim \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & 3 & -1 & 2 \end{bmatrix}.$$

Interchanging R_2 with R_3, we get $\begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The second pivot is missing hence elimination breaks down and a row exchange gives a nonzero second pivot but the third pivot is missing, hence the breakdown is permanent.

Q 16. If columns 1 and 2 are the same, how far can you get with elimination? Which pivot is missing?

$$\begin{aligned} 2x + 2y + z &= 0 \\ 4x + 4y + z &= 0 \\ 6x + 6y + z &= 2 \end{aligned}$$

Ans. The augment matrix of the given system of equations is :

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 4 & 4 & 1 & 0 \\ 6 & 6 & 1 & 2 \end{bmatrix} R_2 - 2R_1 \to R_2 \sim \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 6 & 1 & 2 \end{bmatrix} R_3 - 3R_1 \to R_3 \sim \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & \mathbf{0} & -1 & 0 \\ 0 & 0 & -2 & 2 \end{bmatrix}.$$

The second pivot is missing and the breakdown is permanent.

Matrix notation and Matrix multiplication

Matrix **addition** is compatible If and only if matrix A and B both have same number of rows and same number of columns.

If $C = A + B$ then c_{ij} = ith row jth column element of C

$$c_{ij} = a_{ij} + b_{ij}$$

.

For example, let $A = \begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 16 \end{bmatrix}$ then $C = A + B = \begin{bmatrix} 3 & 5 \\ 9 & 23 \end{bmatrix}$.

Matrix **multiplication** $A \times B$ is compatible if and only if number of columns of A equals to number of rows of B. i.e. $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ then $A \times B$ is possible. In this case,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

For example $C = A \times B = \begin{bmatrix} 1 & 2 \\ \mathbf{5} & \mathbf{7} \end{bmatrix} \times \begin{bmatrix} \mathbf{2} & 3 \\ \mathbf{4} & 16 \end{bmatrix} = \begin{bmatrix} 10 & 51 \\ \mathbf{38} & 127 \end{bmatrix}$, $c_{21} = 5 \times 2 + 7 \times 4 = 38$

Elementary Row Operations

There are three elimintary row operations :

1. $R_i + kR_j \to R_i$ Means kth multiple of jth row is added with ith row.

2. $R_i \leftrightarrow R_j$ Means interchange ith row with jth row.

3. $cR_i \to R_i$ Means ith row is replaced by its c multiple.

Consider the 3×3 identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Now $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ R_2-$\mathbf{3}R_1 \to R_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{E} \implies E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{+3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 \leftrightarrow R_1 \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{F} \implies F^{-1} = F$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\mathbf{5}R_2 \to R_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathbf{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} = G \implies G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathbf{1/5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Then $EA = \begin{bmatrix} 1 & 2 & 3 \\ -31 & -35 & -39 \\ 7 & 8 & 9 \end{bmatrix}$, $FA = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$, $GA = \begin{bmatrix} 1 & 2 & 3 \\ 12 & 15 & 18 \\ 7 & 8 & 9 \end{bmatrix}$.

Q. Give 3 by 3 examples (not just the zero matrix) of

(a) a diagonal matrix : $a_{ij} = 0$ if $i \neq j$.

Ans. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

(b) a symmetric matrix $a_{ij} = a_{ji} \quad \forall \quad i \quad \& \quad j$.

Ans. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 15 & 18 \\ 3 & 18 & 9 \end{bmatrix}$.

(c) an upper triangular matrix $a_{ij} = 0$ if $i > j$.

Ans. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 15 & 18 \\ 0 & 0 & 9 \end{bmatrix}$.

(b) a skew-symmetric matrix $a_{ij} = -a_{ji} \quad \forall \quad i \quad \& \quad j$.

Ans. $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 18 \\ 3 & -18 & 0 \end{bmatrix}$.

Q. The matrix that rotates the xy plane by an angle θ is $A(\theta) = \begin{bmatrix} \cos(\theta) & \text{-}\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Verify that $A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2)$. What is $A(\theta)$ times $A(-\theta)$?

Ans. We know $A(\theta_1) = \begin{bmatrix} \cos(\theta_1) & \text{-}\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$ and $A(\theta_2) = \begin{bmatrix} \cos(\theta_2) & \text{-}\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix}$.

$$A(\theta_1)A(\theta_2) = \begin{bmatrix} \cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2) & \text{-}\cos(\theta_1)sin(\theta_2) - cos(\theta_2)sin(\theta_1) \\ \sin(\theta_1)cos(\theta_2) + cos(\theta_1)sin(\theta_2) & \text{-}\sin(\theta_1)sin(\theta_2) + cos(\theta_1)cos(\theta_2) \end{bmatrix}.$$

$$= \begin{bmatrix} \cos(\theta_1 + \theta_2) & \text{-}\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix} = A(\theta_1 + \theta_2)$$

$$A(\theta)A(-\theta) = A(\theta - \theta) = A(0) = \begin{bmatrix} \cos(0) & \text{-}\sin(0) \\ \sin(0) & \cos(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Q. Compute the products.

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 12+0+5 \\ 0+4+0 \\ 12+0+5 \end{bmatrix} = \begin{bmatrix} 17 \\ 4 \\ 17 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ \text{-}2 \\ 3 \end{bmatrix} == \begin{bmatrix} 5 \\ \text{-}2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Q. For the third one, draw the column vectors $(2, 1)$ end $(0, 3)$. Multiplying by $(1, 1)$ just adds the vectors (do it graphically).

Ans. The graphical representation is :

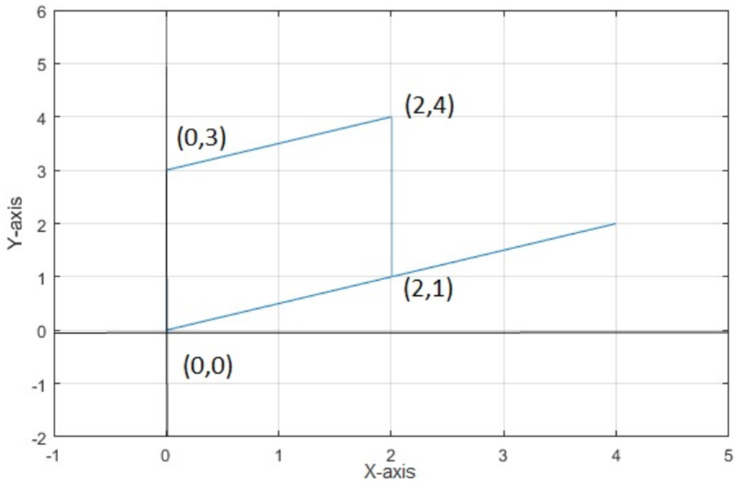


Figure 1: **whole lines of Solution**

Lecture-6

1.5 Triangular Factors and Row Exchanges

Course Outcome: Students will have understanding about the triangular factorization like LU and LDU factorization and permutation matrices that are being used for row exchanges purpose.

Triangular Factorization

$$\text{Given:}\quad \text{A}=\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} \text{R}_2 \leftarrow R_2 - 2R_1(2)$$

$$=\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} R_3 \leftarrow R_3 + R_1(-1)$$

$$=\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \text{R}_3 \leftarrow \text{R}_3 + R_2(-1)$$

$= U$

The elementary matrices are

$$\text{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$E_{32}E_{31}E_{21}A = U$

$\implies MA = U$

where $M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

$M^{-1} = (\mathrm{E}_{32}E_{31}E_{21})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \mathrm{L}$

MA=U $\implies A = M^{-1}U \implies A = LU$,which is known as LU factorization of the matrix A.

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}$

LDU=A, which is known as LDU factorization of the matrix A.

No.2 When an upper triangular matrix is is nonsingular?

Ans: An upper triangular matrix is nonsingular if none of its diagonal elements are zero i.e. it has full set of pivot elements.

No.7 Factor A into LU, and write down the upper triangular system $Ux = c$ which appears after elimination, for

Ax$= \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$

Ans. Ax=b

$$\Longrightarrow \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \mathrm{R}_3 \leftarrow \mathrm{R}_3 - 3R_1$$

$\Longrightarrow$ Ux=c, which is the required upper triangular system.

No.21 What three elimination matrices $\mathrm{E}_{21}, E_{31}, E_{32}$ put A into upper triangular form $E_{21}E_{31}E_{32}A = U$? Multiply by $E_{32}^{-1}, E_{31}^{-1} and E_{21}^{-1}$ to factor A into LU where $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U.

$$\mathrm{A}= \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

Ans. Given: $\mathrm{A}= \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix} R_2 \leftarrow \mathrm{R}_2 - 2R_1(2)$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \mathrm{R}_3 \leftarrow \mathrm{R}_3 - 3R_1(3)$$

$= U$

The elementary matrices are

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathrm{E}_{32}E_{31}E_{21}A = U$

$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$LU = A$

Lecture-7

1.5 Triangular Factors and Row Exchanges

Row Exchanges and Permutation Matrices

During Gaussian elimination in case of breakdown problems, zero is appearing in the pivot place. To make that pivot place zero into nonzero, we are taking the help of row exchange. For this row exchange purpose, we will use permutation matrices.

Permutation Matrices

Order 2: There are 2!=2 permutation matrices of order 2. That are

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

Order 3: There are 3!=6 permutation matrices of order 3. That are

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{21}P_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_{32}P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Points to Remember:

1. Elements of permutation matrices are either 0 or 1.

2. Product of two permutation matrices is again a permutation matrix.

3. There are n! permutation matrices of order n.

Example.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$P_{21}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$P_{32}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix}$$

No.40 Which permutation makes PA upper triangular? Which permutations make P_1AP_2 lower triangular? Multiplying A on the right by P_2

exchanges the what of A?

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

Ans. Given:

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

$$P = P_{32}P_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \text{which is upper triangular.}$$

$$P_1 = P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_2 = P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_1AP_2 = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \text{which is lower triangular.}$$

Multiplying A on the right by P_2 exchanges the columns of A.

Tridiagonal matrix: A square matrix is said to be a tridiagonal matrix if all its elememts are zero except on the main diagonal and the two adjacent diagonals.

No.28 Find the LU and LDU factorization of the following tridiagonal matrix.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Ans. Given:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} R_2 \leftarrow R_2 - R_1(1) \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} R_3 \leftarrow R_3 - R_2(1) \\ &= U \end{aligned}$$

LU factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$LU = A$$

LDU factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$LDU = A$$

Lecture-8

1.6 Inverses and Transposes

Course Outcome: Students will have understanding about existence of inverse, Gauss-Jordan method to find the inverse of square matrices and transpose of matrices.

