Linear Algebra

September 29, 2020

1 Introduction

Linear Algebra is a part of Mathematics and applied in every field of life. With the help of Linear Algebra we can convert our real life problems into various models involving, which are easy to deal with. One can visualize the objects in higher dimensions to get more information. It helps in solving Differential Equations, game developing, Machine Learning, Data Mining, Image Processing, Traffic Controlling, Electrical Circuit problems, Genetics, Cryptography, various economic model and several other fields. With the help of linear transformation concept one can study the properties of different entities in different spaces. Also a complicated geometrical problem can be studied by converting it into a simple algebraic problem. Hence Linear Algebra can be considered as a bridge between Geometry and Algebra.

2 System of Linear Equations

Definition : A linear system of equations is formed when two or more linear equations involving two or more unknowns considered together to represent a problem. For example:

$$
\begin{gathered}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\
\mathrm{a}_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\
... \; ... \; ... \\
\mathrm{a}_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m
\end{gathered}
$$

Which can also be written as $Ax = b$. Where $A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$ is a $m \times n$ matrix having the coefficients of ith unknown as ith column elements and is said to be *coefficient matrix*. $x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$ is a vector of unknowns said to be *solution vector* and

$b = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$ is said to be the *righthand side vector or nonhomogeneous vector.*

The above system of equations can also be represented as

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{m2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}.$$

The values of x_1, x_2, $\cdots$, x_n for which the given equations are satisfied form a vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$ is a *solution* of this system of equations.

A system of equations is said to *singular* if the corresponding coefficient matrix is singular. A matrix is said to be *singular* iff its rows or columns are linearly related to each other. That is a row (or column) can be obtained from the addition of scalar multiples of other rows(or columns).

The system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

is singular if $\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}}$. Otherwise it is said to *nonsingular*. A nonsingular system of equations has a **unique** solution.

A singular system of equations has either infinitely many solutions or no solutions. Hence for the above system of equations

1. if $\frac{a_{11}}{a_{21}} \neq \frac{a_{12}}{a_{22}}$, then it has **unique** solution.
2. if $\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} \neq \frac{b_1}{b_2}$, then it has **no** solution.
3. if $\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} = \frac{b_1}{b_2}$, then it has **infinitely many** solutions.

The following figures give an pictorial representation of the three cases discussed here.

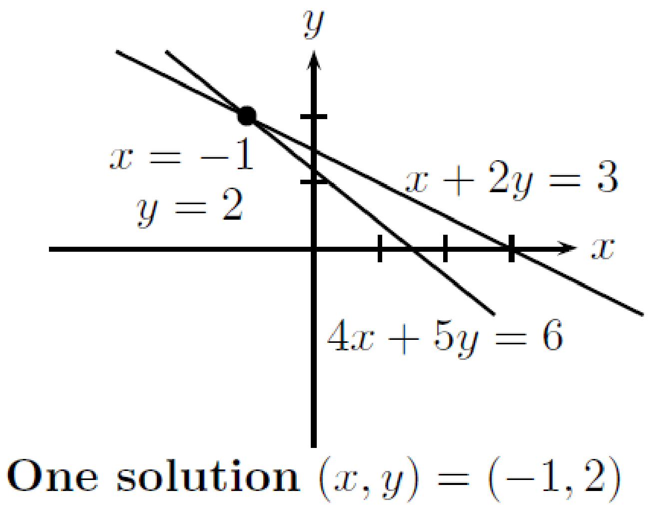


Figure 1: **One Solution**

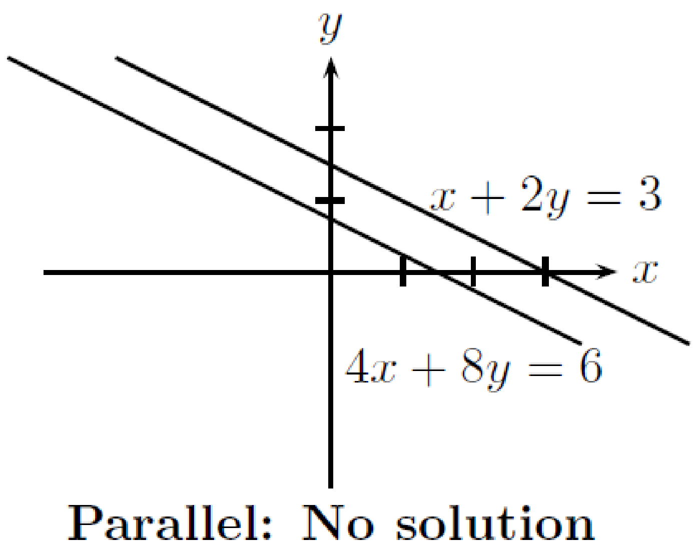


Figure 2: **No Solution**

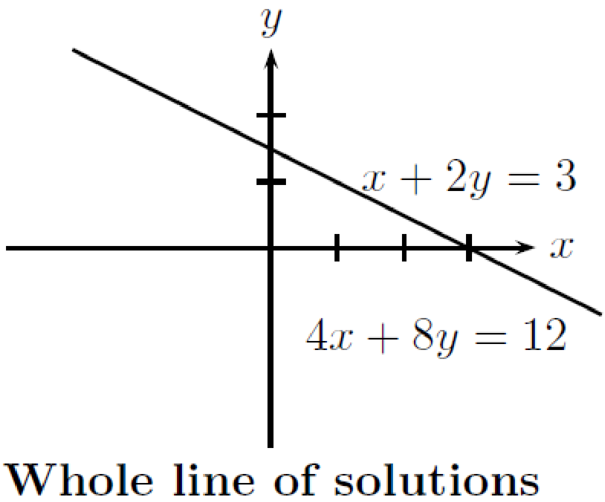


Figure 3: **whole lines of Solution**

THE GEOMETRY OF LINEAR EQUATIONS

To solve a system of equations geometrically two methods are used.

(i) **Row picture method** :

1. Plot the straight lines corresponding to the given equations.
2. Find the points of intersection(s) if exist. The x-coordinate value of the point of intersection represents the value of x and y- coordinate value gives the value of y.
3. Here if the lines are intersecting then unique solution.
4. If they are parallel then no solution.
5. If they represent the same line then every point on the line is a solution of it.

(See the first figure)

(ii) **Column picture method** :

1. Write the given system of equations as a linear combinations of column vectors equal to the rhs vector.

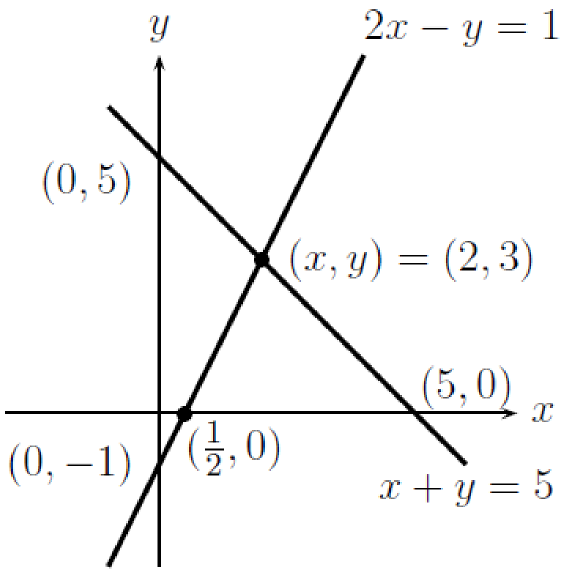
$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} y = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

2. Plot the points $P = (a_{11}, a_{21})$, $Q = (a_{12}, a_{22})$ in xy-plane.
3. Join each point with the origin O. Extend the lines.
4. Plot the point $B = (b_1, b_2)$. Draw a line from B to OP parallel to OQ and get the coordinates of point of intersection (h_1, h_2).
6. Also, draw a line from B to OQ parallel to OP and get the coordinates of point of intersection (k_1, k_2).
7. Find the values of x and y from the equations: $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} y = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$.

(See the second figure)

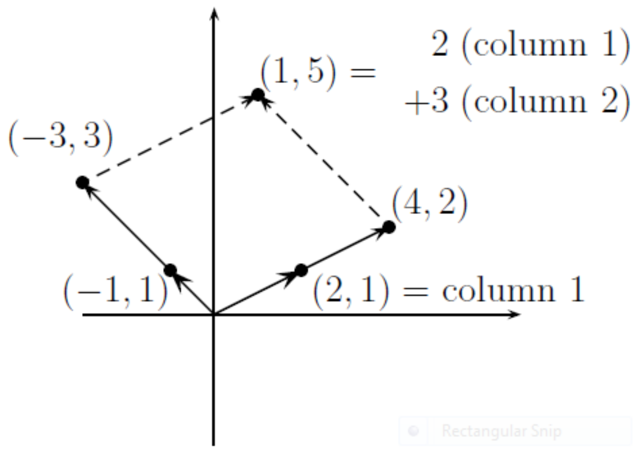
Consider a system of equations

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5. \end{aligned}$$



(a) Lines meet at $x = 2$, $y = 3$

Figure 1: **Row Picture Method**



(b) Columns combine with 2 and 3

Figure 2: **Column Picture Method**

PROBLEM SET 1.2

Q7. Explain why the given system is singular by finding a combination of the three equations that adds up to 0=1. What value should replace the last 0 on the r.h.s. to allow the equations to have solutions and what is one of the solutions?

$$\begin{aligned} u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= 0. \end{aligned}$$

Ans. Here in the left hand side $Row1 + Row3 = Row2$ but not in right hand side. Hence the system is singular but no solution exists. If the last 0 is replaced by -1 then l.h.s. and r.h.s. both satisfy the condition $Row1 + Row3 = Row2$.

Hence solution exists and $\begin{bmatrix} 3+w \\ -1-2w \\ w \end{bmatrix}$ is a general solution. For every value of w it gives a solution of

$$\begin{aligned} u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= -1. \end{aligned}$$

In patricular, $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ is a solution of it.

Q 8. Under what condition on y_1, y_2, y_3 do the points $(0, y_1)$, $(1, y_2)$, $(2, y_3)$ lie on a straight line?

Ans. The points $(0, y_1)$, $(1, y_2)$, $(2, y_3)$ lie on a straight line means the slopes of the line joining the points $(0, y_1)$ and $(1, y_2)$ and the points $(1, y_2)$ and $(2, y_3)$ are equal. That is,

$$\frac{y_2 - y_1}{1 - 0} = \frac{y_3 - y_2}{2 - 1}$$

which implies that $y_1 - 2y_2 + y_3 = 0$. Hence for $y_1 - 2y_2 + y_3 = 0$ the points $(0, y_1)$, $(1, y_2)$, $(2, y_3)$ lie on a straight line.

Q 11. The column picture form of a system is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b.$$

Show that the three columns on the left lie in the same plane by expressing the third column as a combination of the first two. What are all the solutions (u, v, w) if b is the zero vector $(0, 0, 0)$?

Ans. Here it can be observed that $2C_2 - C_1 = C_3$ that is, $2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. Hence the system of the equations

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

has infinitely many solutions. For every value of w the vector $(w, -2w, w)$ represents a solution of it. All solutions of it is represented by the set $\{(w, -2w, w) \in R^3 : w \in R\}$.

Q2. Sketch these three lines and decide if the equations are solvable :

$$\begin{aligned} x + 2y &= 2 \\ x - y &= 2 \\ y &= 1. \end{aligned}$$

What happens if all the right-hand sides are zero? Is there any nonzero choice of right hand sides that allows the three lines to intersect at the same point ?

Ans. The first figure represents the straight lines represented in the question.

$$\begin{aligned} x + 2y &= 2 \\ x - y &= 2 \\ y &= 1. \end{aligned}$$

Which shows that there exist no point common to all straight lines. Hence no solution exists.

The second figure gives the straight lines with r.h.s. vector 0, that is

$$\begin{aligned} x + 2y &= 0 \\ x - y &= 0 \\ y &= 0. \end{aligned}$$

Hence $x = 0$ and $y = 0$ is a solution of it.

The third figure gives the graph of

$$\begin{aligned} x + 2y &= 2 \\ x - y &= 2 \\ y &= 0. \end{aligned}$$

With r.h.s. vector $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ and $x = 2$, $y = 0$ satisfies all the equations. Hence it has a solution at $(2, 0)$.

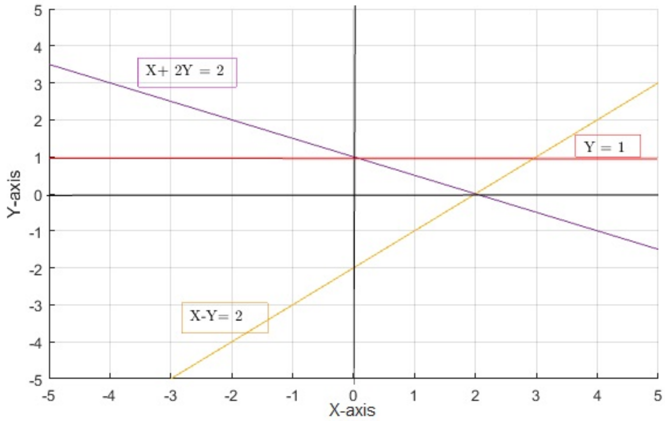


FIG.1

Figure 1: **Solution does not exist**

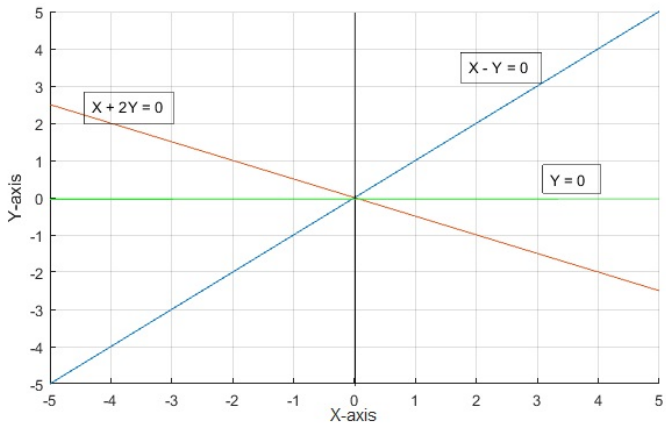


FIG. 2

Figure 2: **Solution exist and is the zero solution**

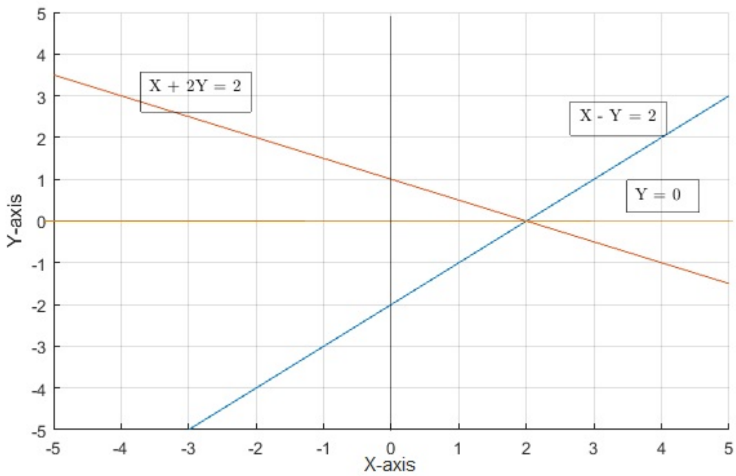


FIG.3

Figure 3: **Nonzero solution exists**

Gaussian Elimination

Consider a system of linear equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ &\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m \end{aligned}$$

Which can also be written as $Ax = b$. Where $A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$ is a $m \times n$ matrix having the coefficients of ith unknown as ith column elements and is said to be **coefficient matrix**. $x = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ . \\ . \\ . \\ \mathrm{x}_n \end{bmatrix}$ is a vector of unknowns said to be **solution vector** and $b = \begin{bmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \\ . \\ . \\ . \\ \mathrm{b}_m \end{bmatrix}$ is said to be the **righthand side vector or nonhomogeneous vector.** The values of $x_1, x_2, \cdots, x_n$ for which the given equations are satisfied form a vector $x = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ . \\ . \\ . \\ \mathrm{x}_n \end{bmatrix}$ is a **solution** of this system of equations.

Let us try to understand elimination method by an example :

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \quad &= -2 \\ -2u + 7v + 2w &= 9. \end{aligned}$$

Here u, v, w are the unknowns.

To eliminate a variable, means making its coefficient 0.
(a) Let us subtract 2 times the first equation from the second
(b) Let us subtract -1 times the first equation from the third to get

$$\begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ 8v + 3w &= 14. \end{aligned}$$

Here the first variable u is eliminated.

Now to eliminate v, let us subtract (-1) times of the second equation from the third to get,

$$\begin{aligned} \mathbf{2}u + v + w &= 5 \\ \mathbf{-8}v - 2w &= -12 \\ \mathbf{1}w &= 2. \end{aligned}$$

These values 2, -8, 1 are called **pivots**. The cofficient of u in the first equation and the coefficient of v in the second equation and the coefficient of w in the third equation in the triangular form are called **1st, 2nd, 3rd pivots** respectively.

In matrix form

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -12 \\ -2 & 7 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Now back substitution yields the complete solution in the opposite order, beginning with the last unknown. The last equation $1w = 2$ gives $w = 2$. Then the second equation $-8v-2w = -12$ gives $v = 1$. Finally, the first equation $2u + v + w = 5$ gives $u = 1$. Hence $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is the required solution.

The Breakdown of Elimination :

If a zero appears in a pivot position, elimination has to stop either temporarily or permanently.

If the zero pivot can be replaced by a nonzero value by row exchange process then the **breakdown of elemination process is temporary or else it is permanent.**
Consider an example of Nonsingular case :

$$\begin{aligned} u + v + w &= - \\ 2u + 2v + 5w &= - \\ 4u + 6v + 8w &= - \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} u + v + w &= - \\ 3w &= - \\ 2v + 4w &= - \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} u + v + w &= - \\ 2v + 4w &= - \\ 3w &= - \end{aligned}$$

The breakdown is **temporary**.

Consider an example of singular case :

$$\begin{aligned} u + v + w &= - \\ 2u + 2v + 5w &= - \\ 4u + 4v + 8w &= - \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} u + v + w &= - \\ 3w &= - \\ 4w &= -. \end{aligned}$$

In this case, there is no exchange of equations that can avoid zero in the second pivot position. Hence the breakdown id **permanent.**

Singular system of equations: A system of linear equations is said to be singular if and only if the corresponding coefficient matrix is singular.

A matrix is **singular** if its one row (column) can be written as a linear combination of other rows (columns).
The breakdown is temporary for a nonsingular system of equations (Having full set of pivots). The breakdown is permanent if the system of linear equations is singular.

Problem set 1.3

Q 1. Choose a r.h.s. which gives no solution and another r.h.s. which gives infinitely many solutions. What are two of those solutions?

$$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= ?. \end{aligned}$$

Ans. Here

$$\frac{3}{6} = \frac{2}{4} = \frac{10}{?} \Longrightarrow ? = 20.$$

Hence, if the r.h.s. $? \neq 20$ then no solution exists. For r.h.s. $? = 20$ the system has infinitely many solutions. Every point on the straight line $3x+2y = 10$ is a solution. In particular, $x = 2$, $y = 2$ and $x = 1$, $y = 3.5$ are two solutions.

Q 3. Choose a coefficient b that makes this system singular.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g \end{aligned}$$

Then choose a r.h.s. g that makes it solvable. Find two solutions in that singular case.

Ans. The system is singular $\iff \frac{2}{4} = \frac{b}{8} \implies b = 4$. The system is singular and solvable $\iff \frac{2}{4} = \frac{b}{8} = \frac{16}{g} \implies b = 4 \quad \& \quad g = 32$. In this case, the system has infinitely many solutions. Every point on the straight line $2x + 4y = 16$ is a solution. In particular, $x = 2$, $y = 3$ and $x = 6$, $y = 1$ are two solutions.

Q 6. What multiple l of equation 1 should be subtracted from equation 2.

$$2x + 3y = 1$$
$$10x + 9y = 1.$$

After this elimination step, write down the upper triangular system and darken the two pivots.

Ans. Here $\frac{10}{2} = 5 = l$. Hence 5 Multiple of equation 1 is subtracted from equation 2 to get the coefficient matrix $\begin{bmatrix} 2 & 3 \\ 10 & 9 \end{bmatrix} \sim \begin{bmatrix} \mathbf{2} & 3 \\ 0 & \mathbf{-6} \end{bmatrix}$. The pivots are 2 and -6.

Q7. What test on b_1 and b_2 decides where these two equations allow a solution? How many solutions will they have? Draw the column pictures.

$$3x - 2y = b_1$$
$$6x - 4y = b_2.$$

Ans. The given system of equations can be written as $x \begin{bmatrix} 3 \\ 6 \end{bmatrix} + y \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Note $\frac{3}{6} = \frac{-2}{-4} \implies 2b_1 = b_2$. Hence the two equations allow a solution and they have infinitely many solutions. If (b_1, b_2) point lies on the straight line joining $(-2, -4)$, $(0, 0)$ and $(3, 6)$. Then the system has infinite number of solutions.

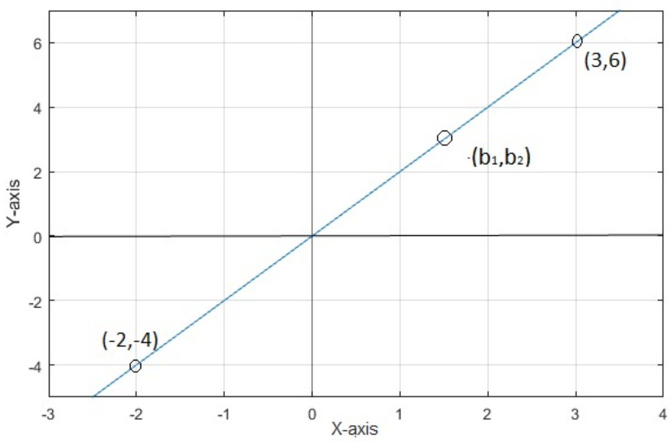


FIG.6

Figure 1: **Column Picture**

Problem set 1.3 continued...

Q 8. For which numbers a does elimination breakdown (a) Permanently (b) Temporarily ?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6. \end{aligned}$$

Ans. Here

$$\frac{a}{4} = \frac{3}{6} \implies a = 2.$$

$\implies$ The system is singular. Hence there is a permanent breakdown of elimination process. $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$. The second pivot is missing and cannot be replaced by any nonzero value by any row exchange process. Hence there exist a breakdown and the breakdown is permanent.

For $a = 0$, $\frac{a}{4} \neq \frac{3}{6}$, the system is nonsingular. Hence the breakdown is temporary. $\begin{bmatrix} 0 & 3 \\ 4 & 6 \end{bmatrix}$ $\implies$ The first pivot is missing, hence the elimination process breaksdown. By interchanging the 1st and 2nd rows, we have $\begin{bmatrix} 4 & 6 \\ 0 & 3 \end{bmatrix}$. Both the pivots 4, 3 are nonzero, hence the breakdown is temporary.

Q 14. Which number q makes this system singular and which right hand side t gives it infinitely many soluions? Find the solution that has $z = 1$

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t. \end{aligned}$$

Ans. The augment matrix of the given system of equations is :

$$\begin{bmatrix} 1 & 4 & -2 & 1 \\ 1 & 7 & -6 & 6 \\ 0 & 3 & q & t \end{bmatrix} R_2 - R_1 \to R_2 \sim \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 3 & q & t \end{bmatrix} R_3 - R_2 \to R_3 \sim \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & q+4 & t-5 \end{bmatrix}.$$

Here $q + 4 = 0$ makes the 3rd pivot missing and hence the system is singular. For $q + 4 = 0$, $t - 5 = 0$ we get infinitely many solutions since

$$\begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & q+4 & t-5 \end{bmatrix}. = \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence $3y - 4z = 5$, $\quad z = 1 \implies y = 3$ and $x + 4y - 2z = 1 \implies x = -9$.

The solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \\ 1 \end{bmatrix}$.

Q 16. If rows 1 and 2 are the same, how far can you get with elimination? Which pivot is missing?

$$\begin{aligned} 2x - y + z &= 0 \\ 2x - y + z &= 0 \\ 4x + y + z &= 2 \end{aligned}$$

Ans. The augment matrix of the given system of equations is :

$$\begin{bmatrix} 2 & -1 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 4 & 1 & 1 & 2 \end{bmatrix} R_2 - R_1 \to R_2 \sim \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 2 \end{bmatrix} R_3 - 2R_1 \to R_3 \sim \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & 3 & -1 & 2 \end{bmatrix}.$$

Interchanging R_2 with R_3, we get $\begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The second pivot is missing hence elimination breaks down and a row exchange gives a nonzero second pivot but the third pivot is missing, hence the breakdown is permanent.

Q 16. If columns 1 and 2 are the same, how far can you get with elimination? Which pivot is missing?

$$\begin{aligned} 2x + 2y + z &= 0 \\ 4x + 4y + z &= 0 \\ 6x + 6y + z &= 2 \end{aligned}$$

Ans. The augment matrix of the given system of equations is :

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 4 & 4 & 1 & 0 \\ 6 & 6 & 1 & 2 \end{bmatrix} R_2 - 2R_1 \to R_2 \sim \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 6 & 1 & 2 \end{bmatrix} R_3 - 3R_1 \to R_3 \sim \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & \mathbf{0} & -1 & 0 \\ 0 & 0 & -2 & 2 \end{bmatrix}.$$

The second pivot is missing and the breakdown is permanent.

Matrix notation and Matrix multiplication

Matrix **addition** is compatible If and only if matrix A and B both have same number of rows and same number of columns.

If $C = A + B$ then c_{ij} = ith row jth column element of C

$$c_{ij} = a_{ij} + b_{ij}$$

.

For example, let $A = \begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 16 \end{bmatrix}$ then $C = A + B = \begin{bmatrix} 3 & 5 \\ 9 & 23 \end{bmatrix}$.

Matrix **multiplication** $A \times B$ is compatible if and only if number of columns of A equals to number of rows of B. i.e. $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ then $A \times B$ is possible. In this case,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

For example $C = A \times B = \begin{bmatrix} 1 & 2 \\ \mathbf{5} & \mathbf{7} \end{bmatrix} \times \begin{bmatrix} \mathbf{2} & 3 \\ \mathbf{4} & 16 \end{bmatrix} = \begin{bmatrix} 10 & 51 \\ \mathbf{38} & 127 \end{bmatrix}$, $c_{21} = 5 \times 2 + 7 \times 4 = 38$

Elementary Row Operations

There are three elimintary row operations :

1. $R_i + kR_j \to R_i$ Means kth multiple of jth row is added with ith row.
2. $R_i \leftrightarrow R_j$ Means interchange ith row with jth row.
3. $cR_i \to R_i$ Means ith row is replaced by its c multiple.

Consider the 3×3 identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Now $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ R_2-$\mathbf{3}R_1 \to R_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{E} \implies E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{+3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 \leftrightarrow R_1 \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{F} \implies F^{-1} = F$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\mathbf{5}R_2 \to R_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathbf{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} = G \implies G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathbf{1/5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Then $EA = \begin{bmatrix} 1 & 2 & 3 \\ -31 & -35 & -39 \\ 7 & 8 & 9 \end{bmatrix}$, $FA = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$, $GA = \begin{bmatrix} 1 & 2 & 3 \\ 12 & 15 & 18 \\ 7 & 8 & 9 \end{bmatrix}$.

Q. Give 3 by 3 examples (not just the zero matrix) of

(a) a diagonal matrix : $a_{ij} = 0$ if $i \neq j$.

Ans. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

(b) a symmetric matrix $a_{ij} = a_{ji} \quad \forall \quad i \quad \& \quad j$.

Ans. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 15 & 18 \\ 3 & 18 & 9 \end{bmatrix}$.

(c) an upper triangular matrix $a_{ij} = 0$ if $i > j$.

Ans. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 15 & 18 \\ 0 & 0 & 9 \end{bmatrix}$.

(b) a skew-symmetric matrix $a_{ij} = -a_{ji} \quad \forall \quad i \quad \& \quad j$.

Ans. $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 18 \\ 3 & -18 & 0 \end{bmatrix}$.

Q. The matrix that rotates the xy plane by an angle θ is $A(\theta) = \begin{bmatrix} \cos(\theta) & \text{-}\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Verify that $A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2)$. What is $A(\theta)$ times $A(-\theta)$?

