Spanning a Subspace:

Let us know what is the meaning of a set of vectors to span a space.

Vector Space $\mathbb{R}^2$:

Minimum two linearly independent vectors are required to span the vector space $\mathbb{R}^2$ i.e. the linear combination of two linearly independent vectors of $\mathbb{R}^2$ can form the whole plane $\mathbb{R}^2$.

More than two vectors of $\mathbb{R}^2$ can also span the whole plane $\mathbb{R}^2$ provided among the vectors two are linearly independent. But one vector of $\mathbb{R}^2$ can not span the whole plane $\mathbb{R}^2$.

Vector Space $\mathbb{R}^3$:

Minimum three linearly independent vectors are required for the spanning of the space $\mathbb{R}^3$ i.e. the linear combination of 03 linearly independent vectors of $\mathbb{R}^3$ can form the whole space $\mathbb{R}^3$.

More than three vectors of $\mathbb{R}^3$ can also span the whole space $\mathbb{R}^3$ provided among the given vectors three are linearly independent. But two linearly independent vectors of $\mathbb{R}^3$ can not span the whole plane $\mathbb{R}^3$.

Notes: 1. The column space of a matrix is spanned by its columns.

2. The row space of a matrix is spanned by its rows.

Ex ÷ Describe the subspace of $\mathbb{R}^2$ spanned by

(a) the vectors (1, 2) and (2, 4).

(b) the vectors (1, 2) and (3, 4).

(c) the vector (1, 2).

(d) the vectors (1, 2), (3, 4) and (1, 1).

Solⁿ ÷ (a) Given: the vectors (1, 2) and (2, 4).

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$, echelon form

Rank of $A = 1$

The required subspace of $\mathbb{R}^2$ spanned by the given two vectors is a line passing through origin.

(b) Given: the vectors (1, 2) and (3, 4).

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$, echelon form

Rank of $A = 2$.

The required subspace of $\mathbb{R}^2$ spanned by the two given vectors is the whole plane $\mathbb{R}^2$ itself.

ⓒ Given: the vector $(1,2)$.

Let $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $R_2 \leftarrow R_2 - 2R_1$ echelon form

Rank of $A = 1$.

The required subspace of $\mathbb{R}^2$ spanned by the given vector is a line passing through origin.

ⓓ Given: the vectors $(1,2)$, $(3,4)$ and $(1,1)$.

Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & -1 \end{bmatrix}$, $R_2 \leftarrow R_2 - 2R_1$ echelon form

Rank of $A = 2$.

The required subspace of $\mathbb{R}^2$ spanned by the three given vectors is the whole plane $\mathbb{R}^2$.

Basis for a Vector Space:

A basis for a vector space V is a subset with a sequence of vectors having two properties at once:

1. The vectors are linearly independent. (not too many vectors)
2. They span the space V. (not too few vectors)

Points to remember:

1. A basis of a vector space is the maximal independent set.
2. A basis of a vector space is also a minimal spanning set.

3. Spanning involves the column space and independence involves the nullspace.
4. No elements of a basis will be wasted.

Ex: Check whether the following sets are bases of $\mathbb{R}^3$ or not.

(a) $B_1 = \{(1,2,2), (-1,2,1), (0,8,0)\}$

(b) $B_2 = \{(1,2,2), (-1,2,1), (0,8,6)\}$

(c) $B_3 = \{(1,2,2), (-1,2,1)\}$

(d) $B_4 = \{(1,0,0), (0,1,0), (0,0,1)\}$

(e) $B_5 = \{(1,1,-1), (2,3,4), (4,1,-1), (0,1,-1)\}$

Solⁿ: (a) Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 0 \end{bmatrix}$

$$|A| = 1(0-8) + 1(-16) = -24 \neq 0$$

The vectors $(1,2,2)$, $(-1,2,1)$ and $(0,8,0)$ are linearly independent.

So, B_1 is a basis of $\mathbb{R}^3$.

