LECTURE-26

4.4 APPLICATIONS OF DETERMINANTS

This section follows four major applications: inverse of A, solving $Ax = 0$ using Cramer's Rule, Area or Volume and pivots.

■ AN INVERSE FORMULA

Let A be invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{detA} adjA$$

EXAMPLE 1. Compute of A^{-1} of 2×2 matrix

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$$

Solution: Since $detA = 2$. Therefore inverse is exist. We know that, $adjA =$ transpose of the cofactor matrix of the matrix A.

$$\therefore \quad adjA = \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}$$

Hence

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}$$

■ CARMER'S RULE

Let A be an invertible $n \times n$ matrix. For any b in R^n, the unique solution x of $Ax = b$ has entries given by

$$x_i = \frac{detA_i}{detA}, \qquad i = 1, 2, \cdots$$

EXAMPLE 2. (Text Q. 14) Use Cramer's rule to solve the system

(a) $2x_1 + 5x_2 = 1$

$x_1 + 4x_2 = 2$

(b) $2x_1 + x_2 = 1$

$x_1 + 2x_2 + x_3 = 0$

$x_2 + 2x_3 = 0$

Solution (a)**:** Given the system as $Ax = b$ where

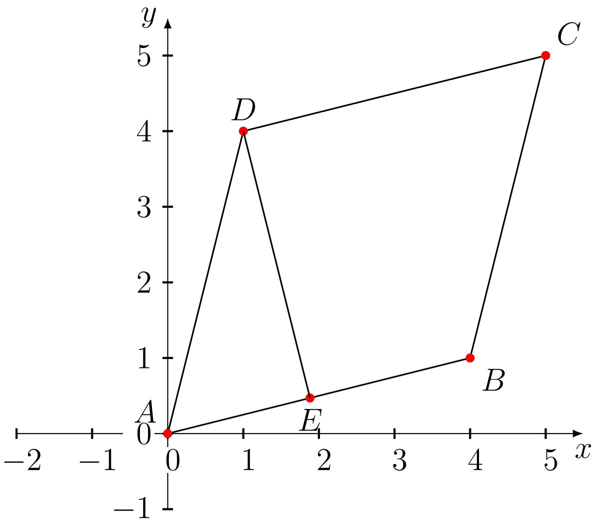
$$A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Since $detA = 3$, the system has a unique solution. By Cramer's rule,

$$x_1 = \frac{detA_1}{detA} = \frac{-6}{3} = -2 \tag{1}$$

$$x_2 = \frac{detA_2}{detA} = \frac{3}{3} = 1 \tag{2}$$

■ AREA AND VOLUME



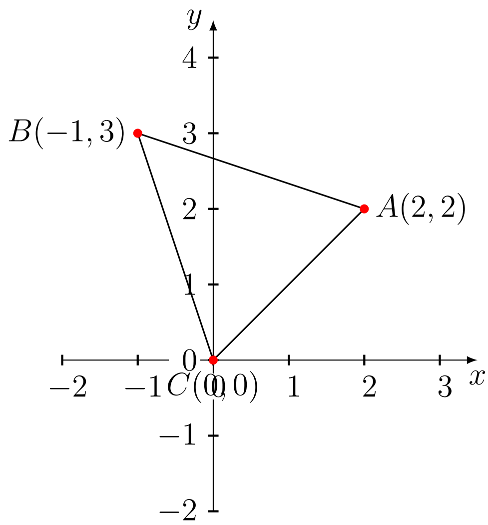
From the figure $OA = l(length)$ and $DE = h(heigh) \quad where \quad h = |E - D|$

The area (Volume) of the parallelogram is $l \times h = |detA|$(determine by the columns of A)

The area of the triangle determine by the columns of A is $\frac{1}{2}$parallelogram=$\frac{1}{2}detA$

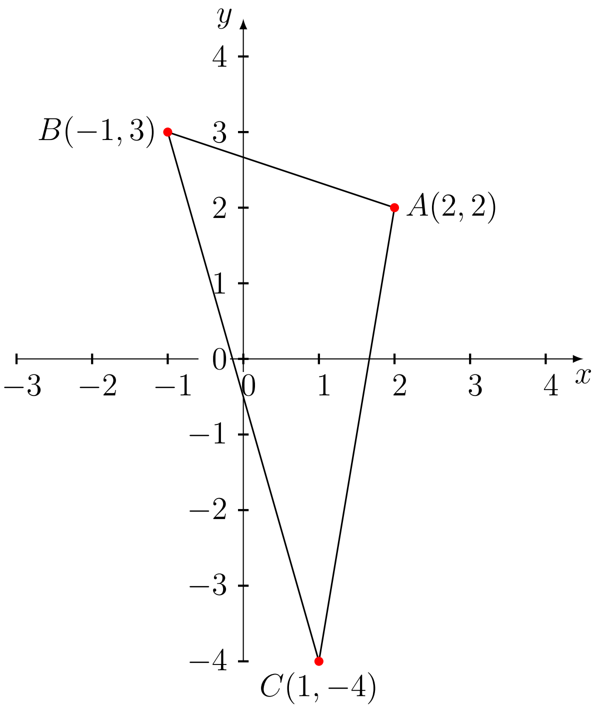
EXAMPLE 3(a)(TEXT Q. 2) Draw the triangle with vertices $A = (2, 2)$, $B = (-1, 3)$ and $C = (0, 0)$. By regarding it as half of a parallelogram , explain why its area equal

$$\text{area}(ABC) = \frac{1}{2}\begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix} = 4$$



