Gaussian Elimination

Consider a system of linear equations.

$$
\begin{gathered}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\
\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m
\end{gathered}
$$

Which can also be written as $Ax = b$. Where $A = \begin{bmatrix} \mathrm{a}_{11} & a_{12} & ... & a_{1n} \\ \mathrm{a}_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ \mathrm{a}_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$ is a $m \times n$ matrix having the coefficients of ith unknown as ith column elements and is said to be **coefficient matrix**. $x = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ . \\ . \\ . \\ \mathrm{x}_n \end{bmatrix}$ is a vector of unknowns said to be **solution vector** and $b = \begin{bmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \\ . \\ . \\ . \\ \mathrm{b}_m \end{bmatrix}$ is said to be the **righthand side vector or nonhomogeneous vector.** The values of $x_1, x_2, \cdots, x_n$ for which the given equations are satisfied form a vector $x = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ . \\ . \\ . \\ \mathrm{x}_n \end{bmatrix}$ is a **solution** of this system of equations.

Let us try to understand elimination method by an example :

$$
\begin{aligned}
2u + v + w &= 5 \\
4u - 6v \quad &= 1 \\
-2u + 7v + 2w &= 9.
\end{aligned}
$$

Here u, v, w are the unknowns.

To eliminate a variable, means making its coefficient 0.
(a) Let us subtract 2 times the first equation from the second
(b) Let us subtract -1 times the first equation from the third to get

$$
\begin{aligned}
2u + v + w &= 5 \\
-8v - 2w &= -12 \\
8v + 3w &= 14.
\end{aligned}
$$

Here the first variable u is eliminated.

Now to eliminate v, let us subtract (-1) times of the second equation from the third to get,

$$\begin{aligned} \mathbf{2}u + v + w &= 5 \\ \mathbf{-8}v - 2w &= -12 \\ \mathbf{1}w &= 2. \end{aligned}$$

These values 2, -8, 1 are called **pivots**. The cofficient of u in the first equation and the coefficient of v in the second equation and the coefficient of w in the third equation in the triangular form are called **1st, 2nd, 3rd pivots** respectively.

In matrix form

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -2 \\ 0 & 8 & 3 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Now back substitution yields the complete solution in the opposite order, beginning with the last unknown. The last equation $1w = 2$ gives $w = 2$. Then the second equation $-8v - 2w = -12$ gives $v = 1$. Finally, the first equation $2u + v + w = 5$ gives $u = 1$. Hence $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is the required solution.

The Breakdown of Elimination :

If a zero appears in a pivot position, elimination has to stop either temporarily or permanently.

If the zero pivot can be replaced by a nonzero value by row exchange process then the **breakdown of elemination process is temporary or else it is permanent.**
Consider an example of Nonsingular case :

$$\begin{aligned} u + v + w &= - \\ 2u + 2v + 5w &= - \\ 4u + 6v + 8w &= - \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} u + v + w &= - \\ 3w &= - \\ 2v + 4w &= - \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} u + v + w &= - \\ 2v + 4w &= - \\ 3w &= - \end{aligned}$$

The breakdown is **temporary**.

Consider an example of singular case :

$$\begin{aligned} u + v + w &= - \\ 2u + 2v + 5w &= - \\ 4u + 4v + 8w &= - \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} u + v + w &= - \\ 3w &= - \\ 4w &= -. \end{aligned}$$

In this case, there is no exchange of equations that can avoid zero in the second pivot position. Hence the breakdown id **permanent.**

Singular system of equations: A system of linear equations is said to be singular if and only if the corresponding coefficient matrix is singular.

A matrix is **singular** if its one row (column) can be written as a linear combination of other rows (columns).
The breakdown is temporary for a nonsingular system of equations (Having full set of pivots). The breakdown is permanent if the system of linear equations is singular.

Problem set 1.3

Q. Choose a r.h.s. which gives no solution and another r.h.s. which gives infinitely many solutions. What are two of those solutions?

$$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= ?. \end{aligned}$$

Ans. Here

$$\frac{3}{6} = \frac{2}{4} = \frac{10}{?} \Longrightarrow ? = 20.$$

Hence, if the r.h.s. $? \neq 20$ then no solution exists. For r.h.s. $? = 20$ the system has infinitely many solutions. Every point on the straight line $3x+2y = 10$ is a solution. In particular, $x = 2$, $y = 2$ and $x = 1$, $y = 3.5$ are two solutions.

Q. Choose a coefficient b that makes this system singular. Then choose a r.h.s. g that makes it solvable. Find two solutions in that singular case.
Ans. The system is singular $\Longleftrightarrow$ $\frac{2}{4} = \frac{b}{8}$ $\Longrightarrow$ $b = 4$. The system is singular and solvable $\Longleftrightarrow$ $\frac{2}{4} = \frac{b}{8} = \frac{16}{g}$ $\Longrightarrow$ $b = 4$ & $g = 32$. In this case, the system has infinitely many solutions. Every point on the straight line $2x + 4y = 16$ is a solution. In particular, $x = 2$, $y = 3$ and $x = 6$, $y = 1$ are two solutions.

Q. What multiple l of equation 1 should be subtracted from equation 2.

$$2x + 3y = 1$$
$$10x + 9y = 1.$$

After this elimination step, write down the upper triangular system and darken the two pivots.

Ans. Here $\frac{10}{2} = 5 = l$. Hence 5 Multiple of equation 1 is subtracted from equation 2 to get the coefficient matrix $\begin{bmatrix} 2 & 3 \\ 10 & 9 \end{bmatrix} \sim \begin{bmatrix} \mathbf{2} & 3 \\ 0 & \mathbf{-6} \end{bmatrix}$. The pivots are 2 and -6.

Q. What test on b_1 and b_2 decides where these two equations allow a solution? How many solutions will they have? Draw the column pictures.

$$3x - 2y = b_1$$
$$6x - 4y = b_2.$$

Ans. The given system of equations can be written as $x \begin{bmatrix} 3 \\ 6 \end{bmatrix} + y \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Note $\frac{3}{6} = \frac{-2}{-4} \implies 2b_1 = b_2$. Hence the two equations allow a solution and they have infinitely many solutions. If (b_1, b_2) point lies on the straight line joining $(-2, -4)$, $(0, 0)$ and $(3, 6)$. Then the system has infinite number of solutions.

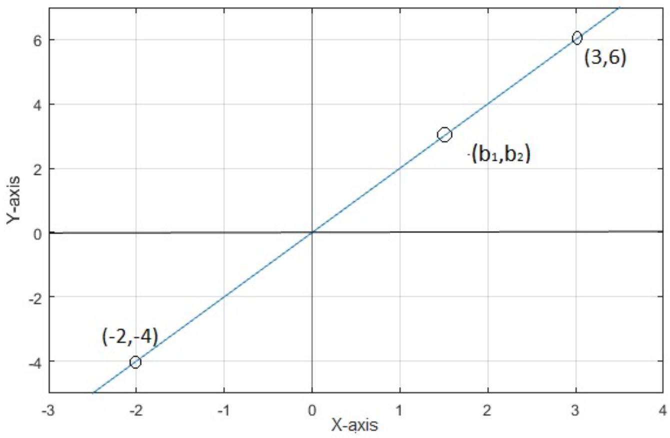

FIG.6

Figure 1: **Column Picture**