Linearly independent and dependent

Linearly independent
A subset $\{v_1, v_2, ...v_n\}$ of a vector space V is said to be linearly independent if whenever $c_1, c_2, ...c_n \in R$ such that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$ then $c_1 = c_2 = ... = c_n = 0$

Linearly dependent
A non empty finite subset $\{v_1, v_2, ...v_n\}$ of a vector space V is said to be linearly dependent if there exists scalars $c_1, c_2, ...c_n \in R$ (**not all zero**)such that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$

Ex 1 if v_1 = zero vector , then the set is linearly dependent . we may choose $c_1 = 1$ and all other $c_i = 0$, this is a non trivial combination that produces zero.
i.e. $1v_1 + 0v_2 + ... + 0v_n = 1 \times 0 + 0 + ... + 0 = 0$

Ex 2 : The Column of the Matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

are linearly dependent , since the 2nd column is 3 times the first,the combination of columns with weights -3,1,0,0 gives the zero vector. i.e. say $A = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \end{bmatrix}$, then $-3C_1 + 1C_2 + 0C_3 + 0C_4 = 0$
The rows are also linearly dependent, row 3 is two times row 2 minus five times row1. i.e. say $A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$, then $R_3 - 2R_2 + 5R_1 = 0$

EX 3
The Column of this Triangular Matrix are Linearly Independent

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

Consider a linear combination of the columnsthat makes zero
Solve $Ac = 0$

$$c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

it means $3c_1 + 4c_2 + 2c_3 = 0$, $0c_1 + 1c_2 + 5c_3 = 0$, $0c_1 + 0c_2 + 2c_3 = 0$ i.e.

$$c_3 = 0, c_2 = 0, c_1 = 0$$

So column of A are Linearly Dependant.
and **null space of A contains only zero vector**

A similar reasoning applies to the rows of A, which are also independent. Suppose

$$c_1(3, 4, 2) + c_2(0, 1, 5) + c_3(0, 0, 2) = (0, 0, 0)$$

. From the first components we find $3c_1 = 0$ or$c_1 = 0$. Then the second components give $c_2 = 0$, and finally $c_3 = 0$.

Note : The columns of A are independent exactly when $N(A) = \{zerovector\}$
Note : It is the columns with pivots that are guaranteed to be independent

Ex 4 The columns of the n by n identity matrix are independent:

$$I = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1 & . & 0 \\ . & . & . & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note : To check any set of vectors $v_1, ..., v_n$ for independence, put them in the columns of A.Then solve the system $Ac = 0$;
1. The vectors are dependent if there is a solution other than c = 0.
2. With no free variables (rank n), there is no nullspace except c = 0; (i.e. $N(A) = \{0\}$)the vectors are independent.
3. If the rank is less than n, at least one free variable can be nonzero and the columns are dependent.

Note :A set of n vectors in R^m must be linearly dependent if $n > m$.

Ex 5 These three column in R^2 can not be independent:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

Sol : To find the combination of the columns producing zero we solve $Ac = 0$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = U$$

If we give the value 1 to the free variable c_3, then back-substitution in $Uc = 0$ gives $c_2 = -1$, $c_1 = 1$
i.e. if $A = \begin{bmatrix} C_1, C_2, C_3 \end{bmatrix}$ then $C_1 - C_2 + C_3 = 0$

Exercise 2.3.1: Choose three independent columns of V, then make two other choices. Do the same for A. You have found bases for which spaces?
$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}$

Solution: Let $U = [U_1 \ \ U_2 \ \ U_3 \ \ U_4]$ $A = [C_1 \ \ C_2 \ \ C_3 \ \ C_4]$ Consider, $A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix} \xrightarrow{R_4 \to R_4 - 2R_1}$
$\begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$
i.e. U is echelon form of A.
Note: Columns of A which have pivot are linearly independent.
Case (i) U_1, U_2, U_4 are L.I.(using the note).
Case (ii) U_1, U_3, U_4 are L.I.
as consider $aU_1 + bU_3 + cU_4 = 0$

$$a \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 4 \\ 7 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a + 4b + c \\ 0a + 7b + 0c \\ 0a + 0b + 9c \\ 0a + 0b + 0c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 9c = 0 \\ 7b = 0 \\ 2a + 4b + c = 0 \end{matrix} \Rightarrow \begin{matrix} c = 0 \\ b = 0 \\ a = 0 \end{matrix}$$

Case (iii) U_1, U_3, U_4 are L.I.
as consider $aU_2 + bU_3 + cU_4 = 0$

$$a \begin{bmatrix} 3 \\ 6 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 4 \\ 7 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3a + 4b + c \\ 6a + 7b + 0c \\ 0a + 0b + 9c \\ 0a + 0b + 0c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 9c = 0 \\ 6a + 7b = 0 \\ 3a + 4b + c = 0 \end{matrix} \Rightarrow \begin{matrix} c = 0 \\ b = 0 \\ a = 0 \end{matrix}$$

