4.4

# Applications of Determinants

① Computation of $A^{-1}$; i.e. inverse of A.

$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$ ; where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

$= \frac{1}{|A|} C^T$ , where C = cofactor matrix

$= \frac{1}{|A|} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}^T$ | $C_{ij} = (-1)^{i+j}$ minor of $a_{ij}$

## Problem Set - 4.4

5) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

~~$\det A = 2(4-1) - 1(0+2) + 0 = 4$~~

$\det A = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix}$

$= 2(4-1) + 1(-2+0) + 0$

$= 4$

$c_{11}$ = cofactor of 2 $= (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$

$c_{12}$ = " " $-1 = (-1)^{1+2}\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 2$

$c_{13}$ = " " $0 = (-1)^{1+3}\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$

$c_{21}$ = " " $-1 = (-1)^{2+1}\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = 2$

$c_{22}$ = " " $2 = (-1)^{2+2}\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$

$c_{23}$ = " " $-1 = (-1)^{2+3}\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} = 2$

$c_{31}$ = " " $0 = (-1)^{3+1}\begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} = 1$

$c_{32}$ = " " $-1 = (-1)^{3+2}\begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} = 2$

$c_{33}$ = " " $2 = (-1)^{3+3}\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$

$\therefore C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} C^T$

$\Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

(Ans)

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\det B = 1\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= 6-4-(3-2)+0$$

$$= 2-1$$

$$= 1$$

$C_{11} = 1\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = 2$ | $C_{21} = -1\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -1$ | $C_{31} = 1\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$

$C_{12} = -1\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -1$ | $C_{22} = 1\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2$ | $C_{32} = -1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1$

$C_{13} = 1\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$ | $C_{23} = -1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1$ | $C_{33} = 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$

$$\therefore A^{-1} = \frac{1}{|A|} C^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{ (Ans:-)}$$

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## (2) Cramer's rule: To solve $Ax = b$

$$Ax = b \Rightarrow x = A^{-1}b$$

The $j^{th}$ component of $x = A^{-1}b$ is the ratio:

$$x_j = \frac{\det B_j}{\det A}; \text{ where } B_j = \begin{bmatrix} a_{11} & a_{12} & b_1 & a_{1n} \\ \vdots & \vdots \quad \cdots & \vdots \quad \cdots & \vdots \\ a_{n1} & a_{n2} & b_n & a_{nn} \end{bmatrix} \text{ has}$$

$b$ in column $j$.

$$\det B_j = b_1 C_{1j} + b_2 C_{2j} + \cdots b_n C_{nj}$$

# Problem Set - 4.4

7) $ax + by = 1$

$cx + dy = 0$

$$x = \frac{\det B_1}{\det A} = \frac{\begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{d}{ad-bc}$$

$$y = \frac{\det B_2}{\det A} = \frac{\begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = -\frac{c}{ad-bc}$$

(Ans:-)

$x + 4y - z = 1$

$x + y + z = 0$

$2x + 3z = 0$

$$\Rightarrow \begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x = \frac{\det B_1}{\det A} = \frac{3}{1} = 3$$

$$\det A = \begin{vmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} - 4\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}$$

$$= 1\cdot 3 - 4(3-2) - 1(-2)$$

$$= 3 - 4 + 2$$

$$= 1$$

$$\det B_1 = \begin{vmatrix} 1 & 4 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 3$$

$$y = \frac{\det B_2}{\det A} = \frac{-1}{1} = -1$$

$$\det B_2 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}$$

$$= 0 - 1 - 0$$

$$= -1$$

$$z = \frac{\det B_3}{\det A} = -2$$

$$\det B_3 = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - 4\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}$$

$$= 0 - 0 + (-2)$$

$$= -2$$

14)

a)
$$2x_1 + 5x_2 = 1$$
$$x_1 + 4x_2 = 2$$

$$\Rightarrow \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3$$

$$x_1 = \frac{\det B_1}{\det A} \quad \Bigg| \quad \det B_1 = \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = 4 - 10 = -6$$

$$= \frac{-6}{3}$$

$$= -2$$

$$x_2 = \frac{\det B_2}{\det A} \quad \Bigg| \quad |B_2| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$= \frac{3}{3}$$

$$= 1$$

$\therefore$ Solⁿ : $x_1 = -2$, $x_2 = 1$ (Ans)

b)
$$2x_1 + x_2 = 1$$
$$x_1 + 2x_2 + x_3 = 0$$
$$x_2 + 2x_3 = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 2.3 - 1.2 + 0$$

$$= 4$$

$$x_1 = \frac{|B_1|}{|A|} \quad \Bigg| \quad |B_1| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} + 0$$

$$= \frac{3}{4} \qquad\qquad = 3$$

$$x_2 = \frac{|B_2|}{|A|} \quad \Bigg| \quad |B_2| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 0$$

$$= -\frac{2}{4} = -\frac{1}{2} \qquad\qquad = 0 - 2 + 0$$

$$= -2$$

$$x_3 = \frac{|B_3|}{|A|} \qquad |B_3| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 0 - 0 + 1$$

$$= 1$$

$$= \frac{1}{4}$$

Soln.: $x_1 = 3/4$ ; $x_2 = -1/2$ ; $x_3 = 1/4$

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(3) <u>Volume of a Box :</u>