Ans. We know $A(\theta_1) = \begin{bmatrix} \cos(\theta_1) & \text{-}\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$ and $A(\theta_2) = \begin{bmatrix} \cos(\theta_2) & \text{-}\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix}$.

$$A(\theta_1)A(\theta_2) = \begin{bmatrix} \cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2) & \text{-}\cos(\theta_1)sin(\theta_2) - cos(\theta_2)sin(\theta_1) \\ \sin(\theta_1)cos(\theta_2) + cos(\theta_1)sin(\theta_2) & \text{-}\sin(\theta_1)sin(\theta_2) + cos(\theta_1)cos(\theta_2) \end{bmatrix}.$$

$$= \begin{bmatrix} \cos(\theta_1 + \theta_2) & \text{-}\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix} = A(\theta_1 + \theta_2)$$

$$A(\theta)A(-\theta) = A(\theta - \theta) = A(0) = \begin{bmatrix} \cos(0) & \text{-}\sin(0) \\ \sin(0) & \cos(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Q. Compute the products.

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 12+0+5 \\ 0+4+0 \\ 12+0+5 \end{bmatrix} = \begin{bmatrix} 17 \\ 4 \\ 17 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ \text{-}2 \\ 3 \end{bmatrix} = = \begin{bmatrix} 5 \\ \text{-}2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Q. For the third one, draw the column vectors $(2, 1)$ end $(0, 3)$. Multiplying by $(1, 1)$ just adds the vectors (do it graphically).

Ans. The graphical representation is :

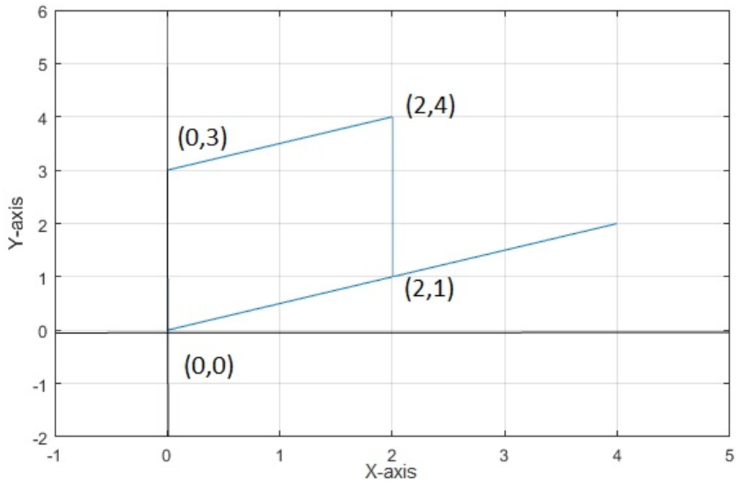


Figure 1: **whole lines of Solution**

Lecture-6

1.5 Triangular Factors and Row Exchanges

Course Outcome: Students will have understanding about the triangular factorization like LU and LDU factorization and permutation matrices that are being used for row exchanges purpose.

Triangular Factorization

$$\text{Given:}\quad \text{A}= \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} \text{R}_2 \leftarrow R_2 - 2R_1(2)$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} R_3 \leftarrow R_3 + R_1(-1)$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \text{R}_3 \leftarrow \text{R}_3 + R_2(-1)$$

$= U$

The elementary matrices are

$$\text{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$E_{32}E_{31}E_{21}A = U$

$\implies MA = U$

where $\quad M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

$M^{-1} =(\mathrm{E}_{32}E_{31}E_{21})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \mathrm{L}$

MA=U $\implies \ A = M^{-1}U \ \implies \ A = LU$,which is known as LU factorization of the matrix A.

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}$

LDU=A, which is known as LDU factorization of the matrix A.

No.2 When an upper triangular matrix is is nonsingular?

Ans: An upper triangular matrix is nonsingular if none of its diagonal elements are zero i.e. it has full set of pivot elements.

No.7 Factor A into LU, and write down the upper triangular system $Ux = c$ which appears after elimination, for

Ax= $\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$

Ans. Ax=b

$$\implies \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \text{R}_3 \leftarrow \text{R}_3 - 3R_1$$

$\implies$ Ux=c, which is the required upper triangular system.

No.21 What three elimination matrices E_{21}, E_{31}, E_{32} put A into upper triangular form $E_{21}E_{31}E_{32}A = U$? Multiply by $E_{32}^{-1}, E_{31}^{-1} and E_{21}^{-1}$ to factor A into LU where $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U.

$$\text{A}= \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

Ans. Given: $\text{A}= \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix} \; R_2 \leftarrow \text{R}_2 - 2R_1(2)$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{R}_3 \leftarrow \text{R}_3 - 3R_1(3)$$

$= U$

The elementary matrices are

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathrm{E}_{32} E_{31} E_{21} A = U$

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$LU = A$

Lecture-7

1.5 Triangular Factors and Row Exchanges

Row Exchanges and Permutation Matrices

During Gaussian elimination in case of breakdown problems, zero is appearing in the pivot place. To make that pivot place zero into nonzero, we are taking the help of row exchange. For this row exchange purpose, we will use permutation matrices.

Permutation Matrices

Order 2: There are 2!=2 permutation matrices of order 2. That are

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

Order 3: There are 3!=6 permutation matrices of order 3. That are

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{21}P_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_{32}P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Points to Remember:

1. Elements of permutation matrices are either 0 or 1.

2. Product of two permutation matrices is again a permutation matrix.

3. There are n! permutation matrices of order n.

Example.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$P_{21}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$P_{32}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix}$$

No.40 Which permutation makes PA upper triangular? Which permutations make P_1AP_2 lower triangular? Multiplying A on the right by P_2

exchanges the what of A?

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

Ans. Given:

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

$$P = P_{32}P_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \text{, which is upper triangular.}$$

$$P_1 = P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_2 = P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_1AP_2 = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix} \text{, which is lower triangular.}$$

Multiplying A on the right by P_2 exchanges the columns of A.

Tridiagonal matrix: A square matrix is said to be a tridiagonal matrix if all its elememts are zero except on the main diagonal and the two adjacent diagonals.

No.28 Find the LU and LDU factorization of the following tridiagonal matrix.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Ans. Given:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} R_2 \leftarrow R_2 - R_1(1) \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} R_3 \leftarrow R_3 - R_2(1) \\ &= U \end{aligned}$$

LU factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$LU = A$$

LDU factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$LDU = A$$

Lecture-8

1.6 Inverses and Transposes

Course Outcome: Students will have understanding about existence of inverse, Gauss-Jordan method to find the inverse of square matrices and transpose of matrices.

Existence of Inverse:

Inverse of a square matrix A exists if it is nonsingular i.e. $|A| \neq 0$. It is denoted by A^{-1}. If a square matrix has full set of pivot elements, then it is also nonsingular. So, inverse of a square matrix exists if it has full set of pivots.

Example.

$$\begin{aligned} Let\ A &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ \Rightarrow |A| &= 4-6 \\ \Rightarrow |A| &= -2 \\ \Rightarrow |A| &\neq 0 \end{aligned}$$

$\Rightarrow A^{-1}$ will exist.

Minor of A = M = $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

Cofactor of A = C = $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

Adjoint of A $(AdjA) = C^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Therefore,

$$\begin{aligned} A^{-1} &= \frac{AdjA}{|A|} \\ &= \frac{C^T}{|A|} \\ &= \frac{-1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \end{aligned}$$

Example.

$$\begin{aligned} Let\ A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \Rightarrow |A| &= ad - bc \\ \Rightarrow |A| &\neq 0 \end{aligned}$$

$$\Rightarrow A^{-1} will exist.$$

Minor of A = M = $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$

Cofactor of A = C = $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Adjoint of A $(AdjA) = C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore,

$$\begin{aligned} A^{-1} &= \frac{Adj A}{|A|} \\ &= \frac{C^T}{|A|} \\ &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

Example. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \implies A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

Example.

$$\begin{aligned} Let\ A &= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \\ \Rightarrow |A| &= 6-2 \\ \Rightarrow |A| &= 4 \\ \Rightarrow |A| &\neq 0 \end{aligned}$$

$$\Rightarrow A^{-1}\ will\ exist.$$

$$\text{Minor of A} = \text{M} = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\text{Cofactor of A} = \text{C} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Adjoint of A $(AdjA) = C^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Therefore,

$$\begin{aligned} A^{-1} &= \frac{AdjA}{|A|} \\ &= \frac{C^T}{|A|} \\ &= \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \end{aligned}$$

Example. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \implies A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$

Points to Remember:

1. The inverse of a square matrix if exits is unique and is known as both sided inverse.

2. $AA^{-1} = I = A^{-1}A$.

3. $(AB)^{-1} = B^{-1}A^{-1}$, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

4. $Ax = b \implies x = A^{-1}b$.

Gauss-Jordan Method: (The calculation of A^{-1})

Example. *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\Rightarrow |A| = -2 \neq 0$

$\Rightarrow A^{-1}$ *will exist.*

Now consider the matrix

$$\begin{aligned}
\left[\begin{array}{c|c} A & I \end{array}\right] \quad i.e, & \\
&= \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] R_2 \leftarrow R_2 - 3R_1 \\
&= \left[\begin{array}{cc|cc} 1 & 0 & -2 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] R_1 \leftarrow R_1 + R_2 \\
&= \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \end{array}\right] R_2 \leftarrow \frac{-1}{2} R_2 \\
&= \left[\begin{array}{c|c} I & A^{-1} \end{array}\right], \\
where \quad A^{-1} &= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \\
&= \frac{-1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}.
\end{aligned}$$

Example.

$$
\begin{aligned}
Let A &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \\
\Rightarrow |A| &= 1(6-4) - 1(3-2) + 1(2-2) \\
&= 2 - 1 \\
&= 1 \\
&\neq 0
\end{aligned}
$$

$\Rightarrow A^{-1}$ *will exist.*

$$
\left[\begin{array}{c|c} A & I \end{array}\right]
$$

$$
\begin{aligned}
&= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] R_2 \leftarrow R_2 - R_1 \\
&= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] R_3 \leftarrow R_3 - R_1
\end{aligned}
$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \mathrm{R}_3 \leftarrow R_3 - R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \mathrm{R}_1 \leftarrow R_1 - R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \mathrm{R}_2 \leftarrow R_2 - R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \mathrm{R}_1 \leftarrow R_1 - R_2$$

$$= \left[\begin{array}{c|c} I & A^{-1} \end{array}\right],$$

$$\text{where} \quad \mathrm{A}^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

No.6 (a) If A is invertible and AB=AC, prove that B=C.

Proof. Let A be invertible and AB=AC.

$AB = AC$

$\implies A^{-1}(AB) = A^{-1}(AC)$

$\implies (A^{-1}A)B) = (A^{-1}A)C)$

$\implies IB = IC$

$\implies B = C$

(b) $If A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, find an example with $AB = AC$ but $B \neq C$.

Sol. Given:A= $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$Let\ B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\ and\ C = \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}.$

$AB = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$ and $AC = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}.$

$AB = AC\ but\ B \neq C.$

No.10 Use the Gauss-Jordan method to find the inverse of $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Sol. Given $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$|A_1| = 1 \neq 0$

So, A_1^{-1} will exist.

$$\left[\begin{array}{c|c} A_1 & I \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \mathrm{R}_2 \leftarrow R_2 - R_1$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \mathrm{R}_2 \leftarrow R_2 - R_3$$

$$= \left[\begin{array}{c|c} I & A_1^{-1} \end{array}\right],$$

where $\quad \mathrm{A}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$

Lecture-9

1.6 Inverses and Transposes

Transpose:

$$\text{A}= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \implies A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Symmetric matrix:

A square matrix A is said to be symmetric if $A^T = A$.

Ex. A= $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, B= $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Skew-symmetric matrix:

A square matrix A is said to be skew-symmetric if $A^T = -A$.

Ex. A= $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, B= $\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$

Points to Remember:

1. $(A + B)^T = A^T + B^T$.

2.$(AB)^T = B^T A^T, \; (ABC)^T = C^T B^T A^T$.

3.$(A^T)^T = A$.

4.$(A^{-1})^T = (A^T)^{-1}$.

If A is any real square matrix, then $A + A^T$ is always symmetric and $A - A^T$ is always skew-symmetric.

Let $B = A + A^T$. Then

$B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = B$

So, B is symmetric.

Let $C = A - A^T$. Then

$C^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -C$

So, C is skew-symmetric.

Now, $A = \dfrac{A + A^T}{2}(symmetric) + \dfrac{A - A^T}{2}(skew - symmetric)$

So, any real square matrix can be expressed as a sum of symmetric and skew-symmetric matrix.

Example. Express the following matrix as a sum of symmetric and skew-symmetric matrix.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Sol. A= $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

$$\implies \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 5/2 \\ 5/2 & 4 \end{bmatrix} \text{(symmetric)} + \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix} \text{(skew-symmetric)}$$

Note: If $A = A^T$can be factored into $A = LDU$ without row exchanges, then U is the transpose of L and $A = LDL^T$.

Example.

Given: A= $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 5 & 18 & 30 \end{bmatrix} R_2 \leftarrow R_2 - 3R_1(3)$

$= \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 3 & 5 \end{bmatrix} R_3 \leftarrow R_3 - 5R_1(5)$

$= \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix} R_3 \leftarrow R_3 - R_2(1)$

=U

LU factorization:

L= $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}$, U= $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

$$LU = A$$

LDU factorization:

L= $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}$, D= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ U= $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$LDU = A$$

Here $U = L^T$. So, $A = LDL^T$.

In this example since $A^T = A$, so LDU=A is $LDL^T = A$.

No.11 If B is square, show that $A = B + B^T$ is always symmetric and $K = B - B^T$ is always skew-symmetric. Find these matrices A and K when $B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ and write B as the sum of a symmetric matrix and a skew-symmetric matrix.

Sol. Let B be a square matrix.

Let $A = B + B^T \textit{ and } K = B - B^T$.

$A^T = (B + B^T)^T = B^T + (B^T)^T = B^T + B = A$

$\implies$ A is symmetric.

Again, $K^T = (B - B^T)^T = B^T - (B^T)^T = B^T - B = -(B - B^T) = K$

$\implies$ K is skew-symmetric.

$$A = B + B^T = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

$$K = B - B^T = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$B = \frac{B + B^T}{2} + \frac{B - B^T}{2}$$

$$\implies \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{(symmetric)} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{(skew-symmetric)}$$

No.17 Give examples of A and B such that

(a) A+B is not invertible although A and B are invertible.

(b) A+B is invertible although A and B are not invertible.

(c) all of A, B and A+B are invertible.

Sol. (a) A= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, B= $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, A+B= $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

A+B is not invertible although A and B are invertible.

(b) A= $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, B= $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, A+B= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A+B is invertible although A and B are not invertible.

(c) A= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, B= $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, A+B= $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

All of A, B and A+B are invertible.

No.41 True or false:

(a) A 4 by 4 matrix with a row of zeros is not invertible.

Ans. True since the matrix is singular.

(b) A matrix with 1s down the main diagonal is invertible.

Ans. False since the matrix is singular.

(c) If A is invertible then A^{-1} is invertible.

Ans. True since A^{-1} is non-singular if A is no-singular.

(d) If A^T is invertible then A is invertible.

Ans. True since A is non-singular if A^T is non-singular.

No.42 For which three numbers c is this matrix not invertible, and why not?

A= $\begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$

Sol.

For c=0,2,7 the matrix is not invertible, as for these three values of c the determinant of the matrix is zero i.e. the matrix is singular.

c=0 $\implies$ zero column(or zero row).

c=2 $\implies$ identical rows.

c=7 $\implies$ identical columns.

Lecture-10

2.1 Vector Spaces and Subspaces

Course Outcomes: Students will have understanding about vector spaces and subspaces. Also, will be acquainted with the column space and the nullspace of different matrices.

In this article we will discuss about the followings:

(i) Vector Space

(ii) Subspaces

(iii) The Column Space

(iv) The Nullspace

Vector Space: A nonempty set V is said to be a vector space if it satisfies the following properties:

1. $x + y = y + x$ (Commutative law of addition)

2. $x + (y + z) = (x + y) + z$ (Associative law of addition)

3. There is a unique vector '0' (zero vector) such that $x + 0 = x$ for all x. (Additive identity property)

4. For each x there is a unique vector $-x$ such that $x + (-x) = 0$.(Additive inverse property)

5.$1x = x$

6. $(c_1c_2)x = c_1(c_2x)$

7. $c(x + y) = cx + cy$

8. $(c_1 + c_2)x = c_1x + c_2x$,

where $x, y, z \in$ V and $c,\ c_1,\ c_2\ \in\ R$.

Since out of the above eight properties first four are coming under vector addition and last four are coming under scalar multiplication, so the following is an alternate definition of vector space.

A nonempty set V is said to be a vector space if it satisfies the following properties:

1. Vector Addition

i.e. $x \in V,\ y \in V \implies x + y \in V$

2. Scalar Multiplication

i.e. $c \in R, x \in V \implies cx \in V$

Examples of vector spaces: R, R^2, R^3,........., R^n.

Example.

Show that R^2 is a vector space.

Proof. R^2 contains infinitely many elements and the elements are pairs like $(0, 0), (1, 2), (-1, 2), \ldots\ldots\ldots$

So, R^2 is nonempty.

1. Vector addition:

Let $x = (x_1, x_2)\ and\ y = (y_1, y_2) \in R^2$.

$x + y = (x_1 + y_1, x_2 + y_2) \in R^2\ as\ x_1 + y_1 \in R\ and\ x_2 + y_2 \in R.$

2. Scalar multiplication:

Let $c \in R$ and $(x_1, x_2) \in R^2$.

$cx = (cx_1, cx_2) \in R^2\ as\ cx_1 \in R\ and\ cx_2 \in R.$

So, R^2 is a vector space.

Example. Verify whether the 1st quadrant of R^2 is a vector space or not.

Proof: Let the set V be the 1st quadrant of R^2.

Then $V = \{(x_1, x_2) \in R^2 : \quad x_1 \geq 0,\ x_1 \geq 0\}$

$= \{(0, 0), (1, 1), (1, 2), (2, 3), \ldots\ldots\ldots\}$

So, V is nonempty.

1. Vector addition:

Let $x = (x_1, x_2)$ *and* $y = (y_1, y_2) \in V.$

Then $\quad x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0.$

$x + y = (x_1 + y_1, x_2 + y_2) \in V \ as \ x_1 + y_1 \geq 0 \ and \ x_2 + y_2 \geq 0.$

2. Scalar multiplication:

Let $c = -2 \in R$ and $x = (1, 2) \in V.$

$cx = (-2, -4) \notin V$

So, V does not satisfy scalar multiplication property.

Hence, V i.e. 1st quadrant is not a vector space.

Example. Verify whether combinedly the 1st and 3rd quadrant of R^2 is a vector space or not.

Proof: Let the set V be the 1st and 3rd quadrant of R^2.

Then $V = \{(1, 1), (1, 2), (-1, -1), (-1, -2), \ldots\ldots\ldots.\}.$

So, V is nonempty.

1. Vector addition:

Let $x = (-1, -2)$ and $y = (2, 1) \in$ V.

$x + y = (1, -1) \notin$ V.

So, V does not satisfy vector addition property.

Hence, V i.e. 1st and 3rd quadrant combinedly is not a vector space.

Subspace: A subspace of a vector space is a nonempty subset that satisfies the requirements for a vector space.

Example. Show that $y = x$ line is a subspace of the vector space R^2.

Proof. Let the set V be $y = x$ line.

Then $V = \{(x_1, x_2) \in R^2 : x_1 = x_2\}$

$= \{(0,0),(1,1),(2,2),(-1,-1),(-2,-2),.........\}$.

So, V is a nonempty subset of R^2.

1. Vector addition:

Let $x = (x_1, x_2)$ *and* $y = (y_1, y_2) \in V$.

Then $x_1 = x_2$ *and* $y_1 = y_2$.

$x + y = (x_1 + y_1, x_2 + y_2) \in V$ *as* $x_1 + y_1 = x_2 + y_2$.

2. Scalar multiplication:

Let $c \in R$ *and* $(x_1, x_2) \in V$.

$cx = (cx_1, cx_2) \in V$ *as* $cx_1 = cx_2$.

So, V i.e. $y = x$ line is a subspace space R^2 .

Vector Space: R^2

Subspaces:

1. R^2

2. Any line passing through origin.

3. Origin i.e. {(0,0)}.

Vector Space: R^3

Subspaces:

1. R^3

2. Any plane passing through origin.

3. Any line passing through origin.

3. Origin i.e. {(0,0,0)}.

Points to remember:

1. Every vector space is a subspace of itself.

2. A subspace is a vector space in its own right.

3. Every vector space is the largest subspace of itself and origin is the smallest

subspace.

No.2 Which of the following subsets of R^3 are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.

(b) The plane of vectors b with $b_1 = 1$.

Sol. (a).

Let $V = \{The\ plane\ of\ vectors\ (b_1, b_2, b_3)\ with\ first\ component\ b_1 = 0\}$.

$= \{(0,0,0), (0,1,0), (0,1,2), \ldots\ldots\ldots\}$

So, V is a nonempty subset of R^3.

1. Vector addition:

Let $b = (b_1, b_2, b_3)\ and\ c = (c_1, c_2, c_3) \in V.\ Then\ b_1 = 0\ and\ c_1 = 0.$

$b + c = (b_1 + c_1, b_2 + c_2, b_3 + c_3) \in V\ as\ b_1 + c_1 = 0.$

2. Scalar multiplication:

Let $\alpha \in R\ and\ b = (b_1, b_2, b_3) \in V.\ Then\ b_1 = 0 \implies \alpha b_1 = 0$

$\alpha b = (\alpha b_1, \alpha b_2, \alpha b_3) \in V\ as\ \ \alpha b_1 = 0$

So, V is a subspace of R^3.

Sol. (b).

Let $V = \{The\ plane\ of\ vectors\ b = (b_1, b_2, b_3)\ with\ first\ component\ b_1 = 1\}$.

$= \{(1,0,0), (1,1,0), (1,1,2), \ldots\ldots\ldots\}$

So, V is a nonempty subset of R^3.

1. Vector addition:

Let $b = (b_1, b_2, b_3)\ and\ c = (c_1, c_2, c_3) \in V.\ Then\ b_1 = 1\ and\ c_1 = 1 \implies$

$b_1 + c_1 = 2.$

$b + c = (b_1 + c_1, b_2 + c_2, b_3 + c_3) \notin V\ as\ b_1 + c_1 \neq 1.$

So, V does not satisfy vector addition property.

Hence, V is not a subspace of R^3.

Column Space and Null Space of a Matrix

Column Space of a matrix is all linear combination of the columns of A. and denoted by C(A).
Column Space of Matrix Let $C_1, C_2, ..., C_n$ be 1st column, 2nd column,....,nth column of the matrix $A_{m\times n}$
then $C(A) = \{a_1C_1 + a_2C_2 + ... + a_nC_n/a_1, a_2, ...a_n \in R\}$
where R is set of real numbers.

Steps for finding C($A_{m\times n}$)
Given: Suppose we are given a matrix A
Output: C(A)
Step 1: find Echelon form of A ,say U is echelon form of A
step 2: find the pivot column in U
step 3: then C(A) is linear combination of those column of A which are corresponding to pivot column of U.
For Ex Let 1st and 5th are only pivot column in U,then $C(A) = \{a_1C_1 + a_5C_5/a_1, a_5 \in R\}$

Note : The system $Ax = b$ is solvable iff the vector b can be expressed as a combination of the columns of A . then b is in the column cpace of A
Note : C(A) is a subspace of R^m

Null Space of Matrix
let A be $m \times n$ matrices.then
Null Space of A consists of all vectors x such that Ax=0
and denoted by N(A).
i.e. $N(A) = \{x \in R^n/Ax = 0\}$
Note : N(A) is subspace of R^n

Exercise 2.1.5 : find the column space and null space of the matrices
(a):

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Sol: Since echelon form of A is itself A. i.e. $U = A$
and first column of U is pivot. so

$$C(A) = \left\{ a \begin{bmatrix} 1 \\ 0 \end{bmatrix} /a \in R \right\}$$

for the null space solve $Ax = 0$

$Aug.matrix = [A \;\; 0] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

since since x_2 is free variable , so assume $x_2 = k$ where k is real number.so $x_1 - k = 0$, $x_1 k$

$N(A) = \left\{ \begin{bmatrix} k \\ k \end{bmatrix} /k \in R \right\} = \left\{ k \begin{bmatrix} 1 \\ 1 \end{bmatrix} /k \in R \right\}$

(b)

$$B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} = U$$

Since first and third columns are pivot in U.

So $C(B) = \left\{ a \begin{bmatrix} 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ 3 \end{bmatrix} \mid a, b \in R \right\}$

for null space solve $BX = 0$

Aug. matrix= $[B \mid 0] = \left[\begin{array}{ccc|c} 0 & 0 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$ Since x_2 is free variable so $x_2 = k, K \in R$

$$0x_1 + 0x_2 + 3x_3 = 0 \Rightarrow x_3 = 0$$
$$x_1 + 2k + 3x_3 = 0$$
$$x_1 + 2k + 3 \times 0 = 0$$
$$x_1 = -2k$$

$N(A) = \left\{ \begin{bmatrix} -2k \\ k \\ 0 \end{bmatrix} \mid k \in R \right\} = \left\{ k \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \mid k \in R \right\}$

(c)

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$C(A) = \left\{ a \begin{bmatrix} 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mid a, b, c \in R \right\}$$
$$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

Since x_1, x_2, x_3 are free variable.
So $x_1 = k_1,\ x_2 = k_2,\ x_3 = k_3,\ k_1, k_2, k_3 \in R$.

So $N(A) = \left\{ \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \mid k_1, k_2, k_3 \in R \right\} = R^3$

Exercise 2.1.24: For which Right hand side(find a condition on b_1, b_2, b_3) are these systems solvable?