Existence of Inverse:

Inverse of a square matrix A exists if it is nonsingular i.e. $|A| \neq 0$. It is denoted by A^{-1}. If a square matrix has full set of pivot elements, then it is also nonsingular. So, inverse of a square matrix exists if it has full set of pivots.

Example.

$$\begin{aligned} Let\ A &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ \Rightarrow |A| &= 4-6 \\ \Rightarrow |A| &= -2 \\ \Rightarrow |A| &\neq 0 \end{aligned}$$

$\Rightarrow$A^{-1} will exist.

Minor of A = M = $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

Cofactor of A = C = $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

Adjoint of A $(AdjA) = C^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Therefore,

$$\begin{aligned} A^{-1} &= \frac{AdjA}{|A|} \\ &= \frac{C^T}{|A|} \\ &= \frac{-1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \end{aligned}$$

Example.

$$\begin{aligned} Let\ A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \Rightarrow |A| &= ad - bc \\ \Rightarrow |A| &\neq 0 \end{aligned}$$

$$\Rightarrow A^{-1} will exist.$$

Minor of A = M = $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$

Cofactor of A = C = $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Adjoint of A $(AdjA) = C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore,

$$\begin{aligned} A^{-1} &= \frac{Adj A}{|A|} \\ &= \frac{C^T}{|A|} \\ &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

Example. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \implies A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

Example.

$$\begin{aligned} Let\ A &= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \\ \Rightarrow |A| &= 6-2 \\ \Rightarrow |A| &= 4 \\ \Rightarrow |A| &\neq 0 \end{aligned}$$

$$\Rightarrow A^{-1}\ will\ exist.$$

$$\text{Minor of A} = \text{M} = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\text{Cofactor of A} = \text{C} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Adjoint of A $(AdjA) = C^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Therefore,

$$\begin{aligned} A^{-1} &= \frac{AdjA}{|A|} \\ &= \frac{C^T}{|A|} \\ &= \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \end{aligned}$$

Example. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \implies A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$

Points to Remember:

1. The inverse of a square matrix if exits is unique and is known as both sided inverse.

2. $AA^{-1} = I = A^{-1}A$.

3. $(AB)^{-1} = B^{-1}A^{-1}$, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

4. $Ax = b \implies x = A^{-1}b$.

Gauss-Jordan Method: (The calculation of A^{-1})

Example. *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\Rightarrow |A| = -2 \neq 0$

$\Rightarrow A^{-1}$ *will exist.*

Now consider the matrix

$$
\begin{aligned}
\left[\begin{array}{c|c} A & I \end{array}\right] \quad i.e, \qquad & \\
&= \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] R_2 \leftarrow R_2 - 3R_1 \\
&= \left[\begin{array}{cc|cc} 1 & 0 & -2 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] R_1 \leftarrow R_1 + R_2 \\
&= \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \end{array}\right] R_2 \leftarrow \frac{-1}{2} R_2 \\
&= \left[\begin{array}{c|c} I & A^{-1} \end{array}\right], \\
where \quad A^{-1} &= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \\
&= \frac{-1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}.
\end{aligned}
$$

Example.

$$
\begin{aligned}
Let A &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \\
\Rightarrow |A| &= 1(6-4) - 1(3-2) + 1(2-2) \\
&= 2 - 1 \\
&= 1 \\
&\neq 0
\end{aligned}
$$

$\Rightarrow A^{-1}$ *will exist.*

$$
\left[\begin{array}{c|c} A & I \end{array}\right]
$$

$$
\begin{aligned}
&= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] R_2 \leftarrow R_2 - R_1 \\
&= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] R_3 \leftarrow R_3 - R_1
\end{aligned}
$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \mathrm{R}_3 \leftarrow R_3 - R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \mathrm{R}_1 \leftarrow R_1 - R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \mathrm{R}_2 \leftarrow R_2 - R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \mathrm{R}_1 \leftarrow R_1 - R_2$$

$$= \left[\begin{array}{c|c} I & A^{-1} \end{array}\right],$$

$$\text{where} \quad \mathrm{A}^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

No.6 (a) If A is invertible and AB=AC, prove that B=C.

Proof. Let A be invertible and AB=AC.

$AB = AC$

$\implies A^{-1}(AB) = A^{-1}(AC)$

$\implies (A^{-1}A)B) = (A^{-1}A)C)$

$\implies IB = IC$

$\implies B = C$

(b) $If A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, find an example with $AB = AC$ but $B \neq C$.

Sol. Given:A= $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$Let\ B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\ and\ C = \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}$.

$AB = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$ and $AC = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$.

$AB = AC\ but\ B \neq C.$

No.10 Use the Gauss-Jordan method to find the inverse of $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Sol. Given $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$|A_1| = 1 \neq 0$

So, A_1^{-1} will exist.

$$\left[\begin{array}{c|c} A_1 & I \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \mathrm{R}_2 \leftarrow R_2 - R_1$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \mathrm{R}_2 \leftarrow R_2 - R_3$$

$$= \left[\begin{array}{c|c} I & A_1^{-1} \end{array}\right],$$

where $\quad \mathrm{A}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$

Lecture-9

1.6 Inverses and Transposes

Transpose:

$$A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} \implies A^T=\begin{bmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{bmatrix}$$

Symmetric matrix:

A square matrix A is said to be symmetric if $A^T = A$.

Ex. A= $\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$, B= $\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 6\\ 3 & 6 & 9\end{bmatrix}$

Skew-symmetric matrix:

A square matrix A is said to be skew-symmetric if $A^T = -A$.

Ex. A= $\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}$, B= $\begin{bmatrix}0 & 2 & 3\\ -2 & 0 & 6\\ -3 & -6 & 0\end{bmatrix}$

Points to Remember:

1. $(A+B)^T = A^T + B^T$.

2.$(AB)^T = B^T A^T,\ (ABC)^T = C^T B^T A^T$.

3.$(A^T)^T = A$.

4.$(A^{-1})^T = (A^T)^{-1}$.

If A is any real square matrix, then $A + A^T$ is always symmetric and $A - A^T$ is always skew-symmetric.

Let $B = A + A^T$. Then

$B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = B$

So, B is symmetric.

Let $C = A - A^T$. Then

$C^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -C$

So, C is skew-symmetric.

Now, $A = \frac{A + A^T}{2}(symmetric) + \frac{A - A^T}{2}(skew - symmetric)$

So, any real square matrix can be expressed as a sum of symmetric and skew-symmetric matrix.

Example. Express the following matrix as a sum of symmetric and skew-symmetric matrix.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Sol. A= $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

$$\implies \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 5/2 \\ 5/2 & 4 \end{bmatrix} \text{(symmetric)} + \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix} \text{(skew-symmetric)}$$

Note: If $A = A^T$can be factored into $A = LDU$ without row exchanges, then U is the transpose of L and $A = LDL^T$.

Example.

Given: A= $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 5 & 18 & 30 \end{bmatrix} R_2 \leftarrow R_2 - 3R_1(3)$

$= \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 3 & 5 \end{bmatrix} R_3 \leftarrow R_3 - 5R_1(5)$

$= \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix} R_3 \leftarrow R_3 - R_2(1)$

=U

LU factorization:

L= $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}$, U= $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

$$LU = A$$

LDU factorization:

L= $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}$, D= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ U= $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$LDU = A$$

Here $U = L^T$. So, $A = LDL^T$.

In this example since $A^T = A$, so LDU=A is $LDL^T = A$.

No.11 If B is square, show that $A = B + B^T$ is always symmetric and $K = B - B^T$ is always skew-symmetric. Find these matrices A and K when $B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ and write B as the sum of a symmetric matrix and a skew-symmetric matrix.

Sol. Let B be a square matrix.

Let $A = B + B^T$ *and* $K = B - B^T$.

$A^T = (B + B^T)^T = B^T + (B^T)^T = B^T + B = A$

$\implies$ A is symmetric.

Again, $K^T = (B - B^T)^T = B^T - (B^T)^T = B^T - B = -(B - B^T) = K$

$\implies$ K is skew-symmetric.

$$A = B + B^T = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

$$K = B - B^T = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$B = \frac{B + B^T}{2} + \frac{B - B^T}{2}$$

$$\implies \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{(symmetric)} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{(skew-symmetric)}$$

No.17 Give examples of A and B such that

(a) A+B is not invertible although A and B are invertible.

(b) A+B is invertible although A and B are not invertible.

(c) all of A, B and A+B are invertible.

Sol. (a) A= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, B= $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, A+B= $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

A+B is not invertible although A and B are invertible.

(b) A= $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, B= $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, A+B= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A+B is invertible although A and B are not invertible.

(c) A= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, B= $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, A+B= $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

All of A, B and A+B are invertible.

No.41 True or false:

(a) A 4 by 4 matrix with a row of zeros is not invertible.

Ans. True since the matrix is singular.

(b) A matrix with 1s down the main diagonal is invertible.

Ans. False since the matrix is singular.

(c) If A is invertible then A^{-1} is invertible.

Ans. True since A^{-1} is non-singular if A is no-singular.

(d) If A^T is invertible then A is invertible.

Ans. True since A is non-singular if A^T is non-singular.

No.42 For which three numbers c is this matrix not invertible, and why not?

A= $\begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$

Sol.

For c=0,2,7 the matrix is not invertible, as for these three values of c the determinant of the matrix is zero i.e. the matrix is singular.

c=0 $\implies$ zero column(or zero row).

c=2 $\implies$ identical rows.

c=7 $\implies$ identical columns.

Lecture-10

2.1 Vector Spaces and Subspaces

Course Outcomes: Students will have understanding about vector spaces and subspaces. Also, will be acquainted with the column space and the nullspace of different matrices.

In this article we will discuss about the followings:

(i) Vector Space

(ii) Subspaces

(iii) The Column Space

(iv) The Nullspace

Vector Space: A nonempty set V is said to be a vector space if it satisfies the following properties:

1. $x + y = y + x$ (Commutative law of addition)

2. $x + (y + z) = (x + y) + z$ (Associative law of addition)

3. There is a unique vector '0' (zero vector) such that $x + 0 = x$ for all x. (Additive identity property)

4. For each x there is a unique vector $-x$ such that $x + (-x) = 0$.(Additive inverse property)

5.$1x = x$

6. $(c_1c_2)x = c_1(c_2x)$

7. $c(x + y) = cx + cy$

8. $(c_1 + c_2)x = c_1x + c_2x$,

where $x, y, z \in$ V and $c,\ c_1,\ c_2\ \in\ R$.

