CENGAGE
Learning

Fourth Edition

LINEAR ALGEBRA
AND ITS APPLICATIONS

Gilbert Strang

there is a chance that b *does* lie in the plane of the columns. In that case there are too many solutions; the three columns can be combined in *infinitely many ways* to produce b. That column picture in Figure 1.6b corresponds to the row picture in Figure 1.5c.

How do we know that the three columns lie in the same plane? One answer is to find a combination of the columns that adds to zero. After some calculation, it is $u = 3$, $v = -1$, $w = -2$. Three times column 1 equals column 2 plus twice column 3. Column 1 is in the plane of columns 2 and 3. Only two columns are independent.

The vector $b = (2, 5, 7)$ is in that plane of the columns—it is column 1 plus column 3—so $(1, 0, 1)$ is a solution. *We can add any multiple of the combination* $(3, -1, -2)$ *that gives* $b = 0$. So there is a whole line of solutions—as we know from the row picture.

The truth is that we *knew* the columns would combine to give zero, because the rows did. That is a fact of mathematics, not of computation—and it remains true in dimension n. ***If the n planes have no point in common, or infinitely many points, then the n columns lie in the same plane.***

If the row picture breaks down, so does the column picture. That brings out the difference between Chapter 1 and Chapter 2. This chapter studies the most important problem—the *nonsingular* case—where there is one solution and it has to be found. Chapter 2 studies the general case, where there may be many solutions or none. In both cases we cannot continue without a decent notation (*matrix notation*) and a decent algorithm (*elimination*). After the exercises, we start with elimination.

Problem Set 1.2

1. Solve to find a combination of the columns that equals b:

Triangular system
$$\begin{aligned} u - v - w &= b_1 \\ v + w &= b_2 \\ w &= b_3. \end{aligned}$$

2. Sketch these three lines and decide if the equations are solvable:

3 by 2 system
$$\begin{aligned} x + 2y &= 2 \\ x - y &= 2 \\ y &= 1. \end{aligned}$$

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

3. For the equations $x + y = 4$, $2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector $(4, 4)$ on the right side).

4. Find two points on the line of intersection of the three planes $t = 0$ and $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

5. (Recommended) Describe the intersection of the three planes $u + v + w + z = 6$ and $u + w + z = 4$ and $u + w = 2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

6. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{aligned} ax + 2y &= 0 \\ 2x + ay &= 0 \end{aligned}$$

7. Explain why the system

$$\begin{aligned} u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= 0 \end{aligned}$$

is singular by finding a combination of the three equations that adds up to $0 = 1$. What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

8. (Recommended) Under what condition on y_1, y_2, y_3 do the points $(0, y_1)$, $(1, y_2)$, $(2, y_3)$ lie on a straight line?

9. When $b = (2, 5, 7)$, find a solution (u, v, w) to equation (4) different from the solution $(1, 0, 1)$ mentioned in the text.

10. Give two more right-hand sides in addition to $b = (2, 5, 7)$ for which equation (4) can be solved. Give two more right-hand sides in addition to $b = (2, 5, 6)$ for which it cannot be solved.

11. The column picture for exercise 7 (singular system) is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b.$$

Show that the three columns on the left lie in the same plane by expressing the third column as a combination of the first two. What are all the solutions (u, v, w) if b is the zero vector $(0, 0, 0)$?

12. Starting with $x + 4y = 7$, find the equation for the parallel line through $x = 0$, $y = 0$. Find the equation of another line that meets the first at $x = 3$, $y = 1$.

Problems 13–15 are a review of the row and column pictures.

13. For two linear equations in three unknowns x, y, z, the row picture will show (2 or 3) (lines or planes) in (two or three)-dimensional space. The column picture is in (two or three)-dimensional space. The solutions normally lie on a ______.

14. For four linear equations in two unknowns x and y, the row picture shows four ______. The column picture is in ______-dimensional space. The equations have no solution unless the vector on the right-hand side is a combination of ______.

15. Draw the two pictures in two planes for the equations $x - 2y = 0$, $x + y = 6$.

16. Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$ and a third point halfway between.

17. In Problem 22 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a "singular case" because the third column is ______. Find two combinations of the columns that give $b = (2, 3, 5)$. This is only possible for $b = (4, 6, c)$ if $c =$ ______.

18. In these equations, the third column (multiplying w) is the *same* as the right side b. The column form of the equations *immediately* gives what solution for (u, v, w)?

$$\begin{aligned} 6u + 7v + 8w &= 8 \\ 4u + 5v + 9w &= 9 \\ 2u - 2v + 7w &= 7. \end{aligned}$$

19. Move the third plane in Problem 22 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line **L**, but the third plane doesn't ______ that line.

20. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?

21. If (a, b) is a multiple of (c, d) with $abcd \neq 0$, *show that* (a, c) *is a multiple of* (b, d). This is surprisingly important: call it a challenge question. You could use numbers first to see how a, b, c, and d are related. The question will lead to:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has dependent rows then it has dependent columns.}$$

22. The first of these equations plus the second equals the third:

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 5. \end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also ______. The equations have infinitely many solutions (the whole line **L**). Find three solutions.

23. Normally 4 "planes" in four-dimensional space meet at a ______. Normally 4 column vectors in four-dimensional space can combine to produce b. What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $b = (3, 3, 3, 2)$? What 4 equations for x, y, z, t are you solving?

1.3 AN EXAMPLE OF GAUSSIAN ELIMINATION

The way to understand elimination is by example. We begin in three dimensions:

$$\textbf{Original system} \qquad \begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9. \end{aligned} \tag{1}$$

The problem is to find the unknown values of u, v, and w, and we shall apply Gaussian elimination. (Gauss is recognized as the greatest of all mathematicians, but certainly not because of this invention, which probably took him ten minutes. Ironically, of two vectors in two-dimensional space would seem to take 8 multiplications, but they can be done in 7. That lowered the exponent from $\log_2 8$, which is 3, to $\log_2 7 \approx 2.8$. This discovery produced tremendous activity to find the smallest possible power of n. The exponent finally fell (at IBM) below 2.376. Fortunately for elimination, the constant C is so large and the coding is so awkward that the new method is largely (or entirely) of theoretical interest. The newest problem is the cost with *many processors in parallel*.

Problem Set 1.3

Problems 1–9 are about elimination on 2 by 2 systems.

1. Choose a right-hand side which gives no solution and another right-hand side which gives infinitely many solutions. What are two of those solutions?

$$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= \_\_. \end{aligned}$$

2. What multiple of equation 2 should be *subtracted* from equation 3?

$$\begin{aligned} 2x - 4y &= 6 \\ -x + 5y &= 0. \end{aligned}$$

After this elimination step, solve the triangular system. If the right-hand side changes to $(-6, 0)$, what is the new solution?

3. Choose a coefficient b that makes this system singular. Then choose a right-hand side g that makes it solvable. Find two solutions in that singular case.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g. \end{aligned}$$

4. What multiple ℓ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} 2x + 3y &= 1 \\ 10x + 9y &= 11. \end{aligned}$$

After this elimination step, write down the upper triangular system and circle the two pivots. The numbers 1 and 11 have no influence on those pivots.

5. Solve the triangular system of Problem 4 by back-substitution, y before x. Verify that x times $(2, 10)$ plus y times $(3, 9)$ equals $(1, 11)$. If the right-hand side changes to $(4, 44)$, what is the new solution?

6. What multiple ℓ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} ax + by &= f \\ cx + dy &= g. \end{aligned}$$

The first pivot is a (assumed nonzero). Elimination produces what formula for the second pivot? What is y? The second pivot is missing when $ad = bc$.

7. What test on b_1 and b_2 decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2. \end{aligned}$$

8. For which numbers a does elimination break down (a) permanently, and (b) temporarily?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6. \end{aligned}$$

Solve for x and y after fixing the second breakdown by a row exchange.

9. For which three numbers k does elimination break down? Which is fixed by a row exchange? In each case, is the number of solutions 0 or 1 or ∞?

$$\begin{aligned} kx + 3y &= 6 \\ 3x + ky &= -6. \end{aligned}$$

Problems 10–19 study elimination on 3 by 3 systems (and possible failure).

