2.2: Solving $Ax=0$ and $Ax=b$

Course Outcomes: Students will have understanding about complete solution, nullspace solution, echelon form, reduced row echelon form and rank of a matrix.

Given: $Ax=b$ ——①

Let $x=x_p$ be a solution of equation ①.

Then
$$Ax_p=b \quad \text{——②}$$

again, $Ax=0$ ——③

Let $x=x_n$ be a solution of equation ③.

Then
$$Ax_n=0 \quad \text{——④}$$

Adding equations ② and ④, we have

$$A(x_p+x_n)=b \quad \text{——⑤}$$

Comparing equations ① and ⑤, we have

$$x=x_p+x_n,$$

which is the complete solution of equation ①, where x_p is the particular solution and x_n is the nullspace solution.

Ex: Write the complete solution of the system

$$y+z=2$$
$$2y+2z=4$$

Soln:

$$y+z=2$$
$$2y+2z=4$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad R_2 \leftarrow R_2 - 2R_1$$

Since 1st column of the coefficient matrix has a pivot element, so the 1st variable y is the pivot variable.

$y \rightarrow$ pivot variable

$z \rightarrow$ free variable

$$y + z = 2$$
$$\Rightarrow y = 2 - z$$

$\therefore x = \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 - z \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the completesolution of the given system.

Ex: Write the nullspace solution of the system

$$y + z = 0$$
$$2y + 2z = 0$$

Soln: Given: $y + z = 0$

$2y + 2z = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad R_2 \leftarrow R_2 - 2R_1$$

$y \rightarrow$ Pivot variable

$z \rightarrow$ Free variable

$$y + z = 0$$
$$\Rightarrow y = -z$$

$\therefore x = \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the nullspace solution of the given system.

Ex: Write the nullspace solution of the system

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solⁿ: $Ax = 0$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + R_1 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - 2R_2$$

Since 1st and 3rd columns of the coefficient matrix have pivot elements, so 1st and 3rd variables u and w are pivot variables.

$u, w \longrightarrow$ Pivot variables

$v, y \longrightarrow$ Free variables

$3w + 3y = 0$

$\Rightarrow w = -y$

$u + 3v + 3w + 2y = 0$

$\Rightarrow u + 3v - 3y + 2y = 0$

$\Rightarrow u + 3v - y = 0$

$\Rightarrow u = -3v + y$

$$\therefore x = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -3v + y \\ v \\ -y \\ y \end{bmatrix} = v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$ is the nullspace solution.

Echelon Form and Reduced Row Echelon Form :

Ex : Convert the following matrix into echelon form and reduced row echelon form.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

Solⁿ : $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + R_1 \end{matrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \leftarrow R_3 + R_2$$

, which is upper triangular form and also the echelon form

$$= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_1 \leftarrow \frac{1}{2} R_1 \\ R_2 \leftarrow -\frac{1}{8} R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_1 \leftarrow R_1 - \frac{1}{2} R_3 \\ R_2 \leftarrow R_2 - \frac{1}{4} R_3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 \leftarrow R_1 - \frac{1}{2} R_2$$

, which is the reduced row echelon form.

Ex : Convert the following matrix into echelon form and reduced row echelon form.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix}$$

Solⁿ: $A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 + R_1$

, which is upper triangular form but not echelon form.

$= \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \leftarrow R_3 - 2R_2$

, which is echelon form.

$= \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_2 \leftarrow \frac{1}{3} R_2$

$= \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_1 \leftarrow R_1 - 3R_2$

, which is reduced row echelon form.

How to get echelon form the upper triangular form of a matrix?

To get the echelon form from the upper triangular form of a matrix, we have to verify the following points:

1. The pivots are the first nonzero entries in their rows.
2. Below each pivot is a column of zeros, obtained by elimination.
3. Each pivot lies to the right of the pivot in the row above.

This produces the staircase pattern and zero rows come last.

How to get reduced row echelon form from the echelon form?

To get reduced row echelon form from the echelon form, we have to follow the following steps:

Step-I Make the pivot elements one by dividing the pivot element with every element of that row.

Step-II Make the elements zero which are present above the pivot places using the pivot place element.

Note :

1. Every echelon form is upper triangular form but every upper triangular form may or may not be echelon form.

2. If A is a nonsingular matrix then the reduced row echelon form of A is an identity matrix of order same as A.

Ex : Construct the 4×4 matrix $A=[a_{ij}]$, where $a_{ij}=(-1)^{ij}$. Also, convert A into echelon form and reduced row echelon form.

Soln : $A=[a_{ij}]$, where $a_{ij}=(-1)^{ij}$ and A is a 4×4 matrix.

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 - R_1 \\ R_4 \leftarrow R_4 + R_1 \end{matrix}$$

$$= \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \leftarrow R_4 - R_2$$

, which is echelon form.

$$= \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_1 \leftarrow -R_1 \\ R_2 \leftarrow \frac{1}{2} R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \leftarrow R_1 + R_2$$

, which is reduced row echelon form.