Since out of the above eight properties first four are coming under vector addition and last four are coming under scalar multiplication, so the following is an alternate definition of vector space.

A nonempty set V is said to be a vector space if it satisfies the following properties:

1. Vector Addition

i.e. $x \in V,\ y \in V \implies x + y \in V$

2. Scalar Multiplication

i.e. $c \in R, x \in V \implies cx \in V$

Examples of vector spaces: R, R^2, R^3,........., R^n.

Example.

Show that R^2 is a vector space.

Proof. R^2 contains infinitely many elements and the elements are pairs like $(0,0),(1,2),(-1,2),\dots\dots\dots$

So, R^2 is nonempty.

1. Vector addition:

Let $x = (x_1, x_2)$ *and* $y = (y_1, y_2) \in R^2$.

$x + y = (x_1 + y_1, x_2 + y_2) \in R^2$ *as* $x_1 + y_1 \in R$ *and* $x_2 + y_2 \in R$.

2. Scalar multiplication:

Let $c \in R$ and $(x_1, x_2) \in R^2$.

$cx = (cx_1, cx_2) \in R^2$ *as* $cx_1 \in R$ *and* $cx_2 \in R$.

So, R^2 is a vector space.

Example. Verify whether the 1st quadrant of R^2 is a vector space or not.

Proof: Let the set V be the 1st quadrant of R^2.

Then $V = \{(x_1, x_2) \in R^2 : \quad x_1 \geq 0,\ x_1 \geq 0\}$

$= \{(0,0),(1,1),(1,2),(2,3),\dots\dots\dots\}$

So, V is nonempty.

1. Vector addition:

Let $x = (x_1, x_2)$ *and* $y = (y_1, y_2) \in V.$

Then $\quad x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0.$

$x + y = (x_1 + y_1, x_2 + y_2) \in V$ *as* $x_1 + y_1 \geq 0$ *and* $x_2 + y_2 \geq 0.$

2. Scalar multiplication:

Let $c = -2 \in R$ and $x = (1, 2) \in V$.

$cx = (-2, -4) \notin V$

So, V does not satisfy scalar multiplication property.

Hence, V i.e. 1st quadrant is not a vector space.

Example. Verify whether combinedly the 1st and 3rd quadrant of R^2 is a vector space or not.

Proof: Let the set V be the 1st and 3rd quadrant of R^2.

Then $V = \{(1, 1), (1, 2), (-1, -1), (-1, -2), \ldots\ldots\ldots\}$.

So, V is nonempty.

1. Vector addition:

Let $x = (-1, -2)$ and $y = (2, 1) \in$ V.

$x + y = (1, -1) \notin$ V.

So, V does not satisfy vector addition property.

Hence, V i.e. 1st and 3rd quadrant combinedly is not a vector space.

Subspace: A subspace of a vector space is a nonempty subset that satisfies the requirements for a vector space.

Example. Show that $y = x$ line is a subspace of the vector space R^2.

Proof. Let the set V be $y = x$ line.

Then $V = \{(x_1, x_2) \in R^2 : x_1 = x_2\}$

$= \{(0,0),(1,1),(2,2),(-1,-1),(-2,-2),.........\}$.

So, V is a nonempty subset of R^2.

1. Vector addition:

Let $x = (x_1, x_2)$ *and* $y = (y_1, y_2) \in V$.

Then $x_1 = x_2$ *and* $y_1 = y_2$.

$x + y = (x_1 + y_1, x_2 + y_2) \in V$ *as* $x_1 + y_1 = x_2 + y_2$.

2. Scalar multiplication:

Let $c \in R$ *and* $(x_1, x_2) \in V$.

$cx = (cx_1, cx_2) \in V$ *as* $cx_1 = cx_2$.

So, V i.e. $y = x$ line is a subspace space R^2 .

Vector Space: R^2

Subspaces:

1. R^2

2. Any line passing through origin.

3. Origin i.e. {(0,0)}.

Vector Space: R^3

Subspaces:

1. R^3

2. Any plane passing through origin.

3. Any line passing through origin.

3. Origin i.e. {(0,0,0)}.

Points to remember:

1. Every vector space is a subspace of itself.

2. A subspace is a vector space in its own right.

3. Every vector space is the largest subspace of itself and origin is the smallest

subspace.

No.2 Which of the following subsets of R^3 are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.

(b) The plane of vectors b with $b_1 = 1$.

Sol. (a).

Let $V = \{The\ plane\ of\ vectors\ (b_1, b_2, b_3)\ with\ first\ component\ b_1 = 0\}$.

$= \{(0,0,0), (0,1,0), (0,1,2), \ldots\ldots\ldots\}$

So, V is a nonempty subset of R^3.

1. Vector addition:

Let $b = (b_1, b_2, b_3)\ and\ c = (c_1, c_2, c_3) \in V.\ Then\ b_1 = 0\ and\ c_1 = 0.$

$b + c = (b_1 + c_1, b_2 + c_2, b_3 + c_3) \in V\ as\ b_1 + c_1 = 0.$

2. Scalar multiplication:

Let $\alpha \in R\ and\ b = (b_1, b_2, b_3) \in V.\ Then\ b_1 = 0 \implies \alpha b_1 = 0$

$\alpha b = (\alpha b_1, \alpha b_2, \alpha b_3) \in V\ as\ \ \alpha b_1 = 0$

So, V is a subspace of R^3.

Sol. (b).

Let $V = \{The\ plane\ of\ vectors\ b = (b_1, b_2, b_3)\ with\ first\ component\ b_1 = 1\}$.

$= \{(1,0,0), (1,1,0), (1,1,2), \ldots\ldots\ldots\}$

So, V is a nonempty subset of R^3.

1. Vector addition:

Let $b = (b_1, b_2, b_3)\ and\ c = (c_1, c_2, c_3) \in V.\ Then\ b_1 = 1\ and\ c_1 = 1 \implies$ $b_1 + c_1 = 2.$

$b + c = (b_1 + c_1, b_2 + c_2, b_3 + c_3) \notin V\ as\ b_1 + c_1 \neq 1.$

So, V does not satisfy vector addition property.

Hence, V is not a subspace of R^3.

Column Space and Null Space of a Matrix

Column Space of a matrix is all linear combination of the columns of A. and denoted by C(A).
Column Space of Matrix Let $C_1, C_2, ..., C_n$ be 1st column, 2nd column,....,nth column of the matrix $A_{m\times n}$
then $C(A) = \{a_1C_1 + a_2C_2 + ... + a_nC_n/a_1, a_2, ...a_n \in R\}$
where R is set of real numbers.

Steps for finding C($A_{m\times n}$)
Given: Suppose we are given a matrix A
Output: C(A)
Step 1: find Echelon form of A ,say U is echelon form of A
step 2: find the pivot column in U
step 3: then C(A) is linear combination of those column of A which are corresponding to pivot column of U.
For Ex Let 1st and 5th are only pivot column in U,then $C(A) = \{a_1C_1 + a_5C_5/a_1, a_5 \in R\}$

Note : The system $Ax = b$ is solvable iff the vector b can be expressed as a combination of the columns of A . then b is in the column cpace of A
Note : C(A) is a subspace of R^m

Null Space of Matrix
let A be $m \times n$ matrices.then
Null Space of A consists of all vectors x such that Ax=0
and denoted by N(A).
i.e. $N(A) = \{x \in R^n/Ax = 0\}$
Note : N(A) is subspace of R^n

Exercise 2.1.5 : find the column space and null space of the matrices
(a):

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Sol: Since echelon form of A is itself A. i.e. $U = A$
and first column of U is pivot. so

$$C(A) = \left\{ a \begin{bmatrix} 1 \\ 0 \end{bmatrix} /a \in R \right\}$$

for the null space solve $Ax = 0$

$Aug.matrix = \begin{bmatrix} A & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

since since x_2 is free variable , so assume $x_2 = k$ where k is real number.so $x_1 - k = 0$, $x_1 k$

$N(A) = \left\{ \begin{bmatrix} k \\ k \end{bmatrix} /k \in R \right\} = \left\{ k \begin{bmatrix} 1 \\ 1 \end{bmatrix} /k \in R \right\}$

(b)

$$B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} = U$$

Since first and third columns are pivot in U.

So $C(B) = \left\{ a \begin{bmatrix} 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ 3 \end{bmatrix} \mid a, b \in R \right\}$

for null space solve $BX = 0$

Aug. matrix= $[B \mid 0] = \left[\begin{array}{ccc|c} 0 & 0 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$ Since x_2 is free variable so $x_2 = k, K \in R$

$$0x_1 + 0x_2 + 3x_3 = 0 \Rightarrow x_3 = 0$$
$$x_1 + 2k + 3x_3 = 0$$
$$x_1 + 2k + 3 \times 0 = 0$$
$$x_1 = -2k$$

$N(A) = \left\{ \begin{bmatrix} -2k \\ k \\ 0 \end{bmatrix} \mid k \in R \right\} = \left\{ k \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \mid k \in R \right\}$

(c)

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C(A) = \left\{ a \begin{bmatrix} 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mid a, b, c \in R \right\}$$

$$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

Since x_1, x_2, x_3 are free variable.

So $x_1 = k_1,\ x_2 = k_2,\ x_3 = k_3,\ k_1, k_2, k_3 \in R.$

So $N(A) = \left\{ \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \mid k_1, k_2, k_3 \in R \right\} = R^3$

Exercise 2.1.24: For which Right hand side(find a condition on b_1, b_2, b_3) are these systems solvable?