(a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Aug. matrix $= [A|b] = \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right] R_1{+}R_3 \to R_3, R_2{-}2R_1 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_1 + b_3 \end{array}\right]$

i.e

$$1x_1 + 4x_2 + 2x_3 = b_1$$

$$0x_1 + 0x_2 + 0x_3 = b_2 - 2b_1$$

$$0x_1 + 0x_2 + 0x_3 = b_1 + b_3$$

solution exist only if $b_2 - 2b_1 = 0$, $b_1 + b_3 = 0$

$\Rightarrow b_2 = 2b_1$ and $b_3 = -b_1$

(b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Aug. matrix $= [A|b] = \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right] R_2 - 2R_1 \to R_2 \sim \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_1 + b_3 \end{array}\right]$

i.e

$$1x_1 + 4x_2 = b_1$$

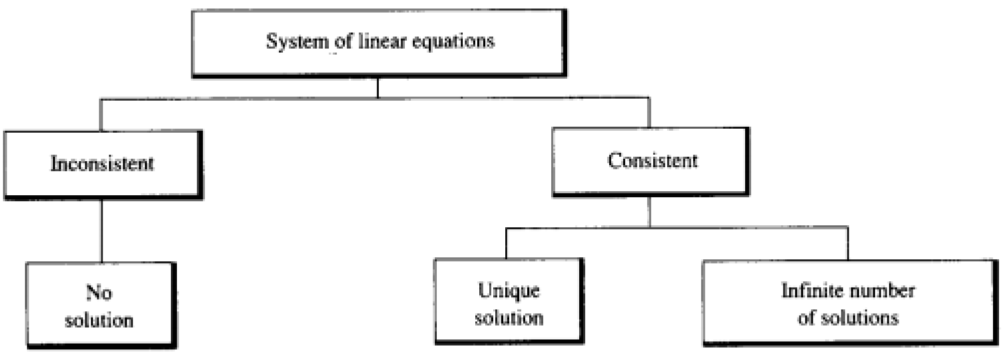
$$0x_1 + 1x_2 = b_2 - 2b_1$$

$$0x_1 + 0x_2 = b_1 + b_3$$

solution exist only if $b_3 + b_1 = 0$

2.2 - Solving Ax=0 And Ax=b

Vocab : Cofficient Mtrix,Augmented Matrix,EchelonForm,Row Reduced Form ,Rank,Pivot Variable,free Variable



Ex 1 - Consider a System of Linear Equation

$$\left.\begin{aligned} & x_1 - 2x_2 + x_3 = 0 \\ & 2x_2 - 8x_3 = 8 \\ & -4x_1 + 5x_2 + 9x_3 = -9 \end{aligned}\right\} \tag{1}$$

Solution: The elimination procedure is shown here with and without matrix notation and the results are placed side by side for comparison:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

Keep x_1 is the first equation and eliminate it from the other equations. To do so, add 4 times equation 1 to equation 3. After some practice, this type of calculation is usually performed mentally:

$$\begin{array}{rc}
4.[\textit{equation 1}]: & 4x_1 - 8x_2 + 4x_3 = 0 \\
+[\textit{equation 3}]: & \underline{-4x_1 + 5x_2 + 9x_3 = -9} \\
\textit{new equation 3}: & -3x_2 + 13x_3 = -9
\end{array}$$

The result of this calculation is written in place of the original third equation:

$$\begin{array}{r} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

Now, multiply equation 2 by 1/2 in order to obtain 1 as the coefficient for x_2. (This calculation will simplify the arithmatic in the next step.)

$$\begin{array}{r} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

Use the x_2 in equation 2 to eliminate the $-3x_2$ in equation 3. The "mental" computation is

$$\begin{array}{rc}
3.[\textit{equation 2}]: & 3x_2 - 12x_3 = 12 \\
+[\textit{equation 3}]: & \underline{-3x_2 + 13x_3 = -9} \\
\textit{new equation 3}: & x_3 = 3
\end{array}$$

The new system has a triangular form.

$$\begin{array}{r} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Eventually, you want to eliminate the $-2x_2$ term from equation 1, but it is more efficient to use the x_3 in equation 3 first, to eliminate the $-4x_3$ and $+x_3$ terms in equation 2 and 1. The two "mental" calculations are

$$\begin{array}{rc}
4.[\textit{equation 3}]: & 4x_3 = 12 \\
+[\textit{equation 2}]: & \underline{x_2 - 4x_3 = 4} \\
\textit{new equation 2}: & x_2 = 16
\end{array} \qquad
\begin{array}{rc}
-1.[\textit{equation 3}]: & -x_3 = -3 \\
+[\textit{equation 1}]: & \underline{x_1 - 2x_2 + x_3 = 0} \\
\textit{new equation 1}: & x_1 - 2x_2 = -3
\end{array}$$

It is convenient to combine the results of these two operations:

$$\begin{array}{r} x_1 - 2x_2 = -3 \\ x_2 = 16 \\ x_3 = 3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Now, having cleaned out the column above the x_3 in equation 3, move back to the x_2 in equation 2 and use it to eliminate the $-2x_2$ above it. Because of the previous work with x_3, there is now no arithmetic involving x_3 terms. Add 2 times equations 2 to equation 1 and obtain the system:

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Ex 2 - A system of linear equations is a list of linear equations with the same unknowns. In particular, a system of 2 linear equations in 2 unknowns x_1, x_2can be put in the standard form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{aligned}\right\} \tag{2}$$

where a_{ij}, b_i are constant.and we can rewrite system (2) as :

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{3}$$

again we can rewrite system (3)(without using unknown,for the simplicity) to as

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right] \tag{4}$$

$$A = \left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right], C = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where A is called **Augmented Matrix** and C is called **Cofficient Matrix** of the system

$$\begin{aligned} \left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right] \frac{1}{a_{11}} R_1 \to R_1 &\sim \left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ a_{21} & a_{22} & b_2 \end{array}\right] \\ R_2 - a_{21}R_1 \to R_2 &\sim \left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & a_{22} - \frac{a_{21}a_{12}}{a_{11}} & b_2 - \frac{a_{21}b_1}{a_{11}} \end{array}\right] \end{aligned} \tag{5}$$

(5) can be written as

$$\left.\begin{aligned} & x_1 + \tfrac{a_{12}}{a_{11}} x_2 = \tfrac{b_1}{a_{11}} \\ & 0x_1 + (a_{22} - \tfrac{a_{21}a_{12}}{a_{11}})x_2 = b_2 - \tfrac{a_{21}b_1}{a_{11}} \end{aligned}\right\} \tag{6}$$

Case 1 : if $a_{22} - \frac{a_{21}a_{12}}{a_{11}} \neq 0$ then $x_2 = \frac{b_2 - \frac{a_{21}b_1}{a_{11}}}{a_{22} - \frac{a_{21}a_{12}}{a_{11}}}$, x_1 ca be calculated from $x_1 + \frac{a_{12}}{a_{11}} x_2 = \frac{b_1}{a_{11}}$
Conclusion- unique solution of (2)

Case 2 : if $a_{22} - \frac{a_{21}a_{12}}{a_{11}} = 0$ and $b_2 - \frac{a_{21}b_1}{a_{11}} = 0$ then second equation of (7) becomes $0x_1 + 0x_2 = 0$
Conclution - infinite solution of (2)

Case 3 : if $a_{22} - \frac{a_{21}a_{12}}{a_{11}} = 0$ and $b_2 - \frac{a_{21}b_1}{a_{11}} \neq 0$ then second equation of (6)becomes $0x_1 + 0x_2 = b_2 - \frac{a_{21}b_1}{a_{11}}$ i.e. $0 = b_2 - \frac{a_{21}b_1}{a_{11}}$ which is not true
Conclution - solution does not exit of (2).

Elementary Row Operations

Suppose A is a matrix with rows R_1, R_2,..., R_m. The following operations on A are called elementary row operations.

$[E_1]$ (Row Interchange): Interchange rows R_i and R_j. This may be written as
"Interchange R_i and R_j" or "$R_i \leftrightarrow R_j$"

$[E_2]$ (Row Scaling): Replace row R_i by a nonzero multiple kR_i of itself. This may be written as
" Replace R_i by $kR_i (k \neq 0)$ or "$kR_i \rightarrow R_i$"

$[E_3]$ (Row Addition): Replace row R_j by the sum of a multiple kR_i of a row R_i and itself. This may be written as
" Replace R_j by $kR_i + R_j$" or "$kR_i + R_j \rightarrow R_j$.

The arrow $\rightarrow$ in E_2 and E_3 may be read as "replaces".

Sometimes (say to avoid fractions when all the given scalars are integers) we may apply $[E_2]$ and $[E_3]$ in one step; that is, we may apply the following operation:

$[E]$ Replace R_j by the sum of a multiple kR_i of a row R_i and a nonzero multiple $k'R_j$ of itself. This may ge written as
"Replace R_j by $kR_i + k'R_j$ $(k' \neq 0)$" or "$kR_i + k'R_j \rightarrow R_j$"

We emphasize that in operations $[E_3]$ and [E] only row R_j is changed.

Echelon Matrices (or in echelon form) U and Row Reduced Form R

Echelon Matrices U

A Matrix U is called an echelon matrix or is said to be in echelon form , if the following two conditions hold :

(1) All zero rows,if any, are at the bottom of the matrix.

(2) Each leading nonzero entry in a row is to the right of the leading nonzero entry in the preceding row.

Row Reduced Form R

A Matrix is said to be in row reduced form R if it is an echelon matrix and if satisfies the following additional two properties:

(3)Each pivot(leading nonzero entry) is equal to 1.

(4) Each pivot is the only nonzero entry in its column.

EX 3 The following is an echelon matrix whose pivots have been circled

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 & 5 & 9 & 0 & 7 \\ 0 & 0 & 0 & 3 & 4 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 5 & 7 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

NOTE 1 - The major difference between an echelon matrix in row reduced form is that in an echelon matrix there must be zeros below the pivots [properties(1)and (2)] but in a matrix in row reduced form , each pivot must also equal 1 [property (3)] and there must also be zeros above the pivots [properties(4)].

Ex-4 The following are echelon matrices whose pivots have been circled

$$\begin{bmatrix} 2 & 3 & 2 & 0 & 4 & 5 & -6 \\ 0 & 0 & 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The Third matrix is also an example of a matrix in row reduced form. the second matrix is not in row reduced form ,since it does not satisfy property(4),taht is,there is a nonzero entry above the second pivot in the third column.The first matrix is not in row reduced form, because it satisfies neither property (3) nor property (4); that is, some pivots are not equal to 1 and there are nonzero entries above the pivots.

Ex-5 The entries of a 5 by 8 echelon matrix U and its reduced form R

$$U = \begin{bmatrix} \bullet & * & * & * & * & * & * & * \\ 0 & \bullet & * & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} \mathbf{1} & \mathbf{0} & * & \mathbf{0} & * & * & * & \mathbf{0} \\ 0 & \mathbf{1} & * & \mathbf{0} & * & * & * & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{1} & * & * & * & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot Variable and Free Variable

Pivot Variable: Pivot Variable are those variable that correspond to columns with pivots.
Free Variable : Free Variable are those variable that correspond to columns without pivots.

Note : If $Ax = 0$ has more unknowns than equations $(n > m)$, it has at least one special solution: There are more solutions than the trivial $x = 0$.
Note : $x_{complete} = x_{particlar} + x_{nullspace}$
Note : if there are n column in a matrix A and there are r pivots then there are r pivot variables and $n - r$ free variable.and this important number r is called **Rank** of a Matrix.

Rank of a Matrix = The rank of a matrix A, written rank(A), is equal to the maximum number of linearly independent columns of A
= number of pivot column in the echelon form of a matrix A
=maximum number of linearly independent rows of A
= dimension of the column space of A
= dimension of the row space of A.

Note : Let A be an n-square matrix. then A is invertible if and only if $rank(A) = n$

Ex 6: Find Rank of A

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Sol. Since Echelon form of A is itself A.and 1st and 3rd column are pivot column.
So Rank of A is 2.

Method for solving System of linear equation

Method-1

Ex 7 - Consider a System of linear equation

$$\left.\begin{aligned} 1x_1 + 2x_2 + 3x_3 + 5x_4 = b_1 \\ 2x_1 + 4x_2 + 8x_3 12x_4 = b_2 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 = b_3 \end{aligned}\right\} \tag{7}$$

Sol.

Step 1: Reduce $Ax = b$ to $Ux = c$

i.e. Reduce Augmented Matrix [A b] to Augmented Matrix [U c]

$$\begin{bmatrix} A & b \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array}\right] R_2 - 2R_1 \to R_2, R_3 - 3R_1 \to R_3 \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & -2 & -2 & b_3 - 3b1 \end{array}\right]$$
$$R_3 + R_2 \to R_3 \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array}\right] \tag{8}$$
$$= \begin{bmatrix} U & c \end{bmatrix}$$

$$\left\{\begin{array}{l} (8) means \\ 1x_1 + 2x_2 + 3x_3 + 5x_4 = b_1 \\ 0x_1 + 0x_2 + 2x_3 + 2x_4 = b_2 - 2b_1 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 = b_3 + b_2 - 5b_1 \\ \textbf{third equation hold only if } b_3 + b_2 - 5b_1 = 0 \\ \text{it means if } b_3 + b_2 - 5b_1 = 0 \text{ then system of equation has infinite solution.} \\ \text{if } b_3 + b_2 - 5b_1 \neq 0 \text{ then system of equation has no solution.} \end{array}\right\}$$

Here

$$U = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 + b_2 - 5b_1 \end{bmatrix}$$

Step 2 :

Find Special Solution : $Ux = 0$

Take particularly $b_1 = 0$,$b_2 = 6$,$b_3 = -6$

$$\begin{bmatrix} U & 0 \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{9}$$

Here x_2 **and** x_4 **are free variables**
Let $x_2 = a$,$x_4 = b$ where a,b belongs to Set of Real Number
Now we can rewrite (10) as
$1x_1 + 2x_2 + 3x_3 + 5x_4 = 0$——(*)
$0x_1 + 0x_2 + 2x_3 + 2x_4 = 0$——(**)
now put the value of x_2 in (**)
$2x_3 + 2b = 0$
i.e. $x_3 = -b$
now put the value of x_3 in (*)
$x_1 + 2a - 3b + 5b = 0$
i.e. $x_1 + 2a + 2b = 0$
i.e. $x_1 = -2a - 2b$
Special Solution $x_n = \begin{bmatrix} -2a-2b \\ a \\ -b \\ b \end{bmatrix}$
$x_n = a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ where a, b belongs to set of real number
Step 3 :
Find Particular Solution x_p**,** $Ux_p = c$ **and put all free variables**$= 0$

So put $x_2 = a = 0, x_4 = b = 0$

$$\begin{bmatrix} U & c \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{10}$$

(10) Can be rewritten as
$1x_1 + 2x_2 + 3x_3 + 5x_4 = 0$——(*)
$0x_1 + 0x_2 + 2x_3 + 2x_4 = 6$——(**)
Now put $b = 0$ in (**)
$2x_3 + 0 = 6$
$x_3 = 3$
Now $a = 0$,$x_3 = 3$,$b = 0$ in (*)
$x_1 0 + 9 + 0 = 0$
$x_1 = -9$

$$x_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Step 4 :

Complete Solution $x = x_n + x_p = a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2a-2b-9 \\ a \\ -b+3 \\ b \end{bmatrix}$

$$= \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2a \\ a \\ 0a \\ 0a \end{bmatrix} + \begin{bmatrix} -2b \\ 0b \\ b \\ -b \end{bmatrix}$$

$$= \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

where a, b belongs to Set of real numbers

Method-2

Ex - Consider a System of linear equation

$$\left.\begin{aligned} 1x_1 + 2x_2 + 3x_3 + 5x_4 &= 0 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= 6 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 &= -6 \end{aligned}\right\} \tag{11}$$

Step 1: Reduce $Ax = b$ to $Ux = c$

i.e. Reduce Augmented Matrix [A b] to Augmented Matrix [U c]

$$\begin{aligned} \begin{bmatrix} A & b \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 2 & 4 & 8 & 12 & 6 \\ 3 & 6 & 7 & 13 & -6 \end{array}\right] R_2 - 2R_1 \rightarrow R_2, R_3 - 3R_1 \rightarrow R_3 \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & -2 & -2 & -6 \end{array}\right] \\ R_3 + R_2 \rightarrow R_3 \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ = \begin{bmatrix} U & c \end{bmatrix} \end{aligned} \tag{12}$$

$$\left\{\begin{array}{l} (12) means \\ 1x_1 + 2x_2 + 3x_3 + 5x_4 = 0 \\ 0x_1 + 0x_2 + 2x_3 + 2x_4 = 6 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 = 0 \end{array}\right\}$$

Here

$$U = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, c = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

Step 2:
Here x_2 **and** x_4 **are free variables**
Let $x_2 = a$,$x_4 = b$ where a,b belongs to Set of Real Number
Now we can rewrite (12) as
$1x_1 + 2x_2 + 3x_3 + 5x_4 = 0$—–(*)
$0x_1 + 0x_2 + 2x_3 + 2x_4 = 6$—–(**)
now put the value of x_2 in (**)
$2x_3 + 2b = 6$
i.e. $x_3 = 3 - b$
now put the value of x_3 in (*)
$x_1 + 2a + 3(3 - b) + 5b = 0$
i.e. $x_1 + 2a + 9 + 2b = 0$
i.e. $x_1 = -9 - 2a - 2b$

Complete Solution $x = x_n + x_p = \begin{bmatrix} -2a - 2b - 9 \\ a \\ -b + 3 \\ b \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2a \\ a \\ 0a \\ 0a \end{bmatrix} + \begin{bmatrix} -2b \\ 0b \\ b \\ -b \end{bmatrix}$

$= \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$
where a, b belongs to Set of real numbers

Exercise 2.2.1 : find the value of c that makes it possible to solve $Ax = b$, and solve it:

$$u + v + 2w = 2$$

$$2u + 3v - w = 5$$

$$3u + 4u + w = c$$

Solution Aug matrix $= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 3 & -1 & 5 \\ 3 & 4 & 1 & c \end{array}\right] R_2 - 2R_1 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 1 & -5 & c \end{array}\right] R_3 - R_2 \to R_2 \sim$ $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & c-7 \end{array}\right]$ Solution Exit only if $c - 7 = 0$ so assume $w = k \in R$

$$v - 5w = 1$$

$$v - 5k = 1$$

$$v = 1 + 5k$$

$$u + v + 2w = 2$$

$$u + (1 + 5k) + 2k = 2$$

$$u = 1 - 7k$$

$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1-7k \\ 1+5k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + k \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}$ where $k \in R$

Exercise 2.2.4 Write the complete solution $x = x_p + x_n$ to these systems ,(as in equation (4))

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix},$$

Solution (1) Aug matrix $= \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 4 & 5 & 4 \end{array}\right] R_2 - 2R_1 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$
Since v is free variable so take $v = k$, $k \in R$

$$w = 2$$

$$u + 2v + 2w = 1$$

$$u + 2k + 4 = 1$$

$$u = -3 - 2k$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -3-2k \\ k \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + k \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \text{ where } k \in R$$

(2) Aug matrix $= \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 4 & 4 & 4 \end{array}\right] R_2 - 2R_1 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$
i.e.

$$u + 2v + 2w = 1$$

$$0u + 0v + 0w = 2$$

i.e. $0 = 2$ which is not true.
So there is no solution.

Exercise 2.2.5 Reduce A and B to echelon form, to find their ranks, which variables are free ?
$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ find the special solutions to $Ax = 0$ and $Bx = 0$. find all solutions.

Solution:(1) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} R_3 - R_1 \to R_2 \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$. Since first two column in U are L.I. So $\rho(A) = 2$.
Now for solving $Ax = 0$.
Aug. matrix $= [A|0] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \end{array}\right] R_3 - R_1 \to R_3 \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$
Since x_3 and x_4 are free variable; so assume $x_3 = k_1$, $x_4 = k_2$, where $k_1, k_2 \in R$
$x_2 + x_3 = 0 \Rightarrow x_2 + k = 0 \Rightarrow x_2 = -k$

$$x_1 + 2x_2 + x_4 = 0$$

$$x_1 - 2k_1 + k_2 = 0$$

$$x_1 = 2k_1 - k_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2k_1 - k_2 \\ -k_1 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $k_1, k_2 \in R$.

This is general solution.

Hence special solutions are $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(2) $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} R_3 - 7R_1 \to R_3, R_2 - 4R_1 \to R_2 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} R_3 - 2R_2 \to R_3 \sim$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} = U$
Since U has two pivot columns, so $\rho(B) = 2$.
for solving $Bx = 0$
Aug. matrix $= [B|0] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] R_3 - 7R_1 \to R_3, R_2 - 4R_1 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] R_3 -$
$2R_2 \to R_3 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$
Since x_3 is free variable.
So $x_3 = k, k \in R$.

$$-3x_2 - 6x_3 = 0$$

$$x_2 = -2k$$

$$x_1 + 2x_2 + 3x_3 = 0$$

$$x_1 - 4k + 3k = 0$$

$$x_1 - k = 0 \Rightarrow x_1 = k.$$

General solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, k \in R$. Special solution is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

PROBLEM SET 2.2 CONTINUED...

Exercise 2.2.13: (a) find the special solutions to $Ux = 0$, Reduce U to R and repeat

$$Ux = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) If the Right hand side is changed from (0,0,0) to (a,b,0) what is solution?

Solution: (a) Aug matrix $= [U|0] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Since x_2, x_4 are free variables. So assume $x_2 = a$, $x_4 = b$, $a, b \in R$.

$$x_3 + 2x_4 = 0$$

$$x_3 = -2b$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

$$x_1 - 2a - 6b + 4b = 0$$

$$x_1 - 2a - 2b = 0$$

$$x_1 = 2a + 2b.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2a + 2b \\ a \\ -2b \\ b \end{bmatrix} = a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}, a, b \in R.$$

Since $U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 - 3R_2 \to R_1 \sim \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$

(b) $Ux = b$

Aug matrix $= [U|b] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & a \\ 0 & 0 & 1 & 2 & b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Since x_2, x_4 are free variables. So assume $x_2 = c$, $x_4 = d$, $c, d \in R$.

$$x_2 + 2x_4 = b$$

$$x_3 + 2d = b \Rightarrow x_3 = b - 2d$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = a$$

$$x_1 + 2c + 3(b - 2d) + 4d = a$$

$$x_1 + 2c + 3b - 6d + 4d = a$$

$$x_1 + 2c + 3b - 2d = a$$

$$x_1 = a - 3b - 2c + 2d$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a - 3b - 2c + 2d \\ c \\ b - 2d \\ d \end{bmatrix} = \begin{bmatrix} a - 3b \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}, c, d \in R.$$

Exercise 2.2.34: What conditions on b_1, b_2, b_3, b_4 make each system solvable? solve for x,

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Solutions:(a) Aug. matrix $= \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{array}\right]$ $R_4 - 3R_1 \to R_4, R_2 - 2R_1 \to R_2, R_3 - 2R_1 \to R_3 \sim$

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 3 & b_4 - 3b_1 \end{array}\right]$$

$$R_4 - 3R_3 \to R_3 \sim \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 0 & b_4 - 3b_3 + 3b_1 \end{array}\right]$$

solution exist only if $\begin{array}{cc} b_2 - 2b_1 = 0, & b_4 - 3b_3 + 3b_1 = 0 \\ b_2 = 2b_1, & b_4 = b_1 + b_3 \end{array}$

Since

$$x_2 = b_3 - 2b_1$$

$$x_1 + 2x_2 = b_1$$

$$x_1 + 2(b_3 - 2b_1) = b_1$$

$$x_1 + 2b_3 - 4b_1 = b_1$$

$$x_1 = 5b_1 - 2b_3$$

General solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix}$

(b)Aug. matrix $= \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{array}\right]$ $R_4 - 3R_1 \to R_4, R_2 - 2R_1 \to R_2,$

$R_3 - 2R_1 \to R_3 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \end{array}\right]$ $R_4 - 3R_3 \to R_4 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 0 & 0 & b_4 - 3b_3 + 3b_1 \end{array}\right]$

solution exist only if $b_2 - 2b_1 = 0$ and $b_4 - 3b_3 + 3b_1 = 0$ for the general solution.
Since x_3 is free variable assume $x_3 = k, k \in R$.

$$x_2 + k = b_3 - 2b_1$$

$$x_2 = b_3 - 2b_1 - k$$

$$x_1 + 2x_2 + 3x_3 = b_1$$

$$x_1 + 2(b_3 - 2b_1 - k) + 3k = b_1$$

$$x_1 + 2b_3 - 4b_1 - 2k + 3k = b_1$$

$$x_1 = 5b_1 - 2b_3 - k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 - k \\ b_3 - 2b_1 - k \\ k \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + k \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

PROBLEM SET 2.2 CONTINUED...

Excercise 2.2.44 Choose the number q so that (if possible) the ranks (a) 1 (b) 2 (c) 3

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}, C = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}$$

Solution (1) $A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} 2R_2 + R_1 \to R_2, 6R_3 - 9R_1 \to R_3 \sim \begin{bmatrix} 6 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 6q-18 \end{bmatrix}$

$R_3 \leftrightarrow R_2 \sim \begin{bmatrix} 6 & 4 & 2 \\ 0 & 0 & 6q-18 \\ 0 & 0 & 0 \end{bmatrix}$

(a) rank of A is 1 if $q = 3$

(b) rank of A is 2 if $q \neq 3$

(c) rank of A is 3 not possible.

(2) $B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix} 3R_2 - qR_1 \to R_2 \sim \begin{bmatrix} 3 & 1 & 3 \\ 0 & 6-q & 0 \end{bmatrix}$

(a) rank of B is 1 if $q = 6$

(b) rank of B is 2 if $q \neq 6$

(c) rank of B is 3 not possible.

Excercise 2.2.54 : True or False? (Give reason if true, or counterexample to show it is false.)

(a) A square matrix has no free variables.

(b) An invertible matrix has no free variables.

(c) An m by n matrix has no more than n pivot variables.

(d) An m by n matrix has no more than m pivot variables.

Solution :

(a) A square matrix has no free variables.

Ans. False , because $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

(b) An invertible matrix has no free variables.

Ans. True , a matrix A is invertible if and only if its columns are linearly independent.so all column has pivot.so invertible matrix has no free variables.

(c) An m by n matrix has no more than n pivot variables.

Ans. True , Since n is number of columns. and each column has atmost 1 pivot. An m by n matrix has no more than n pivot variables.

(d) An m by n matrix has no more than m pivot variables.

Ans. True , Since m is number of rows. and each row has atmost 1 pivot.So An m by n matrix has no more than m pivot variables.

Excercise 2.2.59: The equation $x - 3y - z = 0$ determines a plane in R^3. What is the matrix A in this equation? Which are the free variables? The special solutions are(3,1,0) and (...), the parallel plane $x - 3y - z = 12$ contains the particular point (12,0,0) all points on this plane have the following form (fill in the first component).

Solution: Since $\begin{matrix} x - 3y - z = 0 \\ x = 3y + z, \quad y, z \in R \end{matrix}$ Consider

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y + z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence special solutions are $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ We have to find a matrix A whose special solutions are $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Consider $A \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = 0$

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$3a_{11} + a_{12} = 0 \Rightarrow a_{12} = -3a_{11}$$

$$3a_{21} + a_{22} = 0 \Rightarrow a_{22} = -3a_{21}$$

$$3a_{31} + a_{32} = 0 \Rightarrow a_{32} = -3a_{31}$$

Consider $A\begin{bmatrix}1\\0\\1\end{bmatrix}=0$

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$a_{11}+a_{13}=0 \Rightarrow a_{13}=-a_{11}$$

$$a_{21}+a_{23}=0 \Rightarrow a_{23}=-a_{21}$$

$$a_{31}+a_{33}=0 \Rightarrow a_{33}=-a_{31}$$

So $A=\begin{bmatrix}a_{11} & -3a_{11} & -a_{11}\\ a_{21} & -3a_{21} & -a_{21}\\ a_{31} & -3a_{31} & -a_{31}\end{bmatrix}$

given parallel plane is $x-3y-z=12$

$$x=3y+z+12.$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}12+3y+z\\y\\z\end{bmatrix}=\begin{bmatrix}12\\0\\0\end{bmatrix}+y\begin{bmatrix}3\\1\\0\end{bmatrix}+z\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Linearly independent and dependent

Linearly independent
A subset $\{v_1, v_2, ...v_n\}$ of a vector space V is said to be linearly independent if whenever $c_1, c_2, ...c_n \in R$ such that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$ then $c_1 = c_2 = ... = c_n = 0$

Linearly dependent
A non empty finite subset $\{v_1, v_2, ...v_n\}$ of a vector space V is said to be linearly dependent if there exists scalars $c_1, c_2, ...c_n \in R$ (**not all zero**)such that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$

Ex 1 if v_1 = zero vector , then the set is linearly dependent . we may choose $c_1 = 1$ and all other $c_i = 0$, this is a non trivial combination that produces zero.
i.e. $1v_1 + 0v_2 + ... + 0v_n = 1 \times 0 + 0 + ... + 0 = 0$

Ex 2 : The Column of the Matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

are linearly dependent , since the 2nd column is 3 times the first,the combination of columns with weights -3,1,0,0 gives the zero vector. i.e. say $A = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \end{bmatrix}$, then $-3C_1 + 1C_2 + 0C_3 + 0C_4 = 0$
The rows are also linearly dependent, row 3 is two times row 2 minus five times row1. i.e. say $A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$, then $R_3 - 2R_2 + 5R_1 = 0$

EX 3
The Column of this Triangular Matrix are Linearly Independent

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

Consider a linear combination of the columnsthat makes zero
Solve $Ac = 0$

$$c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

it means $3c_1 + 4c_2 + 2c_3 = 0$, $0c_1 + 1c_2 + 5c_3 = 0$, $0c_1 + 0c_2 + 2c_3 = 0$ i.e.

$$c_3 = 0, c_2 = 0, c_1 = 0$$

So column of A are Linearly Dapandent.
and **null space of A contains only zero vector**

A similar reasoning applies to the rows of A, which are also independent. Suppose

$$c_1(3, 4, 2) + c_2(0, 1, 5) + c_3(0, 0, 2) = (0, 0, 0)$$

. From the first components we find $3c_1 = 0$ or$c_1 = 0$. Then the second components give $c_2 = 0$, and finally $c_3 = 0$.

Note : The columns of A are independent exactly when $N(A) = \{zerovector\}$
Note : It is the columns with pivots that are guaranteed to be independent

Ex 4 The columns of the n by n identity matrix are independent:

$$I = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1 & . & 0 \\ . & . & . & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note : To check any set of vectors $v_1, ..., v_n$ for independence, put them in the columns of A.Then solve the system $Ac = 0$;
1. The vectors are dependent if there is a solution other than c = 0.
2. With no free variables (rank n), there is no nullspace except c = 0; (i.e. $N(A) = \{0\}$)the vectors are independent.
3. If the rank is less than n, at least one free variable can be nonzero and the columns are dependent.

Note :A set of n vectors in R^m must be linearly dependent if $n > m$.

Ex 5 These three column in R^2 can not be independent:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

Sol : To find the combination of the columns producing zero we solve $Ac = 0$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} R_2 - R_1 \to R_2 \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = U$$

If we give the value 1 to the free variable c_3, then back-substitution in $Uc = 0$ gives $c_2 = -1$, $c_1 = 1$
i.e. if $A = \left[C_1, C_2, C_3\right]$then $C_1 - C_2 + C_3 = 0$

Exercise 2.3.1: Choose three independent columns of V, then make two other choices. Do the same for A. You have found bases for which spaces?

