3.3

Least-Squares Fitting of Data :

Suppose we do a series of experiments and expect the output b to be a linear function of the input t. We look for a straight line

$b = C + Dt$.

For example :

At different times we measure the distance to a satellite on its way to Mars. In this case t is the time and b is the distance. Unless the motor was left on or gravity is strong, the satellite should move with nearly constant velocity v : $b = b_0 + vt$.

Ex : Find the best straight line fit (least squares) to the measurements

$b=1$ at $t=-1$, $b=1$ at $t=1$, $b=3$ at $t=2$.

Soln : Let $b = C + Dt$.

$b=1, t=-1 \Rightarrow 1 = C - D$

$b=1, t=1 \Rightarrow 1 = C + D$

$b=3, t=2 \Rightarrow 3 = C + 2D$

$Ax = b$

$$\Rightarrow \begin{aligned} C - D &= 1 \\ C + D &= 1 \\ C + 2D &= 3 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} C \\ D \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\boxed{\hat{x} = (A^TA)^{-1} A^T b}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{14} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{aligned} \hat{x} &= (A^TA)^{-1} A^T b \\ &= \frac{1}{14} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{3}{14} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 9/7 \\ 4/7 \end{bmatrix} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} \end{aligned}$$

The best solution is $\hat{C} = \frac{9}{7}$, $\hat{D} = \frac{4}{7}$ and the best line is $\frac{9}{7} + \frac{4}{7}t$.

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Problem Set 3.3

No. 1

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}, \quad A^Tb = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

$$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\therefore p = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$e = b - p = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$ is perpendicular to both columns.

No. 2

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} u \\ v \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$Ax = b$$

$$\Rightarrow \begin{aligned} u &= 1 \\ v &= 3 \\ u+v &= 4 \end{aligned}$$

$$E^2 = (u-1)^2 + (v-3)^2 + (u+v-4)^2$$

$$\frac{\partial E^2}{\partial u} = 2(u-1) + 2(u+v-4) = 0$$

$$\frac{\partial E^2}{\partial v} = 2(v-3) + 2(u+v-4) = 0.$$

$$\Rightarrow \begin{aligned} 2u + v &= 5 \\ u + 2v &= 7 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \quad (1)$$

same

$$A^Tb = \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \quad A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^TA\hat{x} = A^Tb \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \quad (2)$$

$$(A^TA)^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$p = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

$$e = b - p = 0.$$

$$\Rightarrow b = p.$$

<u>No. 4</u>.

$$Ax = b$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} C - D &= 4 \\ C &= 5 \\ C + D &= 9 \end{aligned} \quad \Rightarrow \quad \begin{aligned} b &= 4 \text{ at } t = -1 \\ b &= 5 \text{ at } t = 0 \\ b &= 9 \text{ at } t = 1 \end{aligned}$$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad A^Tb = \begin{bmatrix} 18 \\ 5 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 18 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ \frac{5}{2} \end{bmatrix},$$

which is the best estimate.

Best line is $6 + \frac{5}{2}t$.

$$p = A\hat{x}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 7/2 \\ 6 \\ 17/2 \end{bmatrix}$$

<u>No. 6</u>.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{22}\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

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$$\hat{x} = (A^TA)^{-1} A^T b$$
$$= \frac{1}{22}\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -11 \\ 27 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 9 \\ 37 \end{bmatrix}$$

$$p = A\hat{x}$$
$$= \frac{1}{22}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 9 \\ 37 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 46 \\ -28 \\ 130 \end{bmatrix} = \begin{bmatrix} 23/11 \\ -14/11 \\ 65/11 \end{bmatrix}$$

$$b = p + q$$
$$\Rightarrow q = b - p = \begin{bmatrix} -12/11 \\ 36/11 \\ 12/11 \end{bmatrix}$$

$$q \perp C(A)$$
$$\Rightarrow q \in N(A^T)$$

No. 9.

$b = 4$ at $t = -2$
$b = 3$ at $t = -1$
$b = 1$ at $t = 0$
$b = 0$ at $t = 2$

Line: $b = C + Dt$

$$\begin{aligned} C - 2D &= 4 \\ C - D &= 3 \\ C &= 1 \\ C + 2D &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow Ax = b.$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{35}\begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix}, \quad A^T b = \begin{bmatrix} 8 \\ -11 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1} A^T b = \begin{bmatrix} 61/35 \\ -36/35 \end{bmatrix}$$

The best line $61/35 - (36/35)t$.

The projection of b onto the column space of A is

$$p = A\hat{x}$$
$$= \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 61/35 \\ -36/35 \end{bmatrix} = \begin{bmatrix} 133/35 \\ 97/35 \\ 61/35 \\ -11/35 \end{bmatrix}$$