10. Which number b leads later to a row exchange? Which b leads to a missing pivot? In that singular case find a nonzero solution x, y, z.

$$\begin{aligned} x + by \phantom{{}-z} &= 0 \\ x - 2y - z &= 0 \\ y + z &= 0. \end{aligned}$$

11. Which number d forces a row exchange, and what is the triangular system (not singular) for that d? Which d makes this system singular (no third pivot)?

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3. \end{aligned}$$

12. Reduce this system to upper triangular form by two row operations:

$$\begin{aligned} 2x + 3y + z &= 8 \\ 4x + 7y + 5z &= 20 \\ -2y + 2z &= 0. \end{aligned}$$

Circle the pivots. Solve by back-substitution for z, y, x.

13. Apply elimination (circle the pivots) and back-substitution to solve

$$\begin{aligned} 2x - 3y \phantom{{}+z} &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned}$$

List the three row operations: Subtract ___ times row ___ from row ___.

14. Which number q makes this system singular and which right-hand side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t. \end{aligned}$$

15. (Recommended) It is impossible for a system of linear equations to have exactly two solutions. *Explain why.*
   (a) If (x, y, z) and (X, Y, Z) are two solutions, what is another one?
   (b) If 25 planes meet at two points, where else do they meet?

16. If rows 1 and 2 are the same, how far can you get with elimination (allowing row exchange)? If columns 1 and 2 are the same, which pivot is missing?

$$\begin{aligned} 2x - y + z &= 0 \\ 2x - y + z &= 0 \\ 4x + y + z &= 2 \end{aligned} \qquad \begin{aligned} 2x + 2y + z &= 0 \\ 4x + 4y + z &= 0 \\ 6x + 6y + z &= 2. \end{aligned}$$

17. (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form and a solution.
   (b) Construct a 3 by 3 system that needs a row exchange to keep going, but breaks down later.

18. Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of A is a ______ of the first two rows. Find a third equation that can't be solved if $x + y + z = 0$ and $x - 2y - z = 1$.

19. Construct a 3 by 3 example that has 9 different coefficients on the left-hand side, but rows 2 and 3 become zero in elimination. How many solutions to your system with $b = (1, 10, 100)$ and how many with $b = (0, 0, 0)$?

Problems 20–22 move up to 4 by 4 and n by n.

20. If you extend Problem 22 following the $1, 2, 1$ pattern or the $-1, 2, -1$ pattern, what is the fifth pivot? What is the nth pivot?

21. Apply elimination and back-substitution to solve

$$\begin{aligned} 2u + 3v \phantom{{}+w} &= 0 \\ 4u + 5v + w &= 3 \\ 2u - v - 3w &= 5. \end{aligned}$$

   What are the pivots? List the three operations in which a multiple of one row is subtracted from another.

22. Find the pivots and the solution for these four equations:

$$\begin{aligned} 2x + y \phantom{{}+z+t} &= 0 \\ x + 2y + z \phantom{{}+t} &= 0 \\ y + 2z + t &= 0 \\ z + 2t &= 5. \end{aligned}$$

23. Solve by elimination the system of two equations

$$\begin{aligned} x - y &= 0 \\ 3x + 6y &= 18. \end{aligned}$$

Draw a graph representing each equation as a straight line in the x-y plane; the lines intersect at the solution. Also, add one more line—the graph of the new second equation which arises after elimination.

24. Find three values of a for which elimination breaks down, temporarily or permanently, in

$$\begin{aligned} au + v &= 1 \\ 4u + av &= 2. \end{aligned}$$

Breakdown at the first step can be fixed by exchanging rows—but not breakdown at the last step.

25. Solve the system and find the pivots when

$$\begin{aligned} 2u - v \qquad\qquad &= 0 \\ -u + 2v - w \qquad &= 0 \\ - v + 2w - z &= 0 \\ - w + 2z &= 5. \end{aligned}$$

You may carry the right-hand side as a fifth column (and omit writing u, v, w, z until the solution at the end).

26. True or false:

(a) If the third equation starts with a zero coefficient (it begins with $0u$) then no multiple of equation 1 will be subtracted from equation 3.
(b) If the third equation has zero as its second coefficient (it contains $0v$) then no multiple of equation 2 will be subtracted from equation 3.
(c) If the third equation contains $0u$ *and* $0v$, then no multiple of equation 1 or equation 2 will be subtracted from equation 3.

27. For the system

$$\begin{aligned} u + v + w &= 2 \\ u + 3v + 3w &= 0 \\ u + 3v + 5w &= 2, \end{aligned}$$

what is the triangular system after forward elimination, and what is the solution?

28. Apply elimination to the system

$$\begin{aligned} u + v + w &= -2 \\ 3u + 3v - w &= 6 \\ u - v + w &= -1. \end{aligned}$$

When a zero arises in the pivot position, exchange that equation for the one below it and proceed. What coefficient of v in the third equation, in place of the present -1, would make it impossible to proceed—and force elimination to break down?

29. Find experimentally the average size (absolute value) of the first and second and third pivots for MATLAB's lu(rand(3, 3)). The average of the first pivot from abs($A(1, 1)$) should be 0.5.

30. For which three numbers a will elimination fail to give three pivots?

$$\begin{aligned} ax + 2y + 3z &= b_1 \\ ax + ay + 4z &= b_2 \\ ax + ay + az &= b_3. \end{aligned}$$

31. (Very optional) Normally the multiplication of two complex numbers

$$(a + ib)(c + id) = (ac - bd) + i(bc + ad)$$

involves the four separate multiplications ac, bd, bc, ad. Ignoring i, can you compute $ac - bd$ and $bc + ad$ with only three multiplications? (You may do additions, such as forming $a + b$ before multiplying, without any penalty.)

32. Use elimination to solve

$$\begin{aligned} u + v + w &= 6 \\ u + 2v + 2w &= 11 \\ 2u + 3v - 4w &= 3 \end{aligned} \quad \text{and} \quad \begin{aligned} u + v + w &= 7 \\ u + 2v + 2w &= 10 \\ 2u + 3v - 4w &= 3. \end{aligned}$$

1.4 MATRIX NOTATION AND MATRIX MULTIPLICATION

With our 3 by 3 example, we are able to write out all the equations in full. We can list the elimination steps, which subtract a multiple of one equation from another and reach a triangular matrix. For a large system, this way of keeping track of elimination would be hopeless; a much more concise record is needed.

We now introduce **matrix notation** to describe the original system, and **matrix multiplication** to describe the operations that make it simpler. Notice that three different types of quantities appear in our example:

Nine coefficients
Three unknowns
Three right-hand sides

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \quad &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned} \tag{1}$$

On the right-hand side is the column vector b. On the left-hand side are the unknowns u, v, w. Also on the left-hand side are nine coefficients (one of which happens to be zero). It is natural to represent the three unknowns by a vector:

$$\text{The unknown is } x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \text{The solution is } x = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

The nine coefficients fall into three rows and three columns, producing a **3 *by* 3 *matrix***:

$$\textbf{Coefficient matrix} \qquad A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}.$$

for elimination. But fortunately *it is the right order for reversing the elimination steps*—which also comes in the next section.

Notice that the product of lower triangular matrices is again lower triangular.

Problem Set 1.4

1. Write down the 2 by 2 matrices A and B that have entries $a_{ij} = i + j$ and $b_{ij} = (-1)^{i+j}$. Multiply them to find AB and BA.

2. Find two inner products and a matrix product:

$$\begin{bmatrix} 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \end{bmatrix}.$$

The first gives the length of the vector (squared).

3. If an m by n matrix A multiplies an n-dimensional vector x, how many separate multiplications are involved? What if A multiplies an n by p matrix B?

4. Give 3 by 3 examples (not just the zero matrix) of
   (a) a diagonal matrix: $a_{ij} = 0$ if $i \neq j$.
   (b) a symmetric matrix: $a_{ij} = a_{ji}$ for all i and j.
   (c) an upper triangular matrix: $a_{ij} = 0$ if $i > j$.
   (d) a skew-symmetric matrix: $a_{ij} = -a_{ji}$ for all i and j.

5. Compute the products

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For the third one, draw the column vectors (2, 1) and (0, 3). Multiplying by (1, 1) just adds the vectors (do it graphically).