$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}$

Solution: Let $U = [U_1 \;\; U_2 \;\; U_3 \;\; U_4]$ $A = [C_1 \;\; C_2 \;\; C_3 \;\; C_4]$ Consider, $A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}$ $R_4 {-} 2R_1 \to$

$R_4 \sim \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$

i.e. U is echelon form of A.
Note: Columns of A which have pivot are linearly independent.
Case (i) U_1, U_2, U_4 are L.I.(using the note).
Case (ii) U_1, U_3, U_4 are L.I.
as consider $aU_1 + bU_3 + cU_4 = 0$

$$a \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 4 \\ 7 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a + 4b + c \\ 0a + 7b + 0c \\ 0a + 0b + 9c \\ 0a + 0b + 0c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 9c = 0 \\ 7b = 0 \\ 2a + 4b + c = 0 \end{matrix} \Rightarrow \begin{matrix} c = 0 \\ b = 0 \\ a = 0 \end{matrix}$$

Case (iii) U_1, U_3, U_4 are L.I.
as consider $aU_2 + bU_3 + cU_4 = 0$

$$a \begin{bmatrix} 3 \\ 6 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 4 \\ 7 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3a + 4b + c \\ 6a + 7b + 0c \\ 0a + 0b + 9c \\ 0a + 0b + 0c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 9c = 0 \\ 6a + 7b = 0 \\ 3a + 4b + c = 0 \end{matrix} \Rightarrow \begin{matrix} c = 0 \\ b = 0 \\ a = 0 \end{matrix}$$

Note: Columns of a matrix A are linearly independent which are corresponding to the pivot column of echelon matrix of A.
Case (i) C_1, C_2, C_4 are L.I.(using the note).
Case (ii) C_1, C_3, C_4 are L.I.
as we can see consider $aC_1 + bC_3 + cC_4 = 0$

$$a\begin{bmatrix}2\\0\\0\\4\end{bmatrix} + b\begin{bmatrix}4\\7\\0\\8\end{bmatrix} + c\begin{bmatrix}1\\0\\9\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}2&4&1\\0&7&0\\0&0&9\\4&8&2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad i.e \;\; S\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Aug. matrix=$[S|0]$ $\left[\begin{array}{ccc|c}2&4&1&0\\0&7&0&0\\0&0&9&0\\4&8&2&0\end{array}\right]$ $R_4 - 2R_1 \to R_4 \sim \left[\begin{array}{ccc|c}2&4&1&0\\0&7&0&0\\0&0&9&0\\0&0&0&0\end{array}\right]$

$$\begin{aligned} 2a+4b+c&=0 & c&=0\\ \Rightarrow \quad 7b&=0 \quad \Rightarrow & b&=0\\ 9c&=0 & a&=0 \end{aligned}$$

Case (iii) C_2, C_3, C_4 are L.I.
consider $aC_2 + bC_3 + cC_4 = 0$

$$\begin{bmatrix}C_2 & C_3 & C_4\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad say \;\; B\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Aug. matrix=$[B|0]$ $\left[\begin{array}{ccc|c}3&4&1&0\\0&6&7&0\\0&0&9&0\\6&8&2&0\end{array}\right]$ $R_4 - 2R_1 \to R_4 \sim \left[\begin{array}{ccc|c}3&4&1&0\\0&6&7&0\\0&0&9&0\\0&0&0&0\end{array}\right]$

$$\begin{aligned} 3a+4b+c&=0 & c&=0\\ \Rightarrow \quad 6b+7c&=0 \quad \Rightarrow & b&=0\\ 9c&=0 & a&=0 \end{aligned}$$

The all three cases, we found bases for $R^{4\times 3}$ space.

Exercise 2.3.3 : Decide the dependence or independence of
(a) the vectors (1,3,2), (2,1,3) and (3,2,1)
(b) the vectors (1,3,-2), (2,1,-3) and (-3,2,1).
Solution:
(a)

$$a\begin{bmatrix}1\\3\\2\end{bmatrix} + b\begin{bmatrix}2\\1\\3\end{bmatrix} + c\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad say \ \ A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Aug. matrix=$[A|0]$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 1 & 2 & 0 \\ 2 & 3 & 1 & 0 \end{array}\right] R_3 - 2R_1 \to R_3, R_2 - 3R_1 \to R_2 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right]$

$-5R_3 + R_2 \to R_3 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & 0 & 18 & 0 \end{array}\right]$

$$\Rightarrow \begin{array}{r} 1a + 2b + 3c = 0 \\ -5b - 7c = 0 \\ 18c = 0 \end{array} \quad \Rightarrow \begin{array}{r} c = 0 \\ b = 0 \\ a = 0 \end{array}$$

Vectors are L.I.

(b) Consider $1(1,-3,2) + 1(2,1,-3) + 1(-3,2,1) = (1+2-3, -3+1+2, 2-3+1) = (0,0,0)$
$\Rightarrow$ Vectors are L.D.

Exercise 2.3.5 : If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$, $v_2 = w_1 - w_3$ and $v_3 = w_1 - w_2$ are dependent. find a combination of $v's$ gives zero.
Solution: Consider $av_1 + bv_2 + cv_3 = 0$
$a(w_2 - w_3) + b(w_1 - w_3) + c(w_1 - w_2) = 0$
$(b+c)w_1 + (a-c)w_2 + (-a-b)w_3 = 0$
Since w_1, w_2, w_3 are L.I.
So $b + c = 0$, $a - c = 0$, $-a - b = 0$
$b = -c$, $a = c$, $b = -a$
$a = -b = c$ take $a = 1$, $b = -1$, $c = 1$
$v_1 - v_2 + v_3 = 0$

Exercise 2.3.8 : Suppose v_1, v_2, v_3, v_4 are vectors in R^3.
(a) thse four vectors are dependent because ......
(b) The two vector v_1 and v_2 will be dependent if .....
(c) The vectors v_1 and (0,0,0) are dependent because .....

Solution: (a) Since $dim(R^3) = 3$
Therefore each base of R^3 contains exactly 3 vectors.
So collection of vectors which are more than 3 are linearly dependent.
So four vectors are L.D.
(b) Let $av_1 + bv_2 = 0$ for $\{v_1, v_2\}$ should be dependent.
So atleast one of a or b is nonzero.
say $a \neq 0$
So $v_1 = \frac{-b}{a}v_2$ So v_1, v_2 are dependent if $\exists \alpha \neq 0$ s.t. $v_1 = \alpha v_2$
(C) Consider $0.v_1 + 1(0,0,0) = (0,0,0)$ $a = 0, b = 1 \neq 0$
So v_1 and (0,0,0) are L.D .

Lecture 16

2.3 Linear Independence, Basis and Dimension

Course Outcomes: Students will have understanding about linear independence,dependence, spanning a subspace, basis and dimension of a vector space.
The aim of this section is to explain and use four ideas:

- Linear Independence or dependence.
- spanning a subspace.
- basis for a subspace.
- dimension of a subspace.

Spanning a Subspace: If $S = \{w_1, w_2, ..., w_l\}$ is a set of vectors in a vector space V, then the **span of S** is the set of all linear combinations of the vectors in S.

$$Span(S) = \{c_1w_1 + c_2w_2 + ... + c_lw_l \mid for\ all\ c_i \in R\}$$

If every vector in a given vector space can be written as the linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.

Notes:

- The column space is spanned by its columns.
- The row space is spanned by its rows.

Basis for a Vector Space: A basis for a vector space V is a subset with a sequence of vectors having two properties at once:

- The vectors are linearly independent.(not too many vectors)
- They span the space V.(not too few vectors)

Notes:

- A basis of a vector space is the maximal independent set.
- A basis of a vector space is also a minimal spanning set.
- Spanning involves the column space and independence involves the null space.
- No elements of a basis will be wasted.

Example: Check whether the following sets are the basis of R^3 or not?

$$\begin{aligned}
(a) B_1 &= \{(1,2,2),(-1,2,1),(0,8,0)\} \\
(b) B_2 &= \{(1,2,2),(-1,2,1),(0,8,6)\} \\
(c) B_3 &= \{(1,2,2),(-1,2,1)\} \\
(d) B_4 &= \{(1,0,0),(0,1,0),(0,0,1)\} \\
(e) B_5 &= \{(1,1,-1),(2,3,4),(4,1,-1),(0,1,-1)\}
\end{aligned}$$

Solution:
(a)

$$\begin{aligned}
Let\ A &= \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 0 \end{bmatrix} \\
\Rightarrow |A| &= -24 \\
\Rightarrow |A| &\neq 0
\end{aligned}$$

The vectors $(1,2,2),(-1,2,1)$ and $(0,8,0)$ are LI.
So, B_1 is a basis of R^3.

Dimension of Vector Spaces: Dimension of a vector space is the maximum number of LI vectors of the vector space.

OR. The no. of elements present in the basis of a vector space is known as the dim. of vector space.

$$dimR = 1, dimR^2 = 2, dimR^3 = 3, ...dimR^n = n$$

Problem Set-2.3

16. Describe the subspace of R^3 (is it a line or a plane or R^3?) spanned by
(a) the two vectors $(1, 1, -1)$ *and* $(-1, -1, 1)$.
(b) the three vectors $(0, 1, 1)$ *and* $(1, 1, 0)$ *and* $(0, 0, 0)$.
(c) the columns of a 3 by 5 echelon matrix with 2 pivots.
(d) all vectors with positive components.
Ans.(a)

$$\begin{aligned} Let\ A &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 + R_1 \end{array} \\ \Rightarrow A &= \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (echelon form) \end{aligned}$$

Here rank of A=1.
$\therefore$ The subspace of R^3 spanned by the two vectors $(1, 1, -1)$ *and* $(-1, -1, 1)$ is a line passing through the origin of R^3.
(b)

$$Let\ A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Interchanging\ R_1\ by\ R_2 we\ have,$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \leftarrow R_3 - R_1 \end{array} \\ \Rightarrow A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \leftarrow R_3 + R_2 \end{array} \\ \Rightarrow A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (echelon form) \end{aligned}$$

Here rank of A=2.
$\therefore$ The subspace of R^3 spanned by the three vectors $(0, 1, 1)$ *and* $(1, 1, 0)$ *and* $(0, 0, 0)$ is a plane of R^3.
(c)

$$Let\ A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here rank of A=2.
$\therefore$ The subspace of R^3 spanned by the the columns of a 3 by 5 echelon matrix with 2 pivots is a plane.

19. Find a basis for the plane $x - 2y + 3z = 0$ in R^3. Then find a basis for the intersection of that plane with the xy-plane. Then find a basis for all vectors perpendicular to the plane.
Ans. The basis for the plane $x - 2y + 3z = 0$ in R^3 is the nullspace of the matrix $A = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$.
The plane $x - 2y + 3z = 0$ in the matrix form can be written as

$$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Here x is the pivot variable and y, z are the free variables.
So, $x = 2y - 3z$

$\therefore x = \begin{bmatrix} 2y - 3z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Basis for the plane is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

A basis for the intersection of the plane $x - 2y + 3z = 0$ with the xy-plane i.e z=0 is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

A basis for all vectors perpendicular to the plane is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}$

23 Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspace.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

Ans.

$$\begin{aligned} Let\ A &= \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \leftarrow R_3 - R_1 \end{matrix} \\ \Rightarrow A &= \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (echelon form) \end{aligned}$$

$$Basis\ for\ C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$Basis\ for\ C(A^T) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

To find basis for nullspace we have,

$$\begin{aligned} Ax &= 0 \\ \Rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow x + 3y + 2z = 0, y + z = 0$$

$$Hence\ x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$Basis\ for\ N(A) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Lecture 17

2.4 The Four Fundamental Subspaces

Course Outcomes: Students will have understanding about four fundamental subspaces of matrices and one sided inverse of rectangular matrices. The four fundamental subspaces of matrices are as follows :

- The column space $C(A)$.
- The null space $N(A)$.
- The row space $C(A^T)$.
- The left null space $N(A^T)$.

The Column Space $C(A)$: The column space of A is denoted by C(A). Its dimension is the rank r.

The Null Space $N(A)$: The nullspace of A is denoted by N(A). Its dimension is $n - r$.

The Row Space $C(A^T)$: The row space of A is the column space of A^T. It is $C(A^T)$, and it is spanned by the rows of A. Its dimension is also r.

The Left Null Space $N(A^T)$: The left nullspace of A is the nullspace of A^T. It contains all vectors y such that $A^T y = 0$, and it is written $N(A^T)$. Its dimension is $m - r$.

Notes

- The nullspace N(A) and row space $C(A^T)$ are subspaces of R^n.
- The left nullspace $N(A^T)$ and column space C(A) are subspaces of R^m.

Existence of Inverses:
let A be a matrix of order $m \times n$ with rank r.
(1) Full row rank r = m. Ax = b has at least one solution x for every b if and only if the columns span R^m. Then A has a right-inverse C such that $AC = I_m$ (m by m). This is possible only if $m \leq n$.

(2) Full column rank r = n. Ax = b has at most one solution x for every b if and only if the columns are linearly independent. Then A has an n by m left-inverse B such that $BA = I_n$. This is possible only if $m \geq n$.

Notes: One-sided inverses are $B = (A^T A)^{-1} A^T \ and \ C = A^T (AA^T)^{-1}$

Example. Find a left-inverse and/or a right-inverse (when they exist) for

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

Ans.

$$Let \ A = \begin{bmatrix} \boxed{4} & 0 & 0 \\ 0 & \boxed{5} & 0 \end{bmatrix}$$

The matrix A is already in Echelon form with 2 pivots.
So, rank of A=r=2.
Here m=2 and n=3.
So, m=r=2
$\Rightarrow A$ is a full row rank matrix.
$\Rightarrow$ right inverse C of A will exist and is given by

$$\begin{aligned} C &= A^T (AA^T)^{-1} \\ \Rightarrow C &= \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Lecture 18
2.4 The Four Fundamental Subspaces

Problem Set-2.4

2. Find the dimension and a basis for the four fundamental subspaces for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

Ans.

$$\begin{aligned} Let\ A &= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \leftarrow R_3 - R_1 \end{array} \\ \Rightarrow A &= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (echelon form) \end{aligned}$$

Rank of $A = r = 2$
Here $m = 3,\ n = 3$
$dim C(A) = r = 2$
$dim C(A^T) = r = 2$
$dim N(A) = n - r = 2$
$dim N(A^T) = m - r = 2$

The Column Space $C(A)$:

$$Basis\ for\ C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

The Null Space $N(A)$:

$$\begin{aligned} Ax &= 0 \\ \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow x_1 + 2x_2 + x_4 = 0, x_2 + x_3 = 0$$

$$\begin{aligned} Hence\ x &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ &= \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} \\ &= x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$Basis\ for\ N(A) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The Row Space $C(A^T)$:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$
\begin{aligned}
Let\ A^T &= \begin{bmatrix} \boxed{1} & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \leftarrow R_2 - 2R_1 \\ R_3 \\ R_4 \leftarrow R_4 - R_1 \end{array} \\
\Rightarrow A^T &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \leftarrow R_3 - R_2 \\ R_4 \end{array} \\
\Rightarrow A^T &= \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (echelon form)
\end{aligned}
$$

$$
Basis\ for\ C(A^T) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}
$$

The Leftnull Space $N(A^T)$:

$$
\begin{aligned}
A^T y &= 0 \\
\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\Rightarrow y_1 + y_3 = 0, y_2 = 0
$$

$$
\begin{aligned}
Hence\ y &= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\
&= \begin{bmatrix} -y_3 \\ 0 \\ y_3 \end{bmatrix} \\
&= y_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
Basis\ for\ N(A^T) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}
$$

13. Find a basis for each of the four subspaces of

$$
A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}
$$

Ans.

$$
\begin{aligned}
Let\ A &= \begin{bmatrix} 0 & \boxed{1} & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \leftarrow R_2 - R_1 \\ R_3 \end{array} \\
\Rightarrow A &= \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \\
\Rightarrow A &= \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$\begin{aligned}
Basis\ for\ C(A) &= \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\} \\
Basis\ for\ C(A^T) &= \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 6 \end{bmatrix} \right\} \\
Basis\ for\ N(A) &= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} \\
Basis\ for\ N(A^T) &= \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}
\end{aligned}$$

18. Find a 1 by 3 matrix whose nullspace consists of all vectors in R^3 such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace.
Ans. A 1 by 3 matrix whose nullspace consists of all vectors in R^3 such that $x_1 + 2x_2 + 4x_3 = 0$ is $A = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$.
Since $x_1 + 2x_2 + 4x_3 = 0$ in the matrix form can be written as

$$\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

A 3 by 3 matrix with that same nullspace is $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 12 \end{bmatrix}$

24. Construct a matrix with the required property, or explain why you can't.

(a) Column space contains $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, row space contains $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

(b) Column space has basis $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, nullspace has basis $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

(c) Dimension of nullspace = 1+ dimension of left nullspace.
(d) Left nullspace contains $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, row space contains $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
(e) Row space = column space, nullspace $\neq$ left nullspace.
Ans.
(a) A matrix with the required property is given by

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Impossible: dimensions $1 + 1 \neq 3$.
(c) A matrix with the required property is given by

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

(d) A matrix with the required property is given by

$$A = \begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$$

(e) Impossible: Row space = column space requires m = n.
Then $m - r = n - r$.

Lecture 19
Chapter 3: Orthogonality

3.1 Orthogonal Vectors and Subspaces

Course Outcomes: Students will be acquainted with orthogonal vectors, orthonormal vectors, orthonormal subspaces, and orthogonal compliment of subspaces.

Length of a vector: It is denoted by $||x||$.
Let $x = (x_1, x_2)$.
Length in 2D = $||x|| = \sqrt{x_1^2 + x_2^2}$.
Length squared=$||x||^2 = x_1^2 + x_2^2$.
Let $x = (x_1, x_2, x_3)$.
Length in 3D = $||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.
Length squared=$||x||^2 = x_1^2 + x_2^2 + x_3^2$
Let $x = (x_1, x_2, ..., x_n)$.
Length in R^n = $||x|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$.
Length squared=$||x||^2 = x_1^2 + x_2^2 + ... + x_n^2$

Inner Product:
Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then the inner product of two vectors x and y is denoted by $x^T y$ and defined as $x^T y = x_1 y_1 + x_2 y_2$
Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then $x^T y = x_1 y_1 + x_2 y_2 + x_3 y_3$
Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$.
Then $x^T y = x_1 y_1 + x_2 y_2 + ... + x_n y_n$

Note:

$$x^T x = \begin{pmatrix} x_1 & x_2 & ......x_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ ...... \\ x_n \end{pmatrix}$$

$$= x_1^2 + x_2^2 + ... + x_n^2 = ||x^2||$$

Hence inner product of a vector with itself is equal to the length square of the vector.

- The inner product $x^T y$ is zero if and only if x and y are orthogonal vectors.
- If $x^T y > 0$, their angle is less than $\deg 90$. If $x^T y < 0$, their angle is greater than $\deg 90$.
- The only vector with length zero and the only vector orthogonal to itself is the zero vector.

Orthogonal Vectors:
Two vectors x and y are said to be orthogonal iff $x^T y = 0$

Orthogonal Subspaces:
Two subspaces V and W of the same space R^n are orthogonal if every vector v in V is orthogonal to every vector w in W i.e $v^T w = 0$ for all v and w.

Examples

- x-axis and y-axis are subspaces of R^2 and every vector of x-axis is orthogonal to every vector in y-axis. So, x-axis $\perp$ y-axis in R^2
- $y = x$ line $\perp$ $y = -x$ line in R^2.
- All the three axes in R^3 are orthogonal to each other.

Notes

- The subspace $\{0\}$ is orthogonal to all subspaces.
- A line can be orthogonal to another line, or it can be orthogonal to a plane, but a plane cannot be orthogonal to a plane.

Fundamental theorem of orthogonality: The row space is orthogonal to the nullspace (in R^n). The column space is orthogonal to the left nullspace (in R^m).

Orthogonal Complement Given a subspace V of R^n, the space of all vectors orthogonal to V is called the orthogonal complement of V. It is denoted by $V^{\perp}$ = "V perp."

Examples

- x-axis is the orthogonal complement of y-axis in R^2.
- $y = x$ line is the orthogonal complement of $y = -x$ line in R^2.
- x-axis is the orthogonal complement of yz-plane in R^3.

Fundamental Theorem of Linear Algebra: The nullspace is the orthogonal complement of the row space in R^n. The left nullspace is the orthogonal complement of the column space in R^m.

Lecture 20

3.1 Orthogonal Vectors and Subspaces

Problem Set-3.1

1. Which pairs are orthogonal among the vectors v_1, v_2, v_3, v_4?

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Ans.

$$\begin{aligned} Here \quad v_1^T v_2 &\neq 0 \\ v_1^T v_3 &= 0 \\ v_1^T v_4 &\neq 0 \\ v_2^T v_3 &= 0 \\ v_2^T v_4 &\neq 0 \\ v_3^T v_4 &\neq 0 \end{aligned}$$

So $v_1 \& v_3$ *and* $v_2 \& v_3$ are orthogonal pairs.

7. Find the lengths and the inner product of $x = (1, 4, 0, 2)$ and $y = (2, -2, 1, 3)$.

Ans.

$$\begin{aligned} x &= (1,4,0,2) \\ \Rightarrow ||x|| &= \sqrt{21} \\ y &= (2,-2,1,3) \\ \Rightarrow ||y|| &= \sqrt{18} \\ x^T y &= 0 \\ \Rightarrow x &\perp y \end{aligned}$$

9. Find a basis for the orthogonal complement of the row space of A:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$$

Split $x = (3,3,3)$ into a row space component x_r and a nullspace component x_n.

Ans. Basis for the orthogonal complement of the row space of A is same as basis for nullspace. i.e. $C(A^T)^{\perp} = N(A)$.

The Null Space $N(A)$:

$$\begin{aligned} Ax &= 0 \\ \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow u + 2w = 0, u + v + 4w = 0$$

$$\Rightarrow u = -2, v = -2, w = 1$$

$$\begin{aligned} Hence\ x &= \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \end{aligned}$$

$$Basis\ for\ N(A) = \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$So, Basis\ for\ C(A^T)^{\perp} = \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$\begin{aligned} x &= x_r + x_n \\ \Rightarrow \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} &= x_r + \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \\ \Rightarrow x_r &= \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix} \end{aligned}$$

12. Show that $x - y$ is orthogonal to $x + y$ if and only if $||x|| = ||y||$
Proof.

$$x - y\ is\ orthogonal\ to\ x + y$$

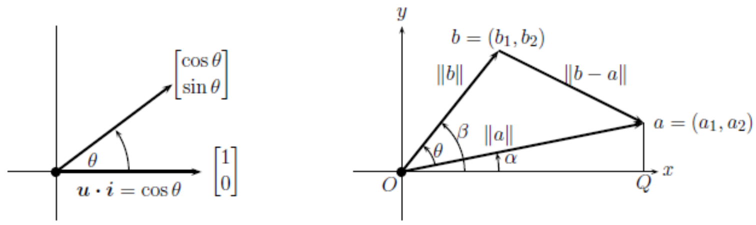
$$\begin{aligned} \Leftrightarrow (x - y) &\perp (x + y) \\ \Leftrightarrow (x - y)^T(x + y) &= 0 \\ \Leftrightarrow (x^T - y^T)(x + y) &= 0 \\ \Leftrightarrow x^T x + x^T y - y^x - y^T y &= 0 \\ \Leftrightarrow ||x||^2 - ||y||^2 &= 0 \quad since\ x^T y = y^T x \\ \Leftrightarrow ||x|| &= ||y|| \end{aligned}$$

3.2 Cosines and Projections Onto Lines

Course outcome: The course outcome of this article is to know about the Cosine angle between two lines and also the projection matrix.

Inner Product and Cosines

The cosine of the angle is directly related to inner product. For this consider the triangle in two dimensional case. Suppose the vectors a and b make angles α and β with $X-$axis as in fig. The length $\| a \|$ is the hypotenuse in the triangle OaQ. So, the sine and cosine of α are $\sin\alpha = \dfrac{a_2}{\| a \|}, \quad \cos\alpha = \dfrac{a_1}{\| a \|}$.



The cosine of the angle $\theta = \beta - \alpha$ using inner products. For angle β, the sine is $\dfrac{b_2}{\| b \|}$ and the cosine is $\dfrac{b_1}{\| b \|}$.

The cosine formula

$$\begin{aligned}\cos\theta &= \cos(\beta-\alpha)\\ &= \cos\beta\cos\alpha + \sin\beta\sin\alpha\\ &= \frac{a_1b_1 + a_2b_2}{\| a \|\| b \|}\\ &= \frac{a^Tb}{\| a \|\| b \|}\\ &= \frac{\langle a, b\rangle}{\| a \|\| b \|}\end{aligned}$$

Law of Cosines: $\| b-a \|^2 = \| b \|^2 + \| a \|^2 - 2\| b \|\| a \|\cos\theta$. When θ is a right angle, we have $\| b-a \|^2 = \| b \|^2 + \| a \|^2$, which is Pythagoras Theorem.

$$
\begin{aligned}
\| b-a \|^2 &= (b-a)^T(b-a) \\
&= (b^T-a^T)(b-a) \\
&= b^Tb - ab^T - a^Tb + a^Ta \\
&= b^Tb - a^Tb - a^Tb + a^Ta \\
&=\| b \|^2 - 2a^Tb + \| b \|^2 \\
&=\| b \|^2 - 2 \| a \| \| b \| \cos\theta + \| b \|^2 .
\end{aligned}
$$

When θ is a right angle, $\cos\theta = 0$. Thus $\| b-a \|^2 = \| a \|^2 + \| b \|^2$.

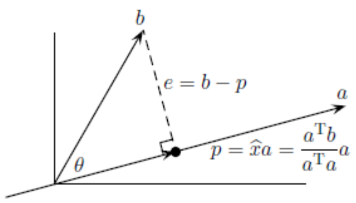
Projection Onto a Line

Suppose that we want to find the distance from a point b to the line in the direction of the vector a. We are looking also that instead of a line for the point p closest to b. The line connecting b to p is perpendicular to a. The situation is the same when we are given a plane or any subspace S instead of a line. Again, the problem is to find the point P on the subspace that is closed to b onto the subspace. Every point on the line is a multiple of a. So,

$$p = \widehat{x}a, \text{ where } \widehat{x} = \frac{a^Tb}{a^Ta}.$$

Hence, the projection of the vector b onto the line in the direction of a is

$$p = \widehat{x}a = \frac{a^Tb}{a^Ta}a = \frac{a^Tb}{\| a \|^2}a.$$



Example 1 Project $b = (1,2,3)$ onto the line through $a = (1,1,1)$ to get $\widehat{x}$ and p.
Solution: $\widehat{x} = \dfrac{a^Tb}{a^Ta} = \dfrac{1+1\times 2+1\times 3}{(\sqrt{1^2+1^2+1^2})^2} = \dfrac{6}{3} = 2$. The projection is $p = \widehat{x}a = 2(1,1,1) = (2,2,2)$. The angle between a and b is $\cos\theta = \dfrac{a^Tb}{\| a \| \| b \|} = \dfrac{6}{\sqrt{3}\sqrt{14}}$.

Schwarz Inequality:

$$| ab | \leq \| a \| \| b \| .$$

Exercise-3.2

1. (a) Given any two positive numbers x and y, choose the vector b equal to $(\sqrt{x}\ ,\sqrt{y})$, and choose $a = \left(\sqrt{y}\ \ ,\sqrt{x}\right))$. Apply the Schwarz inequality to compare the arithmetic mean $\frac{1}{2}(x+y)$ with the geometric mean $\sqrt{xy}$.

Solution: If $a = \begin{pmatrix} \sqrt{y} \\ \sqrt{x} \end{pmatrix}$, $b = \begin{pmatrix} \sqrt{x} \\ \sqrt{y} \end{pmatrix}$. Applying Schwarz inequality, we get

$$\begin{aligned}
& a^T b \leq \| a \| \| b \| \\
& \Rightarrow \begin{pmatrix} \sqrt{y} & \sqrt{x} \end{pmatrix} \begin{pmatrix} \sqrt{x} \\ \sqrt{y} \end{pmatrix} \leq \| (\sqrt{y}, \sqrt{x}) \| \| (\sqrt{x}, \sqrt{y}) \| \\
& \Rightarrow \sqrt{xy} + \sqrt{xy} \leq \sqrt{(\sqrt{y})^2 + (\sqrt{x})^2} \sqrt{(\sqrt{x})^2 + (\sqrt{y})^2} \\
& \Rightarrow 2\sqrt{xy} \leq \sqrt{y+x}\sqrt{x+y} \\
& \Rightarrow \sqrt{xy} \leq \frac{x+y}{2} \quad (G.M. \leq A.M.)
\end{aligned}$$

(b) Using the triangle inequality $\| x + y \| \leq \| x \| + \| y \|$, reduce to Schwarz inequality.