(a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Aug. matrix $= [A|b] = \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right] R_1{+}R_3 \to R_3, R_2{-}2R_1 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_1 + b_3 \end{array}\right]$

i.e

$$1x_1 + 4x_2 + 2x_3 = b_1$$

$$0x_1 + 0x_2 + 0x_3 = b_2 - 2b_1$$

$$0x_1 + 0x_2 + 0x_3 = b_1 + b_3$$

solution exist only if $b_2 - 2b_1 = 0,\ b_1 + b_3 = 0$

$\Rightarrow b_2 = 2b_1$ and $b_3 = -b_1$

(b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Aug. matrix $= [A|b] = \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right] R_2 - 2R_1 \to R_2 \sim \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_1 + b_3 \end{array}\right]$

i.e

$$1x_1 + 4x_2 = b_1$$

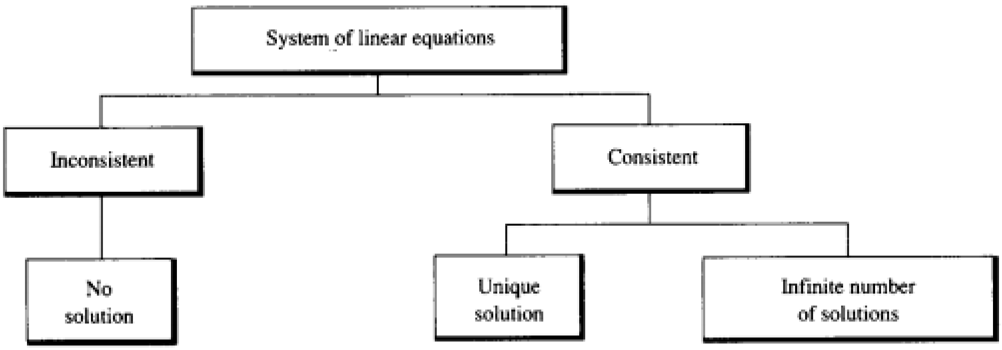
$$0x_1 + 1x_2 = b_2 - 2b_1$$

$$0x_1 + 0x_2 = b_1 + b_3$$

solution exist only if $b_3 + b_1 = 0$

2.2 - Solving Ax=0 And Ax=b

Vocab : Cofficient Mtrix,Augmented Matrix,EchelonForm,Row Reduced Form ,Rank,Pivot Variable,free Variable



Ex 1 - Consider a System of Linear Equation

$$\left.\begin{aligned} &x_1 - 2x_2 + x_3 = 0 \\ &2x_2 - 8x_3 = 8 \\ &-4x_1 + 5x_2 + 9x_3 = -9 \end{aligned}\right\} \tag{1}$$

Solution: The elimination procedure is shown here with and without matrix notation and the results are placed side by side for comparison:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

Keep x_1 is the first equation and eliminate it from the other equations. To do so, add 4 times equation 1 to equation 3. After some practice, this type of calculation is usually performed mentally:

$$\begin{array}{rc} 4.[\textit{equation } 1]: & 4x_1 - 8x_2 + 4x_3 = 0 \\ +[\textit{equation } 3]: & \underline{-4x_1 + 5x_2 + 9x_3 = -9} \\ \hline \textit{new equation } 3: & -3x_2 + 13x_3 = -9 \end{array}$$

The result of this calculation is written in place of the original third equation:

$$\begin{array}{r} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

Now, multiply equation 2 by 1/2 in order to obtain 1 as the coefficient for x_2. (This calculation will simplify the arithmatic in the next step.)

$$\begin{array}{r} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

Use the x_2 in equation 2 to eliminate the $-3x_2$ in equation 3. The "mental" computation is

$$\begin{array}{rc} 3.[\textit{equation } 2]: & 3x_2 - 12x_3 = 12 \\ +[\textit{equation } 3]: & \underline{-3x_2 + 13x_3 = -9} \\ \hline \textit{new equation } 3: & x_3 = 3 \end{array}$$

The new system has a triangular form.

$$\begin{array}{r} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Eventually, you want to eliminate the $-2x_2$ term from equation 1, but it is more efficient to use the x_3 in equation 3 first, to eliminate the $-4x_3$ and $+x_3$ terms in equation 2 and 1. The two "mental" calculations are

$$\begin{array}{rc} 4.[\textit{equation } 3]: & 4x_3 = 12 \\ +[\textit{equation } 2]: & \underline{x_2 - 4x_3 = 4} \\ \hline \textit{new equation } 2: & x_2 = 16 \end{array} \qquad \begin{array}{rc} -1.[\textit{equation } 3]: & -x_3 = -3 \\ +[\textit{equation } 1]: & \underline{x_1 - 2x_2 + x_3 = 0} \\ \hline \textit{new equation } 1: & x_1 - 2x_2 = -3 \end{array}$$

It is convenient to combine the results of these two operations:

$$\begin{array}{r} x_1 - 2x_2 = -3 \\ x_2 = 16 \\ x_3 = 3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Now, having cleaned out the column above the x_3 in equation 3, move back to the x_2 in equation 2 and use it to eliminate the $-2x_2$ above it. Because of the previous work with x_3, there is now no arithmetic involving x_3 terms. Add 2 times equations 2 to equation 1 and obtain the system:

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Ex 2 - A system of linear equations is a list of linear equations with the same unknowns. In particular, a system of 2 linear equations in 2 unknowns x_1, x_2can be put in the standard form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{aligned}\right\} \tag{2}$$

where a_{ij}, b_i are constant.and we can rewrite system (2) as :

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{3}$$

again we can rewrite system (3)(without using unknown,for the simplicity) to as

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right] \tag{4}$$

$$A = \left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right], C = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where A is called **Augmented Matrix** and C is called **Cofficient Matrix** of the system

$$\begin{aligned} \left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right] \frac{1}{a_{11}} R_1 \to R_1 \sim \left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ a_{21} & a_{22} & b_2 \end{array}\right] \\ R_2 - a_{21}R_1 \to R_2 \sim \left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & a_{22} - \frac{a_{21}a_{12}}{a_{11}} & b_2 - \frac{a_{21}b_1}{a_{11}} \end{array}\right] \end{aligned} \tag{5}$$

(5) can be written as

$$\left.\begin{aligned} & x_1 + \tfrac{a_{12}}{a_{11}}x_2 = \tfrac{b_1}{a_{11}} \\ & 0x_1 + (a_{22} - \tfrac{a_{21}a_{12}}{a_{11}})x_2 = b_2 - \tfrac{a_{21}b_1}{a_{11}} \end{aligned}\right\} \tag{6}$$

Case 1 : if $a_{22} - \frac{a_{21}a_{12}}{a_{11}} \neq 0$ then $x_2 = \frac{b_2 - \frac{a_{21}b_1}{a_{11}}}{a_{22} - \frac{a_{21}a_{12}}{a_{11}}}$, x_1 ca be calculated from $x_1 + \frac{a_{12}}{a_{11}}x_2 = \frac{b_1}{a_{11}}$
Conclusion- unique solution of (2)

Case 2 : if $a_{22}-\frac{a_{21}a_{12}}{a_{11}}=0$ and $b_2-\frac{a_{21}b_1}{a_{11}}=0$ then second equation of (7) becomes $0x_1+0x_2=0$
Conclution - infinite solution of (2)

Case 3 : if $a_{22}-\frac{a_{21}a_{12}}{a_{11}}=0$ and $b_2-\frac{a_{21}b_1}{a_{11}}\neq 0$ then second equation of (6)becomes $0x_1+0x_2=b_2-\frac{a_{21}b_1}{a_{11}}$ i.e. $0=b_2-\frac{a_{21}b_1}{a_{11}}$ which is not true
Conclution - solution does not exit of (2).

Elementary Row Operations

Supppose A is a matrix with rows R_1, R_2,..., R_m. The following operations on A are called elementary row operations.

$[E_1]$ (Row Interchange): Interchange rows R_i and R_j. This may be written as

"Interchange R_i and R_j" or "$R_i \leftrightarrow R_j$"

$[E_2]$ (Row Scaling): Replace row R_i by a nonzero multiple kR_i of itself. This may be written as

" Replace R_i by $kR_i(k\neq 0)$ or "$kR_i \rightarrow R_i$"

$[E_3]$ (Row Addition): Replace row R_j by the sum of a multiple kR_i of a row R_i and itself. This may be written as

" Replace R_j by kR_i+R_j" or "$kR_i+R_j\rightarrow R_j$.

The arrow $\rightarrow$ in E_2 and E_3 may be read as "replaces".

Sometimes (say to avoid fractions when all the given scalars are integers) we may apply $[E_2]$ and $[E_3]$ in one step; that is, we may apply the following operation:

$[E]$ Replace R_j by the sum of a multiple kR_i of a row R_i and a nonzero multiple $k'R_j$ of itself. This may ge written as

"Replace R_j by $kR_i+k'R_j$ $(k'\neq 0)$" or "$kR_i+k'R_j\rightarrow R_j$"

We emphasize that in operations $[E_3]$ and [E] only row R_j is changed.

Echelon Matrices (or in echelon form) U and Row Reduced Form R

Echelon Matrices U

A Matrix U is called an echelon matrix or is said to be in echelon form , if the following two conditions hold :

(1) All zero rows,if any, are at the bottem of the matrix.

(2) Each leading nonzero entry in a row is to the right of the leading nonzero entry in the preceding row.

Row Reduced Form R

A Matrix is said to be in row reduced form R if it is an echelon matrix and if satisfies the following additional two properties:

(3)Each pivot(leading nonzero entry) is equal to 1.

(4) Each pivot is the only nonzero entry in its column.

EX 3 The following is an echelon matrix whose pivots have been circled

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 & 5 & 9 & 0 & 7 \\ 0 & 0 & 0 & 3 & 4 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 5 & 7 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

NOTE 1 - The major difference between an echelon matrix in row reduced form is that in an echelon matrix there must be zeros below the pivots [properties(1)and (2)] but in a matrix in row reduced form , each pivot must also equal 1 [property (3)] and there must also be zeros above the pivots [properties(4)].

Ex-4 The following are echelon matrices whose pivots have been circled

$$\begin{bmatrix} 2 & 3 & 2 & 0 & 4 & 5 & -6 \\ 0 & 0 & 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The Third matrix is also an example of a matrix in row reduced form. the second matrix is not in row reduced form ,since it does not satisfy property(4),taht is,there is a nonzero entry above the second pivot in the third column.The first matrix is not in row reduced form, because it satisfies neither property (3) nor property (4); that is, some pivots are not equal to 1 and there are nonzero entries above the pivots.

Ex-5 The entries of a 5 by 8 echelon matrix U and its reduced form R

$$U = \begin{bmatrix} \bullet & * & * & * & * & * & * & * \\ 0 & \bullet & * & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} \mathbf{1} & \mathbf{0} & * & \mathbf{0} & * & * & * & \mathbf{0} \\ 0 & \mathbf{1} & * & \mathbf{0} & * & * & * & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{1} & * & * & * & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Pivot Variable and Free Variable</u>

Pivot Variable: Pivot Variable are those variable that correspond to columns with pivots.