6. Multiply Ax to find a solution vector x to the system $Ax =$ *zero vector*. Can you find more solutions to $Ax = 0$?

$$Ax = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

7. Working a column at a time, compute the products

$$\begin{bmatrix} 4 & 1 \\ 5 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 4 & 3 \\ 6 & 6 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}.$$

8. Do these subroutines multiply Ax *by rows or columns*? Start with $B(I) = 0$:

```
   DO 10 I = 1,N
   DO 10 J = 1,N
10 B(I) = B(I) + A(I,J) * X(J)
```

```
   DO 10 J = 1,N
   DO 10 I = 1,N
10 B(I) = B(I) + A(I,J) * X(J)
```

The outputs $Bx = Ax$ are the same. The second code is slightly more efficient in FORTRAN and much more efficient on a vector machine (the first changes single entries $B(I)$, the second can update whole vectors).

9. The product of two lower triangular matrices is again lower triangular (all its entries above the main diagonal are zero). Confirm this with a 3 by 3 example, and then explain how it follows from the laws of matrix multiplication.

10. Suppose A commutes with every 2 by 2 matrix ($AB = BA$), and in particular

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{commutes with} \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Show that $a = d$ and $b = c = 0$. If $AB = BA$ for all matrices B, then A is a multiple of the identity.

11. True or false? Give a specific counterexample when false.

(a) If columns 1 and 3 of B are the same, so are columns 1 and 3 of AB.
(b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of AB.
(c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of AB.
(d) $(AB)^2 = A^2B^2$.

12. Let x be the column vector $(1, 0, \ldots, 0)$. Show that the rule $(AB)x = A(Bx)$ forces the first column of AB to equal A times the first column of B.

13. Which of the following matrices are guaranteed to equal $(A + B)^2$?

$$A^2 + 2AB + B^2, \quad A(A + B) + B(A + B), \quad (A + B)(B + A), \quad A^2 + AB + BA + B^2.$$

14. If the entries of A are a_{ij}, use subscript notation to write

(a) the first pivot.
(b) the multiplier ℓ_{i1} of row 1 to be subtracted from row i.
(c) the new entry that replaces a_{ij} after that subtraction.
(d) the second pivot.

15. By trial and error find examples of 2 by 2 matrices such that

(a) $A^2 = -I$, A having only real entries.
(b) $B^2 = 0$, although $B \neq 0$.
(c) $CD = -DC$, not allowing the case $CD = 0$.
(d) $EF = 0$, although no entries of E or F are zero.

16. The first row of AB is a linear combination of all the rows of B. What are the coefficients in this combination, and what is the first row of AB, if

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}?$$

17. Describe the rows of EA and the *columns* of AE if

$$E = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}.$$

18. *A fourth way to multiply matrices is columns of A times rows of B:*

$$AB = (\text{column } 1)(\text{row } 1) + \cdots + (\text{column } n)(\text{row } n) = \text{sum of simple matrices.}$$

Give a 2 by 2 example of this important rule for matrix multiplication.

19. Find the powers A^2, A^3 (A^2 times A), and B^2, B^3, C^2, C^3. What are A^k, B^k, and C^k?

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad C = AB = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

20. If A and B are n by n matrices with all entries equal to 1, find $(AB)_{ij}$. Summation notation turns the product AB, and the law $(AB)C = A(BC)$, into

$$(AB)_{ij} = \sum_k a_{ik} b_{kj} \qquad \sum_j \left(\sum_k a_{ik} b_{kj} \right) c_{jl} = \sum_k a_{ik} \left(\sum_j b_{kj} c_{jl} \right).$$

Compute both sides if C is also n by n, with every $c_{jl} = 2$.

21. The matrix that rotates the x-y plane by an angle θ is

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Verify that $A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2)$ from the identities for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$. What is $A(\theta)$ times $A(-\theta)$?

Problems 22–31 are about elimination matrices.

22. Suppose $a_{33} = 7$ and the third pivot is 5. If you change a_{33} to 11, the third pivot is ___. If you change a_{33} to ___, there is zero in the pivot position.

23. What matrix E_{31} subtracts 7 times row 1 from row 3? To reverse that step, R_{31} should ______ 7 times row ___ to row ___. Multiply E_{31} by R_{31}.

24. If every column of A is a multiple of (1, 1, 1), then Ax is always a multiple of (1, 1, 1). Do a 3 by 3 example. How many pivots are produced by elimination?

25. In Problem 26, applying E_{21} and then E_{32} to the column $b = (1, 0, 0)$ gives $E_{32}E_{21}b =$ ___. Applying E_{32} before E_{21} gives $E_{21}E_{32}b =$ ___. When E_{32} comes first, row ___ feels no effect from row ___.

26. Write down the 3 by 3 matrices that produce these elimination steps:

(a) E_{21} subtracts 5 times row 1 from row 2.

(b) E_{32} subtracts -7 times row 2 from row 3.

(c) P exchanges rows 1 and 2, then rows 2 and 3.

27. Which three matrices E_{21}, E_{31}, E_{32} put A into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad E_{32}E_{31}E_{21}A = U.$$

Multiply those E's to get one matrix M that does elimination: $MA = U$.

You have to be careful with L. Suppose elimination subtracts row 1 from row 2, creating $\ell_{21} = 1$. Then suppose it exchanges rows 2 and 3. If that exchange is done in advance, the multiplier will change to $\ell_{31} = 1$ in $PA = LU$.

Example 7

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} = U. \tag{10}$$

That row exchange recovers LU—but now $\ell_{31} = 1$ and $\ell_{21} = 2$:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad PA = LU. \tag{11}$$

In MATLAB, A([r k],:) exchanges row k with row r below it (where the kth pivot has been found). We update the matrices L and P the same way. At the start, $P = I$ and sign $= +1$:

```
A([r k], : ) = A([k r], : );
L([r k],1:k-1) = L([k r],1:k-1);
P([r k], : ) = P([k r], : );
sign = -sign
```

The "**sign**" of P tells whether the number of row exchanges is even (sign $= +1$) or odd (sign $= -1$). A row exchange reverses sign. The final value of sign is the **determinant of** P and it does not depend on the order of the row exchanges.

To summarize: A good elimination code saves L and U and P. Those matrices carry the information that originally came in A—and they carry it in a more usable form. $Ax = b$ reduces to two triangular systems. This is the practical equivalent of the calculation we do next—*to find the inverse matrix* A^{-1} *and the solution* $x = A^{-1}b$.

Problem Set 1.5

1. Multiply the matrix $L = E^{-1}F^{-1}G^{-1}$ in equation (6) by GFE in equation (3):

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{times} \quad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

Multiply also in the opposite order. *Why are the answers what they are?*

2. When is an upper triangular matrix nonsingular (a full set of pivots)?

3. What multiple ℓ_{32} of row 2 of A will elimination subtract from row 3 of A? Use the factored form

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix}.$$

What will be the pivots? Will a row exchange be required?

4. Find the products FGH and HGF if (with upper triangular zeros omitted)

$$F = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

5. (a) Under what conditions is the following product nonsingular?

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Solve the system $Ax = b$ starting with $Lc = b$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = b.$$

6. (*Second proof of* $A = LU$) The third row of U comes from the third row of A by subtracting multiples of rows 1 and 2 (*of* U!):

$$\text{row 3 of } U = \text{row 3 of } A - \ell_{31}(\text{row 1 of } U) - \ell_{32}\,(\text{row 2 of } U).$$

(a) Why are rows of U subtracted off and not rows of A? Answer: Because by the time a pivot row is used, ___.

(b) The equation above is the same as

$$\text{row 3 of } A = \ell_{31}(\text{row 1 of } U) + \ell_{32}(\text{row 2 of } U) + 1\,(\text{row 3 of } U).$$

Which rule for matrix multiplication makes this row 3 of L times U?

The other rows of LU agree similarly with the rows of A.

7. Factor A into LU, and write down the upper triangular system $Ux = c$ which appears after elimination, for

$$Ax = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}.$$

8. (a) Why does it take approximately $n^2/2$ multiplication–subtraction steps to solve each of $Lc = b$ and $Ux = c$?

(b) How many steps does elimination use in solving 10 systems with the same 60 by 60 coefficient matrix A?