Solution: From the triangle inequality, we know

$$\begin{aligned}
& \| x + y \| \leq \| x \| + \| y \| \\
& \Rightarrow \| x + y \|^2 \leq (\| x \| + \| y \|)^2 \\
& \Rightarrow (x+y)^T (x+y) \leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 \\
& \Rightarrow (x^T + y^T)(x+y) \leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 \\
& \Rightarrow x^T x + x^T y + y^T x + y^T y \leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 \\
& \Rightarrow \| x \|^2 + 2x^T y + \| y \|^2 \leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 \\
& \Rightarrow x^T y \leq \| x \| \| y \| \quad \text{(Schwarz inequality).}
\end{aligned}$$

3. By using the correct b in the Schwarz Inequality, prove that

$$(a_1 + a_2 + \cdots + a_n)^2 \leq n(a_1^2 + a_2^2 + \cdots a_n^2).$$

When does equality hold?

Solution: Let $b = (1, 1, \cdots, 1)$ and $a = (a_1, a_2, \cdots, a_n)$. Using Schwarz inequality, we have

$$
\begin{aligned}
& a^T b \leq \| a \| \| b \| \\
& \Rightarrow \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\sqrt{1^2 + 1^2 + \cdots + 1^2} \\
& \Rightarrow a_1 + a_2 + \cdots a_n \leq \sqrt{n}\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \\
& \Rightarrow (a_1 + a_2 + \cdots a_n)^2 \leq n(a_1^2 + a_2^2 + \cdots + a_n^2).
\end{aligned}
$$

The inequality becomes equality, if $a_i = a_j$ for $i = j = 1, 2, \ldots, n$.

Assignments

Exercise-3.2, Q. 5,9.

Projection Matrix of Rank 1

The projection onto a line is carried out by a projection matrix P which is a symmetric matrix and $P^2 = P,\ P = \dfrac{aa^T}{a^Ta}$.

Example 1 The matrix that projects onto the line through $a = (1, 1, 1)$ is

$$P = \frac{aa^T}{a^Ta} = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1 & 1 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{pmatrix}.$$

Here, $rank(P) = 1$.

Remark 1

1. Projection matrix $\dfrac{aa^T}{a^Ta}$ is same if a is doubled $a = (2, 2, 2,),$.i.e,

$$P = \frac{aa^T}{a^Ta} = \frac{1}{12}\begin{pmatrix}2\\2\\2\end{pmatrix}\begin{pmatrix}2 & 2 & 2\end{pmatrix} = \begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{pmatrix}.$$

2. To project b onto a, multiply it by the projection matrix $P : p = Pb, \quad P = \dfrac{aa^T}{a^Ta}$

Exercise-3.2

8. Prove that the trace of $P = \dfrac{aa^T}{a^Ta}$, which is the sum of its diagonal entries, always 1.

Solution: Let $a = (a_1 a_2 \cdots a_n)^T$. Then

$$
\begin{aligned}
P &= \frac{aa^T}{a^Ta} \\
&= \frac{\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}}{\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}} \\
&= \frac{\begin{pmatrix} a_1^2 & a_1a_2 & \cdots & a_1a_n \\ a_1a_2 & a_2^2 & \cdots & a_2a_n \\ \vdots & \vdots & \vdots & \vdots \\ a_na_1 & a_na_2 & \cdots & a_n^2 \end{pmatrix}}{a_1^2 + a_2^2 + \cdots + a_n^2}
\end{aligned}
$$

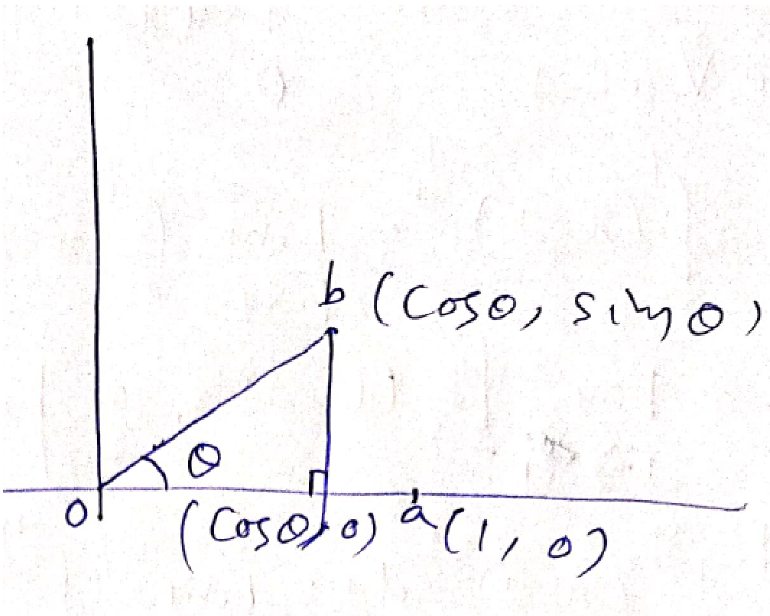
Hence $tr(P) = \dfrac{a_1^2 + a_2^2 + \cdots + a_n^2}{a_1^2 + a_2^2 + \cdots + a_n^2} = 1.$

17. Draw the projection of b onto a and also compute it from $p = \widehat{x}a$:

(a) $b = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution:

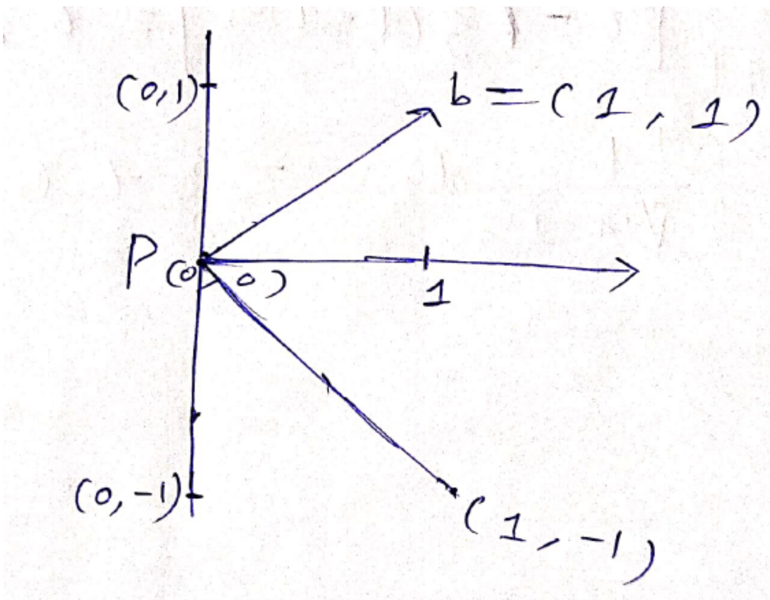
$$
\begin{aligned}
P &= \widehat{x}a \\
&= \frac{a^T b}{a^T a} a \\
&= \frac{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}}{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{\cos\theta}{1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta \\ 0 \end{pmatrix}
\end{aligned}
$$



(b) $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $a = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solution:

$$
\begin{aligned}
P &= \widehat{x}a \\
&= \frac{a^T b}{a^T a} a \\
&= \frac{\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \frac{0}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$



Assignments

Exercise-3.2, Q. 11,19.

3.3 Projections and Least Squares

A system of equations $Ax = b$ has a solution iff $b \in C(A)$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

Equation (1) can be written as

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} x_2 + \cdots \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

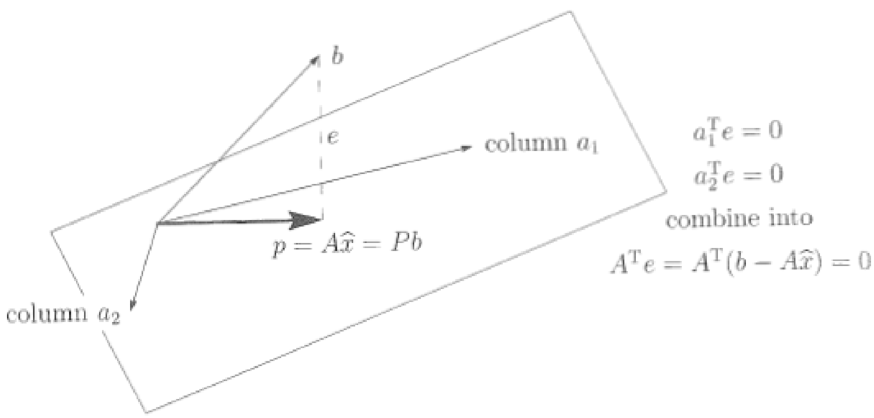
Let the co-efficient vector of x_i be a_i. Now, $b \in C(A)$,i.e, there exists $c_1, c_2, \cdots, c_n$ such that $b = \sum_{i=1}^{n} c_i a_i$. Then for $x_i = c_i$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is a solution of (1).

If x is a solution of (1), then $c_i = x_i$, i.e., b is a linear combination of columns of A, i.e., $b \in C(A)$.

Consider a system of linear equations

$$Ax = b \tag{2}$$

with $b \notin C(A)$. Then (2) has no solution. Then one must choose $\widehat{x}$ a best fit solution of it.



As $b \notin C(A)$, let us obtain a column vector p in $C(A)$, which is the most closest vector of b in $C(A)$ such that the error $e = b - p$ is of minimum length .i.e,$\| e \|^2 = \| b - p \|^2$ is minimum. Hence it is called least square approximation.

$$\begin{aligned}
&\| e \|^2 \quad \text{is minimum} \\
&\iff e \perp C(A) \\
&\iff e \in N(A^T) \\
&\iff A^T e = 0 \\
&\iff A^T(b - p) = 0 \\
&\iff A^T(b - A\widehat{x}) = 0 \\
&\iff A^T b = A^T A\widehat{x}
\end{aligned}$$

i.e., Normal equations $A^T A\widehat{x} = A^T b$.
Best estimate: $\widehat{x} = (A^T A)^{-1} A^T b$.
Projection $p = A\widehat{x} = A(A^T A)^{-1} A^T b$.
Projection matrix $P = A(A^T A)^{-1} A^T$.

Remark 1 1. *Suppose b is actually in the column space of A it is a combination $b = Ax$ of the columns. Then the projection of b is still b: b in column space*

$$p = A(A^T A)^{-1} A^T Ax = Ax = b :$$

The closest point p is just b itself, which is obvious.

2. *At the other extreme, suppose b is perpendicular to every column, so $A^T b = 0$. In this case b projects to the zero vector: b in left nullspace, i.e., $b \in N(A^T)$*

$$p = A(A^T A)^{-1} A^T b = A(A^T A)^{-1} 0 = 0 :$$

3. *When A is square and invertible, the column space is the whole space. Every vector projects to itself, p equals b, and $\widehat{x} = x$: If A is invertible*

$$p = A(A^T A)^{-1} A^T b = AA^{-1}(A^T)^{-1} A^T b = b :$$

This is the only case when we can take apart $(A^T A)^{-1}$, and write it as $A^{-1}(A^T)^{-1}$. When A is rectangular that is not possible.

4. *Suppose A has only one column, containing a. Then the matrix $A^T A$ is the number $a^T a$ and $\widehat{x}$ is $a^T b / a^T a$.*

5. *A^TA has the same nullspace as A.*

   If $Ax = 0$, then $A^TAx = 0$. Vectors x in the nullspace of A are also in the nullspace of A^TA. To go in the other direction, start by supposing that $A^TAx = 0$, and take the inner product with x to show that $Ax = 0$:

   $$x^TA^TAx = 0 \Rightarrow \| Ax \|^2 = 0 \Rightarrow Ax = 0.$$

   Hence, the two nullspaces are identical.

6. *If A has independent columns , then $N(A^TA) = N(A) = \{0\}$.*

7. *If A has independent columns, then A^TA is square, symmetric, and invertible.*

Projection Matrices

$$\begin{aligned} P &= \text{Projection matrix, which projects a vector onto C(A)} \\ &= A(A^TA)^{-1}A^T \end{aligned}$$

i.e, $p = Pb$.
Note that

$$\begin{aligned} P^2 &= A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T \\ &= A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T \\ &= A(A^TA)^{-1}A^T \\ &= P \end{aligned}$$

and

$$\begin{aligned} P^T &= [A(A^TA)^{-1}A^T]^T \\ &= (A^T)^T[(A^TA)^{-1}]^TA^T \\ &= A(A^TA)^{-1}A^T \\ &= P. \end{aligned}$$

Suppose that A is invertible. Then,

$$\begin{aligned} P &= A(A^TA)^{-1}A^T \\ &= AA^{-1}(A^T)^{-1}A^T \\ &= I. \end{aligned}$$

Hence, $p = Pb = Ib = b$. Therefore, $error = p - b = 0$.

1. Solve $Ax = b$ by least squares and find $p = A\widehat{x}$, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

   Verify that the error $b - p$ is perpendicular to the columns of A.

   Solution:

$$\begin{aligned} \widehat{x} &= (A^TA)^{-1}A^Tb \\ &= \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} \\ \\ \frac{1}{3} \end{pmatrix} \end{aligned}$$

   and

$$\begin{aligned} P &= A\widehat{x} \\ &= (A^TA)^{-1}A^Tb \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} \\ \\ \frac{1}{3} \\ \\ \frac{2}{3} \end{pmatrix}. \end{aligned}$$

   Here, Error$= b - p = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{3} \\ \\ \frac{1}{3} \\ \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \\ \frac{2}{3} \\ \\ \frac{-2}{3} \end{pmatrix}.$

   $b - p$ is perpendicular to the column of the matrix.

$$(b-p)^T C_1 = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{-2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{2}{3} - \frac{2}{3} = 0 \text{ and}$$

$$(b-p)^T C_2 = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{-2}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{2}{3} - \frac{2}{3} = 0.$$

4. The following system has no solution: $Ax = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 9 \end{pmatrix} = b.$

Sketch and solve a straight-line fit that leads to the minimization of the quadratic $(C-D-4)^2 + (C-5)^2 + (C+D-9)^2$. What is the projection of b onto the column space of A?

Solution:

$$\begin{aligned}
\widehat{x} &= (A^T A)^{-1} A^T b \\
&= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 9 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 9 \end{pmatrix} \\
&= \begin{pmatrix} 6 \\ \frac{5}{2} \end{pmatrix}.
\end{aligned}$$

Now,

$$\begin{aligned}
P &= A\widehat{x} \\
&= \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ \frac{5}{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{7}{2} \\ 6 \\ \frac{17}{2} \end{pmatrix}.
\end{aligned}$$

Assignments

Exercise-3.3, Q. 2,9,12,24.

Least Squares Fitting a Straight Line

$$b = C + Dt$$

or

$$\begin{aligned} C + Dt_1 &= b_1 \\ C + Dt_2 &= b_2 \\ &\vdots \\ C + Dt_m &= b_m \end{aligned}$$

or $\begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \\ 1 & t_m \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ or $Ax = b,\ \widehat{x} = (\widehat{C}, \widehat{D})$.

$E^2 = \| b - Ax \|^2 = \sum_{i=1}^{m} (b_i - C - Dt_i)^2$, where $\| E \|^2$ is minimum for some C and D.

$$\frac{\partial E^2}{\partial C} = 0 \tag{1}$$

$$\frac{\partial E^2}{\partial D} = 0 \tag{2}$$

Solving (1) and (2) for C and D we get the required straight line

$$b = C + Dt.$$

or

$$
\begin{aligned}
& A^T A\widehat{x} = A^T b, \quad A = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \\ 1 & t_m \end{pmatrix} \\
\Longleftrightarrow\ & A^T A \begin{pmatrix} \widehat{C} \\ \widehat{D} \end{pmatrix} = A^T b \\
\Longleftrightarrow\ & \begin{pmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_m \end{pmatrix} \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \\ 1 & t_m \end{pmatrix} \begin{pmatrix} \widehat{C} \\ \widehat{D} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_m \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \\
\Longleftrightarrow\ & \begin{pmatrix} m & \sum_{i=1}^{m} t_i \\ \sum_{i=1}^{m} t_i & \sum_{i=1}^{m} t_i^2 \end{pmatrix} \begin{pmatrix} \widehat{C} \\ \widehat{D} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{m} b_i \\ \sum_{i=1}^{m} t_i b_i \end{pmatrix}
\end{aligned}
$$

Exercise-3.3

6. Find the projection of b onto the column space of A :

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix}, \ b = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}.$$

Split b into $p + q$, with p in the column space and q perpendicular to that space. Which of the four subspaces contains q?

Solution:

$$
\begin{aligned}
p &= A\widehat{x} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} \frac{18}{44} & \frac{8}{44} \\ \frac{8}{44} & \frac{6}{44} \end{pmatrix} \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \\
&= \begin{pmatrix} \frac{26}{44} & \frac{14}{44} \\ \frac{10}{44} & \frac{2}{44} \\ \frac{-4}{44} & \frac{8}{44} \end{pmatrix} \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \\
&= \begin{pmatrix} \frac{40}{44} & \frac{12}{44} & \frac{4}{44} \\ \frac{12}{44} & \frac{8}{44} & \frac{-12}{44} \\ \frac{4}{44} & \frac{-12}{44} & \frac{40}{44} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \\
&= \frac{1}{44} \begin{pmatrix} 92 \\ -56 \\ 260 \end{pmatrix}.
\end{aligned}
$$

Let find the vector q such that $b = p + q \Rightarrow q = p - b = \frac{1}{44} \begin{pmatrix} 92 \\ -56 \\ 260 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \frac{1}{44} \begin{pmatrix} 48 \\ -144 \\ -48 \end{pmatrix}$. Thus, $q^T A = \frac{1}{44} \begin{pmatrix} 48 \\ -144 \\ -48 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Hence, q belongs to the left null space.

Assignments

Exercise- 3.3, Q. 12, 24.

4.2 Properties of the Determinant

1. $det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

2. $det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

3. The determinant of the identity matrix is 1, eg-$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ and $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

4. The determinant changes sign when two rows are interchanged, i.e.,$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$.

5. The determinant depends linearly on the first row, i.e., $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$ and $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

6. If any two rows of A are equal then $det(A) = 0$, eg, $\begin{vmatrix} a & c \\ a & c \end{vmatrix} = 0.$

7. Substracting a multiple of one row from another row leaves the same determinant, i.e., $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a-tc & b-td \\ c & d \end{vmatrix}$.

8. If A is singular then $det(A) = 0$ and if A is nonsingular then $det(A) \neq 0$.

9. $det(AB) = det(A)det(B)$

10. $det(A) = det(A^T)$

Exercise-4.2

4. By applying row operations to produce an upper triangular U, compute

(i) $det \begin{pmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{pmatrix}$.

Solution:

$$\begin{vmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 2 & 5 & 3 \end{vmatrix} \quad R_2 \leftarrow R_2 - 2R_1,\ R_3 \leftarrow R_3 + R_1$$

$$= \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & 5 \end{vmatrix} \quad R_4 \leftarrow R_4 + 2R_2$$

$$= \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{vmatrix} \quad R_4 \leftarrow R_4 + \frac{5}{2}R_3$$

$$= 1 \times (-1) \times (-2) \times 10 = 20.$$

(ii) $det \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{pmatrix}$.

Solution:

$$\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{vmatrix} \quad R_2 \leftarrow R_2 + \frac{1}{2}R_1$$

$$= \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & -2 \end{vmatrix} \quad R_3 \leftarrow R_3 + \frac{2}{3}R_2$$

$$= \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{vmatrix} \quad R_4 \leftarrow R_4 + \frac{3}{4}R_3$$

$$= 2 \times \frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} = 5.$$

5. Find the determinants of:

(a) a rank one matrix $A = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \end{pmatrix}$

(b) the upper triangular matrix $U = \begin{pmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

(c) the lower triangular matrix U^T.

(d) the inverse matrix U^{-1}.

(e) the reverse-triangular matrix that results from row exchanges, $M = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{pmatrix}$

Solution:

(a) $A = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 \\ 8 & -4 & 8 \\ 4 & -2 & 4 \end{pmatrix}$. Hence, $det(A) = \begin{vmatrix} 2 & -1 & 2 \\ 8 & -4 & 8 \\ 4 & -2 & 4 \end{vmatrix} =$ $\begin{vmatrix} 2 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$

(b) $\mid U \mid = 4 \times 1 \times 2 \times 2 = 16$

(c) $\mid U^T \mid = \mid U \mid = 16.$

(d) $\mid U^{-1} \mid = \dfrac{1}{\mid U \mid} = \dfrac{1}{16}.$

(e) $\mid M \mid = \begin{vmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{vmatrix} = 4 \times 1 \times 2 \times 2 = 16.$

Assignments

Exercise-4.2, Q. 2,6,13.

LECTURE-26

4.4 APPLICATIONS OF DETERMINANTS

This section follows four major applications: inverse of A, solving $Ax = 0$ using Cramer's Rule, Area or Volume and pivots.

■ AN INVERSE FORMULA

Let A be invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{detA} adjA$$

EXAMPLE 1. Compute of A^{-1} of 2×2 matrix

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$$

Solution: Since $detA = 2$. Therefore inverse is exist. We know that, $adjA =$ transpose of the cofactor matrix of the matrix A.

$$\therefore \quad adjA = \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}$$

Hence

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}$$

■ CARMER'S RULE

Let A be an invertible $n \times n$ matrix. For any b in R^n, the unique solution x of $Ax = b$ has entries given by

$$x_i = \frac{detA_i}{detA}, \qquad i = 1, 2, \cdots$$

EXAMPLE 2. (Text Q. 14) Use Cramer's rule to solve the system

(a) $2x_1 + 5x_2 = 1$

$x_1 + 4x_2 = 2$

(b) $2x_1 + x_2 = 1$

$x_1 + 2x_2 + x_3 = 0$

$x_2 + 2x_3 = 0$

Solution (a)**:** Given the system as $Ax = b$ where

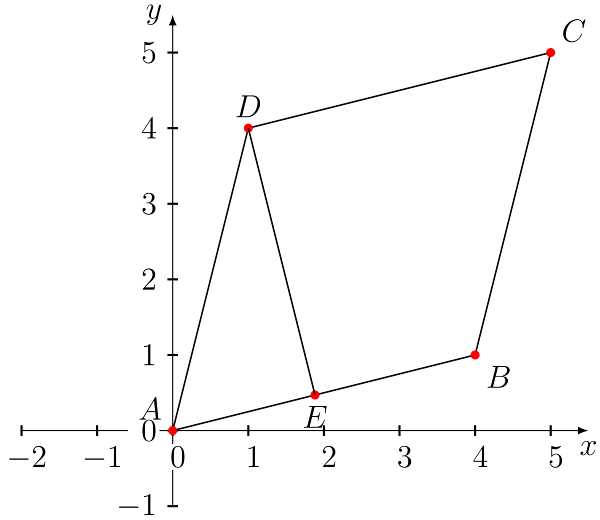
$$A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Since $det A = 3$, the system has a unique solution. By Cramer's rule,

$$x_1 = \frac{det A_1}{det A} = \frac{-6}{3} = -2 \quad (1)$$

$$x_2 = \frac{det A_2}{det A} = \frac{3}{3} = 1 \quad (2)$$

■ AREA AND VOLUME



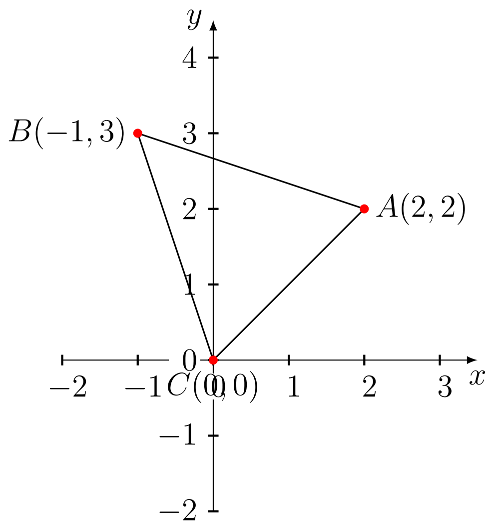
From the figure $OA = l(length)$ and $DE = h(heigh) \quad where \quad h = |E - D|$

The area (Volume) of the parallelogram is $l \times h = |detA|$(determine by the columns of A)

The area of the triangle determine by the columns of A is $\frac{1}{2}$parallelogram=$\frac{1}{2}detA$

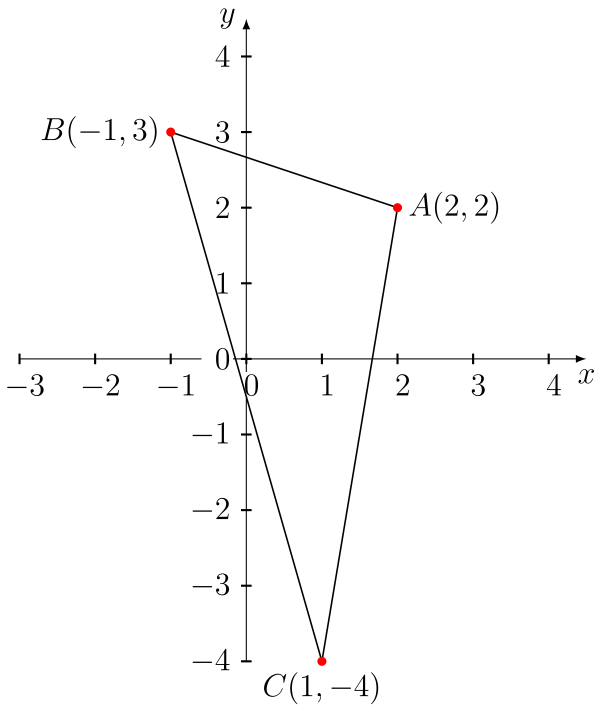
EXAMPLE 3(a)(TEXT Q. 2) Draw the triangle with vertices $A = (2, 2)$, $B = (-1, 3)$ and $C = (0, 0)$. By regarding it as half of a parallelogram , explain why its area equal

$$\text{area}(ABC) = \frac{1}{2}\begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix} = 4$$



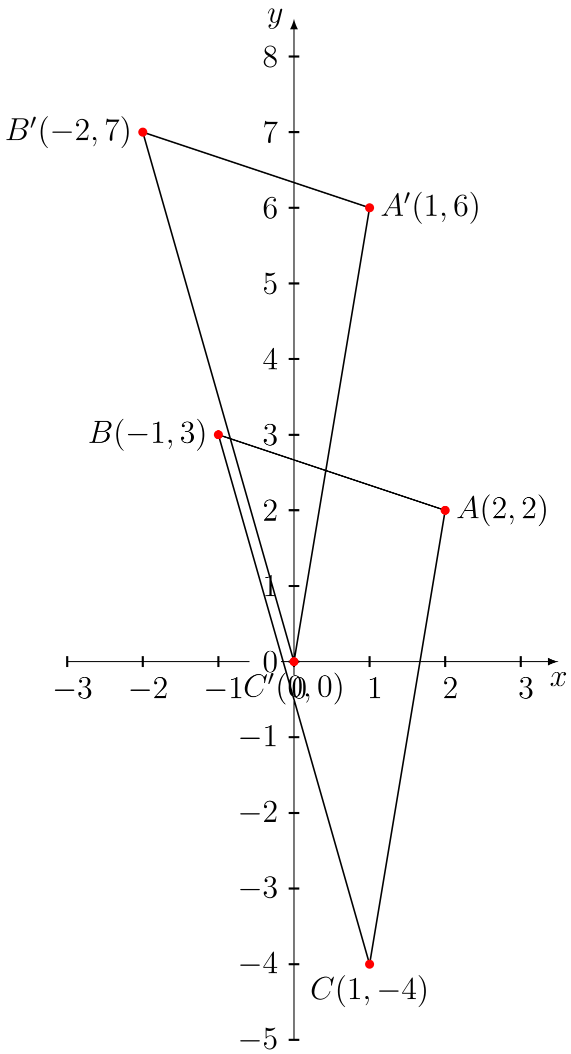
EXAMPLE 3(b) Move the third vertex to $C = (1, -4)$ and justify the formula

$$\text{area}(ABC) = \frac{1}{2}det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \frac{1}{2}det\begin{bmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -4 & 1 \end{bmatrix} = \frac{19}{2}$$



EXAMPLE 3(c) Sketch $A' = (1,6)$, $B' = (-2,7)$, and $C' = (0,0)$ and their relation to A, B, C.

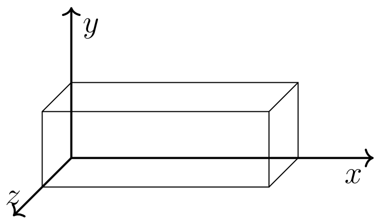
$$\text{area}(A'B'C') = \frac{1}{2}det\begin{bmatrix} 1 & 6 & 1 \\ -2 & 7 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{2}det\begin{bmatrix} 1 & 6 \\ -2 & 7 \end{bmatrix} = \frac{19}{2}$$



The formula is justified because $A' = (1,6)$, $B' = (-2,7)$, and $C' = (0,0)$ are translations of the vertices $A = (2,2)$, $B = (-1,3)$ and $C = (1,4)$.