(b) Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 6 \end{bmatrix}$

$$|A| = 1(12-8) + 1(12-16) = 4 - 4 = 0$$

The vectors of B_2 are linearly dependent.

So, B_2 is not a basis of $\mathbb{R}^3$.

(c) B_3 is not a basis of $\mathbb{R}^3$ as it does not contain the maximum number of linearly independent vectors of $\mathbb{R}^3$.

(d) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$|A| = 1 \neq 0$

The vectors of B_4 are linearly independent

So, B_4 is a basis of $\mathbb{R}^3$ and is known as standard basis of $\mathbb{R}^3$.

(e) The elements of B_5 are linearly dependent as four or more vectors in $\mathbb{R}^3$ are always linearly dependent.

So, B_5 is not a basis of $\mathbb{R}^3$.

Dimension of Vector Spaces:

Dimension of a vector space is the maximum no. of linearly independent vectors of the vector space.

OR

The no. of elements present in the basis of a vector space is known as dimension of the vector space.

$\dim. \mathbb{R} = 1$, $\dim. \mathbb{R}^2 = 2$, $\dim. \mathbb{R}^3 = 3, \ldots,$

$\dim. \mathbb{R}^n = n$.

Dimension of the vector space $\mathbb{R}^3 = 3$.

Any plane passing through origin are 2-dimensional subspaces of $\mathbb{R}^3$.

Any line passing through origin are 1-dimensional subspace of $\mathbb{R}^3$.

The origin $\{(0,0,0)\}$ is a 0-dimensional subspace of $\mathbb{R}^3$.

Note : A vector space has multiple bases.

Problem Set 2.3

No. 3. (a) Given: vectors $(1,3,2)$, $(2,1,3)$ and $(3,2,1)$.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

$$|A| = 1(1-6) - 2(3-4) + 3(9-2)$$
$$= -5 + 2 + 21$$
$$= 18 \neq 0$$

So, the given vectors are linearly independent.

(b) Given: vectors $(1,-3,2)$, $(2,1,-3)$ and $(-3,2,1)$.

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix}$

$$|A| = 1(1+6) - 2(-3-4) - 3(9-2)$$
$$= 7 + 14 - 21$$
$$= 21 - 21 = 0$$

So, the given vectors are linearly dependent

No. 5. Let w_1, w_2, w_3 are linearly independent vectors.

Let $v_1 = w_2 - w_3$, $v_2 = w_1 - w_3$ and $v_3 = w_1 - w_2$.

$$v_1 - v_2 + v_3 = w_2 - w_3 - w_1 + w_3 + w_1 - w_2$$
$$= 0$$

$\Rightarrow v_1, v_2$ and v_3 are linearly dependent.

No. 8. Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^3$

(a) These four vectors are dependent because four or more vectors in $\mathbb{R}^3$ are always dependent

(b) The two vectors v_1 and v_2 will be dependent if one is a multiple of other.

(c) The vectors v_1 and $(0,0,0)$ are dependent because $0 \cdot v_1 + c(0,0,0) = 0$ has a nonzero solution for any $c \neq 0$.

No. 9. Given: Plane: $x + 2y - 3z - t = 0$ in $\mathbb{R}^4$.

(i) $x = -2y + 3z + t$

$y=0,\ z=0,\ t=1 \Rightarrow x = 1$

$y=0,\ z=1,\ t=0 \Rightarrow x = 3$

So, $(1,0,0,1)$ and $(3,0,1,0)$ are two independent vectors.

(ii) $y=1,\ z=0,\ t=0 \Rightarrow x = -2$

So, $(1,0,0,1)$, $(3,0,1,0)$ and $(-2,1,0,0)$ are three independent vectors.

(iii) Since there are 3 free variables, so there will be not four independent vectors on the plane.

(iv) The plane is the nullspace of the matrix $A = \begin{bmatrix} 1 & 2 & -3 & -1 \end{bmatrix}$.