When the adjacent edges of the box are right-angled:

volume : $l_1 l_2 \cdots l_n$ (product of the edge lengths)

Write the edge as the rows of A.

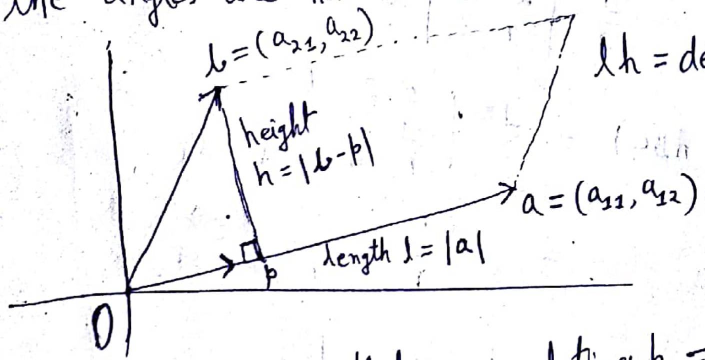
$$\therefore A A^T = \begin{bmatrix} \boxed{\quad l_1 \quad} \\ \boxed{\quad l_2 \quad} \\ \vdots \\ \boxed{\quad l_n \quad} \end{bmatrix} \begin{bmatrix} \boxed{l_1} & \boxed{l_2} & \cdots & \boxed{l_n} \end{bmatrix}$$

$$= \begin{bmatrix} l_1^2 & 0 & 0 & \cdots & 0 \\ 0 & l_2^2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & & l_n^2 \end{bmatrix}$$

$$\therefore l_1^2 l_2^2 \cdots l_n^2 = \det(AA^T) = (\det A)(\det A^T) = (\det A)^2$$

$$\therefore \det A = l_1 l_2 \cdots l_n = \text{volume of the box.}$$

When the angles are not 90°:




$$lh = \det\begin{bmatrix} a \\ b-p \end{bmatrix} = \det\begin{bmatrix} a \\ b \end{bmatrix}$$

Volume (Area) of the parallelogram = $l$ times $h$ = $|\det A|$

$\therefore$ Area of the triangle $= \frac{1}{2}$ (area of the parallelogram)

$$= \tfrac{1}{2} (l \text{ times } h)$$

$$= \frac{1}{2} \det A$$

$$= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

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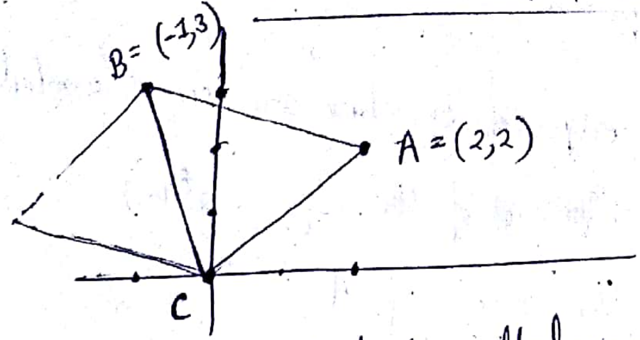
Problem Set- 4.4

2) a)




The area of that parallelogram is $= \begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix}$

$$= 6 - 2 = 4$$

$\therefore$ area (ABC) $= \frac{1}{2} \det A$

$$= \tfrac{1}{2} 4 = 2$$

$$\text{area}(ABC) = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix} = 2$$

b) $$\text{area}(ABC) = \frac{1}{2} \begin{vmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -4 & 1 \end{vmatrix}$$