EXAMPLE 4 (Text Q.29) A box has edges from $(0,0,0)$ to $(3,1,1)$ and $(1,1,3)$. Find its volume and also find the area of each parallelogram face?

Solution:(First Part)



$$\overrightarrow{OA} = \overrightarrow{u} = (3, 1, 1)$$

$$\overrightarrow{OB} = \overrightarrow{v} = (1, 3, 1)$$

$$\overrightarrow{OC} = \overrightarrow{w} = (1, 1, 3)$$

Volume of the box is $V = (\overrightarrow{u} \times \overrightarrow{v} \cdot \overrightarrow{u}) = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} = 20$

(Second Part) Area of the face

$$OAEB = ||(\overrightarrow{OA} \times \overrightarrow{OB})|| = ||(\overrightarrow{u} \times \overrightarrow{v})||$$

Now,

$$(\overrightarrow{u} \times \overrightarrow{v}) = \begin{bmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} = -2i - 2j + 8k$$

Hence,

$$||(\overrightarrow{u} \times \overrightarrow{v})|| = \sqrt{(-2)^2 + (-2)^2 + 8^2} = \sqrt{72} = 6\sqrt{2}$$

Similarly,

$$OBGC = ||(\overrightarrow{OB} \times \overrightarrow{OC})|| = ||(\overrightarrow{v} \times \overrightarrow{w})||$$

Now,

$$(\overrightarrow{v} \times \overrightarrow{w}) = \begin{bmatrix} i & j & k \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} = 8i - 2j - 2k$$

Hence,

$$||(\overrightarrow{v} \times \overrightarrow{w})|| = \sqrt{8^2 + (-2)^2 + (-2)^2} = \sqrt{72} = 6\sqrt{2}$$

$$OADC = ||(\overrightarrow{OA} \times \overrightarrow{OC})|| = ||(\overrightarrow{u} \times \overrightarrow{w})||$$

Now,

$$(\overrightarrow{u} \times \overrightarrow{w}) = \begin{bmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} = 2i - 8j + 2k$$

Hence,

$$||(\overrightarrow{u} \times \overrightarrow{w})|| = \sqrt{2^2 + (-8)^2 + (2)^2} = \sqrt{72} = 6\sqrt{2}$$

Q.27 The Parallelogram with sides $(2, 1)$ and $(2, 3)$ has the same area as the parallelogram with sides $(2, 2)$ and $(1, 3)$. Find those areas from 2 by 2 determinants and say why they must be equal.

LECTURE-27

5 EIGENVALUES AND EIGENVECTORS

5.1 INTRODECTION

Eigenvalues and Eigenvectors are most important and interesting topic in Linear Algebra. Some basic concept of determinant and Matrices will be use to study the Eigenvalue and Eigenvector problems. One important thing is that, Eigenvalues and Eigenvectors we can find only from the square matrix.

Eigenvalues and Eigenvectors will help to solve many problems in Linear Algebra. However the basic concepts of Eigenvalues and Eigenvectors are useful throughout pure and applied mathematics. Eigenvalues are also used to study differential equations and continuous dynamics systems, they provide critical information in engineering design and they arise naturally in fields such as physics and chemistry.

Let us consider A be square matrix and x be the vector multiply with A as a in goes vector then outcomes vector is Ax (i.e. gradient of A). It is like a function, x is in goes then then $f(x)$ is outcomes function. Since Ax is a vector, so, question will be arise about the direction of Ax. This outcome vector may be goes different direction. But some particular vector of x, Ax will be parallel to x. Then we can write

$$Ax = \lambda x.$$

This special vector of x is called eigenvector of the matrix A and this scalar value of λ is called the eigenvalue of the matrix A.

Example 1. Let

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \qquad u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad and \quad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Are both u and v are eigenvectors of A?

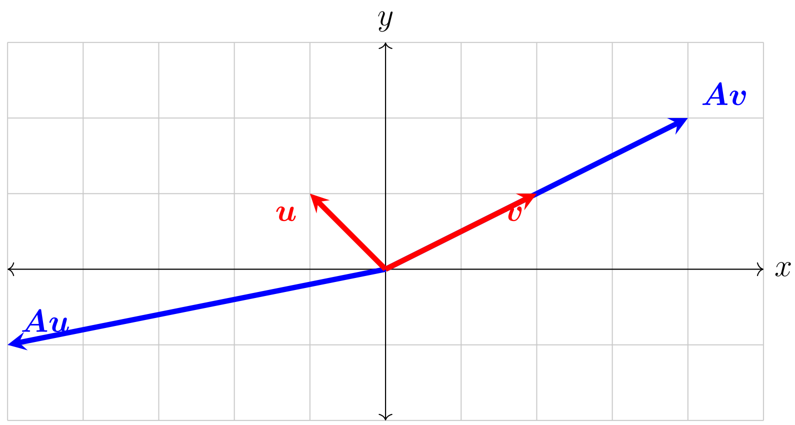
Solution:

$$Au = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} 5 \\ 1 \end{bmatrix} \neq \lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix} \neq \lambda u$$

where $\lambda = -1$. Therefore u is not an eigenvector of A, because Au is not a multiple of u. Again,

$$Av = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = (2) \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda v$$

Where $\lambda = 2$. Thus v is an eigenvector of A corresponding to an eigenvalue $\lambda = 2$, and Av is a multiple of v only.



NOTE: If λ is positive, then Ax and x are in same direction.

Example 2. Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, \qquad u = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \quad and \quad v = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

Are both u and v are eigenvectors of A?

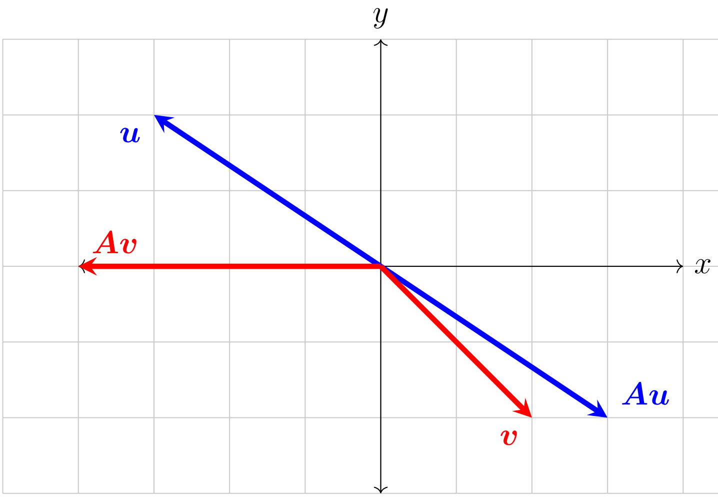
Solution:

$$Au = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} = (-1) \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \lambda u$$

where $\lambda = -1$. Therefore u is an eigenvector corresponding to an eigenvalue of (-1), and Au is a multiple of u only. Again,

$$Av = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ -2 \end{bmatrix} \neq \lambda v$$

Thus v is not an eigenvector of A, because Av is not a multiple of v.



NOTE: If λ is negative, then Ax and x are in opposite direction.

Above two examples, it is clear to see that some special vectors only the eigenvector corresponding to eigenvalue λ of the any matrix A. This scalar value λ may be zero or nonzero(i.e. positive, negative or imaginary). The numbers of eigenvalue (i.e. λ)and eigenvectors are depends on the order of the matrix. If the order of the matrix is $n \times n$ then we will get n eigenvalues and for each corresponding eigenvalues will get n linearly independent eigenvectors. But some times eigenvalue may repeat. In that case all eigenvalue are not linearly independent.

DEFINATION

An **eigenvector** of an $n \times n$ matrix A is anonzero vector x such that $Ax = \lambda x$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution x of $Ax = \lambda x$; such an x is called an eigenvector corresponding to λ.

Possible two cases of λ of the Special equation $Ax = \lambda x$

Case-1 The solution of $Ax = 0$,

If $\lambda = 0$, then the special equation becomes $Ax = 0$ and x is in the nullspace of A. Since x is a vector so A will be zero. i.e. A is a singular matrix. i.e. $det(A) = 0$

NOTE: If $\lambda = 0$, then A is a singular matrix and if A is a singular matrix then $\lambda = 0$.

Case-2 The solutions of $Ax = \lambda x$

If $\lambda \neq 0$ then the equation $Ax = \lambda x$ is becomes $(A - \lambda \cdot I)x = 0$. Again same condition, x is a nullspace of $(A - \lambda \cdot I)$. Since x is vector so $(A - \lambda \cdot I)$ will be zero. i.e. $(A - \lambda \cdot I)$ is a singular matrix. i.e. $det(A - \lambda \cdot I) = 0$. This is the key equation to find the eigenvalue, and is called **eigenvalue equation or**

characteristic equation.

EXAMPLES OF EIGENVALUES AND EIGENVECTORS

The first step is to understand how eigenvalues can be useful. One of their application is to ordinary differential equations.

Example 3. Let us consider the coupled pair of equations

$$\begin{aligned} \frac{dv}{dt} &= 4v - 5w, \quad v = 8 \quad t = 0 \\ \frac{dv}{dt} &= 2v - 3w, \quad w = 5 \quad t = 0. \end{aligned}$$

The system of matrix form is

$$\frac{du}{dt} = Au, \quad with \quad u = u(0)att = 0 \tag{1}$$

where

$$u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}, \qquad att = 0, \quad u(0) = \begin{bmatrix} 8) \\ 5 \end{bmatrix}, \qquad A = \begin{bmatrix} 4 & -5) \\ 2 & -3 \end{bmatrix}$$

The pure exponential solution of the above the equation (1) is $u(t) = e^{at}u(0)$.

Now we are going to find the eigenvalue and eigenvector of the equation (1) with the help of coefficient matrix A.

The characteristic equation is $det(A - \lambda I) = 0$.

$$i.e. \quad det(A - \lambda I) = 0$$

$$i.e. \quad \begin{vmatrix} 4-\lambda & -5) \\ 2 & -3-\lambda \end{vmatrix} = 0$$

$$i.e. \quad \lambda^2 - \lambda - 2 = 0$$

$$i.e. \quad \lambda = -1 \quad or \quad \lambda = 2$$

Therefor $\lambda = -1$ and $lambda = 2$ are the eigenvalue of the matrix A

NOTE: A matrix with zero determinant is singular, so there must be nonzero vector x in its nullspace. In fact the nullspace contains a whole line of eigenvectors. It is a subspace.

Let x_1 be the eigenvector of the corresponding eigenvalue $\lambda = -1$.

$$(A + I)x = 0$$

$$i.e. \quad \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution (the First eigenvalue)is any nonzero multiple of x_1

Eigenvaetor for $\lambda_1 = -1$,

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let x_2 be the eigenvector of the corresponding eigenvalue $\lambda_2 = 2$.

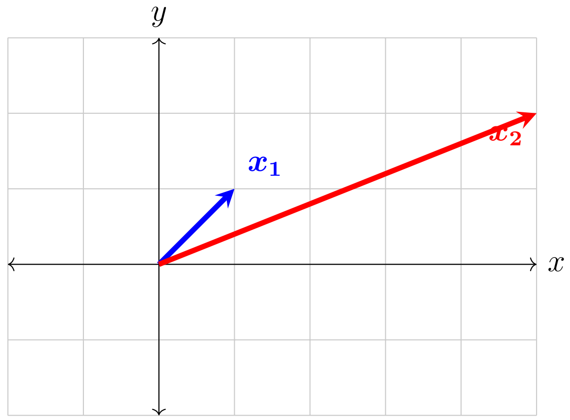
$$(A - 2I)x = 0$$

$$i.e. \quad \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution (the second eigenvector)is any nonzero multiple of x_2

Eigenvaetor for $\lambda_2 = 2$,

$$x_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$



The special solution of equation (1) is $u = e^{\lambda t}x$. Therefore the complete solution equation (1) is

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2. \tag{2}$$

Where c_1 and c_2 are two free parameter. The initial condition of the system is $u = u(0)$ at $t = 0$.

At $t = 0$, the equation (2) becomes

$$c_1 x_1 + c_2 x_2 = 0$$

$$i.e. \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

The constant are $c_1 = 3$ and $c_2 = 1$ Therefor the solution to the original equation is

$$u(t) = 3e^{-t} x_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 5 \\ 2 \end{bmatrix} \tag{3}$$

where $u(t) = 3e^{-1} + 5e^{2t}$, $w(t) = 3e^{-1} + 2e^{2t}$

Hence at $t = 0$, $v(0) = 8$ and $w(0) = 5$

LECTURE-28

Continue from the previous lecture...

Example 4 The eigenvalues of the diagonal are main diagonal element of the matrix.

Solution:

Let

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad has \quad \lambda_1 = 3 \quad and \quad \lambda_2 = 2.(try \quad yourself)$$

Let x_1 be the eigenvector of the corresponding eigenvalue $\lambda_1 = 3$.

$$(A - 3I)x = 0$$

$$i.e. \quad \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvaetor for $\lambda_1 = 3$,

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

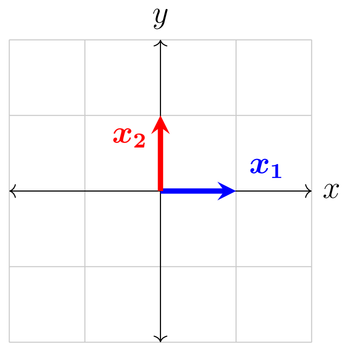
Let x_2 be the eigenvector of the corresponding eigenvalue $\lambda_2 = 2$.

$$(A - 2I)x = 0$$

$$i.e. \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvaetor for $\lambda_2 = 2$,

$$x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Example 5. The eigenvalues of a projection matrix are 1 or 0.

Solution:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad has \quad \lambda_1 = 1 \quad and \quad \lambda_2 = 0.(check \quad yourself)$$

Let x_1 be the eigenvector of the corresponding eigenvalue $\lambda_1 = 1$.

$$(A - I)x = 0$$

$$i.e. \quad \begin{bmatrix} \frac{1}{2} - 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad R_2 \leftarrow R_2 + R_1$$

$$i.e. \quad \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvaetor for $\lambda_1 = 1$,

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let x_2 be the eigenvector of the corresponding eigenvalue $\lambda_2 = 0$.

$$(A - 0 \cdot I)x = 0$$

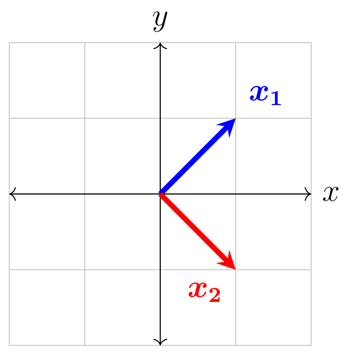
$$i.e. \quad \begin{bmatrix} \frac{1}{2} - 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_2 \leftarrow R_2 - R_1$$

$$i.e. \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvaetor for $\lambda_2 = 0$,

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



Example 6. The eigenvalues are on the main diagonal when A is triangular.

Solution: Let

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & \frac{3}{4} & 6 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

The characteristic equation is

$$det(A - \lambda I) = 0$$

$$i.e. \quad \begin{vmatrix} 1-\lambda & 4 & 5 \\ 0 & \frac{3}{4}-\lambda & 6 \\ 0 & 0 & \frac{1}{2}-\lambda = 0 \end{vmatrix}$$

$$i.e. \quad (1-\lambda)(\frac{3}{4}-\lambda)(\frac{1}{2}-\lambda = 0)$$

$$i.e. \quad \lambda = 1, \quad \lambda = \frac{3}{4} \quad or \quad \frac{1}{2}$$

The above eigenvalues are the main diagonal of the given triangular matrix.

PROPERTIES OF EIGENVALUES

1. The sum of the eigenvalue of a matrix is equal to the trace of the matrix.(Trace means sum of the principal diagonal of the matrix)

e.g. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since A is a 2×2 matirx. Therefore the numbers eigenvalues will be two.

Let λ_1 and λ_2 are the eigenvalue of the matrix A.

$$i.e. \quad \lambda_1 + \lambda_2 = trace of the matrix A = a + d$$

2. The product of the eigenvalue of a matrix A is equal to the determinate of A

$$\lambda_1 \cdot \lambda_2 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

3. Any square matrix A and its transpose A^T have the same eigenvalue.

4. Eigenvalue of the **diagonal matrix or scalar matrix or triangular matrix** are the principal diagonal elements of the matrix.

5. Eigenvalue of the projection matrix are always 1 or 0. If numbers of eigenvalues $\lambda = 1$ repeated r times of the $n \times n$ matrix, then $\lambda = 0$ repeated $n - r$ times.

6. If λ is an eigenvalue of A, then eigenvalue of $A \pm kI$ is $\lambda \pm k$.

7. Eigenvalues of symmetric matrix are always distinct.

8. If λ is an eigenvalue of an **orthogonal matrix** A. Then $\frac{1}{\lambda}$ is also an eigenvalue of the same matrix.

$$Ax = \lambda x \quad (\lambda \quad is \quad an \quad eigenvalue \quad of \quad A)$$

$$i.e. \quad A^T Ax = \lambda A^T x$$

$$i.e. \quad Ix = \lambda A^T x$$

$$i.e. \quad \frac{1}{\lambda} x = A^T x$$

$$i.e. \quad A^T x = \frac{1}{\lambda} x$$

$$i.e. \quad \frac{1}{\lambda} is an eigenvalue of A^T$$

we know that A and A^T have same eigenvalue. Therefore λ and $\frac{1}{\lambda}$ is an eigenvalue of A as well as A^T

NOTE: If A is an orthogonal matrix then λ or $\frac{1}{\lambda}$ are an eigenvalue of A as well as A^T.

9. If $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_m$ are an eigenvalue of the matrix A then,

(a) The eigenvalue of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, ..., \frac{1}{\lambda_m}$

(b) The eigenvalue of A^n are $\lambda_1{}^n, \lambda_2{}^n, \lambda_3{}^n, ..., \lambda_m{}^n$

(c) The eigenvalue of kA are $k\lambda_1, k\lambda_2, k\lambda_3, ..., k\lambda_m$

(d) The eigenvalue of $A \pm kI$ $\lambda_1 \pm k, \lambda_2 \pm k, \lambda_3 \pm k, ..., \lambda_m \pm k$

PROPERTIES OF EIGENVECTOR

1. Eigenvector of A and A^T are always different.

2. Eigenvector of A and $A \pm kI$ are always same.

3. Eigenvector of symmetric matrix for corresponding eigenvalue are orthogonal.

Let A is asymmetric matrix of order 3. Let $\lambda_1, \lambda_2, \lambda_3$ are three eigenvalue of the matrix A. For corresponding eigenvalue $\lambda_1, \lambda_2, \lambda_3$, we have three eigenvector x_1, x_2, x_3 then

$$x_1{}^T \cdot x_2 = 0 \qquad x_2{}^T \cdot x_3 = 0 \qquad x_3{}^T \cdot x_1 = 0$$

i.e. x_1, x_2, x_3 are orthogonal.

Problem Set 5.1

Q.1 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}.$$

Verify that the trace equals the sum of the eigenvalues, and the determinant equals their product.

Solution: The characteristic equation is

$$det.(A - \lambda.I) = 0$$

$$i.e. \quad \begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$i.e. \quad (1-\lambda)(4-\lambda)+2=0$$

$$i.e \quad \lambda^2-5\lambda+6=0$$

$$i.e. \quad \lambda=2 \quad or \quad \lambda=3$$

Let x_1 be the eigenvector of the corresponding eigenvalue $\lambda_1 = 2$.

$$(A-2I)x=0$$

$$i.e. \quad \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad R_2 \leftarrow R_2 + 2R_1$$

Eigenvaetor for $\lambda_1 = 2$,

$$x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let x_2 be the eigenvector of the corresponding eigenvalue $\lambda_2 = 3$.

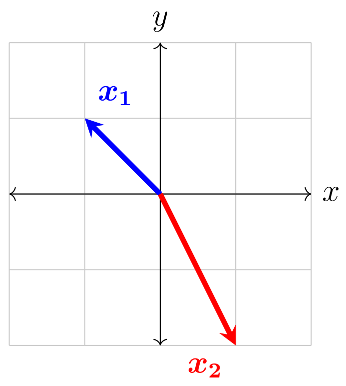
$$(A-3I)x=0$$

$$i.e. \quad \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_2 \leftarrow R_2 + R_1$$

Eigenvaetor for $\lambda_2 = 3$,

$$x_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$



Second part

Trace of A=sum of diagonal of the matrix A =$1 + 4$ =5

and Sum of eigenvalue= $\lambda_1 + \lambda_2$ =$2 + 3$ =5

Therefore Trace of A=$\lambda_1 + \lambda_2$

Again, Product of the eigenvalue =$\lambda_1 \times \lambda_2$ =$2 \cdot 3$ =6

Determinant of A,

$$\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = 6$$

Therefore Determinant of A=Product of the eigenvalue.

Q.2. With the same matrix, Solve the differential equation $\frac{du}{dt} = Au$,

$u(0) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$, What are the two pure exponential solutions?

Solution: The given coefficient matrix is

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

The pure exponential solution the differential equation $\frac{du}{dt} = Au$ is $u = e^{\lambda t}x$,

and the two special solutions are

$$u(t) = e^{\lambda_1 t}x_1 = e^2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad and \qquad u(t) = e^{\lambda_2 t}x_1 = e^3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

These two special solution gives the complete solution. Therefore the complete solution $\frac{du}{dt} = Au$ is

$$u(t) = c_1 e^{\lambda_1 t}x_1 + c_2 e^{\lambda_2 t}x_2.$$

$$i.e. \quad u(t) = c_1 e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

At initial condition $t = 0$, $u(t) = u(0)$

$$c_1 x_1 + c_2 x_2 = 0$$

$$i.e. \quad \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix} \qquad R_2 \leftarrow R_2 + R_1$$

$$i.e. \quad \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

The constant are $c_1 = 2$ and $c_2 = 3$ Therefor the solution to the original equation is

$$u(t) = 2e^{2t}x_1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

LECTURE-29

continue from the previous lecture...

Q.6. Shoe that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into

$$det(A - \lambda t) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)....(\lambda_n - \lambda)$$

and making a cleaver choice of λ.

Solution:

If $\lambda = 0$ then,

$$det(A - 0) = (\lambda_1 - 0)(\lambda_2 - 0)....(\lambda_n - 0)$$

$$detA = \lambda_1\lambda_2\lambda_3....\lambda_n$$

Q.7. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 0 \end{bmatrix}$$

Check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace and $\lambda_1\lambda_2\lambda_3$ equals the determinant.

Solution:

Since the A is triangular matrix. Therefore the eigenvalues of the matrix A are $\lambda_1 = 3, \quad \lambda_2 = 1 \quad and \quad \lambda_3 = 0$

Sum of the eigenvalue

$$\lambda_1 + \lambda_2 + \lambda_3 = 3 + 1 + 0 = 4$$

Trace of of the matrix A=$3+1+0$=4

Therefore Sum of the eigenvalue is equal to the trace of of the matrix A.

Product of the eigenvalue

$$\lambda_1\lambda_2\lambda_3 = 3\cdot 1\cdot 0 = 0$$

and determinant of the matrix A is

$$\begin{vmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 3(0-0) - 4(0-0) + 2(0-0) = 0$$

Therefore Product of the eigenvalue is equal to the determinant of the matrix A.

Second part

Let Let x_1 be the eigenvector of the corresponding eigenvalue $\lambda_1 = 3$.

$$(A-3I)x = 0$$

$$i.e. \quad \begin{bmatrix} 3-3 & 4 & 2 \\ 0 & 1-3 & 2 \\ 0 & 0 & 0-3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \leftarrow R_2 + \frac{1}{2}R_1$$

$$i.e. \quad \begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \leftarrow R_3 + R_2$$

$$i.e. \quad \begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here x is free variable since x's pivot is missing.

Eigenvaetor for $\lambda_1 = 3$, is

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let x_2 be the eigenvector of the corresponding eigenvalue $\lambda_2 = 3$.

$$(A - I)x = 0$$

$$i.e. \quad \begin{bmatrix} 3-1 & 4 & 2 \\ 0 & 1-1 & 2 \\ 0 & 0 & 0-1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \leftarrow R_3 + \frac{1}{2}R_2$$

$$i.e. \quad \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here y is free variable since y's pivot is missing. Here $z = 0$ and $x = 2y$

Eigenvaetor for $\lambda_2 = 1$, is

$$x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Let x_3 be the eigenvector of the corresponding eigenvalue $\lambda_3 = 0$.

$$(A - 0)x = 0$$

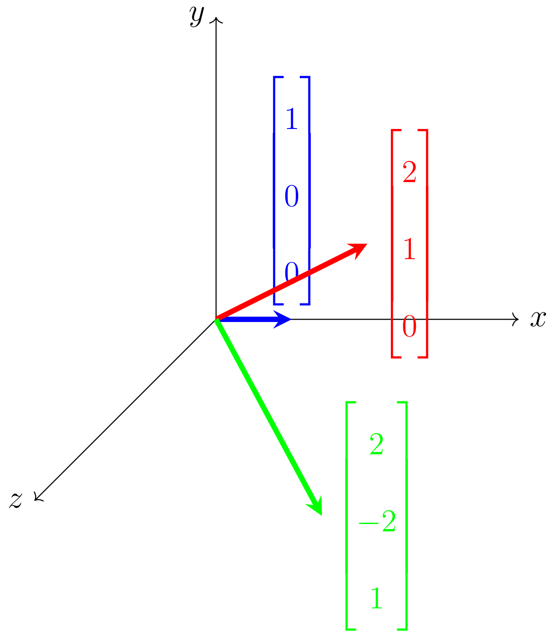
$$i.e. \quad \begin{bmatrix} 3-0 & 4 & 2 \\ 0 & 1-0 & 2 \\ 0 & 0 & 0-0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here z is the free variable since z's pivot is missing. Here $y = -2z$ and $x = 2z$

Eigenvaetor for $\lambda_3 = 0$, is

$$x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$



Q. 15. Find the eigenvalue and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \qquad and \quad B = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

Solution:

We know that, sum of the eigenvalue of the matrix is equal to trace of the matrix.

Since A is 2×2 matrix, therefore Matrix A has two eigenvalue. Let $\lambda_1 \quad and \quad \lambda_2$.

$$\lambda_1 + \lambda_2 = 3 - 3 = 0 \tag{1}$$

$$\lambda_1 \times \lambda_2 = \begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix} = -25 \tag{2}$$

Solving Eq. (1) and (2), $\quad \lambda_1 = 5 \quad or \quad \lambda_2 = -5$ Let x_1 be the eigenvector of the corresponding eigenvalue $\lambda_1 = 5$.

$$(A - 5I)x = 0$$

$$i.e. \quad \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_2 \leftarrow R_2 + 2R_1$$

$$i.e. \quad \begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvaetor for $\lambda_1 = 5$,

$$x_1 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Let x_2 be the eigenvector of the corresponding eigenvalue $\lambda_2 = -5$.

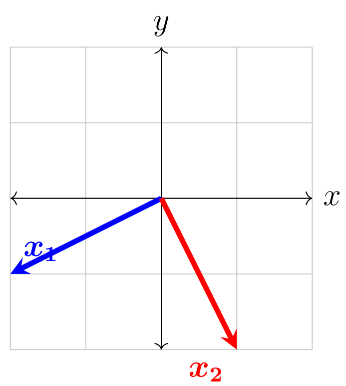
$$(A + 5I)x = 0$$

$$i.e. \quad \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i.e. \quad \begin{bmatrix} 8 & 4 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_2 \leftarrow R_2 - \frac{1}{2}R_1$$

Eigenvaetor for $\lambda_2 = -5$,

$$x_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$



Q.17. The Powers A^k of this matrix A approaches a limit as $k \to \infty$

$$A = \begin{vmatrix} .8 & .3 \\ .2 & .7 \end{vmatrix} \quad and \quad A^2 = \begin{vmatrix} .70 & .45 \\ .30 & .55 \end{vmatrix} \quad and \quad A^\infty = \begin{vmatrix} .6 & .6 \\ .4 & .4 \end{vmatrix}$$

The matrix A^2 is halfway between A and A^∞. Explain why $A^2 = \frac{1}{2}(A + A^\infty)$ from the eigenvalues and eigenvectors of these three matrices.

