2.2 Rank of Matrices

Rank of a matrix: The number of pivot elements in the echelon form of a matrix is known as rank of the matrix.

Ex: $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 + R_1$

$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ $R_3 \leftarrow R_3 + R_2$

, which is the echelon form.

∴ The no. of pivot elements in the echelon form is 3. So, rank of $A = 3$.

Ex: $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 + R_1$

$= \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \leftarrow R_3 - 2R_2$

, is the echelon form.

Since the no. of pivots in the echelon form is 2, so rank of $A = 2$.

Points to remember:

1. Rank of a zero matrix is always zero.
2. Rank of a nonsingular matrix is equals with its order.

3. If A is a nonzero matrix of order $m\times n$, then rank of $A \le \min(m,n)$.

4. Rank of a nonzero matrix is at least one.

Problem Set 2.2

No. 1

$u + v + 2w = 2$

$2u + 3v - w = 5$

$3u + 4v + w = c$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \\ 3 & 4 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 1 & -5 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ c-6 \end{bmatrix} \qquad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ c-7 \end{bmatrix} \qquad R_3 \leftarrow R_3 - R_2$$

The system is solvable for

$$c - 7 = 0$$

$$\Rightarrow \boxed{c = 7}$$

No. 4

$Ax = b$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad R_2 \leftarrow R_2 - 2R_1$$

u, w are pivot variables

v is the free variable.

$w = 2$

$u + 2v + 2w = 1$

$\Rightarrow u + 2v = -3 \Rightarrow u = -3 - 2v$

$$\therefore x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -3-2v \\ v \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + v\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$ is the complete solution.

$Ax = b$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad R_2 \leftarrow R_2 - 2R_1$$

The system has no solution.

No. 5

(i) $$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_1$$

, is the echelon form

Since there are two pivot elements, so rank of $A = 2$.

(ii) $$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 4R_1 \\ R_3 \leftarrow R_3 - 7R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - 2R_2$$

, is the echelon form.

Since there are two pivots, so rank of $A = 2$.

No. 7

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (Ax = b)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - 2b_1 \end{bmatrix} \quad R_3 \leftarrow R_3 - 2R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - 3b_2 - 2b_1 \end{bmatrix} \quad R_3 \leftarrow R_3 - 3R_2$$

$\boxed{b_3 - 3b_2 - 2b_1 = 0}$, is the required constraints on b that turn the third equation into $0 = 0$.

$b_3 = 3b_2 + 2b_1$

$b_1 = 0, b_2 = 0 \Rightarrow b_3 = 0$

$b_1 = 1, b_2 = 0 \Rightarrow b_3 = 2$

$b_1 = 0, b_2 = 1 \Rightarrow b_3 = 3$

$b_1 = 1, b_2 = 1 \Rightarrow b_3 = 5$

$\vdots$

The attainable right hand sides i.e. the column space is

$$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}, \ldots \right\}$$

Since there are two pivots in the echelon form of the matrix A, so rank of $A = 2$.

<u>No. 13.</u>

(a) $Ux = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_3 + 2x_4 = 0$.

$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$

x_1, x_3 are pivot variables

x_2, x_4 are free variables.

$\Rightarrow x_3 = -2x_4$

$x_1 + 2x_2 - 2x_4 = 0$

$\Rightarrow x_1 = -2x_2 + 2x_4$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + 2x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

is the nullspace solution of $Ux = 0$.

$Ux = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_1 \leftarrow R_1 - 3R_2$$

$\Rightarrow$ $Rx = 0$, where R is the reduced row echelon form.

$x_1, x_3 \rightarrow$ Pivot variables

$x_2, x_4 \rightarrow$ Free variables

$x_3 + 2x_4 = 0$

$x_1 + 2x_2 - 2x_4 = 0$

$\Rightarrow$ $x_3 = -2x_4$

$x_1 = -2x_2 + 2x_4$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + 2x_4 \\ x_2 + 0 \cdot x_4 \\ 0 \cdot x_2 - 2x_4 \\ 0 \cdot x_2 + x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$ is

the nullspace solution of $Rx = 0$.

No. 44

(i) $$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & q-3 \end{bmatrix} \qquad \begin{matrix} R_2 \leftarrow R_2 + \frac{1}{2}R_1 \\ R_3 \leftarrow R_3 - \frac{3}{2}R_1 \end{matrix}$$

$$= \begin{bmatrix} 6 & 4 & 2 \\ 0 & 0 & q-3 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_2 \leftrightarrow R_3$$

, is the echelon form.

For $q = 3$, rank of $A = 1$

For $q \neq 3$, rank of $A = 2$.

Rank of A will be never 3.

(ii) $B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}$

$= \begin{bmatrix} 3 & 1 & 3 \\ 0 & 2-\frac{q}{3} & 0 \end{bmatrix}$ $R_2 \leftarrow R_2 - \frac{q}{3}R_1$, echelon form

$2 - \frac{q}{3} = 0$

$\Rightarrow q = 6$

For $q = 6$, rank of $A = 1$

For $q \neq 6$, rank of $A = 2$

No. 36.

$x + 3y + 3z = 1$
$2x + 6y + 9z = 5$
$-x - 3y + 3z = 5$

$\Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 + R_1$

$\Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ $R_3 \leftarrow R_3 - 2R_2$

x, z are pivot variables

y is the free variable

$3z = 3 \Rightarrow z = 1$

$x + 3y + 3z = 1 \Rightarrow x = -3y - 3z + 1$
$= -2 - 3y$

$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2-3y \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ is the complete solution.

(ii) $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad R_2 \leftarrow R_2 - 2R_1$

$\Rightarrow \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad R_3 \leftarrow R_3 - R_2$

x, z are pivot variables.

y, t are free variables.

$2z + 4t = 1 \Rightarrow z = -\frac{1}{2} - 2t$

$x + 3y + z + 2t = 1 \Rightarrow x = -3y - z - 2t + 1$

$= -3y + \frac{1}{2} + 2t - 2t + 1$

$= \frac{3}{2} - 3y$

$\therefore X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - 3y + 0\cdot t \\ 0 + y + 0\cdot t \\ -\frac{1}{2} + 0\cdot y - 2t \\ 0 + 0\cdot y + 1\cdot t \end{bmatrix}$

$= \begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix},$

is the complete solution.