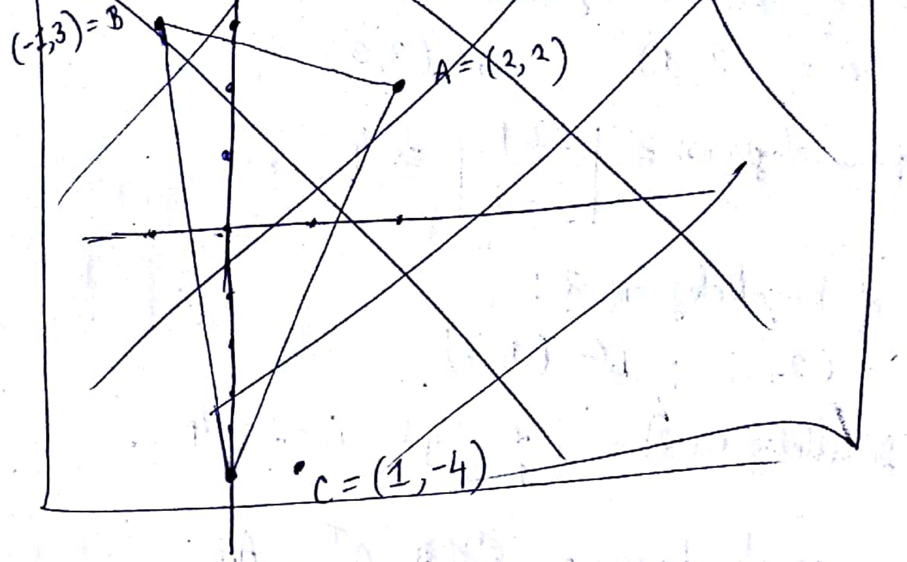
$$= \frac{1}{2} \begin{vmatrix} 1 & 6 & 0 \\ -2 & 7 & 0 \\ 1 & -4 & 1 \end{vmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - R_3 \\ R_1 \rightarrow R_1 - R_3 \end{matrix}$$