Solution: Given

$$A = \begin{vmatrix} .8 & .3 \\ .2 & .7 \end{vmatrix}$$

The characteristic equation for the matrix A is

$$det(A - \lambda I) = \begin{vmatrix} .8 - \lambda & .3 \\ .2 & .7 - \lambda \end{vmatrix}$$

$$= (.8 - \lambda)(.7 - \lambda) - 0.03 \cdot 0.2 = 0$$

$$= \lambda^2 - 1.5k + 0.5$$

$$= \lambda_1 = 1 \quad or \quad \lambda_2 = 0.5$$

Therefore the eigenvalue of A^2 are $\mu_1 = 1$ and $\mu_2 = 0.25$

Similarly, the eigenvalue of A^∞ are $\nu_1 = 1$ and $\nu_2 = 0$

Again,

$$\mu_i = \frac{1}{2} \cdot (\lambda_i + \nu_i)$$
$$\mu_i x_i = \frac{1}{2} \cdot (\lambda_i mu_i + \nu_i) x_i$$
$$A^2 x_i = \frac{1}{2} \cdot (A + A^{\infty}) x_i$$
$$A^2 x_1 + A^2 x_2 = \frac{1}{2} \cdot (A + A^{\infty}) x_1 + \frac{1}{2} \cdot (A + A^{\infty}) x_i$$
$$A^2 (x_1 + X_2) = \frac{1}{2} (A + A^{\infty})(x_1 + X_2)$$
$$A^2 = \frac{1}{2} (A + A^{\infty})$$

Q.3. If you shift to $A - 7I$, what are the eigenvalue and eigenvectors and how are they related to those of A?

Q.9 The eigenvalue of A equal the eigenvalue of A^T. This is because $det(A - \lambda I)$ equals $det(A^T - \lambda I)$. That is true because .... Show by an example that the eigenvectors of A and AT are not the same.

Q.39. When $a + b = c + d$ show that $(1, 1)$ is an eigenvector and find both eigenvalues

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Solution: Show that $(1, 1)$ is a vector of the matrix A for some eigenvalue λ. So, we have to find the eigenvalue for the given eigenvector.

Let $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$Ax = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \end{bmatrix} = (a+b) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad since \quad a+b = c+d$$

1.e. $\quad Ax = \lambda x, \quad where \quad \lambda = a+b$

$\therefore (1,1)$ is the eigenvector for the corresponding eigenvalue $\lambda = a+b$.

LECTURE-30

5.2 DIAGONALIZATION OF A MATRIX

Suppose the n by n matrix A has n linearly independent eigenvectors. If these eigenvectors are the columns of a matrix S, then $S^{-1}AS$ is a diagonal matrix Λ. The eigenvalues of A are on the diagonal of Λ:

Diagonalization $$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \textbf{(1)}$$

We call **S** the "eigenvector matrix" and Λ the "eigenvalue matrix"—using a capital lambda because of the small lambdas for the eigenvalues on its diagonal.

Where the eigenvectors x_i in the columns of S, and

$$S = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_n \\ | & | & & | \end{bmatrix}$$

$$AS = S\Lambda \xrightarrow[\text{from the left}]{\text{multiplying by } S^{-1}} \boxed{S^{-1}AS = \Lambda} \quad (2)$$

Eigenvdecomposition

From $AS = S\Lambda \xrightarrow[\text{from the right}]{\text{multiplying by } S^{-1}} \boxed{A = S\Lambda S^{-1}}$

S is invertible, because its columns(the eigenvector) were assumed to be independents.

Note that depending on which side you multiply by the inverse of "S", you get the equations for "diagonalization" and "eigendecomposition".

Underlying assumption behind the diagonalization and eigendecomposition

In order for the matrix "A" to be either diagonalized or eigendecomposed, it has to meet the following critera:

- A Must be a Square matrix.
- A has linearly independent eigenvectors (i.e. eigenvector matrix S has to be invertible).

Remark 1: If the matrix A has no repeated eigenvalues—the numbers $\lambda_1, \lambda_2, ..., \lambda_n$ are distinct—then its n eigenvectors are automatically independent. Therefore any matrix with distinct eigenvalues can be diagonalized.

Example 1 The eigenvalues of the diagonal are main diagonal element of the matrix.

Solution:

Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad has \quad \lambda_1 = 3 \quad and \quad \lambda_2 = 2.(try \quad yourself)$

Let x_1 be the eigenvector of the corresponding eigenvalue $\lambda_1 = 3$.

$$(A - 3I)x = 0$$

$$i.e. \quad \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvaetor for $\lambda_1 = 3$,

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

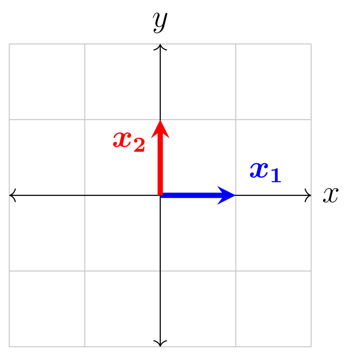
Let x_2 be the eigenvector of the corresponding eigenvalue $\lambda_2 = 2$.

$$(A - 2I)x = 0$$

$$i.e. \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvaetor for $\lambda_2 = 2$,

$$x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



From the above figure it is clear that, the eigenvector of matrix A are independent.

Therefore the eigenvector matrix is

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad and \quad S^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence the matrix A is diagoanlized.

Example 2 The eigenvalues of the diagonal are main diagonal element of the matrix.

Solution:

Let $B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \quad has \quad \lambda_1 = 3 \quad and \quad \lambda_2 = 3.(try \quad yourself)$

Let x_1 be the eigenvector of the corresponding eigenvalue $\lambda_1 = 3$.

$$(A - 3I)x = 0$$

$$i.e. \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

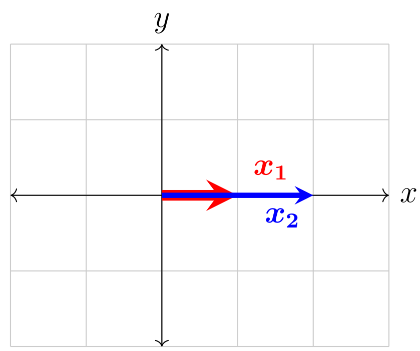
Eigenvaetor for $\lambda_1 = 3$,

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let x_2 be the eigenvector of the corresponding eigenvalue $\lambda_2 = 3$.

Eigenvaetor for $\lambda_2 = 3$,

$$x_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$



From the above figure it is clear that, the eigenvector of matrix A are not independent.

Therefore the eigenvector matrix is

$$S = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Since the matrix S is singular, therefore inverse of the this matrix is not exist.

Hence the matrix B is not diagoanlized.

Remark 2: The diagonalizing matrix S is not unique. An eigenvector x can be multiplied by a constant, and remains an eigenvector. We can multiply the columns of S by any nonzero constants, and produce a new diagonalizing S. Repeated eigenvalues leave even more freedom in S. For the trivial example $A = I$, any invertible S will do: $S^{-1}IS$ is always diagonal (Λ is just I). All vectors are eigenvectors of the identity.

Remark 3: Other matrices S will not produce a diagonal Λ. Suppose the first column of S is y. Then the first column of $S\Lambda$ is $\lambda_1 y$. If this is to agree with the first column of AS, which by matrix multiplication is Ay, then y must be an eigenvector: $Ay = \lambda_1 y$. The order of the eigenvectors in S and the eigenvalues in Λ is automatically the same.

Remark 4: Not all matrices possess n linearly independent eigenvectors, so not all matrices are diagonalizable.

Example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Its eigenvalues are $\lambda_1 = \lambda_2 = 0$, since it is triangular with zeros on the diagonal:

$$\left|A - \lambda I\right| = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 \ .$$

All eigenvectors of this A are multiples of the vector (1,0):

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{x, or x} = \begin{bmatrix} c \\ 0 \end{bmatrix}.$$

$\lambda = 0$ is a double eigenvalue—its algebraic multiplicity is 2. But the geometric multiplicity is 1—there is only one independent eigenvector. We can't construct S.

Note: Algebraic multiplicity is the number repeated eigen value of a particular matrix A and geometric multiplicity is the numberof independent vectors. That failure of diagonalization was not a result of λ = 0. It came from $\lambda_1 = \lambda_2$.

Repeated eigenvalues $\quad A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

Their eigenvalues are 3, 3 and 1, 1. They are not singular! The problem is the shortage of eigenvectors—which are needed for S.

$\boxed{\text{Diagonalizability of A depends on enough eigenvectors}}$.

$\boxed{\text{Invertibility of A depends on nonzero eigenvalues.}}$

Diagonalization can fail only if there are repeated eigenvalues. Even then, it does not always fail. $A = I$ has repeated eigenvalues 1,1,...,1 but it is already diagonal! There is no shortage of eigenvectors in that case.

The test is to check, for an eigenvalue that is repeated p times, whether there are p independent eigenvectors—in other words, whether $A - \lambda I$ has rank $n - p$. To complete that circle of ideas, we have to show that distinct eigenvalues present no problem.

Q.8. Which of these matrices cannot be diagonalized?

$$A_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}, \quad and \quad A_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

Solution:

The eigen value of A_1 matrix are $\lambda_1 = 0$ and $\lambda_2 = 0$

i.e. repeated eigenvalue. Hence A_1 cannot be diagonaled.

The eigen value of A_2 matrix are $\lambda_1 = 2$ and $\lambda_2 = -2$

i.e. non repeated eigenvalue. Hence A_2 is a diagonaled.

The eigen value of A_2 matrix are $\lambda_1 = 2$ and $\lambda_2 = 2$

i.e. repeated eigenvalue. Hence A_3 cannot be diagonaled.

Lecture-31

Repeated eigenvalues $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

Their eigenvalues are 3, 3 and 1, 1. They are not singular! The problem is the shortage of eigenvectors—which are needed for S.

$\boxed{\text{Diagonalizability of A depends on enough eigenvectors}}$.

$\boxed{\text{Invertibility of A depends on nonzero eigenvalues.}}$

Diagonalization can fail only if there are repeated eigenvalues. Even then, it does not always fail. $A = I$ has repeated eigenvalues 1,1,...,1 but it is already diagonal! There is no shortage of eigenvectors in that case.

The test is to check, for an eigenvalue that is repeated p times, whether there are p independent eigenvectors—in other words, whether $A - \lambda I$ has rank $n - p$. To complete that circle of ideas, we have to show that distinct eigenvalues present no problem .

■ **If eigenvectors $x_1, ..., x_k$ correspond to different eigenvalues $\lambda_1, ..., \lambda_k$,then those eigenvectors are linearly independent.**

Suppose first that k = 2, and that some combination of x_1 and x_2 produces zero: $c_1x_1 + c_2x_2 = 0$. Multiplying by A, we find $c_1\lambda_1x_1 + c_2\lambda_2x_2 = 0$. Subtracting λ_2 times the previous equation, the vector x_2 disappears: $c_1(\lambda_1 - \lambda_2)x_1 = 0$. Since $\lambda_1 \neq \lambda_2$ and $x_1 \neq 0$, we are forced into $c_1 = 0$. Similarly $c_2 = 0$, and the two vectors are independent; only the trivial combination gives zero.

This same argument extends to any number of eigenvectors: If some combination produces zero, multiply by A, subtract λ_k times the original combination, and x_k disappears— leaving a combination of $x_1, ..., x_{k-1}$, which produces zero. By repeating the same steps (this is really mathematical induction) we end up with a multiple of x_1 that produces zero. This forces $c_1 = 0$, and ultimately every $c_i = 0$. Therefore eigenvectors that come from distinct eigenvalues are automatically independent.

A matrix with n distinct eigenvalues can be diagonalized.

Example **If a 3 by 3 upper triangular matrix has diagonal entries 1, 2, 7, how do you know it can be diagonalized? What is Λ?**

Answer- $A = \begin{bmatrix} 1 & * & * \\ 0 & 2 & * \\ 0 & 0 & 7 \end{bmatrix}$. Then $det(A - \lambda I) = \begin{vmatrix} 1-\lambda & \bullet & \bullet \\ 0 & 2-\lambda & \bullet \\ 0 & 0 & 7-\lambda \end{vmatrix}$

$= (1-\lambda)(2-\lambda)(7-\lambda)$.

$det(A - \lambda I) = 0 \Rightarrow \lambda = 1, 2, 7$

Since all the eigenvalues are distinct, the matrix A is diagonalized and the diagonal matrix

$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$.

Powers and Products: A^k and AB

There is one more situation in which the calculations are easy. The eigenvalue of A^2 are exactly $\lambda_1^2, ..., \lambda_n^2$, and every eigenvector of A is also an eigenvector of A^2. We start from $Ax = \lambda x$, and multiply again by A:

$A^2x = A\lambda x = \lambda Ax = \lambda^2 x$.

Thus λ^2 is an eigenvalue of A^2, with the same eigenvector x. If the first multiplication by A leaves the direction of x unchanged, then so does the second. The same result comes from diagonalization, by squaring $S^{-1}AS = \Lambda$:

Eigenvalues of A^2 $\quad (S^{-1}AS)(S^{-1}AS) = \Lambda^2$ or $S^{-1}A^2S = \Lambda^2$.
The matrix A^2 is diagonalized by the same S, so the eigenvectors are unchanged. The eigenvalues are squared. This continues to hold for any power of A:

■ **The eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, ..., \lambda_n^k$ and each eigenvector of A is still an eigenvector of A^k.**

When S diagonalizes A, it also diagonalizes A^k:
$\Lambda^k = (S^{-1}AS)(S^{-1}AS)...(S^{-1}AS) = (S^{-1}A^kS)$
Each S^{-1}cancels an S, except for the first S^{-1} the last S.

If A is invertible this rule also applies to its inverse (the power $k = -1$). The eigenvalues of A^{-1} are $\frac{1}{\lambda_i}$. That can be seen even without diagonalizing:
if $Ax = \lambda x$ then $x = \lambda A^{-1}x$ and $\frac{1}{\lambda}x = A^{-1}x$.
■ **Diagonalizable matrices share the same eigenvector matrix S if and only if $AB = BA$.**

Proof. If the same S diagonalizes both $A = S\Lambda_1S^{-1}$ and $B = S\Lambda_2S^{-1}$, we can multiply in either order:
$AB = S\Lambda_1S^{-1}S\Lambda_2S^{-1} = S\Lambda_1\Lambda_2S^{-1}$ and $BA = S\Lambda_2S^{-1}S\Lambda_1S^{-1} = S\Lambda_2\Lambda_1S^{-1}$.
Since $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$ (diagonal matrices always commute) we have $AB = BA$.

In the opposite direction, suppose $AB = BA$. Starting from $Ax = \lambda x$, we have
$ABx = BAx = B\lambda x = \lambda Bx$.
Thus x and Bx are both eigenvectors of A, sharing the same λ (or else $Bx = 0$). If we assume for convenience that the eigenvalues of A are distinct—the eigenspaces are all one-dimensional—then Bx must be a multiple of x. in other words x is an eigenvector of B as well as A. The proof with repeated eigenvalues is a little longer.

■

Problem Set 5.2

Q.3 Factor the following matrices into $S\Lambda S^{-1}$:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

Answer : $\quad |A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda - 2) = 0 \Rightarrow \lambda = 2, 0.$$

For $\lambda = 2$, solve $(A - \lambda I)x = 0$.

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Eigenvector corresponding to $\lambda = 2$ is : $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = 0$, solve $(A - \lambda I)x = 0$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Eigenvector corresponding to $\lambda = 0$ is : $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

So $\quad A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

Similarly,

$|B - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda - 2) = 0 \Rightarrow \lambda = 2, 0.$$

For $\lambda = 2$, solve $(B - \lambda I)x = 0$.

$$\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2 = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Eigenvector corresponding to $\lambda = 2$ is : $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For $\lambda = 0$, solve $(B - \lambda I)x = 0$.

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_2 = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Eigenvector corresponding to $\lambda = 0$ is : $x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

$S = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

So $\quad A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}^{-1}$

Lecture-32

Q.4 **If** $A = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$, find A^{100} by diagonalizing A.

Answer:

$\lambda_1 + \lambda_2 = 4 + 2 = 6$

$\lambda_1 \lambda_2 = 5 \Rightarrow \lambda = 5, 1.$

For $\lambda = 5$, solve $(A - \lambda I)x = 0$.

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + 3x_2 = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Eigenvector corresponding to $\lambda = 5$ is : $x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

For $\lambda = 1$, solve $(A - \lambda I)x = 0$.

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Eigenvector corresponding to $\lambda = 1$ is : $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$S = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

So $\quad A = S\Lambda S^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

$$A^{100} = S\Lambda S^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5^{100} & 0 \\ 0 & 1^{100} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$= \frac{1}{4} \begin{bmatrix} 3 \times 5^{100} + 1 & 3 \times 5^{100} - 3 \\ 5^{100} - 1 & 5^{100} + 3 \end{bmatrix}.$$

Q.6 **Find all the eigenvalues and eigenvectors of**

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

and write two diagonalizing matrices S.

Answer:

We first find its eigenvalues by solving the characteristic equation:

$$0 = det(A{-}\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_2 + R_3 + R_4} \begin{vmatrix} 4-\lambda & 4-\lambda & 4-\lambda & 4-\lambda \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix}$$

$$= (4{-}\lambda) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} \xrightarrow[R_4 \leftarrow R_4 - R_1]{R_2 \leftarrow R_2 - R_1,\, R_3 \leftarrow R_3 - R_1} (4{-}\lambda) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix}$$

$$= \lambda^3(\lambda - 4) \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \\ \lambda_4 = 4 \end{cases}$$

We now find the eigenvectors corresponding to $\lambda = 0$:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow a + b + c + d = 0$$

$$\Longrightarrow x = \begin{pmatrix} -b-c-d \\ b \\ c \\ d \end{pmatrix} = a \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

SO $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ are the eigen vectors with respect to the eigen value 0.

By orthonormalizing them, we obtain

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

We finally find the eigenvector corresponding to $\lambda = 4$:

Notice that

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Longrightarrow \lambda_4 = 4$ and the corresponding eigen vector is: $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

By normalizing it we get $\left\{ \frac{1}{\sqrt{4}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

The first diagonalizing matrix

$$S_1 = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and the second diagonalizing matrix

$$S_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{4}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{4}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{4}} \end{pmatrix}$$

ASSIGNMENT PROBLEMS

Q.8 Which of these matrices can not be diagonalized?

$$A_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

Q.12 If the eigenvalues of A are 1, 1, 2, which of the following are certain to be true? Give a reason if true or counterexample if false:

(a) A is invertible.

(b) A is diagonalizable.

(c) A is not diagonalizable.

Q.16 Factor these two matrices into $A = S\Lambda S^{-1}$:

$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$**.**

Q.17 True or false: If the n columns of S (eigenvectors of A) are independent, then

(a) A **is invertible.**

(b) A **is diagonalizable.**

(c) S **is invertible.**

(d) S **is diagonalizable.**

Q.32 Diagonalize B and compute $S\Lambda^k S^{-1}$ to prove this formula for B^k :

$B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ **has** $B^k = \begin{bmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{bmatrix}$.

Lecture-33

5.5 Complex Matrices

Lengths and Transposes in the Complex Case

By definition, the complex vector space C^n contains all vectors **x** with n complex components:

$$\textbf{Complex vector} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{with components } x_j = a_j + ib_j.$$

Vectors x and y are still added component by component. Scalar multiplication cx is now done with complex numbers c. The vectors $v_1, ..., v_k$ are linearly dependent if some nontrivial combination gives $c_1v_1 + ... + c_kv_k = 0$; the cj may now be complex. The unit coordinate vectors are still in C^n; they are still independent; and they still form a basis. Therefore C^n is a complex vector space of dimension n.

Length squared $\qquad \|x\|^2 = |x_1|^2 + |x_2|^2 + ... + |x_n|^2$

Example $\qquad x = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\|x\|^2 = 2$; $y = \begin{bmatrix} 2+i \\ 2-4i \end{bmatrix}$ and $\|y\|^2 = 25$.

For complex vector, $\|x\|^2 = \bar{x}^T x$. $\qquad$ (For real vectors, $\|x\|^2 = x^T x$.)

Inner product $\qquad \bar{x}^T y = \bar{x_1} y_1 + \bar{x_2} y_2 + ... + \bar{x_n} y_n$.

If we take the inner product of x with itself, we are back to $\|x\|^2$.

Note that $\bar{y}^T x$ is different from $\bar{x}^T y$; we have to watch the order of the vectors. For vectors and matrices, a superscript H (or a star) combines both operations - conjugate and transpose. This matrix $\bar{A}^T = A^H = A^*$ is called "A Hermitian":

Conjugate transpose $\begin{bmatrix} 2+i & 3i \\ 4-i & 5 \\ 0 & 0 \end{bmatrix}^H = \begin{bmatrix} 2-i & 4+i & 0 \\ -3i & 5 & 0 \end{bmatrix}.$

Note-

1. The inner product of x and y is $x^H y$. Orthogonal vectors have $x^H y$.

2. The squared length of x is $\|x\|^2 = x^H = |x_1|^2 + |x_2|^2 + ... + |x_n|^2$.

3. Conjugating $(AB)^T = B^T A^T$ produces $(AB)^H = B^H A^H$

Hermitian Matrices A matrix A is Hermitian if $\bar{A}^T = A$.

Example $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} = A^H.$

- The diagonal entries of a Hermitian matrix must be real.

- A real symmetric matrix is certainly Hermitian.

- For real matrices there is no difference between A^T and A^H.

Properties of Hermitian matricec

Property 1. **If $A = A^H$, then for all complex vectors x, the number $x^H Ax$ is real.**

Proof. $(x^H Ax)^H$ is conjugate of 1 by 1 matrix $x^H Ax$ but we actually get the same number back again:$(x^H Ax)^H = x^H A^H x^{HH} = x^H Ax$. So the number must be real. ■

Property 2. **If $A = A^H$, every eigenvalue is real.**

Proof. Suppose $Ax = \lambda x$. The trick is to multiply by x^H: $x^H Ax = \lambda x^H x$. The left-hand side is real by Property 1, and the right-hand side $x^H x = \|x\|^2$ is real and positive, because $x \neq 0$. Therefore $\lambda = \dfrac{x^H Ax}{x^T x}$ must be real. ■

Property 3. **Two eigenvectors of a real symmetric matrix or a Hermitian matrix, if they come from different eigenvalues, are orthogonal to one another.**

Proof. The proof starts with $Ax = \lambda_1 x, Ay = \lambda_1 y$, and $A = A^H$:
$(\lambda_1 x)^H y = (Ax)^H y = x^H Ay = x^H(\lambda_2 y)$.

The outside numbers are $\lambda_1 x^H y = \lambda_2 x^H y$, since the λ's are real. Now we use the assumption $\lambda_1 \neq \lambda_2$, which forces the conclusion that $x^H y = 0$. $\blacksquare$

Lecture-34

Spectral Theorem: A real symmetric matrix can be factored into $A = Q\Lambda Q^T$. Its orthonormal eigenvectors are in the orthogonal matrix Q and its eigenvalues are in Λ.

$$A = Q\Lambda Q^T = \begin{bmatrix} | & & | \\ x_1 & \dots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} -x_1^T- \\ \vdots \\ -x_n^T- \end{bmatrix}$$

$$= \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + ... + \lambda_n x_n x_n^T$$

Unitary Matrices A complex matrix with orthonormal columns is called a unitary matrix. We denote it by U.

Unitary Matrix $\quad U^H U = UU^H = I$, and $U^H = U^{-1}$.

Example. Show that the following matrix is Unitary.

$$A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

Solution Since $AA^H = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = I_4$.

We conclude that $A^H = A^{-1}$. Therefor, A is unitary matrix.

Property 1. $(Ux)^H(Uy = x^H U^H U y = x^H y$ and the lengths are preserved by U:

Length Unchanged $\|Ux\|^2 = x^H U^H Ux = \|x\|^2$.

Property 2. Every eigenvalue of U has absolute value $|\lambda| = 1$.

This follows directly from $Ux = \lambda x$, by comparing the lengths of the two sides: $\|Ux\| = \|x\|$ by Property-1, and always $\|\lambda x\| = |\lambda|\|x\|$. Therefore $|\lambda| = 1$.

Property 3. Eigenvectors corresponding to different eigenvalues are orthonormal .

Start with $Ux = \lambda_1 x$ and $Uy = \lambda_2 y$, and take inner product by Property-1:

$x^H y = (Ux)^H (Uy) = (\lambda_1 x)^H (\lambda_2 y) = \bar{\lambda_1} \lambda_2 x^H y$.

Comparing the left to the right,$\bar{\lambda_1}\lambda_2 = 1$ or $x^H y = 0$.

But Property-2 is $\bar{\lambda_1}\lambda_1 = 1$, so we cannot also have $\bar{\lambda_1}\lambda_2 = 1$.

Thus $x^H y = 0$ and the eigenvectors are orthogonal.

Skew-Hermitian Matrices: A matrix K is skew-Hermitian if $K^H = -K$.

$K = \begin{bmatrix} 2i & 3+3i \\ -3+3i & 5i \end{bmatrix}$.

If A is Hermitian then $K = iA$ is skew-Hermitian.

Theorem A Hermitian matrix can be factored into $K = U\Lambda U^H$. Its orthonormal eigenvectors are in the unitary matrix U and its eigenvalues are in Λ.

Q.15 Show that if U and V are unitary, so is UV. Use the criterion $U^H U = I$.

Answer : When U and V are unitary,

$(UV)(UV)^H = (UV)(V^H U^H) = UIU^H = UU^H = I$;

and likewise

$$(UV)^H (UV) = (V^H U^H)(UV) = VIV^H = VV^H = I$$

So indeed $(UV)^H = (UV)^{-1}$, and thus UV is unitary.

Q.2 Write out the matrix A^H and compute $C = A^H A$ if

$$A = \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix}.$$

What is the relation between C and C^H? Does it hold when C is constructed from some $A^H A$?

Answer :

$$C = A^H A = \begin{bmatrix} 1 & -i \\ -i & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - i^2 & i & -i \\ -i & -i^2 & 0 \\ i & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & i & -i \\ -i & 1 & 0 \\ i & 0 & 1 \end{bmatrix}$$

$$C^H = \begin{bmatrix} 2 & i & -i \\ -i & 1 & 0 \\ i & 0 & 1 \end{bmatrix}$$

This is always true since

$$C^H = (A^H A)^H = A^H (A^H)^H = A^H A = C.$$

Lecture-35

Q.33 Diagonalize A and K to reach $U\Lambda U^H$:

$$A = \begin{bmatrix} 0 & 1-i \\ 1+i & 3 \end{bmatrix} \qquad K = \begin{bmatrix} 0 & -1+i \\ 1+i & i \end{bmatrix}$$

Answer: Diagonalization of matrix A

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 0-\lambda & 1-i \\ 1+i & 1-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(1-\lambda)-(1-i)(1+i) = 0\lambda^2-\lambda-2 = 0 \Rightarrow \lambda = 2, -1$$

For $\lambda = 2$, solve $Ax = \lambda x$ or $(A - \lambda I)x = 0$ for x(nontrivial).

$$\begin{bmatrix} -2 & 1-i \\ 1+i & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x_1 + (1-i)x_2 = 0$

$(1+i)x_1 = x_2$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ (1+i) \end{bmatrix}$. Normalized vector, $u = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ (1+i) \end{bmatrix}$.