9. Apply elimination to produce the factors L and U for

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}.$$

10. Find E^2 and E^8 and E^{-1} if

$$E = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}.$$

11. Decide whether the following systems are singular or nonsingular, and whether they have no solution, one solution, or infinitely many solutions:

$$\begin{aligned} v - w &= 2 \\ u - v \quad &= 2 \\ u \quad - w &= 2 \end{aligned} \quad\text{and}\quad \begin{aligned} v - w &= 0 \\ u - v \quad &= 0 \\ u \quad - w &= 0 \end{aligned} \quad\text{and}\quad \begin{aligned} v + w &= 1 \\ u + v \quad &= 1 \\ u \quad + w &= 1. \end{aligned}$$

12. Write down all six of the 3 by 3 permutation matrices, including $P = I$. Identify their inverses, which are also permutation matrices. The inverses satisfy $PP^{-1} = I$ and are on the same list.

13. Find a 4 by 4 permutation matrix that requires three row exchanges to reach the end of elimination (which is $U = I$).

14. The less familiar form $A = LPU$ exchanges rows only at the end:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \to L^{-1}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} = PU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}.$$

What is L is this case? Comparing with $PA = LU$ in Box 1J, the multipliers now stay in place (ℓ_{21} is 1 and ℓ_{31} is 2 when $A = LPU$).

15. How could you factor A into a product UL, upper triangular times lower triangular? Would they be the same factors as in $A = LU$?

16. Which numbers a, b, c lead to row exchanges? Which make the matrix singular?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ a & 8 & 3 \\ 0 & b & 5 \end{bmatrix} \quad\text{and}\quad A = \begin{bmatrix} c & 2 \\ 6 & 4 \end{bmatrix}.$$

17. Solve by elimination, exchanging rows when necessary:

$$\begin{aligned} u + 4v + 2w &= -2 \\ -2u - 8v + 3w &= 32 \\ v + w &= 1 \end{aligned} \quad\text{and}\quad \begin{aligned} v + w &= 0 \\ u + v \quad &= 0 \\ u + v + w &= 1. \end{aligned}$$

Which permutation matrices are required?

18. Solve as two triangular systems, without multiplying LU to find A:

$$LUx = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

19. Find the $PA = LDU$ factorizations (and check them) for

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad\text{and}\quad A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Problems 20–31 compute the factorization $A = LU$ (and also $A = LDU$).

20. What are the 3 by 3 triangular systems $Lc = b$ and $Ux = c$ from Problem 24? Check that $c = (5, 2, 2)$ solves the first one. Which x solves the second one?

21. What three elimination matrices E_{21}, E_{31}, E_{32} put A into upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1} and E_{21}^{-1} to factor A into LU where $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}.$$

22. When zero appears in a pivot position, $A = LU$ *is not possible*! (We need nonzero pivots d, f, i in U.) Show directly why these are both impossible:

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \ell & 1 & \\ m & n & 1 \end{bmatrix}\begin{bmatrix} d & e & g \\ & f & h \\ & & i \end{bmatrix}.$$

23. Forward elimination changes $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}x = b$ to a triangular $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}x = c$:

$$\begin{matrix} x + \ y = 5 \\ x + 2y = 7 \end{matrix} \quad \longrightarrow \quad \begin{matrix} x + \ y = 5 \\ y = 2 \end{matrix} \qquad \begin{bmatrix} 1 & 1 & \mathbf{5} \\ 1 & 2 & \mathbf{7} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & \mathbf{5} \\ 0 & 1 & \mathbf{2} \end{bmatrix}.$$

That step subtracted ℓ_{21} = ______ times row 1 from row 2. The reverse step *adds* ℓ_{21} times row 1 to row 2. The matrix for that reverse step is L = ______. Multiply this L times the triangular system $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}x = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to get ______ = ______. In letters, L multiplies $Ux = c$ to give ______.

24. (Move to 3 by 3) Forward elimination changes $Ax = b$ to a triangular $Ux = c$:

$$\begin{aligned} x + \ y + \ z &= 5 \\ x + 2y + 3z &= 7 \\ x + 3y + 6z &= 11 \end{aligned} \qquad \begin{aligned} x + \ y + \ z &= 5 \\ y + 2z &= 2 \\ 2y + 5z &= 6 \end{aligned} \qquad \begin{aligned} x + \ y + \ z &= 5 \\ y + 2z &= 2 \\ z &= 2. \end{aligned}$$

The equation $z = 2$ in $Ux = c$ comes from the original $x + 3y + 6z = 11$ in $Ax = b$ by subtracting ℓ_{31} = __ times equation 1 and ℓ_{32} = __ times the *final* equation 2. Reverse that to recover $[1 \ \ 3 \ \ 6 \ \ 11]$ in $[A \ \ b]$ from the final $[1 \ \ 1 \ \ 1 \ \ 5]$ and $[0 \ \ 1 \ \ 2 \ \ 2]$ and $[0 \ \ 0 \ \ 1 \ \ 2]$ in $[U \ \ c]$:

$$\text{Row 3 of } \begin{bmatrix} A & b \end{bmatrix} = (\ell_{31} \text{ Row } 1 + \ell_{32} \text{ Row } 2 + 1 \text{ Row } 3) \text{ of } \begin{bmatrix} U & c \end{bmatrix}.$$

In matrix notation this is multiplication by L. So $A = LU$ and $b = Lc$.

25. What two elimination matrices E_{21} and E_{32} put A into upper triangular form $E_{32}E_{21}A = U$? Multiply by E_{32}^{-1} and E_{21}^{-1} to factor A into $LU = E_{21}^{-1}E_{32}^{-1}U$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix}.$$

26. Which number c leads to zero in the second pivot position? A row exchange is needed and $A = LU$ is not possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

27. (Recommended) Compute L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

28. *Tridiagonal matrices* have zero entries except on the main diagonal and the two adjacent diagonals. Factor these into $A = LU$ and $A = LDV$:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix}.$$

29. Solve $Lc = b$ to find c. Then solve $Ux = c$ to find x. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

30. A and B are symmetric across the diagonal (because $4 = 4$). Find their triple factorizations LDU and say how U is related to L for these symmetric matrices:

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}.$$

31. Solve the triangular system $Lc = b$ to find c. Then solve $Ux = c$ to find x:

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 11 \end{bmatrix}.$$

For safety find $A = LU$ and solve $Ax = b$ as usual. Circle c when you see it.

32. What are L and D for this matrix A? What is U in $A = LU$ and what is the new U in $A = LDU$?

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}.$$

33. Find L and U for the nonsymmetric matrix

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Find the four conditions on a, b, c, d, r, s, t to get $A = LU$ with four pivots.

Problem Set 1.6

1. Suppose elimination fails because there is no pivot in column 3:

$$\textbf{Missing pivot} \qquad A = \begin{bmatrix} 2 & 1 & 4 & 6 \\ 0 & 3 & 8 & 5 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix}.$$

Show that A cannot be invertible. The third row of A^{-1}, multiplying A, should give the third row $[0 \;\; 0 \;\; 1 \;\; 0]$ of $A^{-1}A = I$. Why is this impossible?

2. If the inverse of A^2 is B, show that the inverse of A is AB. (Thus A is invertible whenever A^2 is invertible.)

3. Find three 2 by 2 matrices, other than $A = I$ and $A = -I$, that are their own inverses: $A^2 = I$.

4. Find the inverses (in any legal way) of

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}.$$

5. Show that $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ has no inverse by solving $Ax = 0$, and by failing to solve

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

6. (a) If A is invertible and $AB = AC$, prove quickly that $B = C$.
   (b) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, find an example with $AB = AC$ but $B \neq C$.

7. (a) Find the inverses of the permutation matrices

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

   (b) Explain for permutations why P^{-1} *is always the same as* P^{T}. Show that the 1s are in the right places to give $PP^{\mathrm{T}} = I$.

8. From $AB = C$ find a formula for A^{-1}. Also find A^{-1} from $PA = LU$.

9. Find the inverses (no special system required) of

$$A_1 = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

10. Use the Gauss–Jordan method to invert

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

11. If B is square, show that $A = B + B^{\mathrm{T}}$ is always symmetric and $K = B - B^{\mathrm{T}}$ is always *skew-symmetric*—which means that $K^{\mathrm{T}} = -K$. Find these matrices A and K when $B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$, and write B as the sum of a symmetric matrix and a skew-symmetric matrix.

12. Compute the symmetric LDL^{T} factorization of

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

13. (a) If $A = LDU$, with 1s on the diagonals of L and U, what is the corresponding factorization of A^{T}? Note that A and A^{T} (square matrices with no row exchanges) share the same pivots.