Free Variable : Free Variable are those variable that correspond to columns without pivots.

Note : If $Ax = 0$ has more unknowns than equations ($n > m$), it has at least one special solution: There are more solutions than the trivial $x = 0$.

Note : $x_{complete} = x_{particlar} + x_{nullspace}$

Note : if there are n column in a matrix A and there are r pivots then there are r pivot variables and $n - r$ free variable.and this important number r is called **Rank** of a Matrix.

Rank of a Matrix = The rank of a matrix A, written rank(A), is equal to the maximum number of linearly independent columns of A

= number of pivot column in the echelon form of a matrix A

=maximum number of linearly independent rows of A

= dimension of the column space of A

= dimension of the row space of A.

Note : Let A be an n-square matrix. then A is invertible if and only if $rank(A) = n$

Ex 6: Find Rank of A

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Sol. Since Echelon form of A is itself A.and 1st and 3rd column are pivot column.

So Rank of A is 2.

Method for solving System of linear equation

Method-1

Ex 7 - Consider a System of linear equation

$$\left.\begin{aligned} 1x_1+2x_2+3x_3+5x_4=b_1\\ 2x_1+4x_2+8x_3 12x_4=b_2\\ 3x_1+6x_2+7x_3+13x_4=b_3 \end{aligned}\right\} \tag{7}$$

Sol.

Step 1: Reduce $Ax=b$ to $Ux=c$

i.e. Reduce Augmented Matrix [A b] to Augmented Matrix [U c]

$$\begin{bmatrix} A & b \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array}\right] R_2-2R_1 \to R_2, R_3-3R_1 \to R_3 \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2-2b_1 \\ 0 & 0 & -2 & -2 & b_3-3b1 \end{array}\right]$$
$$R_3+R_2 \to R_3 \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3+b_2-5b_1 \end{array}\right] \tag{8}$$
$$= \begin{bmatrix} U & c \end{bmatrix}$$

$$\left\{\begin{array}{l} (8) means \\ 1x_1+2x_2+3x_3+5x_4=b_1 \\ 0x_1+0x_2+2x_3+2x_4=b_2-2b_1 \\ 0x_1+0x_2+0x_3+0x_4=b_3+b_2-5b_1 \\ \textbf{third equation hold only if } b_3+b_2-5b_1=0 \\ \text{it means if } b_3+b_2-5b_1=0 \text{ then system of equation has infinite solution.} \\ \text{if } b_3+b_2-5b_1\neq 0 \text{ then system of equation has no solution.} \end{array}\right\}$$

Here

$$U=\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C=\begin{bmatrix} b_1 \\ b_2-2b_1 \\ b_3+b_2-5b_1 \end{bmatrix}$$

Step 2 :

Find Special Solution : $Ux=0$

Take particularly $b_1=0$,$b_2=6$,$b_3=-6$

$$\begin{bmatrix} U & 0 \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{9}$$

Here x_2 **and** x_4 **are free variables**

Let $x_2 = a$,$x_4 = b$ where a,b belongs to Set of Real Number

Now we can rewrite (10) as

$1x_1 + 2x_2 + 3x_3 + 5x_4 = 0$—–(*)

$0x_1 + 0x_2 + 2x_3 + 2x_4 = 0$—–(**)

now put the value of x_2 in (**)

$2x_3 + 2b = 0$

i.e. $x_3 = -b$

now put the value of x_3 in (*)

$x_1 + 2a - 3b + 5b = 0$

i.e. $x_1 + 2a + 2b = 0$

i.e. $x_1 = -2a - 2b$

Special Solution $x_n = \begin{bmatrix} -2a - 2b \\ a \\ -b \\ b \end{bmatrix}$

$x_n = a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ where a, b belongs to set of real number

Step 3 :

Find Particular Solution x_p**,** $Ux_p = c$ **and put all free variables**$= 0$

So put $x_2 = a = 0, x_4 = b = 0$

$$\begin{bmatrix} U & c \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{10}$$

(10) Can be rewritten as

$1x_1 + 2x_2 + 3x_3 + 5x_4 = 0$—–(*)

$0x_1 + 0x_2 + 2x_3 + 2x_4 = 6$—–(**)

Now put $b = 0$ in (**)

$2x_3 + 0 = 6$

$x_3 = 3$

Now $a = 0$,$x_3 = 3$,$b = 0$ in (*)

$x_10 + 9 + 0 = 0$

$x_1 = -9$

$$x_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Step 4 :

Complete Solution $x = x_n + x_p = a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2a - 2b - 9 \\ a \\ -b + 3 \\ b \end{bmatrix}$

$$= \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2a \\ a \\ 0a \\ 0a \end{bmatrix} + \begin{bmatrix} -2b \\ 0b \\ b \\ -b \end{bmatrix}$$

$$= \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

where a, b belongs to Set of real numbers

Method-2

Ex - Consider a System of linear equation

$$\left.\begin{aligned} 1x_1 + 2x_2 + 3x_3 + 5x_4 &= 0 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= 6 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 &= -6 \end{aligned}\right\} \tag{11}$$

Step 1: Reduce $Ax = b$ to $Ux = c$

i.e. Reduce Augmented Matrix [A b] to Augmented Matrix [U c]

$$\begin{bmatrix} A & b \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 2 & 4 & 8 & 12 & 6 \\ 3 & 6 & 7 & 13 & -6 \end{array}\right] R_2 - 2R_1 \to R_2, R_3 - 3R_1 \to R_3 \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & -2 & -2 & -6 \end{array}\right]$$
$$R_3 + R_2 \to R_3 \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{12}$$
$$= \begin{bmatrix} U & c \end{bmatrix}$$

$$\left\{\begin{array}{l} (12) means \\ 1x_1 + 2x_2 + 3x_3 + 5x_4 = 0 \\ 0x_1 + 0x_2 + 2x_3 + 2x_4 = 6 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 = 0 \end{array}\right\}$$

Here

$$U = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, c = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

Step 2:
Here x_2 **and** x_4 **are free variables**
Let $x_2 = a$,$x_4 = b$ where a,b belongs to Set of Real Number
Now we can rewrite (12) as
$1x_1 + 2x_2 + 3x_3 + 5x_4 = 0$——(*)
$0x_1 + 0x_2 + 2x_3 + 2x_4 = 6$——(**)
now put the value of x_2 in (**)
$2x_3 + 2b = 6$
i.e. $x_3 = 3 - b$
now put the value of x_3 in (*)
$x_1 + 2a + 3(3 - b) + 5b = 0$
i.e. $x_1 + 2a + 9 + 2b = 0$
i.e. $x_1 = -9 - 2a - 2b$

Complete Solution $x = x_n + x_p = \begin{bmatrix} -2a - 2b - 9 \\ a \\ -b + 3 \\ b \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2a \\ a \\ 0a \\ 0a \end{bmatrix} + \begin{bmatrix} -2b \\ 0b \\ b \\ -b \end{bmatrix}$

$= \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$

where a, b belongs to Set of real numbers

Exercise 2.2.1 : find the value of c that makes it possible to solve $Ax = b$, and solve it:

$$u + v + 2w = 2$$

$$2u + 3v - w = 5$$

$$3u + 4u + w = c$$

Solution Aug matrix $= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 3 & -1 & 5 \\ 3 & 4 & 1 & c \end{array}\right] R_2 - 2R_1 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 1 & -5 & c \end{array}\right] R_3 - R_2 \to R_2 \sim$ $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & c-7 \end{array}\right]$ Solution Exit only if $c - 7 = 0$ so assume $w = k \in R$

$$v - 5w = 1$$

$$v - 5k = 1$$

$$v = 1 + 5k$$

$$u + v + 2w = 2$$

$$u + (1 + 5k) + 2k = 2$$

$$u = 1 - 7k$$

$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1-7k \\ 1+5k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + k \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}$ where $k \in R$

Exercise 2.2.4 Write the complete solution $x = x_p + x_n$ to these systems ,(as in equation (4))

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix},$$

Solution (1) Aug matrix $= \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 4 & 5 & 4 \end{array}\right] R_2 - 2R_1 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$
Since v is free variable so take $v = k$, $k \in R$

$$w = 2$$

$$u + 2v + 2w = 1$$

$$u + 2k + 4 = 1$$

$$u = -3 - 2k$$

$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -3-2k \\ k \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + k \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, where $k \in R$

(2) Aug matrix $= \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 4 & 4 & 4 \end{array}\right] R_2 - 2R_1 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$
i.e.

$$u + 2v + 2w = 1$$

$$0u + 0v + 0w = 2$$

i.e. $0 = 2$ which is not true.
So there is no solution.

Exercise 2.2.5 Reduce A and B to echelon form, to find their ranks, which variables are free ?
$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ find the special solutions to $Ax = 0$ and $Bx = 0$. find all solutions.

Solution:(1) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} R_3 - R_1 \to R_2 \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$. Since first two column in U are L.I. So $\rho(A) = 2$.
Now for solving $Ax = 0$.
Aug. matrix $= [A|0] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \end{array}\right] R_3 - R_1 \to R_3 \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$
Since x_3 and x_4 are free variable; so assume $x_3 = k_1$, $x_4 = k_2$, where $k_1, k_2 \in R$
$x_2 + x_3 = 0 \Rightarrow x_2 + k = 0 \Rightarrow x_2 = -k$

$$x_1 + 2x_2 + x_4 = 0$$

$$x_1 - 2k_1 + k_2 = 0$$

$$x_1 = 2k_1 - k_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2k_1 - k_2 \\ -k_1 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $k_1, k_2 \in R$.

This is general solution.

Hence special solutions are $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(2) $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} R_3 - 7R_1 \to R_3, R_2 - 4R_1 \to R_2 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} R_3 - 2R_2 \to R_3 \sim$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} = U$
Since U has two pivot columns, so $\rho(B) = 2$.
for solving $Bx = 0$
Aug. matrix $= [B|0] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] R_3 - 7R_1 \to R_3, R_2 - 4R_1 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] R_3 -$ $2R_2 \to R_3 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$
Since x_3 is free variable.
So $x_3 = k, k \in R$.

$$-3x_2 - 6x_3 = 0$$

$$x_2 = -2k$$

$$x_1 + 2x_2 + 3x_3 = 0$$

$$x_1 - 4k + 3k = 0$$

$$x_1 - k = 0 \Rightarrow x_1 = k.$$

General solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, k \in R$. Special solution is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

PROBLEM SET 2.2 CONTINUED...