$$= \frac{1}{2} \begin{vmatrix} 1 & 6 \\ -2 & 7 \end{vmatrix}$$

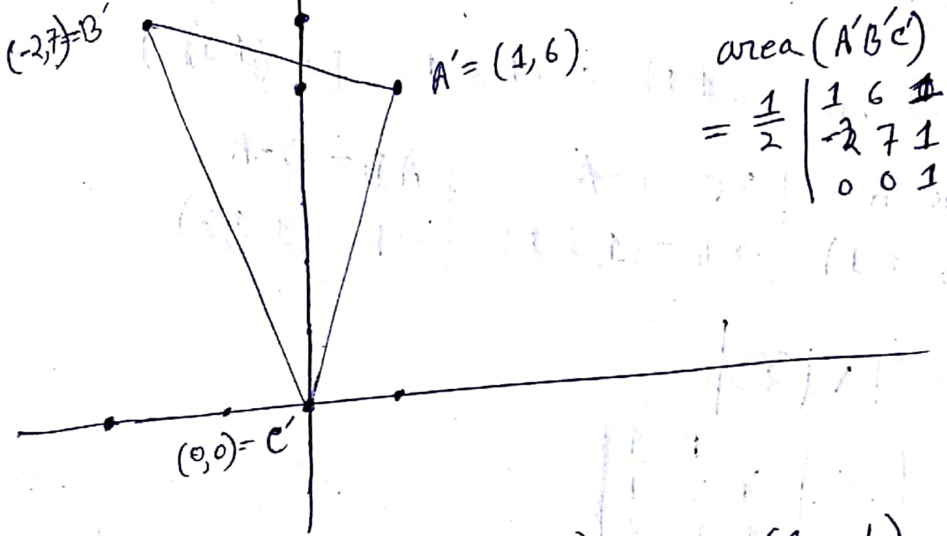
$$= \frac{19}{2}$$

allelogram

$A = (2,2)$ , $B = (-1,3)$ , $C = (1,-4)$




$A' = (1,6)$ , $B' = (-2,7)$ , $C' = (0,0)$

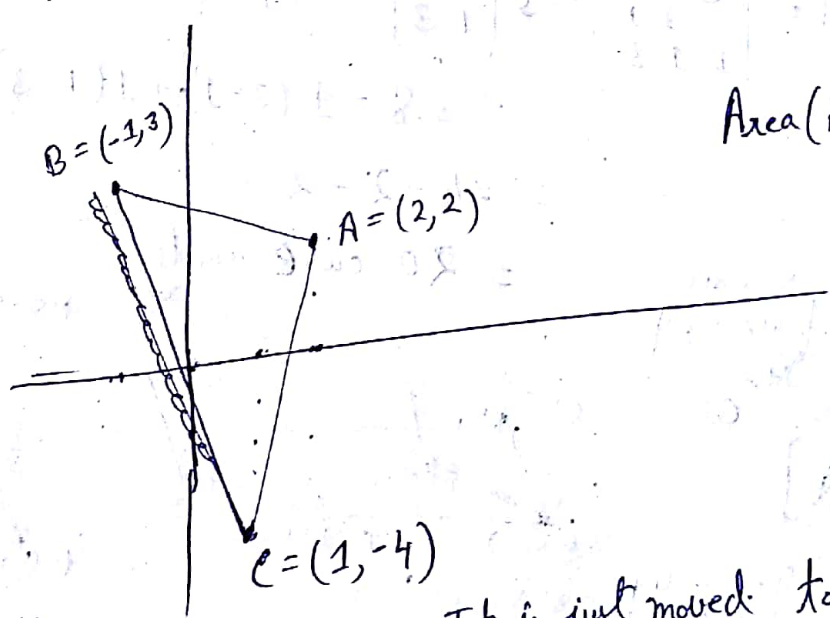



area$(A'B'C')$

$$= \frac{1}{2}\begin{vmatrix} 1 & 6 & 1 \\ -2 & 7 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 6 \\ -2 & 7 \end{vmatrix}$$

$$= 19/2$$

$A = (2,2)$ , $B = (-1,3)$ , $C = (1,-4)$




Area$(ABC) = 19/2$

Both areas are same. It is just moved to origin.

27> Sides of parallelogram 1:

$a = (2,1)$ ; $b = (2,3)$

$$\text{area(parallelogram1)} = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 4$$

Sides of parallelogram 2:

$a' = (2,2)$ ; $b' = (1,3)$

$$\text{area (parallelogram 2)} = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 6-2 = 4$$

They are equal because $A_2^T = A_1$

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29> $A = (0,0,0)$ , $B = (3,1,1)$, $C = (1,3,1)$ , $D = (1,1,3)$

$\therefore$ $AB = B - A$ $\Rightarrow l = (3,1,1)$ | $AC = C - A$ $\Rightarrow b = (1,3,1)$ | $AD = D - A$ $\Rightarrow h = (1,1,3)$

$$V = \begin{bmatrix} l \\ b \\ h \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

$$\text{Volume} = |V| = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 3\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= 3.8 - 1.(3-1) + 1(1-3)$$

$$= 24 - 2 - 2$$

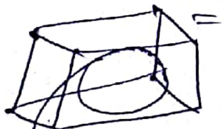
$$= 20 \text{ cube units}$$

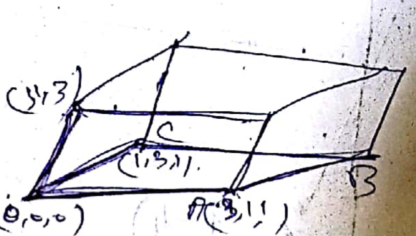
Area of the face OABC

$= |OA \times OC|$

$= \sqrt{72}$

$= 6\sqrt{2}$



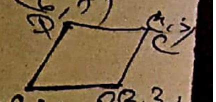



$$OA \times OC = \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} \quad OA = (3,1,1) \quad OC = (1,3,1)$$

$= -2i - 2j + 8k$

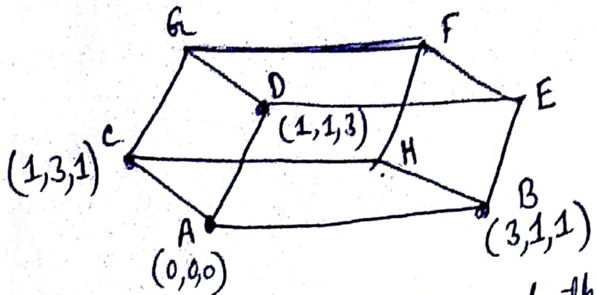
Area of ABCD

$= \begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix}$




$AB = (1,2)$

$AD = (5,6)$




Area of the face ABED = Area of the face CHFG

" " " " ADGC = " " " BEFH

" " " " ABHC = " " " DEFG

$$\text{Area}(ABED) = \|AB \times AD\|$$
$$= \sqrt{2^2 + 8^2 + 2^2}$$
$$= \sqrt{72}$$
$$= 6\sqrt{2} \text{ sq. unit}$$

$$AB \times AD = \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 1 & 3 \end{vmatrix}$$
$$= 2i - 8j + 2k$$

$$\text{Area}(ABHC) = \|AB \times AC\|$$
$$= \sqrt{(-2)^2 + 2^2 + 8^2}$$
$$= \sqrt{72}$$
$$= 6\sqrt{2} \text{ sq. unit}$$

$$AB \times AC = \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix}$$
$$= -2i - 2j + 8k$$

$$\text{Area}(ADGC) = \|AD \times AC\|$$
$$= \sqrt{(-8)^2 + (-2)^2 + 2^2}$$
$$= \sqrt{72}$$
$$= 6\sqrt{2} \text{ sq. unit.}$$

$$AD \times AC = \begin{vmatrix} i & j & k \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{vmatrix}$$
$$= -8i + 2j + 2k$$