(b) What triangular systems will give the solution to $A^{\mathrm{T}}y = b$?

14. Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

15. Under what conditions on their entries are A and B invertible?

$$A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}.$$

16. If $A = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, compute $A^{\mathrm{T}}B$, $B^{\mathrm{T}}A$, AB^{T}, and BA^{T}.

17. Give examples of A and B such that

(a) $A + B$ is not invertible although A and B are invertible.

(b) $A + B$ is invertible although A and B are not invertible.

(c) all of A, B, and $A + B$ are invertible.

In the last case use $A^{-1}(A + B)B^{-1} = B^{-1} + A^{-1}$ to show that $C = B^{-1} + A^{-1}$ is also invertible—and find a formula for C^{-1}.

18. (a) How many entries can be chosen independently in a symmetric matrix of order n?

(b) How many entries can be chosen independently in a skew-symmetric matrix ($K^{\mathrm{T}} = -K$) of order n? The diagonal of K is zero!

19. If A is invertible, which properties of A remain true for A^{-1}? (a) A is triangular. (b) A is symmetric. (c) A is tridiagonal. (d) All entries are whole numbers. (e) All entries are fractions (including numbers like $\frac{3}{1}$).

20. If $A = L_1D_1U_1$ and $A = L_2D_2U_2$, prove that $L_1 = L_2$, $D_1 = D_2$, and $U_1 = U_2$. If A is invertible, the factorization is unique.

(a) Derive the equation $L_1^{-1}L_2D_2 = D_1U_1U_2^{-1}$, and explain why one side is lower triangular and the other side is upper triangular.

(b) Compare the main diagonals and then compare the off-diagonals.

21. (Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

(a) Explain why $Ax = (1, 0, 0)$ cannot have a solution.

(b) Which right-hand sides (b_1, b_2, b_3) might allow a solution to $Ax = b$?

(c) What happens to row 3 in elimination?

22. Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible? How would you find B^{-1} from A^{-1}?

23. (a) What matrix E has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.

(b) What single matrix L has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

24. (Remarkable) If A and B are square matrices, show that $I - BA$ is invertible if $I - AB$ is invertible. Start from $B(I - AB) = (I - BA)B$.

25. If the product $M = ABC$ of three square matrices is invertible, then A, B, C are invertible. Find a formula for B^{-1} that involves M^{-1} and A and C.

26. Find the numbers a and b that give the inverse of 5 * eye(4) − ones(4,4):

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}.$$

What are a and b in the inverse of 6 * eye(5) − ones(5,5)?

27. Show that $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has no inverse by trying to solve for the column (x, y):

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x & t \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{must include} \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

28. Multiply $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ times $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. What is the inverse of each matrix if $ad \neq bc$?

29. Solve for the columns of $A^{-1} = \begin{bmatrix} x & t \\ y & z \end{bmatrix}$:

$$\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

30. If A has column 1 + column 2 = column 3, show that A is not invertible:

(a) Find a nonzero solution x to $Ax = 0$. The matrix is 3 by 3.

(b) Elimination keeps column 1 + column 2 = column 3. Explain why there is no third pivot.

31. Find the inverses (directly or from the 2 by 2 formula) of A, B, C:

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

32. Prove that a matrix with a column of zeros cannot have an inverse.

33. There are sixteen 2 by 2 matrices whose entries are 1s and 0s. How many of them are invertible?

34. Show that A = 4 * eye(4) − ones(4,4) is *not* invertible: Multiply A * ones(4,1).

Problems 35–39 are about the Gauss–Jordan method for calculating A^{-1}.

35. Use Gauss–Jordan elimination on $[A \quad I]$ to solve $AA^{-1} = I$:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & \\ x_1 & x_2 & x_3 \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

36. Exchange rows and continue with Gauss–Jordan to find A^{-1}:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}.$$

37. Follow the 3 by 3 text example but with plus signs in A. Eliminate above and below the pivots to reduce $[A \quad I]$ to $[I \quad A^{-1}]$:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

38. Change I into A^{-1} as you reduce A to I (by row operations):

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix}.$$

39. Invert these matrices A by the Gauss–Jordan method starting with $[A \quad I]$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

40. Prove that A is invertible if $a \neq 0$ and $a \neq b$ (find the pivots and A^{-1}):

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}.$$

41. True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.
(b) A matrix with 1s down the main diagonal is invertible.
(c) If A is invertible then A^{-1} is invertible.
(d) If A^T is invertible then A is invertible.

42. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

43. Use inv(S) to invert MATLAB's 4 by 4 symmetric matrix S = pascal(4). Create Pascal's lower triangular A = abs(pascal(4,1)) and test inv(S) = inv(A') * inv(A).

44. This matrix has a remarkable inverse. Find A^{-1} by elimination on $[A \quad I]$. Extend to a 5 by 5 "alternating matrix" and guess its inverse:

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

45. M^{-1} shows the change in A^{-1} (useful to know) when a matrix is subtracted from A. Check part 3 by carefully multiplying MM^{-1} to get I:

1. $M = I - uv^{\mathrm{T}}$ and $M^{-1} = I + uv^{\mathrm{T}}/(1 - v^{\mathrm{T}}u)$.
2. $M = A - uv^{\mathrm{T}}$ and $M^{-1} = A^{-1} + A^{-1}uv^{\mathrm{T}}A^{-1}/(1 - v^{\mathrm{T}}A^{-1}u)$.
3. $M = I - UV$ and $M^{-1} = I_k + U(I_n - VU)^{-1}V$.
4. $M = A - UW^{-1}V$ and $M^{-1} = A^{-1} + A^{-1}U(W - VA^{-1}U)^{-1}VA^{-1}$.

The four identities come from the 1, 1 block when inverting these matrices:

$$\begin{bmatrix} I & u \\ v^{\mathrm{T}} & 1 \end{bmatrix} \qquad \begin{bmatrix} A & u \\ v^{\mathrm{T}} & 1 \end{bmatrix} \qquad \begin{bmatrix} I_k & U \\ V & I_n \end{bmatrix} \qquad \begin{bmatrix} A & U \\ V & W \end{bmatrix}.$$

46. If $A =$ ones(4,4) and $b =$ rand(4,1), how does MATLAB tell you that $Ax = b$ has no solution? If $b =$ ones(4,1), which solution to $Ax = b$ is found by $A\backslash b$?

47. If B has the columns of A in reverse order, solve $(A - B)x = 0$ to show that $A - B$ is not invertible. An example will lead you to x.

48. Find and check the inverses (assuming they exist) of these block matrices:

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}.$$

Problems 49–55 are about the rules for transpose matrices.

49. (a) The matrix $((AB)^{-1})^{\mathrm{T}}$ comes from $(A^{-1})^{\mathrm{T}}$ and $(B^{-1})^{\mathrm{T}}$. *In what order?*
(b) If U is upper triangular then $(U^{-1})^{\mathrm{T}}$ is ______ triangular.

50. Find A^{T} and A^{-1} and $(A^{-1})^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix} \quad \text{and also} \quad A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}.$$

51. Show that $A^2 = 0$ is possible but $A^{\mathrm{T}}A = 0$ is not possible (unless $A =$ zero matrix).

52. Verify that $(AB)^{\mathrm{T}}$ equals $B^{\mathrm{T}}A^{\mathrm{T}}$ but those are different from $A^{\mathrm{T}}B^{\mathrm{T}}$:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

In case $AB = BA$ (not generally true!), how do you prove that $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$?

other solutions. (There always are, if there are more unknowns than equations, $n > m$.) ***The solutions to $Ax = 0$ form a vector space—the nullspace of A.***

> The ***nullspace*** of a matrix consists of all vectors x such that $Ax = 0$. It is denoted by $N(A)$. It is a subspace of $\mathbf{R}^n$, just as the column space was a subspace of $\mathbf{R}^m$

Requirement (i) holds: If $Ax = 0$ and $Ax' = 0$, then $A(x + x') = 0$. Requirement (ii) also holds: If $Ax = 0$ then $A(cx) = 0$. Both requirements fail if the right-hand side is not zero! Only the solutions to a *homogeneous* equation ($b = 0$) form a subspace. The nullspace is easy to find for the example given above; it is as small as possible:

$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The first equation gives $u = 0$, and the second equation then forces $v = 0$. The nullspace contains only the vector $(0, 0)$. This matrix has "independent columns"—a key idea that comes soon.