Exercise 2.2.13: (a) find the special solutions to $Ux = 0$, Reduce U to R and repeat

$$Ux = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) If the Right hand side is changed from (0,0,0) to (a,b,0) what is solution?

Solution: (a) Aug matrix $= [U|0] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Since x_2, x_4 are free variables. So assume $x_2 = a$, $x_4 = b$, $a, b \in R$.

$$x_3 + 2x_4 = 0$$

$$x_3 = -2b$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

$$x_1 - 2a - 6b + 4b = 0$$

$$x_1 - 2a - 2b = 0$$

$$x_1 = 2a + 2b.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2a + 2b \\ a \\ -2b \\ b \end{bmatrix} = a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}, a, b \in R.$$

Since $U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 - 3R_2 \to R_1 \sim \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$

(b) $Ux = b$

Aug matrix $= [U|b] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & a \\ 0 & 0 & 1 & 2 & b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Since x_2, x_4 are free variables. So assume $x_2 = c$, $x_4 = d$, $c, d \in R$.

$$x_2 + 2x_4 = b$$

$$x_3 + 2d = b \Rightarrow x_3 = b - 2d$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = a$$

$$x_1 + 2c + 3(b - 2d) + 4d = a$$

$$x_1 + 2c + 3b - 6d + 4d = a$$

$$x_1 + 2c + 3b - 2d = a$$

$$x_1 = a - 3b - 2c + 2d$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a - 3b - 2c + 2d \\ c \\ b - 2d \\ d \end{bmatrix} = \begin{bmatrix} a - 3b \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}, c, d \in R.$$

Exercise 2.2.34: What conditions on b_1, b_2, b_3, b_4 make each system solvable? solve for x,
$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$

Solutions:(a) Aug. matrix $= \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{array}\right]$ $R_4 - 3R_1 \to R_4, R_2 - 2R_1 \to R_2, R_3 - 2R_1 \to R_3 \sim$

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 3 & b_4 - 3b_1 \end{array}\right]$$

$R_4 - 3R_3 \to R_3 \sim \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 0 & b_4 - 3b_3 + 3b_1 \end{array}\right]$

solution exist only if $\begin{array}{ll} b_2 - 2b_1 = 0, & b_4 - 3b_3 + 3b_1 = 0 \\ b_2 = 2b_1, & b_4 = b_1 + b_3 \end{array}$

Since
$$x_2 = b_3 - 2b_1$$

$$x_1 + 2x_2 = b_1$$

$$x_1 + 2(b_3 - 2b_1) = b_1$$

$$x_1 + 2b_3 - 4b_1 = b_1$$

$$x_1 = 5b_1 - 2b_3$$

General solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix}$

(b)Aug. matrix $= \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{array}\right]$ $R_4 - 3R_1 \to R_4, R_2 - 2R_1 \to R_2,$

$R_3 - 2R_1 \to R_3 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \end{array}\right]$ $R_4 - 3R_3 \to R_4 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 0 & 0 & b_4 - 3b_3 + 3b_1 \end{array}\right]$

solution exist only if $b_2 - 2b_1 = 0$ and $b_4 - 3b_3 + 3b_1 = 0$ for the general solution.

Since x_3 is free variable assume $x_3 = k, k \in R$.

$$x_2 + k = b_3 - 2b_1$$

$$x_2 = b_3 - 2b_1 - k$$

$$x_1 + 2x_2 + 3x_3 = b_1$$

$$x_1 + 2(b_3 - 2b_1 - k) + 3k = b_1$$

$$x_1 + 2b_3 - 4b_1 - 2k + 3k = b_1$$

$$x_1 = 5b_1 - 2b_3 - k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 - k \\ b_3 - 2b_1 - k \\ k \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + k \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

PROBLEM SET 2.2 CONTINUED...

Excercise 2.2.44 Choose the number q so that (if possible) the ranks (a) 1 (b) 2 (c) 3

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}, C = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}$$

Solution (1) $A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} 2R_2 + R_1 \to R_2, 6R_3 - 9R_1 \to R_3 \sim \begin{bmatrix} 6 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 6q-18 \end{bmatrix}$
$R_3 \leftrightarrow R_2 \sim \begin{bmatrix} 6 & 4 & 2 \\ 0 & 0 & 6q-18 \\ 0 & 0 & 0 \end{bmatrix}$

(a) rank of A is 1 if $q = 3$
(b) rank of A is 2 if $q \neq 3$
(c) rank of A is 3 not possible.

(2) $B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix} 3R_2 - qR_1 \to R_2 \sim \begin{bmatrix} 3 & 1 & 3 \\ 0 & 6-q & 0 \end{bmatrix}$

(a) rank of B is 1 if $q = 6$
(b) rank of B is 2 if $q \neq 6$
(c) rank of B is 3 not possible.

Excercise 2.2.54 : True or False? (Give reason if true, or counterexample to show it is false.)
(a) A square matrix has no free variables.
(b) An invertible matrix has no free variables.
(c) An m by n matrix has no more than n pivot variables.
(d) An m by n matrix has no more than m pivot variables.
Solution :
(a) A square matrix has no free variables.
Ans. False , because $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

(b) An invertible matrix has no free variables.

Ans. True , a matrix A is invertible if and only if its columns are linearly independent.so all column has pivot.so invertible matrix has no free variables.

(c) An m by n matrix has no more than n pivot variables.

Ans. True , Since n is number of columns. and each column has atmost 1 pivot. An m by n matrix has no more than n pivot variables.

(d) An m by n matrix has no more than m pivot variables.

Ans. True , Since m is number of rows. and each row has atmost 1 pivot.So An m by n matrix has no more than m pivot variables.

Excercise 2.2.59: The equation $x - 3y - z = 0$ determines a plane in R^3. What is the matrix A in this equation? Which are the free variables? The special solutions are(3,1,0) and (...), the parallel plane $x - 3y - z = 12$ contains the particular point (12,0,0) all points on this plane have the following form (fill in the first component).

Solution: Since $\begin{array}{c} x - 3y - z = 0 \\ x = 3y + z, \quad y, z \in R \end{array}$ Consider

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y + z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence special solutions are $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ We have to find a matrix A whose special solutions are $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Consider $A \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = 0$

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$3a_{11} + a_{12} = 0 \Rightarrow a_{12} = -3a_{11}$$

$$3a_{21} + a_{22} = 0 \Rightarrow a_{22} = -3a_{21}$$

$$3a_{31} + a_{32} = 0 \Rightarrow a_{32} = -3a_{31}$$

Consider $A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$a_{11} + a_{13} = 0 \Rightarrow a_{13} = -a_{11}$$

$$a_{21} + a_{23} = 0 \Rightarrow a_{23} = -a_{21}$$

$$a_{31} + a_{33} = 0 \Rightarrow a_{33} = -a_{31}$$

So $A = \begin{bmatrix} a_{11} & -3a_{11} & -a_{11} \\ a_{21} & -3a_{21} & -a_{21} \\ a_{31} & -3a_{31} & -a_{31} \end{bmatrix}$

given parallel plane is $x - 3y - z = 12$

$$x = 3y + z + 12.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 + 3y + z \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Linearly independent and dependent

Linearly independent
A subset $\{v_1, v_2, ...v_n\}$ of a vector space V is said to be linearly independent if whenever $c_1, c_2, ...c_n \in R$ such that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$ then $c_1 = c_2 = ... = c_n = 0$

Linearly dependent
A non empty finite subset $\{v_1, v_2, ...v_n\}$ of a vector space V is said to be linearly dependent if there exists scalars $c_1, c_2, ...c_n \in R$ (**not all zero**)such that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$

Ex 1 if v_1 = zero vector , then the set is linearly dependent . we may choose $c_1 = 1$ and all other $c_i = 0$, this is a non trivial combination that produces zero.
i.e. $1v_1 + 0v_2 + ... + 0v_n = 1 \times 0 + 0 + ... + 0 = 0$

Ex 2 : The Column of the Matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

are linearly dependent , since the 2nd column is 3 times the first,the combination of columns with weights -3,1,0,0 gives the zero vector. i.e. say $A = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \end{bmatrix}$, then $-3C_1 + 1C_2 + 0C_3 + 0C_4 = 0$
The rows are also linearly dependent, row 3 is two times row 2 minus five times row1. i.e. say $A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$, then $R_3 - 2R_2 + 5R_1 = 0$

EX 3
The Column of this Triangular Matrix are Linearly Independent

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

Consider a linear combination of the columnsthat makes zero
Solve $Ac = 0$

$$c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

it means $3c_1 + 4c_2 + 2c_3 = 0$, $0c_1 + 1c_2 + 5c_3 = 0$, $0c_1 + 0c_2 + 2c_3 = 0$ i.e.

$$c_3 = 0, c_2 = 0, c_1 = 0$$

So column of A are Linearly Depandent.
and **null space of A contains only zero vector**

A similar reasoning applies to the rows of A, which are also independent. Suppose

$$c_1(3,4,2) + c_2(0,1,5) + c_3(0,0,2) = (0,0,0)$$

. From the first components we find $3c_1 = 0$ or$c_1 = 0$. Then the second components give $c_2 = 0$, and finally $c_3 = 0$.

Note : The columns of A are independent exactly when $N(A) = \{zerovector\}$
Note : It is the columns with pivots that are guaranteed to be independent

Ex 4 The columns of the n by n identity matrix are independent:

$$I = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1 & . & 0 \\ . & . & . & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note : To check any set of vectors $v_1, ..., v_n$ for independence, put them in the columns of A.Then solve the system $Ac = 0$;
1. The vectors are dependent if there is a solution other than c = 0.
2. With no free variables (rank n), there is no nullspace except c = 0; (i.e. $N(A) = \{0\}$)the vectors are independent.
3. If the rank is less than n, at least one free variable can be nonzero and the columns are dependent.

Note :A set of n vectors in R^m must be linearly dependent if $n > m$.