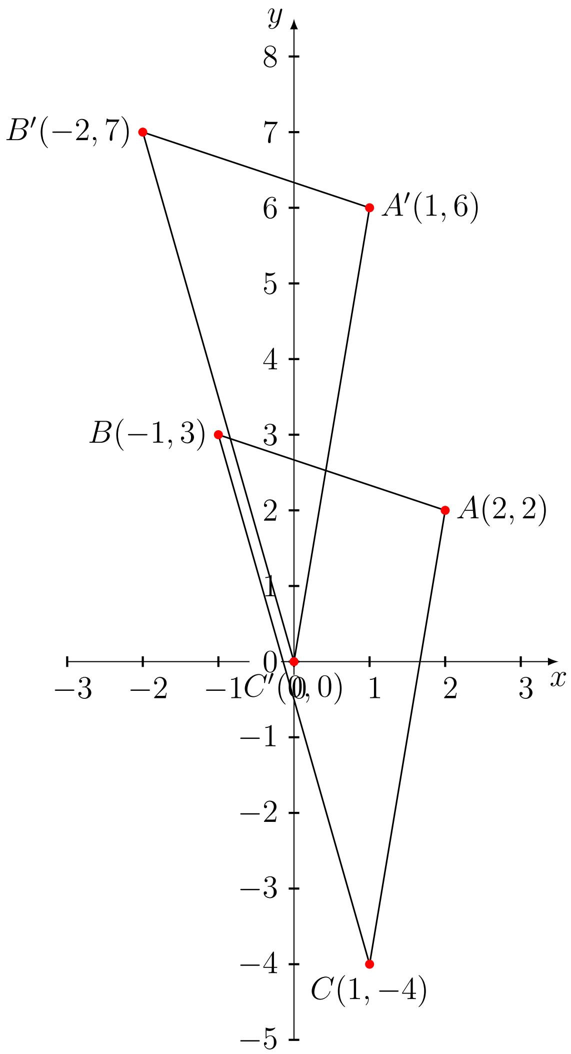
EXAMPLE 3(b) Move the third vertex to $C = (1, -4)$ and justify the formula

$$\text{area}(ABC) = \frac{1}{2}det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \frac{1}{2}det\begin{bmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -4 & 1 \end{bmatrix} = \frac{19}{2}$$



EXAMPLE 3(c) Sketch $A' = (1,6)$, $B' = (-2,7)$, and $C' = (0,0)$ and their relation to A, B, C.

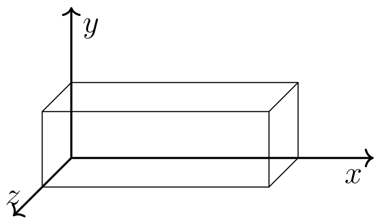
$$\text{area}(A'B'C') = \frac{1}{2} det \begin{bmatrix} 1 & 6 & 1 \\ -2 & 7 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{2} det \begin{bmatrix} 1 & 6 \\ -2 & 7 \end{bmatrix} = \frac{19}{2}$$



The formula is justified because $A' = (1, 6)$, $B' = (-2, 7)$, and $C' = (0, 0)$ are translations of the vertices $A = (2, 2)$, $B = (-1, 3)$ and $C = (1, 4)$.

EXAMPLE 4 (Text Q.29) A box has edges from $(0, 0, 0)$ to $(3, 1, 1)$ and $(1, 1, 3)$. Find its volume and also find the area of each parallelogram face?

Solution:(First Part)



$$\overrightarrow{OA} = \overrightarrow{u} = (3, 1, 1)$$

$$\overrightarrow{OB} = \overrightarrow{v} = (1, 3, 1)$$

$$\overrightarrow{OC} = \overrightarrow{w} = (1, 1, 3)$$

Volume of the box is $V = (\overrightarrow{u} \times \overrightarrow{v} \cdot \overrightarrow{u}) = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} = 20$

(Second Part) Area of the face

$$OAEB = ||(\overrightarrow{OA} \times \overrightarrow{OB})|| = ||(\overrightarrow{u} \times \overrightarrow{v})||$$

Now,

$$(\overrightarrow{u} \times \overrightarrow{v}) = \begin{bmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} = -2i - 2j + 8k$$

Hence,

$$||(\overrightarrow{u} \times \overrightarrow{v})|| = \sqrt{(-2)^2 + (-2)^2 + 8^2} = \sqrt{72} = 6\sqrt{2}$$

Similarly,

$$OBGC = ||(\overrightarrow{OB} \times \overrightarrow{OC})|| = ||(\overrightarrow{v} \times \overrightarrow{w})||$$

Now,

$$(\overrightarrow{v} \times \overrightarrow{w}) = \begin{bmatrix} i & j & k \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} = 8i - 2j - 2k$$

Hence,

$$||(\overrightarrow{v} \times \overrightarrow{w})|| = \sqrt{8^2 + (-2)^2 + (-2)^2} = \sqrt{72} = 6\sqrt{2}$$

$$OADC = ||(\overrightarrow{OA} \times \overrightarrow{OC})|| = ||(\overrightarrow{u} \times \overrightarrow{w})||$$

Now,

$$(\overrightarrow{u} \times \overrightarrow{w}) = \begin{bmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} = 2i - 8j + 2k$$

Hence,

$$||(\overrightarrow{u} \times \overrightarrow{w})|| = \sqrt{2^2 + (-8)^2 + (2)^2} = \sqrt{72} = 6\sqrt{2}$$

Q.27 The Parallelogram with sides $(2,1)$ and $(2,3)$ has the same area as the parallelogram with sides $(2,2)$ and $(1,3)$. Find those areas from 2 by 2 determinants and say why they must be equal.