The situation is changed when a third column is a combination of the first two:

$$\textbf{Larger nullspace} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}.$$

B has the same column space as A. The new column lies in the plane of Figure 2.1; it is the sum of the two column vectors we started with. But the nullspace of B contains the vector $(1, 1, -1)$ and automatically contains any multiple $(c, c, -c)$:

$$\textbf{Nullspace is a line} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} c \\ c \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The nullspace of B is the line of all points $x = c$, $y = c$, $z = -c$. (The line goes through the origin, as any subspace must.) We want to be able, for any system $Ax = b$, to find $C(A)$ and $N(A)$: all attainable right-hand sides b and all solutions to $Ax = 0$.

The vectors b are in the column space and the vectors x are in the nullspace. We shall compute the dimensions of those subspaces and a convenient set of vectors to generate them. We hope to end up by understanding all *four* of the subspaces that are intimately related to each other and to A—the column space of A, the nullspace of A, and their two perpendicular spaces.

Problem Set 2.1

1. Construct a subset of the x-y plane $\mathbf{R}^2$ that is
   (a) closed under vector addition and subtraction, but not scalar multiplication.
   (b) closed under scalar multiplication but not under vector addition.

   Hint: Starting with u and v, add and subtract for (a). Try cu and cv for (b).

2. Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?
   (a) The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.

(b) The plane of vectors b with $b_1 = 1$.
(c) The vectors b with $b_2 b_3 = 0$ (this is the union of two subspaces, the plane $b_2 = 0$ and the plane $b_3 = 0$).
(d) All combinations of two given vectors $(1, 1, 0)$ and $(2, 0, 1)$.
(e) The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.

3. What is the smallest subspace of 3 by 3 matrices that contains all symmetric matrices *and* all lower triangular matrices? What is the largest subspace that is contained in both of those subspaces?

4. Which of the following are subspaces of $\mathbf{R}^\infty$?
   (a) All sequences like $(1, 0, 1, 0, \ldots)$ that include infinitely many zeros.
   (b) All sequences $(x_1, x_2, \ldots)$ with $x_j = 0$ from some point onward.
   (c) All decreasing sequences: $x_{j+1} \le x_j$ for each j.
   (d) All convergent sequences: the x_j have a limit as $j \to \infty$.
   (e) All arithmetic progressions: $x_{j+1} - x_j$ is the same for all j.
   (f) All geometric progressions $(x_1, kx_1, k^2x_1, \ldots)$ allowing all k and x_1.

5. Describe the column space and the nullspace of the matrices

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

6. Addition and scalar multiplication are required to satisfy these eight rules:
   1. $x + y = y + x$.
   2. $x + (y + z) = (x + y) + z$.
   3. There is a unique "zero vector" such that $x + 0 = x$ for all x.
   4. For each x there is a unique vector $-x$ such that $x + (-x) = 0$.
   5. $1x = x$.
   6. $(c_1c_2)x = c_1(c_2x)$.
   7. $c(x + y) = cx + cy$.
   8. $(c_1 + c_2)x = c_1x + c_2x$.

   (a) Suppose addition in $\mathbf{R}^2$ adds an extra 1 to each component, so that $(3, 1) + (5, 0)$ equals $(9, 2)$ instead of $(8, 1)$. With scalar multiplication unchanged, which rules are broken?
   (b) Show that the set of all positive real numbers, with $x + y$ and cx redefined to equal the usual xy and x^c, is a vector space. What is the "zero vector"?
   (c) Suppose $(x_1, x_2) + (y_1, y_2)$ is defined to be $(x_1 + y_2, x_2 + y_1)$. With the usual $cx = (cx_1, cx_2)$, which of the eight conditions are not satisfied?

7. Let $\mathbf{P}$ be the plane in 3-space with equation $x + 2y + z = 6$. What is the equation of the plane $\mathbf{P}_0$ through the origin parallel to $\mathbf{P}$? Are $\mathbf{P}$ and $\mathbf{P}_0$ subspaces of $\mathbf{R}^3$?

8. Which of the following descriptions are correct? The solutions x of

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

form

(a) a plane.
(b) a line.
(c) a point.
(d) a subspace.
(e) the nullspace of A.
(f) the column space of A.

9. (a) Describe a subspace of $\mathbf{M}$ that contains $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.
(b) If a subspace of $\mathbf{M}$ contains A and B, must it contain I?
(c) Describe a subspace of $\mathbf{M}$ that contains no nonzero diagonal matrices.

10. Show that the set of nonsingular 2 by 2 matrices is not a vector space. Show also that the set of *singular* 2 by 2 matrices is not a vector space.

11. Describe the smallest subspace of the 2 by 2 matrix space $\mathbf{M}$ that contains

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. (d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

12. $\mathbf{P}_0$ is the plane through $(0, 0, 0)$ parallel to the plane $\mathbf{P}$ in Problem 17. What is the equation for $\mathbf{P}_0$? Find two vectors in $\mathbf{P}_0$ and check that their sum is in $\mathbf{P}_0$.

13. The functions $f(x) = x^2$ and $g(x) = 5x$ are "vectors" in the vector space $\mathbf{F}$ of all real functions. The combination $3f(x) - 4g(x)$ is the function $h(x) =$ ______. Which rule is broken if multiplying $f(x)$ by c gives the function $f(cx)$?

14. The four types of subspaces of $\mathbf{R}^3$ are planes, lines, $\mathbf{R}^3$ itself, or $\mathbf{Z}$ containing only $(0, 0, 0)$.
(a) Describe the three types of subspaces of $\mathbf{R}^2$.
(b) Describe the five types of subspaces of $\mathbf{R}^4$.

15. (a) The intersection of two planes through $(0, 0, 0)$ is probably a ______ but it could be a ______. It can't be the zero vector $\mathbf{Z}$!
(b) The intersection of a plane through $(0, 0, 0)$ with a line through $(0, 0, 0)$ is probably a ______ but it could be a ______.
(c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{R}^5$, their intersection $\mathbf{S} \cap \mathbf{T}$ (vectors in both subspaces) is a subspace of $\mathbf{R}^5$. *Check the requirements on $x + y$ and cx.*

16. If the sum of the "vectors" $f(x)$ and $g(x)$ in $\mathbf{F}$ is defined to be $f(g(x))$, then the "zero vector" is $g(x) = x$. Keep the usual scalar multiplication $cf(x)$, and find two rules that are broken.

17. Let $\mathbf{P}$ be the plane in $\mathbf{R}^3$ with equation $x + y - 2z = 4$. The origin $(0, 0, 0)$ is not in $\mathbf{P}$! Find two vectors in $\mathbf{P}$ and check that their sum is not in $\mathbf{P}$.

18. The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space $\mathbf{M}$ of all 2 by 2 matrices. Write the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A?

19. True or false for $\mathbf{M}$ = all 3 by 3 matrices (check addition using an example)?

(a) The skew-symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} = -A$) form a subspace.
(b) The unsymmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} \neq A$) form a subspace.
(c) The matrices that have $(1, 1, 1)$ in their nullspace form a subspace.

20. Suppose $\mathbf{P}$ is a plane through $(0, 0, 0)$ and $\mathbf{L}$ is a line through $(0, 0, 0)$. The smallest vector space containing both $\mathbf{P}$ and $\mathbf{L}$ is either ______ or ______.

Problems 21–31 are about column spaces $C(A)$ and the equation $Ax = b$.

21. Adding row 1 of A to row 2 produces B. Adding column 1 to column 2 produces C. A combination of the columns of ______ is also a combination of the columns of A. Which two matrices have the same column ______?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

22. (Recommended) If we add an extra column b to a matrix A, then the column space gets larger unless ______. Give an example in which the column space gets larger and an example in which it doesn't. Why is $Ax = b$ solvable exactly when the column space *doesn't* get larger by including b?

23. If A is any 8 by 8 invertible matrix, then its column space is ______. Why?

24. For which right-hand sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\text{(a)} \quad \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad \text{(b)} \quad \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

25. The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A*. Give an example where the column spaces of A and AB are not equal.

26. Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

27. True or false (with a counterexample if false)?

(a) The vectors b that are not in the column space $C(A)$ form a subspace.
(b) If $C(A)$ contains only the zero vector, then A is the zero matrix.
(c) The column space of $2A$ equals the column space of A.
(d) The column space of $A - I$ equals the column space of A.