Ex 5 These three column in R^2 can not be independent:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

Sol : To find the combination of the columns producing zero we solve $Ac = 0$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} R_2 - R_1 \to R_2 \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = U$$

If we give the value 1 to the free variable c_3, then back-substitution in $Uc = 0$ gives $c_2 = -1$, $c_1 = 1$
i.e. if $A = \left[C_1, C_2, C_3\right]$then $C_1 - C_2 + C_3 = 0$

Exercise 2.3.1: Choose three independent columns of V, then make two other choices. Do the same for A. You have found bases for which spaces?
$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}$

Solution: Let $U = [U_1 \;\; U_2 \;\; U_3 \;\; U_4]$ $A = [C_1 \;\; C_2 \;\; C_3 \;\; C_4]$ Consider, $A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}$ $R_4 - 2R_1 \rightarrow$

$R_4 \sim \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$

i.e. U is echelon form of A.
Note: Columns of A which have pivot are linearly independent.
Case (i) U_1, U_2, U_4 are L.I.(using the note).
Case (ii) U_1, U_3, U_4 are L.I.
as consider $aU_1 + bU_3 + cU_4 = 0$

$$a \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 4 \\ 7 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a + 4b + c \\ 0a + 7b + 0c \\ 0a + 0b + 9c \\ 0a + 0b + 0c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 9c = 0 \\ 7b = 0 \\ 2a + 4b + c = 0 \end{matrix} \Rightarrow \begin{matrix} c = 0 \\ b = 0 \\ a = 0 \end{matrix}$$

Case (iii) U_1, U_3, U_4 are L.I.
as consider $aU_2 + bU_3 + cU_4 = 0$

$$a \begin{bmatrix} 3 \\ 6 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 4 \\ 7 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3a + 4b + c \\ 6a + 7b + 0c \\ 0a + 0b + 9c \\ 0a + 0b + 0c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 9c = 0 \\ 6a + 7b = 0 \\ 3a + 4b + c = 0 \end{matrix} \Rightarrow \begin{matrix} c = 0 \\ b = 0 \\ a = 0 \end{matrix}$$

Note: Columns of a matrix A are linearly independent which are corresponding to the pivot column of echelon matrix of A.
Case (i) C_1, C_2, C_4 are L.I.(using the note).
Case (ii) C_1, C_3, C_4 are L.I.
as we can see consider $aC_1 + bC_3 + cC_4 = 0$

$$a\begin{bmatrix}2\\0\\0\\4\end{bmatrix} + b\begin{bmatrix}4\\7\\0\\8\end{bmatrix} + c\begin{bmatrix}1\\0\\9\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}2&4&1\\0&7&0\\0&0&9\\4&8&2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad i.e \;\; S\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Aug. matrix=$[S|0]$ $\left[\begin{array}{ccc|c}2&4&1&0\\0&7&0&0\\0&0&9&0\\4&8&2&0\end{array}\right] R_4 - 2R_1 \to R_4 \sim \left[\begin{array}{ccc|c}2&4&1&0\\0&7&0&0\\0&0&9&0\\0&0&0&0\end{array}\right]$

$$\begin{aligned} 2a+4b+c&=0 & c&=0\\ \Rightarrow \quad 7b&=0 \quad \Rightarrow & b&=0\\ 9c&=0 & a&=0\end{aligned}$$

Case (iii) C_2, C_3, C_4 are L.I.
consider $aC_2 + bC_3 + cC_4 = 0$

$$\begin{bmatrix}C_2 & C_3 & C_4\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad say \;\; B\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Aug. matrix=$[B|0]$ $\left[\begin{array}{ccc|c}3&4&1&0\\0&6&7&0\\0&0&9&0\\6&8&2&0\end{array}\right] R_4 - 2R_1 \to R_4 \sim \left[\begin{array}{ccc|c}3&4&1&0\\0&6&7&0\\0&0&9&0\\0&0&0&0\end{array}\right]$

$$\begin{aligned} 3a+4b+c&=0 & c&=0\\ \Rightarrow \quad 6b+7c&=0 \quad \Rightarrow & b&=0\\ 9c&=0 & a&=0\end{aligned}$$

The all three cases, we found bases for $R^{4\times 3}$ space.

Exercise 2.3.3 : Decide the dependence or independence of
(a) the vectors (1,3,2), (2,1,3) and (3,2,1)
(b) the vectors (1,3,-2), (2,1,-3) and (-3,2,1).
Solution:
(a)

$$a\begin{bmatrix}1\\3\\2\end{bmatrix} + b\begin{bmatrix}2\\1\\3\end{bmatrix} + c\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 3\\3 & 1 & 2\\2 & 3 & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad say \;\; A\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Aug. matrix=$[A|0]$ $\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\3 & 1 & 2 & 0\\2 & 3 & 1 & 0\end{array}\right] R_3 - 2R_1 \to R_3, R_2 - 3R_1 \to R_2 \sim \left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\0 & -5 & -7 & 0\\0 & -1 & -5 & 0\end{array}\right]$

$-5R_3 + R_2 \to R_3 \sim \left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\0 & -5 & -7 & 0\\0 & 0 & 18 & 0\end{array}\right]$

$$\Rightarrow \begin{array}{r}1a + 2b + 3c = 0\\-5b - 7c = 0\\18c = 0\end{array} \quad \Rightarrow \begin{array}{c}c = 0\\b = 0\\a = 0\end{array}$$

Vectors are L.I.

(b) Consider $1(1, -3, 2) + 1(2, 1, -3) + 1(-3, 2, 1) = (1 + 2 - 3, -3 + 1 + 2, 2 - 3 + 1) = (0, 0, 0)$
$\Rightarrow$ Vectors are L.D.

Exercise 2.3.5 : If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$, $v_2 = w_1 - w_3$ and $v_3 = w_1 - w_2$ are dependent. find a combination of $v's$ gives zero.
Solution: Consider $av_1 + bv_2 + cv_3 = 0$
$a(w_2 - w_3) + b(w_1 - w_3) + c(w_1 - w_2) = 0$
$(b + c)w_1 + (a - c)w_2 + (-a - b)w_3 = 0$
Since w_1, w_2, w_3 are L.I.
So $b + c = 0$, $a - c = 0$, $-a - b = 0$
$b = -c$, $a = c$, $b = -a$
$a = -b = c$ take $a = 1$, $b = -1$, $c = 1$
$v_1 - v_2 + v_3 = 0$

Exercise 2.3.8 : Suppose v_1, v_2, v_3, v_4 are vectors in R^3.
(a) thse four vectors are dependent because ......
(b) The two vector v_1 and v_2 will be dependent if .....
(c) The vectors v_1 and (0,0,0) are dependent because .....

Solution: (a) Since $dim(R^3) = 3$
Therefore each base of R^3 contains exactly 3 vectors.
So collection of vectors which are more than 3 are linearly dependent.
So four vectors are L.D.
(b) Let $av_1 + bv_2 = 0$ for $\{v_1, v_2\}$ should be dependent.
So atleast one of a or b is nonzero.
say $a \neq 0$
So $v_1 = \frac{-b}{a}v_2$ So v_1, v_2 are dependent if $\exists \alpha \neq 0$ s.t. $v_1 = \alpha v_2$
(C) Consider $0.v_1 + 1(0, 0, 0) = (0, 0, 0)$ $a = 0, b = 1 \neq 0$
So v_1 and (0,0,0) are L.D .

Lecture 16
2.3 Linear Independence, Basis and Dimension

Course Outcomes: Students will have understanding about linear independence,dependence, spanning a subspace, basis and dimension of a vector space.
The aim of this section is to explain and use four ideas:

- Linear Independence or dependence.
- spanning a subspace.
- basis for a subspace.
- dimension of a subspace.

Spanning a Subspace: If $S = \{w_1, w_2, ..., w_l\}$ is a set of vectors in a vector space V, then the **span of S** is the set of all linear combinations of the vectors in S.

$$Span(S) = \{c_1w_1 + c_2w_2 + ... + c_lw_l \mid for\ all\ c_i \in R\}$$

If every vector in a given vector space can be written as the linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.

Notes:

- The column space is spanned by its columns.
- The row space is spanned by its rows.

Basis for a Vector Space: A basis for a vector space V is a subset with a sequence of vectors having two properties at once:

- The vectors are linearly independent.(not too many vectors)
- They span the space V.(not too few vectors)

Notes:

- A basis of a vector space is the maximal independent set.
- A basis of a vector space is also a minimal spanning set.
- Spanning involves the column space and independence involves the null space.
- No elements of a basis will be wasted.

Example: Check whether the following sets are the basis of R^3 or not?

$$\begin{aligned}
(a) B_1 &= \{(1,2,2),(-1,2,1),(0,8,0)\}\\
(b) B_2 &= \{(1,2,2),(-1,2,1),(0,8,6)\}\\
(c) B_3 &= \{(1,2,2),(-1,2,1)\}\\
(d) B_4 &= \{(1,0,0),(0,1,0),(0,0,1)\}\\
(e) B_5 &= \{(1,1,-1),(2,3,4),(4,1,-1),(0,1,-1)\}
\end{aligned}$$

Solution:
(a)

$$\begin{aligned}
Let\ A &= \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 0 \end{bmatrix}\\
\Rightarrow |A| &= -24\\
\Rightarrow |A| &\neq 0
\end{aligned}$$

The vectors $(1,2,2),(-1,2,1)$ and $(0,8,0)$ are LI.
So, B_1 is a basis of R^3.

Dimension of Vector Spaces: Dimension of a vector space is the maximum number of LI vectors of the vector space.

OR. The no. of elements present in the basis of a vector space is known as the dim. of vector space.

$$dimR = 1, dimR^2 = 2, dimR^3 = 3, ...dimR^n = n$$

Problem Set-2.3

16. Describe the subspace of R^3 (is it a line or a plane or R^3?) spanned by
(a) the two vectors $(1, 1, -1)$ *and* $(-1, -1, 1)$.
(b) the three vectors $(0, 1, 1)$ *and* $(1, 1, 0)$ *and* $(0, 0, 0)$.
(c) the columns of a 3 by 5 echelon matrix with 2 pivots.
(d) all vectors with positive components.
Ans.(a)

$$\begin{aligned} Let\ A &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 + R_1 \end{array} \\ \Rightarrow A &= \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (echelon form) \end{aligned}$$

Here rank of A=1.
$\therefore$ The subspace of R^3 spanned by the two vectors $(1, 1, -1)$ *and* $(-1, -1, 1)$ is a line passing through the origin of R^3.
(b)

$$Let\ A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Interchanging R_1 by R_2 we have,

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \leftarrow R_3 - R_1 \end{array} \\ \Rightarrow A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \leftarrow R_3 + R_2 \end{array} \\ \Rightarrow A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (echelon form) \end{aligned}$$

Here rank of A=2.
$\therefore$ The subspace of R^3 spanned by the three vectors $(0,1,1)$ *and* $(1,1,0)$ *and* $(0,0,0)$ is a plane of R^3.
(c)

$$Let\ A \quad = \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here rank of A=2.
$\therefore$ The subspace of R^3 spanned by the the columns of a 3 by 5 echelon matrix with 2 pivots is a plane.