Similarly, for $\lambda = -1$, solve $Ax = \lambda y$ or $(A - \lambda I)y = 0$

$$\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$y_1 + (1-i)y_2 = 0$

$y_1 = (-1+i)y_2$ $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$ Normalized vector, $v = \frac{1}{\sqrt{3}} \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$

$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1+i \\ 1+i & 1 \end{bmatrix}$

$U^H = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ -1-i & 1 \end{bmatrix}$

$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

Hence $$A = U\Lambda U^H = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1+i \\ 1+i & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ -1-i & 1 \end{bmatrix}$$

Diagonalization of matrix K

$|K - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 0-\lambda & -1+i \\ 1+i & i-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(i-\lambda) - (1+i)(-1+i) = 0\lambda^2 - i\lambda + 2 = 0 \Rightarrow \lambda =$

$2i, -i$

For $\lambda = 2i$, solve $Kx = \lambda x$ or $(K - \lambda I)x = 0$ for x(nontrivial).

$$\begin{bmatrix} -2i & -1+i \\ 1+i & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2ix_1 + x_2(-1+i) = 0$

$(1-i)x_1 = x_2$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ (1-i) \end{bmatrix}$ Normalized vector, $u = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ (1-i) \end{bmatrix}$

Similarly, for $\lambda = -i$, solve $Kx = \lambda y$ or $(K - \lambda I)y = 0$

$$\begin{bmatrix} i & -1+i \\ 1+i & 2i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$iy_1 + y_2(-1+i) = 0$

$y_1 = (-1-i)y_2$ $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ Normalized vector, $v = \frac{1}{\sqrt{3}} \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$

$$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1-i \\ 1-i & 1 \end{bmatrix}$$

$$U^H = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ -1+i & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 2i & 0 \\ 0 & -i \end{bmatrix}$$

Hence $$A = U\Lambda U^H = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1-i \\ 1-i & 1 \end{bmatrix} \begin{bmatrix} 2i & 0 \\ 0 & -i \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ -1+i & 1 \end{bmatrix}.$$

ASSIGNMENT PROBLEMS

Q.3 (a) **If $x = re^{i\theta}$ what are x^2, x^{-1} and $\bar{x}$ in polar coordinates? Where are the complex numbers that have $x^{-1} = \bar{x}$?**

Q.3 (b) **At $t = 0$, the complex number $e^{(-1+i)t}$ equals one. Sketch its path in complex plane as t increases from 0 to 2π .**

Q.10 **Find the lengths and inner product of**

$$x = \begin{bmatrix} 2-4i \\ 4i \end{bmatrix} \text{ and } \quad y = \begin{bmatrix} 2+4i \\ 4i \end{bmatrix}.$$

Q.22 **Prove that A^HA is always a Hermitian matrix. Compute A^HA and AA^H :**

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}$$

Q.43 43 A matrix with orthonormal eigenvectors has the form $A = U\Lambda U^{-1} = U\Lambda U^H$. Prove that $AA^H = A^H A$. These are exactly normal matrices.

POSITIVE DEFINITE MATRIX

lecturenote36

1 TEST FOR POSITIVE DEFINITE MATRIX

Now question arise (i) which symmetricx matrix have the property $x^T Ax > 0$ for all nonozero vectors x
(ii) Signs of eigenvalues that gave place to the tests on a, b, c of a symmetric matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \tag{1}$$

is positive definite when $a > 0$ and $ac - b^2 > 0$.
(iii) For every large symmetric matrix A, it's impractical to compute eigenvalues of A, we can compute the pivots, and the signs of the pivots of a symmetric matrix are same as the signs of the eigen values.
(iv) Also using the determinant to test the positive definite. If $a = c = -1$ and $b = 0$ then det$A$=1 but $A = -I$ = negative definite. The determinant test is applied not only to A itself, giving $ac - b^2 > 0$, but also to the 1 by 1 submatrix a in the upper-hand corner.

2 Positive Definite Matrices

This tests is a necessary and sufficient condition for the real symmetric matrix A to be positive definite:
(I) $x^T Ax > 0$ for all nonzero real vectors x.
(II) All the eigen values of A satisfy $\lambda_i > 0$
(III) All the upper left submatrices A_k have positive determinants.
(IV) All the pivots (without row exchanges) satisfy $d_k > 0$

Example:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{2}$$

3 Positive SemiDefinite Matrices

This tests is a necessary and sufficient condition for the real symmetric matrix A to be positive semidefinite:

(I) $x^T Ax \geq 0$ for all nonzero real vectors x.
(II) All the eigen values of A satisfy $\lambda_i \geq 0$
(III)No principal submatrices have negative determinants..
(IV) No pivots negative.

Example:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{3}$$

4 Negative Definite Matrices

This tests is a necessary and sufficient condition for the real symmetric matrix A to be negative definite:
(I) $x^T Ax < 0$ for all nonzero real vectors x.
(II) All the eigen values of A satisfy $\lambda_i < 0$
(III) All the upper left submatrices A_k have negative determinants.
(IV) All the pivots (without row exchanges) satisfy $d_k < 0$

Example:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \tag{4}$$

5 Negative SemiDefinite Matrices

This tests is a necessary and sufficient condition for the real symmetric matrix A to be negative semidefinite:
(I) $x^T Ax \leq 0$ for all nonzero real vectors x.
(II) All the eigen values of A satisfy $\lambda_i \leq 0$
(III)No principal submatrices have positive determinants..
(IV) No pivots positive.

Example:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{5}$$

Note: This is another way to test positive definite, the symmetric matrix A is positive definite if and only if there is a matrix R with independent columns such that $A = R^T R$

Elimination $A = LDL^T = (L\sqrt{D})(\sqrt{D}L^T)$ so take $R = \sqrt{D}L^T$

Eigenvalues $A = Q\Lambda Q^T = (Q\sqrt{\Lambda})(\sqrt{\Lambda}Q^T)$ so take $R = \sqrt{\Lambda}Q^T$

6 Problem Set 6.2

1. Decide for or against the positive definiteness of

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Answer: det$A$=0 implies since the determinant is the product of its eigenvalue it follows that atleast of eigenvalues of A is equal to zero.Thus A is not positive definite.
det$(B - \lambda I)$=0
$\Rightarrow (\lambda - 1)^2(\lambda - 4)$=0
$\Rightarrow \lambda = 1, 1, 4 > 0$. Thus B is positive definite.
det C_2 = -1, where $C_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$\Rightarrow$ det $C = -1 < 0$. Thus C is not positive definite.

3. If A and B are positive definite,m then $A + B$ is positive definite. Pivots and eigenvalues are not convenient for $A + B$. Much better to prove $x^T(A + B)x > 0$.

Proof: A and B are positive definite.
$\Rightarrow x^T Ax$ >0 and $x^T Bx$ >0 $\forall$ real x
$\Rightarrow x^T(A + B)x = x^T Ax + x^T Bx > 0$ $\forall$ real x
$\Rightarrow A + B$ is positive definite.

11. If $A = R^T R$ prove the generalized Schwarz inequality

$$|x^T Ay|^2 \leq (x^T Ax)(y^T Ay)$$

Answer: Given $A = R^T R$
$\Rightarrow |x^T Ay|^2 = |x^T R^T Ry|^2 = |(Rx)^T(Ry)|^2 \leq \|Rx\|^2\|Ry\|^2$
$= [(Rx)^T(Rx)][(Ry)^T(Ry)]$=$(x^T R^T Rx)(y^T R^T Ry)$= $(x^T Ax)(y^T Ay)$.
Hence proved.

Lecturenote37

SINGULAR VALUE DECOMPOSITION

Any m by n matrix A can be factored into $A = U\Sigma V^T$
$A = U\Sigma V^T$ is known as the "SVD" or the singular value decomposition, where U is orthogonal, Σ is diagonal and V is orthogonal.
We know that if A is symmetric positive definite its eigenvectors are orthogonal and we can write $A = Q\Lambda Q^T$. This is a special case of SVD with $U = V = Q$

1 Work Step

To find U, we follow the following steps:
1. Find eigen values of AA^T. Arrange them in decreasing order i.e.

$$\lambda_1 > \lambda_2 > ....$$

2. Find eigen vectors $x_1, x_2, ...x_n$ for $\lambda_1, \lambda_2, ...\lambda_n$

3. Normalize the eigenvectors $u_i = \frac{x_i}{\|x_i\|}$

4. Write the orthogonal matrix $U = \begin{bmatrix} u_1 & u_2 & .... & u_m \end{bmatrix}$ with u_i as its ith column.

To find V^T, we need the steps as follows:
1. Find the eigenvalues of A^TA. Name them such that $\lambda_1 > \lambda_2 > ....$

2. Find eigenvectors y_i for λ_i of A^TA

3. Normalize the eigenvectors $v_i = \frac{y_i}{\|y_i\|}$

4. Write the orthogonal matrix $V^T = \begin{bmatrix} v_1 \\ v_2 \\ .... \\ v_n \end{bmatrix}$ with v_i as its ith row.

To find Σ, we follow the following steps:

1. The diagonal (but rectangular) matrix Σ has eigen values from A^TA
Those positive entries will be $\sigma_1, \sigma_2, ...\sigma_r$. They are the singular values of A. They fill the the first r places on the main diagonal of Σ, the rest are 0.

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & 0 & \cdots & 0 & \\ 0 & 0 & \sigma_3 & 0 & \cdots . & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & \end{bmatrix} \text{ with } \sigma_i = \sqrt{\lambda}_i \text{ as its dogonal.}$$

NOTE: All nonzero eigenvalues of AA^T and A^TA are same.

SVD of

$$A_{mxn} = U_{mxm}\Sigma_{mxn}V^T_{nxn}$$

If $A = U\Sigma V^T$ (the SVD), then its pseudoinverse is $A^+ = V\Sigma^+U^T$, where $\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 & \cdots & 0 & \\ 0 & \frac{1}{\sigma_2} & 0 & \cdots & 0 & \\ 0 & 0 & \frac{1}{\sigma_3} & 0 & \cdots . & \\ \cdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & \end{bmatrix}$

It is obtained from Σ^T by replacing all the nonzero singular values by their reciprocals.
If A^{-1} exist then $A^{-1} = A^+$

2 Problem Set-6.3

Question 1. (a) Compute AA^T and its eigenvalues ${\sigma_1}^2$, 0 and unit eigenvectors u_1, u_2.
(b) Choose signs so that $Av_1 = \sigma_1 u_1$ and verify the SVD:

$$\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$$

.
(c) Which four vectors give orthonormal bases for $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$
Answer: (a) Let's compute AA^T and its eigen value

$$AA^T = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix}$$

det(AA^T- λI) = 0

$$\Rightarrow \begin{vmatrix} 17 - \lambda & 34 \\ 34 & 68 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (17 - \lambda)(68 - \lambda) - 34 = 0$$

$$\Rightarrow \lambda^2 - 85\lambda + 1156 - 1156 = 0$$

$$\Rightarrow \lambda^2 - 85\lambda = 0$$

$\Rightarrow \lambda = 0,\ 85$

$\Rightarrow {\sigma_1}^2 = \lambda_1 = 85,\ \lambda_2 = 0$

Now let's find corresponding unit eigen vectors for eigenvalues of AA^T.

$(AA^T - \lambda_1 \text{ I}) \text{ x} = 0$

$$\Rightarrow \begin{bmatrix} -68 & 34 \\ 34 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 68x_1 + 34x_2 = 0$
$34x_1 - 17x_2 = 0$

$\Rightarrow 2x_1 - x_2 = 0$

$\Rightarrow 2x_1 = x_2$

$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2}$

$\therefore x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 85$

$\Rightarrow u_1 = \frac{x}{\|x\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$ (unit eigenvectors corresponding to λ_1=85)

$(AA^T - \lambda_2 \text{ I}) \text{ x} = 0$

$$\Rightarrow \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 17x_1 + 34x_2 = 0$
$34x_1 + 68x_2 = 0$

$\Rightarrow x_1 + 2x_2 = 0$

$\Rightarrow x_1 = -2x_2$

$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1}$

$\therefore x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_2 = 0$

$\Rightarrow u_2 = \frac{x}{\|x\|} = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ (unit eigenvectors corresponding to λ_2= 0)

(b) If we take $v_1 = \begin{bmatrix} \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \end{bmatrix}$ from the previous problem, let's choose a sign for σ_1 so that $Av_1 = \sigma_1 u_1$ and verify SVD for the matrix$A$.

$$Av_1 = \begin{bmatrix}1 & 4\\ 2 & 8\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{17}}\\ \frac{4}{\sqrt{17}}\end{bmatrix} = \begin{bmatrix}\sqrt{17}\\ 2\sqrt{17}\end{bmatrix} = \sqrt{8}5 \begin{bmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{bmatrix} = \sqrt{8}5\ u_1 = \sigma_1\ u_1$$

$$\begin{bmatrix}u_1 & u_2\end{bmatrix}\begin{bmatrix}\sigma_1 & \\ & 0\end{bmatrix}\begin{bmatrix}v_1 & v_2\end{bmatrix}^T = \frac{1}{\sqrt{5}}\begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix}\begin{bmatrix}\sqrt{8}5 & 0\\ 0 & 0\end{bmatrix}\frac{1}{\sqrt{17}}\begin{bmatrix}1 & -4\\ 4 & 1\end{bmatrix} = \begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & -4\\ 4 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ 2 & 0\end{bmatrix}\begin{bmatrix}1 & -4\\ 4 & 1\end{bmatrix} = \begin{bmatrix}1 & 4\\ 2 & 8\end{bmatrix} = A$$

(c) Since the rank of A and consequently of A^TA and AA^T are equal to 1, it follows that u_1 forms a basis for $C(A)$, while u_2 forms a basis for the $N(A^T)$. On the otherhand, v_1 forms a basis for $C(A^T)$, while v_2 forms a basis for $N(A)$.

Question 4. Find the SVD from the eigenvectors v_1, v_2 of A^TA and $Av_i = \sigma_i u_i$:

Fibonacci matrix $A = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$

Answer: In this case A^T= $\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$=$A$ then

$$A^TA = A^2 = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$$

det(AA^T- λI) = 0

$$\Rightarrow \begin{vmatrix}2-\lambda & 1\\ 1 & 1-\lambda\end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)1-\lambda) - 1 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 1 = 0$$

$\Rightarrow \lambda = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$ are the eigenvalues.

Now we determine eigenvectors.

$$A^TAx = \lambda x$$

$$\Rightarrow \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}\lambda x_1\\ \lambda x_2\end{bmatrix}$$

$$\Rightarrow x_1(\lambda - 2) = x_2$$

$$\iff x_1 = \frac{x_2}{\lambda-2}$$

$$(x_2 = 1),\ x_1 = \frac{1}{\lambda-2} = \frac{2}{3\pm\sqrt{5}-4} = \frac{2}{-1\pm\sqrt{5}} = \frac{1\pm\sqrt{5}}{2}$$

$\therefore v_1$=$\begin{bmatrix}\frac{1+\sqrt{5}}{2}\\ 1\end{bmatrix}$ and v_2=$\begin{bmatrix}\frac{1-\sqrt{5}}{2}\\ 1\end{bmatrix}$ which norms are

$\|v_1\| = \frac{5+\sqrt{5}}{2}$ and $\|v_2\| = \frac{5-\sqrt{5}}{2}$

the unit vectors are $\hat{v_1} = \begin{bmatrix} \sqrt{\frac{5+\sqrt{5}}{10}} \\ \sqrt{\frac{2}{5+\sqrt{5}}} \end{bmatrix}$ and $\hat{v_2} = \begin{bmatrix} -\sqrt{\frac{5-\sqrt{5}}{10}} \\ \sqrt{\frac{2}{5-\sqrt{5}}} \end{bmatrix}$

On the otherhand we have $\sigma_1 = \sqrt{\frac{3+\sqrt{5}}{2}}$ and $\sigma_2 = \sqrt{\frac{3-\sqrt{5}}{2}}$

$u_1 = \frac{1}{\sigma_1} A\hat{v_1} = \frac{1}{\sqrt{\frac{3+\sqrt{5}}{2}}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{5+\sqrt{5}}{10}} \\ \sqrt{\frac{2}{5+\sqrt{5}}} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{5+\sqrt{5}}{10}} \\ \sqrt{\frac{2}{5+\sqrt{5}}} \end{bmatrix} = \hat{v_1}$

$u_2 = \frac{1}{\sigma_2} A\hat{v_2} = \frac{1}{\sqrt{\frac{3-\sqrt{5}}{2}}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -\sqrt{\frac{5-\sqrt{5}}{10}} \\ \sqrt{\frac{2}{5-\sqrt{5}}} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{5-\sqrt{5}}{10}} \\ -\sqrt{\frac{2}{5-\sqrt{5}}} \end{bmatrix} =$
$-\hat{v_2}$

Finally the SVD decomposition of the matrix A result:

$$A = U\Sigma V^T = \begin{bmatrix} \sqrt{\frac{5+\sqrt{5}}{10}} & \sqrt{\frac{5-\sqrt{5}}{10}} \\ \sqrt{\frac{2}{5+\sqrt{5}}} & -\sqrt{\frac{2}{5-\sqrt{5}}} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{3+\sqrt{5}}{2}} & 0 \\ 0 & \sqrt{\frac{3-\sqrt{5}}{2}} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{5+\sqrt{5}}{10}} & \sqrt{\frac{2}{5+\sqrt{5}}} \\ -\sqrt{\frac{5-\sqrt{5}}{10}} & \sqrt{\frac{2}{5-\sqrt{5}}} \end{bmatrix}$$

Lecturenote38

SINGULAR VALUE DECOMPOSITION

A as a linear transformation taking a vector v_1 in its row space to a vector $u_1 = Av_1$ in its columnspace. The SVD arises from finding an orthogonal basis for the row space that gets transformed into an orthogonal basis for the column space: $Av_i = \sigma_i u_i$
We have to find an orthonormal basis $v_1, v_2, ...v_r$ for the row space of A for which A

$$\begin{bmatrix} v_1 & v_2 & .... & v_r \end{bmatrix} = \begin{bmatrix} \sigma_1 u_1 & \sigma_2 u_2 & .... & \sigma_r u_r \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ .... & & \\ 0 & 0 & \sigma_r \end{bmatrix}$$

with $u_1, u_2, ...u_r$ an orthonormal basis for the column space of A. Once we add in the null spaces, this equation will become $AV = U\Sigma$. (We have the orthonormal bases $v_1, v_2, ...v_r$ and $u_1, u_2, ...u_r$ to orthonormal bases for the entire space. Since $v_{r+1}, v_{r+2}, ...v_n$ will be in the null space of A, the diagonal entries $\sigma_{r+1}, \sigma_{r+2}, ..., \sigma_n$ will be 0). The columns of U and V are bases for the row and column spaces respectively.

1 Problem Set- 6.3

Question.14: Find the SVD and the pseudoinverse $V\Sigma^+U^T$ of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Answer: Let's find out the eigen values and eigen vectors of AA^T and A^TA

$$AA^T = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \end{bmatrix}$$

det(AA^T- λI) = 0

$\Rightarrow \begin{vmatrix} 4 - \lambda \end{vmatrix} = 0$
$\lambda = 4$
and $x = 1$ is an eigen vector.

$$A^TA = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\end{bmatrix}$$

det(A^TA- λI) $= 0$
The eigen values of A^TA are 4,0,0,0.
For$\lambda = 4$, $(A^TA - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix}-3 & 1 & 1 & 1\\1 & -3 & 1 & 1\\1 & 1 & -3 & 1\\1 & 1 & 1 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \implies \begin{bmatrix}1 & -3 & 1 & 1\\0 & -8 & 4 & 4\\0 & 0 & -2 & -2\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$\Rightarrow -2x_3 + 2x_4 = 0$
$-8x_2 + 4x_3 + 4x_4 = 0$
$x_1 - 3x_2 + x_3 + x_4 = 0$

$\Rightarrow x_3 = x_4$, let $x_4 = 1$
$\Rightarrow x_2 = 1$, $x_1 = 1$

$$\therefore x= \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \text{ or } x= \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$$

For$\lambda = 0$, $(A^TA - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix}1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$\Rightarrow x_1 + x_2 + x_3 + x_4 = 0$

$$\therefore x = \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$$

The eigen vectors for $\lambda = 0$ are $\begin{bmatrix}\frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\\0\end{bmatrix}, \begin{bmatrix}\frac{-1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}, \begin{bmatrix}\frac{-1}{\sqrt{2}}\\0\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$

$$\therefore SVD \text{ of } A = U\Sigma V^T = \begin{bmatrix}1\end{bmatrix}\begin{bmatrix}2 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & 0\\\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0\\\frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}^T$$

$$\therefore A^+ = V\Sigma^+U^T = \begin{bmatrix}\frac{1}{2} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & 0\\\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0\\\frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\0\\0\\0\end{bmatrix}\begin{bmatrix}1\end{bmatrix} = \begin{bmatrix}\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}$$

Now B matrix SVD will find out:

$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $B^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$

$BB^T = B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

det(BB^T- λI) $= 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$\lambda = 1, 1$

and eigen vectoros BB^T are$(BB^T - \lambda I)x = 0$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ the eigenvectors are $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Hence $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$B^T B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

det($B^T B$- λI) $= 0$

The eigen values of $B^T B$ are 1,1,0,.

For$\lambda = 1$, $(B^T B - \lambda I)x = 0$

$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For$\lambda = 0$, $(B^T B - \lambda I)x = 0$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 0, x_2 = 0, x_3 = 1(freevariable)$ so the eigen vector is $x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Here$\lambda$ is repeated three times, but we got only one eigen vector. So we have to consider orthornmal eigen vectors.

$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

SVD of $B = U\Sigma V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$

Pseuodinverse is $B^+ = V\Sigma^+U^T = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} = \begin{bmatrix}0&1\\1&0\\0&0\end{bmatrix}$

Now C matrix SVD will find out:

$C = \begin{bmatrix}1&1\\0&0\end{bmatrix}$, $C^T = \begin{bmatrix}1&0\\1&0\end{bmatrix}$

$CC^T = \begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\1&0\end{bmatrix} = \begin{bmatrix}2&0\\0&0\end{bmatrix}$

det(CC^T- λI) = 0

The eigen values of CC^T are 2,0

For$\lambda = 2$, $(CC^T - \lambda I)x = 0$

$\Rightarrow \begin{bmatrix}0&0\\0&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\Rightarrow -2x_2 = 0$

$\Rightarrow x_2 = 0$ and $x_1 = 1$ (free variable)

$\therefore x = \begin{bmatrix}1\\0\end{bmatrix}$

For$\lambda = 0$ $(CC^T - \lambda I)x = 0$

$\Rightarrow \begin{bmatrix}2&0\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\Rightarrow 2x_1 = 0$

$\Rightarrow x_1 = 0$ and $x_2 = 1$(free variable)

$\therefore x = \begin{bmatrix}0\\1\end{bmatrix}$

$U = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

$C^TC = \begin{bmatrix}1&0\\1&0\end{bmatrix}\begin{bmatrix}1&1\\0&0\end{bmatrix} = \begin{bmatrix}1&1\\1&1\end{bmatrix}$

det(C^TC- λI) = 0

The eigen values of C^TC are 2,0

For$\lambda = 2$, $(C^TC - \lambda I)x = 0$

$\Rightarrow \begin{bmatrix}-1&1\\1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\Rightarrow x_1 = x_2$, the eigenvector is x=$\begin{bmatrix}1\\1\end{bmatrix}$

. So $v_1 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$

For$\lambda = 0$, $(C^TC - \lambda I)x = 0$

$\Rightarrow \begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\Rightarrow x_1 = -x_2$, the eigenvector is x=$\begin{bmatrix}1\\-1\end{bmatrix}$

. So $v_2 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{bmatrix}$.

Hence $V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

SVD of $C = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}^T$

Pseuodinverse of C is $C^+ = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$

Lecturenote39

MATRIX NORM AND CONDITION NORM

For a positive definite matrix, A. The norm of A is denoted as $\|A\|$ which is $\lambda_{max} of A$

The norm of A^{-1} is denoted as $\|A^{-1}\|$ which is $\frac{1}{\lambda_{min}} of A$

The condition number is denoted as $c = \|A\| \ \|A^{-1}\| = \frac{\lambda_{max}}{\lambda_{min}}$ of A.

For any given matrix A .(A may not be positive definite matrix).

$\|A\|$ = Norm of $A = \sqrt{\lambda_{max} of A^T A}$

$\|A^{-1}\|$ = Norm of $A^{-1} = \frac{1}{\sqrt{\lambda_{min} of A^T A}}$

c = Condition number = $\|A\| \ \|A^{-1}\|$

$= \frac{\sqrt{\lambda_{max} of A^T A}}{\sqrt{\lambda_{min} of A^T A}}$

1 Problem set- 7.2

Question 15. Find the norms λ_{max} and condition numbers $\frac{\lambda_{max}}{\lambda_{min}}$ of these positive definite matrices:

$\begin{bmatrix} 100 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$

Answer: $A = \begin{bmatrix} 100 & 0 \\ 0 & 2 \end{bmatrix}$

$\Rightarrow det(A - \lambda I)$=0

$\Rightarrow (100 - \lambda)(2 - \lambda)$=0

$\Rightarrow \lambda = 100, 2$

$\|A\| = \lambda_{max} of A$=100

$\|A^{-1}\|$= $\frac{1}{\lambda_{min}} of A = \frac{1}{2}$

$c = \|A\| \ \|A^{-1}\| = 100 * \frac{1}{2}$= 50

Now $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
$\Rightarrow det(B - \lambda I)$=0
$\Rightarrow (2-\lambda)^2 - 1$=0
$\Rightarrow \lambda = 3, 1$

$\|B\| = \lambda_{max} of B$=3

$\|B^{-1}\|$= $\frac{1}{\lambda_{min}} of B = 1$

$c = \|B\| \; \|B^{-1}\| = 3$

Now let $D = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$
$\Rightarrow det(D - \lambda I)$=0
$\Rightarrow (3-\lambda)(1-\lambda) - 1$=0
$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$
$\Rightarrow \lambda = 2 \pm \sqrt{2}$

$\|D\| = \lambda_{max} of D$=$2 + \sqrt{2}$

$\|D^{-1}\|$= $\frac{1}{\lambda_{min}} of D = \frac{1}{2-\sqrt{2}}$

$c = \|D\| \; \|D^{-1}\| = \frac{2+\sqrt{2}}{2-\sqrt{2}}$

Question-17: Find the norms and condition numbers from the square roots of $\lambda_{max}(A^T A)$ and $\lambda_{min}(A^T A)$:

$\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Answer: Let us name A= $\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$, $A^T = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow A^T A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$
The eigen values are 4,4.

$\|A\| = \sqrt{\lambda_{max} of A^T A} = 2$

$\|A^{-1}\| = \frac{1}{\sqrt{\lambda_{min} of A^T A}} = \frac{1}{2}$

$c = \|A\| \; \|A^{-1}\| = 2 * \frac{1}{2} = 1$

Next matrix name as $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow B^T B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
The eigen values are 2.0

$\|B\| = \sqrt{\lambda_{max} of B^T B} = \sqrt{2}$

$\|B^{-1}\| = \frac{1}{\sqrt{\lambda_{min} of B^T B}} = \frac{1}{0} = \infty$

$c = \|B\| \ \|B^{-1}\| = 2 * \frac{1}{0} = \infty$

Next matrix name as $D = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $D^T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow D^T D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
The eigen values are 2,2.

$\|D\| = \sqrt{\lambda_{max} of D^T D} = \sqrt{2}$

$\|D^{-1}\| = \frac{1}{\sqrt{\lambda_{min} of D^T D}} = \frac{1}{\sqrt{2}}$

$c = \|D\| \ \|D^{-1}\| = \sqrt{2} * \frac{1}{\sqrt{2}} = 1$

lecturenote40
ITERATIVE METHODS FOR $Ax = b$

We know that the solution of the system for $Ax = b$, required finite number of steps for a full matrix by Gaussian elimination, but when it is large, we may have to set for an approximate x that can be obtained more quickly.
An iterative method is easy to invent, by splittin the matrix A. Put $A = S - T$ in $Ax = b$.
$\Rightarrow Sx = Tx + b$ Iterative from x_k to $x_{k+1} \Rightarrow Sx_{k+1} = Tx_k + b$
The iterative method in the above equation is convergent if and only if every eigen value of $S^{-1}T$ satisfies $|\lambda| < 1$. Its rate of convergence depends on the maximum of $|\lambda|$.

Working Procedure:
1. Write A as sum of Lower trianguar L, Diagonal D, and Upper triangular U matrix form i.e. $A = L + D + U$
2. to select S = digonal part of A (Jacobi's method) i.e. $S = D$
S = triangular part of A (Gauss-Seidel method) i.e. $S = D + L$
3.T= (-L-U) for (Jacobi's method), T= (-U) for (Gauss-Seidel method)
4. Find $S^{-1}T$=$D^{-1}(-L - U)$ for (Jacobi's method)
$S^{-1}T$=$(D + L)^{-1}(-U)$ for (Gauss-Seidel method)

2 Problem set- 7.4

Question-2: The matrix has eigenvalues $2 - \sqrt{2}, 2, and 2 + \sqrt{2}$:
$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$
. Find the Jacobi matrix $D^{-1}(-L - U)$ and the Gauss-Seidel matrix $(D + L)^{-1}(-U)$

Answer: $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \text{L+D+U} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Jacobi matrix $D^{-1}(-L-U) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$

Gauss-Seidel matrix $(D+L)^{-1}(-U) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{8} & \frac{1}{4} \end{bmatrix}$

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Lecturenote40

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1 Problem set- 7.4

Question-2: The matrix has eigenvalues $2 - \sqrt{2}, 2, and 2 + \sqrt{2}$:
$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$
. Find the Jacobi matrix $D^{-1}(-L - U)$ and the Gauss-Seidel matrix $(D + L)^{-1}(-U)$

$$\text{Answer: } A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \text{L+D+U} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Jacobi matrix } D^{-1}(-L - U) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$$\text{Gauss-Seidel matrix } (D + L)^{-1}(-U) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{8} & \frac{1}{4} \end{bmatrix}$$

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