28. For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

That final matrix $[R \quad d]$ **is** $\text{rref}([A \quad b]) = \text{rref}([U \quad c])$. The numbers 2 and 0 and 2 and 1 in the free columns of R have opposite sign in the special solutions (the nullspace matrix N). Everything is revealed by $Rx = d$.

Problem Set 2.2

1. Find the value of c that makes it possible to solve $Ax = b$, and solve it:

$$\begin{aligned} u + \ v + 2w &= 2 \\ 2u + 3v - \ \ w &= 5 \\ 3u + 4v + \ \ w &= c. \end{aligned}$$

2. Find the echelon form U, the free variables, and the special solutions:

$$A = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

$Ax = b$ is consistent (has a solution) when b satisfies $b_2 =$ ______. Find the complete solution in the same form as equation (4).

3. Construct a system with more unknowns than equations, but no solution. Change the right-hand side to zero and find all solutions x_n.

4. Write the complete solutions $x = x_p + x_n$ to these systems, as in equation (4):

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

5. Reduce A and B to echelon form, to find their ranks. Which variables are free?

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Find the special solutions to $Ax = 0$ and $Bx = 0$. Find all solutions.

6. Carry out the same steps as in the previous problem to find the complete solution of $Mx = b$:

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

7. Describe the set of attainable right-hand sides b (in the column space) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

by finding the constraints on b that turn the third equation into $0 = 0$ (after elimination). What is the rank, and a particular solution?

8. Find R for each of these (block) matrices, and the special solutions:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} A & A \end{bmatrix} \qquad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}.$$

9. Find a 2 by 3 system $Ax = b$ whose complete solution is

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Find a 3 by 3 system with these solutions exactly when $b_1 + b_2 = b_3$.

10. Which of these rules give a correct definition of the *rank* of A?

(a) The number of nonzero rows in R.
(b) The number of columns minus the total number of rows.
(c) The number of columns minus the number of free columns.
(d) The number of 1s in R.

11. If the r pivot variables come first, the reduced R must look like

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \qquad \begin{matrix} I \text{ is } r \text{ by } r \\ F \text{ is } r \text{ by } n-r \end{matrix}$$

What is the nullspace matrix N containing the special solutions?

12. Under what conditions on b_1 and b_2 (if any) does $Ax = b$ have a solution?

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Find two vectors in the nullspace of A, and the complete solution to $Ax = b$.

13. (a) Find the special solutions to $Ux = 0$. Reduce U to R and repeat:

$$Ux = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) If the right-hand side is changed from $(0, 0, 0)$ to $(a, b, 0)$, what are all solutions?

14. Write a 2 by 2 system $Ax = b$ with many solutions x_n but no solution x_p. (Therefore the system has no solution.) Which b's allow an x_p?

15. Find the reduced row echelon forms R and the rank of these matrices:

(a) The 3 by 4 matrix of all 1s.
(b) The 4 by 4 matrix with $a_{ij} = (-1)^{ij}$.
(c) The 3 by 4 matrix with $a_{ij} = (-1)^j$.

16. If A is 2 by 3 and C is 3 by 2, show from its rank that $CA \neq I$. Give an example in which $AC = I$. For $m < n$, a right inverse is not a left inverse.

17. Find the ranks of AB and AM (rank 1 matrix times rank 1 matrix):

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1.5 & 6 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & b \\ c & bc \end{bmatrix}.$$

18. *If A has rank r, then it has an r by r submatrix S that is invertible.* Find that submatrix S from the pivot rows and pivot columns of each A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

19. If A has r pivot columns, then A^{T} has r pivot columns. Give a 3 by 3 example for which the column numbers are different for A and A^{T}.

20. Multiplying the rank 1 matrices $A = uv^{\mathrm{T}}$ and $B = wz^{\mathrm{T}}$ gives uz^{T} times the number ___. AB has rank 1 unless ___ $= 0$.

21. (Important) Suppose A and B are n by n matrices, and $AB = I$. Prove from $\operatorname{rank}(AB) \le \operatorname{rank}(A)$ that the rank of A is n. So A is invertible and B must be its two-sided inverse. Therefore $BA = I$ (*which is not so obvious!*).

22. Suppose A and B have the *same* reduced-row echelon form R. Explain how to change A to B by elementary row operations. So B equals an ______ matrix times A.

23. Suppose all r pivot variables come *last*. Describe the four blocks in the m by n reduced echelon form (the block B should be r by r):

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

What is the nullspace matrix N of special solutions? What is its shape?

24. Explain why the pivot rows and pivot columns of A (not R) always give an r by r invertible submatrix of A.

25. (Silly problem) Describe all 2 by 3 matrices A_1 and A_2 with row echelon forms R_1 and R_2, such that $R_1 + R_2$ is the row echelon form of $A_1 + A_2$. Is it true that $R_1 = A_1$ and $R_2 = A_2$ in this case?

26. What are the special solutions to $Rx = 0$ and $R^{\mathrm{T}}y = 0$ for these R?

$$R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

27. Every column of AB is a combination of the columns of A. Then the dimensions of the column spaces give $\mathbf{rank}(AB) \le \mathbf{rank}(A)$. Problem: Prove also that $\mathbf{rank}(AB) \le \mathbf{rank}(B)$.

28. Every m by n matrix of rank r reduces to (m by r) times (r by n):

$$A = (\text{pivot columns of } A)(\text{first } r \text{ rows of } R) = (\mathbf{COL})(\mathbf{ROW}).$$

Write the 3 by 4 matrix A at the start of this section as the product of the 3 by 2 matrix from the pivot columns and the 2 by 4 matrix from R:

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

29. (Recommended) Execute the six steps following equation (6) to find the column space and nullspace of A and the solution to $Ax = b$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}.$$

30. What is the nullspace matrix N (of special solutions) for A, B, C?

$$A = \begin{bmatrix} I & I \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} I & I & I \end{bmatrix}.$$

31. Suppose A is an m by n matrix of rank r. Its reduced echelon form is R. Describe exactly the *reduced row echelon form of* R^{T} (not A^{T}).

32. For every c, find R and the special solutions to $Ax = 0$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}.$$

Problems 33–36 are about the solution of $Ax = b$. Follow the steps in the text to x_p and x_n. Reduce the augmented matrix $[A \;\; b]$.

33. Which vectors (b_1, b_2, b_3) are in the column space of A? Which combinations of the rows of A give zero?

$$\text{(a)}\; A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix} \qquad \text{(b)}\; A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}.$$

34. What conditions on b_1, b_2, b_3, b_4 make each system solvable? *Solve for* x:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

35. Under what condition on b_1, b_2, b_3 is the following system solvable? Include b as a fourth column in $[A \;\; b]$. Find all solutions when that condition holds:

$$\begin{aligned} x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3. \end{aligned}$$

36. Find the complete solutions of

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5 \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

37. If you know x_p (free variables = 0) and all special solutions for $Ax = b$, find x_p and all special solutions for these systems:

$$Ax = 2b \qquad \begin{bmatrix} A & A \end{bmatrix}\begin{bmatrix} x \\ X \end{bmatrix} = b \qquad \begin{bmatrix} A \\ A \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix}.$$

38. If $Ax = b$ has infinitely many solutions, why is it impossible for $Ax = B$ (new right-hand side) to have only one solution? Could $Ax = B$ have no solution?

39. Why can't a 1 by 3 system have $x_p = (2, 4, 0)$ and $x_n =$ any multiple of $(1, 1, 1)$?

40. (a) If $Ax = b$ has two solutions x_1 and x_2, find two solutions to $Ax = 0$.
(b) Then find another solution to $Ax = b$.

41. Explain why all these statements are false:

(a) The complete solution is any linear combination of x_p and x_n.
(b) A system $Ax = b$ has at most one particular solution.
(c) The solution x_p with all free variables zero is the shortest solution (minimum length $\|x\|$). (Find a 2 by 2 counterexample.)
(d) If A is invertible there is no solution x_n in the nullspace.

42. Give examples of matrices A for which the number of solutions to $Ax = b$ is

(a) 0 or 1, depending on b.
(b) ∞, regardless of b.
(c) 0 or ∞, depending on b.
(d) 1, regardless of b.

43. Write all known relations between r and m and n if $Ax = b$ has

(a) no solution for some b.
(b) infinitely many solutions for every b.
(c) exactly one solution for some b, no solution for other b.
(d) exactly one solution for every b.

