Projection Matrix of Rank 1

The projection onto a line is carried out by a projection matrix P which is a symmetric matrix and $P^2 = P$, $P = \dfrac{aa^T}{a^Ta}$.

Example 1 The matrix that projects onto the line through $a = (1, 1, 1)$ is

$$P = \frac{aa^T}{a^Ta} = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1 & 1 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{pmatrix}.$$

Here, $rank(P) = 1$.

Remark 1

1. Projection matrix $\dfrac{aa^T}{a^Ta}$ is same if a is doubled $a = (2, 2, 2,)$,.i.e,

$$P = \frac{aa^T}{a^Ta} = \frac{1}{12}\begin{pmatrix}2\\2\\2\end{pmatrix}\begin{pmatrix}2 & 2 & 2\end{pmatrix} = \begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{pmatrix}.$$

2. To project b onto a, multiply it by the projection matrix $P : p = Pb$, $\quad P = \dfrac{aa^T}{a^Ta}$

Exercise-3.2

8. Prove that the trace of $P = \dfrac{aa^T}{a^Ta}$, which is the sum of its diagonal entries, always 1.

   Solution: Let $a = (a_1 a_2 \cdots a_n)^T$. Then

$$
\begin{aligned}
P &= \frac{aa^T}{a^Ta} \\
&= \frac{\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}}{\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}} \\
&= \frac{\begin{pmatrix} a_1^2 & a_1a_2 & \cdots & a_1a_n \\ a_1a_2 & a_2^2 & \cdots & a_2a_n \\ \vdots & \vdots & \vdots & \vdots \\ a_na_1 & a_na_2 & \cdots & a_n^2 \end{pmatrix}}{a_1^2 + a_2^2 + \cdots + a_n^2}
\end{aligned}
$$

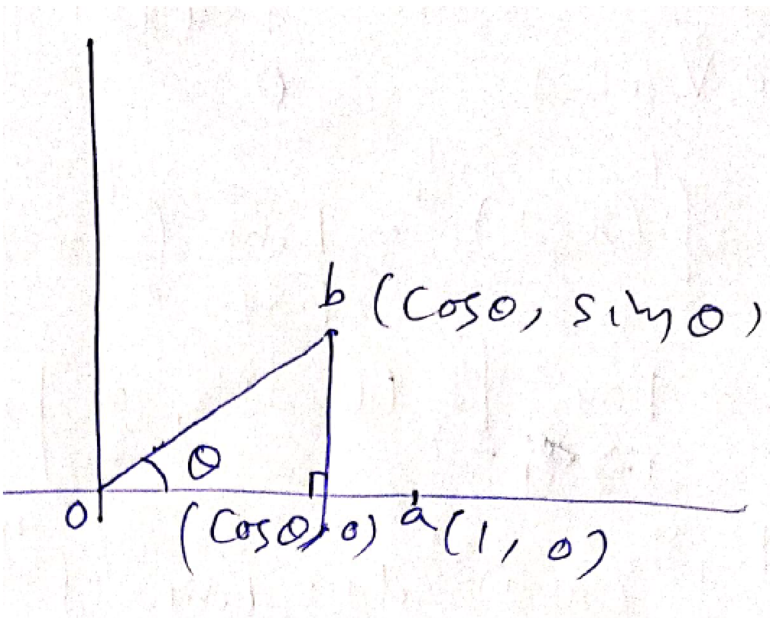
   Hence $tr(P) = \dfrac{a_1^2 + a_2^2 + \cdots + a_n^2}{a_1^2 + a_2^2 + \cdots + a_n^2} = 1.$

17. Draw the projection of b onto a and also compute it from $p = \widehat{x}a$:

   (a) $b = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution:

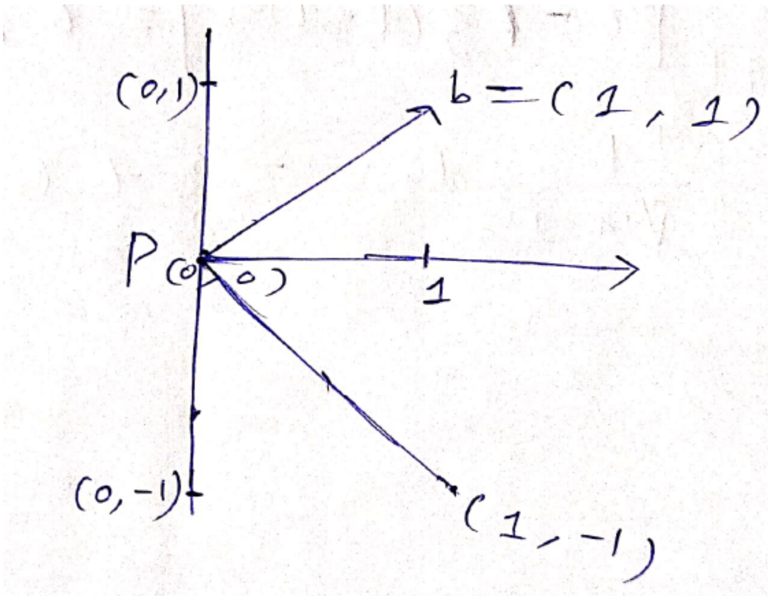
$$
\begin{aligned}
P &= \widehat{x}a \\
&= \frac{a^Tb}{a^Ta}a \\
&= \frac{\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}}{\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix}}\begin{pmatrix}1 \\ 0\end{pmatrix} \\
&= \frac{\cos\theta}{1}\begin{pmatrix}1 \\ 0\end{pmatrix} \\
&= \begin{pmatrix}\cos\theta \\ 0\end{pmatrix}
\end{aligned}
$$



(b) $b = \begin{pmatrix}1 \\ 1\end{pmatrix}$ and $a = \begin{pmatrix}1 \\ -1\end{pmatrix}$

Solution:

$$
\begin{aligned}
P &= \widehat{x}a \\
&= \frac{a^T b}{a^T a} a \\
&= \frac{\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \frac{0}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$



Assignments

Exercise-3.2, Q. 11,19.