Least Squares Fitting a Straight Line

$$b = C + Dt$$

or

$$\begin{aligned} C + Dt_1 &= b_1 \\ C + Dt_2 &= b_2 \\ &\vdots \\ C + Dt_m &= b_m \end{aligned}$$

or $\begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \\ 1 & t_m \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ or $Ax = b,\ \widehat{x} = (\widehat{C}, \widehat{D})$.

$E^2 = \parallel b - Ax \parallel^2 = \sum_{i=1}^{m} (b_i - C - Dt_i)^2$, where $\parallel E \parallel^2$ is minimum for some C and D.

$$\frac{\partial E^2}{\partial C} = 0 \tag{1}$$

$$\frac{\partial E^2}{\partial D} = 0 \tag{2}$$

Solving (1) and (2) for C and D we get the required straight line

$$b = C + Dt.$$

or

$$
\begin{aligned}
&A^T A\widehat{x} = A^T b, \quad A = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \\ 1 & t_m \end{pmatrix} \\
&\Longleftrightarrow A^T A \begin{pmatrix} \widehat{C} \\ \widehat{D} \end{pmatrix} = A^T b \\
&\Longleftrightarrow \begin{pmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_m \end{pmatrix} \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \\ 1 & t_m \end{pmatrix} \begin{pmatrix} \widehat{C} \\ \widehat{D} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_m \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \\
&\Longleftrightarrow \begin{pmatrix} m & \sum_{i=1}^{m} t_i \\ \sum_{i=1}^{m} t_i & \sum_{i=1}^{m} t_i^2 \end{pmatrix} \begin{pmatrix} \widehat{C} \\ \widehat{D} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{m} b_i \\ \sum_{i=1}^{m} t_i b_i \end{pmatrix}
\end{aligned}
$$

Exercise-3.3

6. Find the projection of b onto the column space of A :

$$
A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix}, \; b = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}.
$$

Split b into $p + q$, with p in the column space and q perpendicular to that space. Which of the four subspaces contains q?

Solution:

$$
\begin{aligned}
p &= A\widehat{x} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} \frac{18}{44} & \frac{8}{44} \\ \frac{8}{44} & \frac{6}{44} \end{pmatrix} \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \\
&= \begin{pmatrix} \frac{26}{44} & \frac{14}{44} \\ \frac{10}{44} & \frac{2}{44} \\ \frac{-4}{44} & \frac{8}{44} \end{pmatrix} \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \\
&= \begin{pmatrix} \frac{40}{44} & \frac{12}{44} & \frac{4}{44} \\ \frac{12}{44} & \frac{8}{44} & \frac{-12}{44} \\ \frac{4}{44} & \frac{-12}{44} & \frac{40}{44} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \\
&= \frac{1}{44} \begin{pmatrix} 92 \\ -56 \\ 260 \end{pmatrix}.
\end{aligned}
$$

Let find the vector q such that $b = p + q \Rightarrow q = p - b = \frac{1}{44}\begin{pmatrix} 92 \\ -56 \\ 260 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \frac{1}{44}\begin{pmatrix} 48 \\ -144 \\ -48 \end{pmatrix}$. Thus, $q^T A = \frac{1}{44}\begin{pmatrix} 48 \\ -144 \\ -48 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Hence, q belongs to the left null space.

Assignments

Exercise- 3.3, Q. 12, 24.