Problem Set - 7.2

2) Given $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is positive definite.

$\therefore \|A\| = \lambda_{max}(A)$

$\therefore \|A^{-1}\| = \dfrac{1}{\lambda_{min}(A)}$

Eigenvalues of A,

$|A - \lambda I| = 0$

$\Rightarrow (2-\lambda)^2 - 1 = 0$

$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$\therefore \lambda_1 = 1 \;|\; \lambda_2 = 3$

$\therefore \|A^{-1}\| = \dfrac{1}{\lambda_1} = \dfrac{1}{1} = 1$

$\therefore \|A\| = \lambda_2 = 3$

$c(A) = \dfrac{\lambda_2}{\lambda_1} = \dfrac{3}{1} = 3 = \|A\|\,\|A^{-1}\|$

(Ans)

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10) A^TA and AA^T have the same ~~non zero~~ eigenvalues.

$A^TAx = \lambda x$

$\Rightarrow A(A^TAx) = A(\lambda x)$

$\Rightarrow (AA^T)(Ax) = \lambda(Ax)$

$\Rightarrow (AA^T)x' = \lambda x'$

$\therefore \lambda_{max}(A^TA) = \lambda_{max}(AA^T)$

$\therefore \|A^T\| = \sqrt{\lambda_{max}((A^T)^TA^T)} = \sqrt{\lambda_{max}(AA^T)}$

$= \sqrt{\lambda_{max}(A^TA)}$

$= \|A\|$

$\therefore \|A\| = \|A^T\|$ (Proved)

15)

$$A = \begin{bmatrix} 100 & 0 \\ 0 & 2 \end{bmatrix}$$

Eigenvalues of A:

$\lambda_1 = 100 \mid \lambda_2 = 2$

$\therefore \|A\| = \lambda_{max}(A) = 100$

$\therefore c(A) = \dfrac{\lambda_{max}(A)}{\lambda_{min}(A)} = \dfrac{100}{2} = 50$ (Ans)

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Eigenvalues of B:

$|B - \lambda I| = 0$

$\Rightarrow (2-\lambda)^2 - 1 = 0$

$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$\therefore \lambda_1 = 3 \mid \lambda_2 = 1$

$\therefore \|B\| = \lambda_{max}(B) = 3$

$\therefore c(B) = \dfrac{\lambda_{max}(B)}{\lambda_{min}(B)} = \dfrac{3}{1} = 3$ (Ans)

$$D = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

Eigenvalues of D:

$|D - \lambda I| = 0$

$\Rightarrow (3-\lambda)(1-\lambda) - 1 = 0$

$\Rightarrow \lambda^2 - 4\lambda + 2 = 0$

$\therefore \lambda = \dfrac{4 \pm \sqrt{16-8}}{2}$

$\therefore \lambda_1 = 2+\sqrt{2} \mid \lambda_2 = 2-\sqrt{2}$

$\therefore \|D\| = \lambda_{max}(D) = 2+\sqrt{2}$

$\therefore c(D) = \dfrac{\lambda_{max}(D)}{\lambda_{min}(D)} = \dfrac{2+\sqrt{2}}{2-\sqrt{2}} = \dfrac{(2+\sqrt{2})^2}{4-2} = \dfrac{6+4\sqrt{2}}{2}$

$= 3+2\sqrt{2}$ (Ans)

17)

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

A is not positive definite.

$$\therefore A^TA = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

Eigenvalues of A^TA:

$$\lambda_1 = \lambda_2 = 4$$

$$\therefore \|A\| = \sqrt{\lambda_{max}(A^TA)} = \sqrt{4} = 2$$

$$\therefore c(A) = \sqrt{\frac{\lambda_{max}(A^TA)}{\lambda_{min}(A^TA)}} = \sqrt{\frac{4}{4}} = 1$$

(Ans:)

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

B is not positive definite.

$$\therefore B^TB = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Eigenvalues of B^TB:

$$|B^TB - \lambda I| = 0$$

$$\Rightarrow (1-\lambda)^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda = 0$$

$$\Rightarrow \lambda(\lambda - 2) = 0$$

$$\therefore \lambda_1 = 0 \mid \lambda_2 = 2$$

$$\therefore \|B\| = \sqrt{\lambda_{max}(B^TB)}$$

$$= \sqrt{2}$$

$$\therefore c(B) = \sqrt{\frac{\lambda_{max}(B^TB)}{\lambda_{min}(B^TB)}}$$

$$= \sqrt{\frac{2}{0}}$$

$$= \infty$$

(Ans:)

$$D = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\therefore D^TD = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Eigenvalues of D^TD:

$$\lambda_1 = 2 \mid \lambda_2 = 2$$

$$\therefore \|D\| = \sqrt{\lambda_{max}(D^TD)} = \sqrt{2}$$

$$\therefore c(D) = \sqrt{\frac{\lambda_{max}(D^TD)}{\lambda_{min}(D^TD)}} = \sqrt{\frac{2}{2}} = 1$$

(Ans:)

Important Notes :

1) Norm of A $= \|A\| = \max \frac{\|Ax\|}{\|x\|} = \sqrt{\lambda_{max}(A^TA)}$

2) Norm of $A^{-1} = \|A^{-1}\| = \frac{1}{\sqrt{\lambda_{min}(A^TA)}}$

Because :

$$\|A^{-1}\| = \sqrt{\lambda_{max}((A^{-1})^T A^{-1})}$$

$$= \sqrt{\lambda_{max}[(AA^T)^{-1}]}$$

$$= \frac{1}{\sqrt{\lambda_{min}(AA^T)}}$$

$$= \frac{1}{\sqrt{\lambda_{min}(A^TA)}} \quad [\because A^TA \ \& \ AA^T \text{ have same eigenvalue}]$$

$$(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1}$$

3) Condition Number of A :

$$c(A) = \sqrt{\frac{\lambda_{max}(A^TA)}{\lambda_{min}(A^TA)}} = \|A\| \cdot \|A^{-1}\|$$

If A is symmetric positive definite matrix,

$$\Rightarrow A^T = A$$

$$\therefore \|A\| = \sqrt{\lambda_{max}(A^TA)} = \sqrt{\lambda_{max}(A^2)} = \sqrt{(\lambda_{max}(A))^2} = \lambda_{max}(A)$$

$$\boxed{\therefore \|A\| = \lambda_{max}(A)}$$

$$\therefore \|A^{-1}\| = \frac{1}{\sqrt{\lambda_{min}(A^TA)}} = \frac{1}{\sqrt{\lambda_{min}(A^2)}} = \frac{1}{\sqrt{(\lambda_{min}(A))^2}} = \frac{1}{\lambda_{min}(A)}$$

$$\boxed{\therefore \|A^{-1}\| = \frac{1}{\lambda_{min}(A)}}$$

$\therefore$ Condition number :

$$\boxed{c(A) = \|A\| \cdot \|A^{-1}\| = \frac{\lambda_{max}(A)}{\lambda_{min}(A)}}$$

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