19. Find a basis for the plane $x - 2y + 3z = 0$ in R^3. Then find a basis for the intersection of that plane with the xy-plane. Then find a basis for all vectors perpendicular to the plane.
Ans. The basis for the plane $x - 2y + 3z = 0$ in R^3 is the nullspace of the matrix $A = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$.
The plane $x - 2y + 3z = 0$ in the matrix form can be written as

$$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Here x is the pivot variable and y, z are the free variables.
So, $x = 2y - 3z$

$$\therefore x = \begin{bmatrix} 2y - 3z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Basis for the plane is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

A basis for the intersection of the plane $x - 2y + 3z = 0$ with the xy-plane i.e z=0 is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

A basis for all vectors perpendicular to the plane is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}$

23 Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspace.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

Ans.

$$\begin{aligned} Let\ A &= \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \leftarrow R_3 - R_1 \end{matrix} \\ \Rightarrow A &= \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (echelon form) \end{aligned}$$

$$Basis\ for\ C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$Basis\ for\ C(A^T) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

To find basis for nullspace we have,

$$\begin{aligned} Ax &= 0 \\ \Rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow x + 3y + 2z = 0, y + z = 0$$

$$Hence\ x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$Basis\ for\ N(A) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Lecture 17

2.4 The Four Fundamental Subspaces

Course Outcomes: Students will have understanding about four fundamental subspaces of matrices and one sided inverse of rectangular matrices. The four fundamental subspaces of matrices are as follows :

- The column space $C(A)$.
- The null space $N(A)$.
- The row space $C(A^T)$.
- The left null space $N(A^T)$.

The Column Space $C(A)$: The column space of A is denoted by C(A). Its dimension is the rank r.

The Null Space $N(A)$: The nullspace of A is denoted by N(A). Its dimension is $n - r$.

The Row Space $C(A^T)$: The row space of A is the column space of A^T. It is $C(A^T)$, and it is spanned by the rows of A. Its dimension is also r.

The Left Null Space $N(A^T)$: The left nullspace of A is the nullspace of A^T. It contains all vectors y such that $A^T y = 0$, and it is written $N(A^T)$. Its dimension is $m - r$.

Notes

- The nullspace N(A) and row space $C(A^T)$ are subspaces of R^n.
- The left nullspace $N(A^T)$ and column space C(A) are subspaces of R^m.

Existence of Inverses:
let A be a matrix of order $m \times n$ with rank r.
(1) Full row rank r = m. Ax = b has at least one solution x for every b if and only if the columns span R^m. Then A has a right-inverse C such that $AC = I_m$ (m by m). This is possible only if $m \leq n$.

(2) Full column rank r = n. Ax = b has at most one solution x for every b if and only if the columns are linearly independent. Then A has an n by m left-inverse B such that $BA = I_n$. This is possible only if $m \geq n$.

Notes: One-sided inverses are $B = (A^TA)^{-1}A^T \; and \; C = A^T(AA^T)^{-1}$

Example. Find a left-inverse and/or a right-inverse (when they exist) for

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

Ans.

$$Let\ A = \begin{bmatrix} \boxed{4} & 0 & 0 \\ 0 & \boxed{5} & 0 \end{bmatrix}$$

The matrix A is already in Echelon form with 2 pivots.
So, rank of A=r=2.
Here m=2 and n=3.
So, m=r=2
$\Rightarrow A$ is a full row rank matrix.
$\Rightarrow$ right inverse C of A will exist and is given by

$$\begin{aligned} C &= A^T(AA^T)^{-1} \\ \Rightarrow C &= \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Lecture 18
2.4 The Four Fundamental Subspaces

Problem Set-2.4

2. Find the dimension and a basis for the four fundamental subspaces for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

Ans.

$$\begin{aligned} Let\ A &= \begin{bmatrix} 1 & \boxed{2} & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \leftarrow R_3 - R_1 \end{matrix} \\ \Rightarrow A &= \begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (echelon form) \end{aligned}$$

Rank of $A = r = 2$
Here $m = 3,\ n = 3$
$dimC(A) = r = 2$
$dimC(A^T) = r = 2$
$dimN(A) = n - r = 2$
$dimN(A^T) = m - r = 2$

The Column Space $C(A)$:

$$Basis\ for\ C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

The Null Space $N(A)$:

$$\begin{aligned} Ax &= 0 \\ \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow x_1 + 2x_2 + x_4 = 0, x_2 + x_3 = 0$$

$$\begin{aligned} Hence\ x &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ &= \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} \\ &= x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$Basis\ for\ N(A) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The Row Space $C(A^T)$:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
Let\ A^T &= \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \leftarrow R_2 - 2R_1 \\ R_3 \\ R_4 \leftarrow R_4 - R_1 \end{matrix} \\
\Rightarrow A^T &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \leftarrow R_3 - R_2 \\ R_4 \end{matrix} \\
\Rightarrow A^T &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (echelonform)
\end{aligned}$$

$$Basis\ for\ C(A^T) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

The Leftnull Space $N(A^T)$:

$$\begin{aligned}
A^T y &= 0 \\
\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

$$\Rightarrow y_1 + y_3 = 0, y_2 = 0$$

$$
\begin{aligned}
Hence\ y &= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\
&= \begin{bmatrix} -y_3 \\ 0 \\ y_3 \end{bmatrix} \\
&= y_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
Basis\ for\ N(A^T) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}
$$

13. Find a basis for each of the four subspaces of

$$
A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}
$$

Ans.

$$
\begin{aligned}
Let\ A &= \begin{bmatrix} 0 & \boxed{1} & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \leftarrow R_2 - R_1 \\ R_3 \end{matrix} \\
\Rightarrow A &= \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \\
\Rightarrow A &= \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$\begin{aligned}
Basis\ for\ C(A) &= \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\} \\
Basis\ for\ C(A^T) &= \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 6 \end{bmatrix} \right\} \\
Basis\ for\ N(A) &= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} \\
Basis\ for\ N(A^T) &= \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}
\end{aligned}$$

18. Find a 1 by 3 matrix whose nullspace consists of all vectors in R^3 such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace.
Ans. A 1 by 3 matrix whose nullspace consists of all vectors in R^3 such that $x_1 + 2x_2 + 4x_3 = 0$ is $A = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$.
Since $x_1 + 2x_2 + 4x_3 = 0$ in the matrix form can be written as

$$\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

A 3 by 3 matrix with that same nullspace is $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 12 \end{bmatrix}$

24. Construct a matrix with the required property, or explain why you can't.

(a) Column space contains $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, row space contains $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

(b) Column space has basis $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, nullspace has basis $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

(c) Dimension of nullspace = 1+ dimension of left nullspace.
(d) Left nullspace contains $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, row space contains $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
(e) Row space = column space, nullspace $\neq$ left nullspace.
Ans.
(a) A matrix with the required property is given by

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Impossible: dimensions $1 + 1 \neq 3$.
(c) A matrix with the required property is given by

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

(d) A matrix with the required property is given by

$$A = \begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$$

(e) Impossible: Row space = column space requires m = n.
Then $m - r = n - r$.

Lecture 19
Chapter 3: Orthogonality

3.1 Orthogonal Vectors and Subspaces

Course Outcomes: Students will be acquainted with orthogonal vectors, orthonormal vectors, orthonormal subspaces, and orthogonal compliment of subspaces.

Length of a vector: It is denoted by $||x||$.
Let $x = (x_1, x_2)$.
Length in 2D = $||x|| = \sqrt{x_1^2 + x_2^2}$.
Length squared=$||x||^2 = x_1^2 + x_2^2$.
Let $x = (x_1, x_2, x_3)$.
Length in 3D = $||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.
Length squared=$||x||^2 = x_1^2 + x_2^2 + x_3^2$
Let $x = (x_1, x_2, ..., x_n)$.
Length in R^n = $||x|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$.
Length squared=$||x||^2 = x_1^2 + x_2^2 + ... + x_n^2$

Inner Product:
Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then the inner product of two vectors x and y is denoted by $x^T y$ and defined as $x^T y = x_1 y_1 + x_2 y_2$
Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then $x^T y = x_1 y_1 + x_2 y_2 + x_3 y_3$
Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$.
Then $x^T y = x_1 y_1 + x_2 y_2 + ... + x_n y_n$

Note:

$$x^Tx = \begin{pmatrix} x_1 & x_2 & ......x_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ ...... \\ x_n \end{pmatrix}$$

$$= x_1^2 + x_2^2 + ... + x_n^2 = ||x^2||$$

Hence inner product of a vector with itself is equal to the length square of the vector.

- The inner product x^Ty is zero if and only if x and y are orthogonal vectors.
- If $x^Ty > 0$, their angle is less than $\deg 90$. If $x^Ty < 0$, their angle is greater than $\deg 90$.
- The only vector with length zero and the only vector orthogonal to itself is the zero vector.

Orthogonal Vectors:
Two vectors x and y are said to be orthogonal iff $x^Ty = 0$

Orthogonal Subspaces:
Two subspaces V and W of the same space R^n are orthogonal if every vector v in V is orthogonal to every vector w in W i.e $v^Tw = 0$ for all v and w.

Examples

- x-axis and y-axis are subspaces of R^2 and every vector of x-axis is orthogonal to every vector in y-axis. So, x-axis $\perp$ y-axis in R^2
- $y = x$ line $\perp$ $y = -x$ line in R^2.
- All the three axes in R^3 are orthogonal to each other.

Notes

- The subspace $\{0\}$ is orthogonal to all subspaces.
- A line can be orthogonal to another line, or it can be orthogonal to a plane, but a plane cannot be orthogonal to a plane.

Fundamental theorem of orthogonality: The row space is orthogonal to the nullspace (in R^n). The column space is orthogonal to the left nullspace (in R^m).

Orthogonal Complement Given a subspace V of R^n, the space of all vectors orthogonal to V is called the orthogonal complement of V. It is denoted by $V^{\perp}$ = "V perp."

Examples

- x-axis is the orthogonal complement of y-axis in R^2.
- $y = x$ line is the orthogonal complement of $y = -x$ line in R^2.
- x-axis is the orthogonal complement of yz-plane in R^3.

Fundamental Theorem of Linear Algebra: The nullspace is the orthogonal complement of the row space in R^n. The left nullspace is the orthogonal complement of the column space in R^m.