Note: Columns of a matrix A are linearly independent which are corresponding to the pivot column of echelon matrix of A.
Case (i) C_1, C_2, C_4 are L.I.(using the note).
Case (ii) C_1, C_3, C_4 are L.I.
as we can see consider $aC_1 + bC_3 + cC_4 = 0$

$$a\begin{bmatrix}2\\0\\0\\4\end{bmatrix} + b\begin{bmatrix}4\\7\\0\\8\end{bmatrix} + c\begin{bmatrix}1\\0\\9\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}2&4&1\\0&7&0\\0&0&9\\4&8&2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad i.e \;\; S\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\text{Aug. matrix}=[S|0]\left[\begin{array}{ccc|c}2&4&1&0\\0&7&0&0\\0&0&9&0\\4&8&2&0\end{array}\right] \xrightarrow{R_4\to R_4-2R_1} \left[\begin{array}{ccc|c}2&4&1&0\\0&7&0&0\\0&0&9&0\\0&0&0&0\end{array}\right]$$

$$\begin{aligned}2a+4b+c&=0 & & c=0\\ \Rightarrow \quad 7b&=0 & \Rightarrow\; & b=0\\ 9c&=0 & & a=0\end{aligned}$$

Case (iii) C_2, C_3, C_4 are L.I.
consider $aC_2 + bC_3 + cC_4 = 0$

$$\begin{bmatrix}C_2 & C_3 & C_4\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad say \;\; B\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\text{Aug. matrix}=[B|0]\left[\begin{array}{ccc|c}3&4&1&0\\0&6&7&0\\0&0&9&0\\6&8&2&0\end{array}\right] \xrightarrow{R_4\to R_4-2R_1} \left[\begin{array}{ccc|c}3&4&1&0\\0&6&7&0\\0&0&9&0\\0&0&0&0\end{array}\right]$$

$$\begin{aligned}3a+4b+c&=0 & & c=0\\ \Rightarrow \quad 6b+7c&=0 & \Rightarrow\; & b=0\\ 9c&=0 & & a=0\end{aligned}$$

The all three cases, we found bases for $R^{4\times 3}$ space.

Exercise 2.3.3 : Decide the dependence or independence of
(a) the vectors (1,3,2), (2,1,3) and (3,2,1)
(b) the vectors (1,3,-2), (2,1,-3) and (-3,2,1).
Solution:
(a)

$$a\begin{bmatrix}1\\3\\2\end{bmatrix} + b\begin{bmatrix}2\\1\\3\end{bmatrix} + c\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1&2&3\\3&1&2\\2&3&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad say \;\; A\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Aug. matrix=$[A|0]\left[\begin{array}{ccc|c}1&2&3&0\\3&1&2&0\\2&3&1&0\end{array}\right]\xrightarrow[R_3\to R_3-2R_1]{R_2\to R_2-3R_1}\left[\begin{array}{ccc|c}1&2&3&0\\0&-5&-7&0\\0&-1&-5&0\end{array}\right]\xrightarrow{R_3\to -5R_3+R_2}\left[\begin{array}{ccc|c}1&2&3&0\\0&-5&-7&0\\0&0&18&0\end{array}\right]$

$$\begin{array}{rl} & 1a+2b+3c=0 \\ \Rightarrow & -5b-7c=0 \\ & 18c=0\end{array} \qquad \begin{array}{rl} & c=0 \\ \Rightarrow & b=0 \\ & a=0\end{array}$$

Vectors are L.I.

(b) Consider $1(1,-3,2)+1(2,1,-3)+1(-3,2,1)=(1+2-3,-3+1+2,2-3+1)=(0,0,0)$
$\Rightarrow$ Vectors are L.D.

Exercise 2.3.5 : If w_1,w_2,w_3 are independent vectors, show that the differences $v_1=w_2-w_3$, $v_2=w_1-w_3$ and $v_3=w_1-w_2$ are dependent. find a combination of $v's$ gives zero.
Solution: Consider $av_1+bv_2+cv_3=0$
$a(w_2-w_3)+b(w_1-w_3)+c(w_1-w_2)=0$
$(b+c)w_1+(a-c)w_2+(-a-b)w_3=0$
Since w_1,w_2,w_3 are L.I.
So $b+c=0,\ a-c=0,\ -a-b=0$
$b=-c,\ a=c\ ,\ b=-a$
$a=-b=c$ take $a=1,\ b=-1,\ c=1$
$v_1-v_2+v_3=0$

Exercise 2.3.8 : Suppose v_1,v_2,v_3,v_4 are vectors in R^3.
(a) thse four vectors are dependent because ......
(b) The two vector v_1 and v_2 will be dependent if .....
(c) The vectors v_1 and (0,0,0) are dependent because .....

Solution: (a) Since $dim(R^3)=3$
Therefore each base of R^3 contains exactly 3 vectors.
So collection of vectors which are more than 3 are linearly dependent.
So four vectors are L.D.
(b) Let $av_1+bv_2=0$ for $\{v_1,v_2\}$ should be dependent.
So atleast one of a or b is nonzero.
say $a\neq 0$
So $v_1=\frac{-b}{a}v_2$ So v_1,v_2 are dependent if $\exists\alpha\neq 0$ s.t. $v_1=\alpha v_2$
(C) Consider $0.v_1+1(0,0,0)=(0,0,0)$ $a=0,b=1\neq 0$
So v_1 and (0,0,0) are L.D .