44. Choose the number q so that (if possible) the ranks are (a) 1, (b) 2, (c) 3:

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}.$$

45. Apply Gauss–Jordan elimination (right-hand side becomes extra column) to $Ux = 0$ and $Ux = c$. Reach $Rx = 0$ and $Rx = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix}.$$

Solve $Rx = 0$ to find x_n (its free variable is $x_2 = 1$). Solve $Rx = d$ to find x_p (its free variable is $x_2 = 0$).

46. Suppose column 5 of U has no pivot. Then x_5 is a ______ variable. The zero vector (is) (is not) the only solution to $Ax = 0$. If $Ax = b$ has a solution, then it has ______ solutions.

47. Find A and B with the given property or explain why you can't.

(a) The only solution to $Ax = \begin{bmatrix}1\\2\\3\end{bmatrix}$ is $x = \begin{bmatrix}0\\1\end{bmatrix}$.

(b) The only solution to $Bx = \begin{bmatrix}0\\1\end{bmatrix}$ is $x = \begin{bmatrix}1\\2\\3\end{bmatrix}$.

48. Is there a 3 by 3 matrix with no zero entries for which $U = R = I$?

49. Reduce these matrices A and B to their ordinary echelon forms U:

$$\text{(a) } A = \begin{bmatrix}1 & 2 & 2 & 4 & 6\\1 & 2 & 3 & 6 & 9\\0 & 0 & 1 & 2 & 3\end{bmatrix} \qquad \text{(b) } B = \begin{bmatrix}2 & 4 & 2\\0 & 4 & 4\\0 & 8 & 8\end{bmatrix}.$$

Find a special solution for each free variable and describe every solution to $Ax = 0$ and $Bx = 0$. Reduce the echelon forms U to R, and draw a box around the identity matrix in the pivot rows and pivot columns.

50. Apply elimination with the extra column to reach $Rx = 0$ and $Rx = d$:

$$\begin{bmatrix}U & 0\end{bmatrix} = \begin{bmatrix}3 & 0 & 6 & \mathbf{0}\\0 & 0 & 2 & \mathbf{0}\\0 & 0 & 0 & \mathbf{0}\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}U & c\end{bmatrix} = \begin{bmatrix}3 & 0 & 6 & \mathbf{9}\\0 & 0 & 2 & \mathbf{4}\\0 & 0 & 0 & \mathbf{5}\end{bmatrix}.$$

Solve $Rx = 0$ (free variable $= 1$). What are the solutions to $Rx = d$?

51. Suppose column 4 of a 3 by 5 matrix is all 0s. Then x_4 is certainly a ______ variable. The special solution for this variable is the vector $x =$ ______.

52. Put as many 1s as possible in a 4 by 7 echelon matrix U and in a *reduced* form R whose pivot columns are 2, 4, 5.

53. The nullspace of a 3 by 4 matrix A is the line through $(2, 3, 1, 0)$.

(a) What is the *rank* of A and the complete solution to $Ax = 0$?
(b) What is the exact row reduced echelon form R of A?

54. True or False? (Give reason if true, or counterexample to show it is false.)

(a) A square matrix has no free variables. False
(b) An invertible matrix has no free variables. True
(c) An m by n matrix has no more than n pivot variables. True
(d) An m by n matrix has no more than m pivot variables. True

55. The complete solution to $Ax = \begin{bmatrix}1\\3\end{bmatrix}$ is $x = \begin{bmatrix}1\\0\end{bmatrix} + c\begin{bmatrix}0\\1\end{bmatrix}$. Find A.

56. Reduce to $Ux = c$ (Gaussian elimination) and then $Rx = d$:

$$Ax = \begin{bmatrix}1 & 0 & 2 & 3\\1 & 3 & 2 & 0\\2 & 0 & 4 & 9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}2\\5\\10\end{bmatrix} = b.$$

Find a particular solution x_p and all nullspace solutions x_n.

four-dimensional ***subspace***; an example is the set of vectors in $\mathbf{R}^6$ whose first and last components are zero. The members of this four-dimensional subspace are six-dimensional vectors like $(0, 5, 1, 3, 4, 0)$.

One final note about the language of linear algebra. We never use the terms "basis of a matrix" or "rank of a space" or "dimension of a basis." These phrases have no meaning. It is *the dimension of the column space* that equals *the rank of the matrix*, as we prove in the coming section.

Problem Set 2.3

Problems 1–10 are about linear independence and linear dependence.

1. Choose three independent columns of U. Then make two other choices. Do the same for A. You have found bases for which spaces?

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}.$$

2. Prove that if $a = 0$, $d = 0$, or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

3. Decide the dependence or independence of

(a) the vectors $(1, 3, 2)$, $(2, 1, 3)$, and $(3, 2, 1)$.

(b) the vectors $(1, -3, 2)$, $(2, 1, -3)$, and $(-3, 2, 1)$.

4. Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve $c_1 v_1 + \cdots + c_4 v_4 = 0$ or $Ac = 0$. The v's go in the columns of A.

5. If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$, $v_2 = w_1 - w_3$, and $v_3 = w_1 - w_2$ are *dependent*. Find a combination of the v's that gives zero.

6. If a, d, f in Problem 2 are all nonzero, show that the only solution to $Ux = 0$ is $x = 0$. Then U has independent columns.

7. Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

This number is the ______ of the space spanned by the v's.

8. Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbf{R}^3$.

(a) These four vectors are dependent because $Ax = 0$

(b) The two vectors v_1 and v_2 will be dependent if rank = 0 or 1

(c) The vectors v_1 and $(0, 0, 0)$ are dependent because it has non-zero sol^n

9. Find two independent vectors on the plane $x + 2y - 3z - t = 0$ in $\mathbf{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

10. If w_1, w_2, w_3 are independent vectors, show that the sums $v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$, and $v_3 = w_1 + w_2$ are *independent*. (Write $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ in terms of the w's. Find and solve equations for the c's.)

Problems 11–18 are about the space *spanned* by a set of vectors. Take all linear combinations of the vectors.

11. The vector b is in the subspace spanned by the columns of A when there is a solution to ______. The vector c is in the row space of A when there is a solution to ______. *True or false*: If the zero vector is in the row space, the rows are dependent.

12. $v + w$ and $v - w$ are combinations of v and w. Write v and w as combinations of $v + w$ and $v - w$. The two pairs of vectors ______ the same space. When are they a basis for the same space?

13. Decide whether or not the following vectors are linearly independent, by solving $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Decide also if they span $\mathbf{R}^4$, by trying to solve $c_1 v_1 + \cdots + c_4 v_4 = (0, 0, 0, 1)$.

14. Suppose the vectors to be tested for independence are placed into the rows instead of the columns of A. How does the elimination process from A to U decide for or against independence?

15. Find the dimensions of (a) the column space of A, (b) the column space of U, (c) the row space of A, (d) the row space of U. Which two of the spaces are the same?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

16. Describe the subspace of $\mathbf{R}^3$ (is it a line or a plane or $\mathbf{R}^3$?) spanned by

(a) the two vectors $(1, 1, -1)$ and $(-1, -1, 1)$.

(b) the three vectors $(0, 1, 1)$ and $(1, 1, 0)$ and $(0, 0, 0)$.

(c) the columns of a 3 by 5 echelon matrix with 2 pivots.

(d) all vectors with positive components.

17. Choose $x = (x_1, x_2, x_3, x_4)$ in $\mathbf{R}^4$. It has 24 rearrangements like (x_2, x_1, x_3, x_4) and (x_4, x_3, x_1, x_2). Those 24 vectors, including x itself, span a subspace $\mathbf{S}$. Find specific vectors x so that the dimension of $\mathbf{S}$ is: (a) 0, (b) 1, (c) 3, (d) 4.

18. To decide whether b is in the subspace spanned by $w_1, \dots, w_n$, let the vectors w be the columns of A and try to solve $Ax = b$. What is the result for

(a) $w_1 = (1, 1, 0)$, $w_2 = (2, 2, 1)$, $w_3 = (0, 0, 2)$, $b = (3, 4, 5)$?

(b) $w_1 = (1, 2, 0)$, $w_2 = (2, 5, 0)$, $w_3 = (0, 0, 2)$, $w_4 = (0, 0, 0)$, and any b?

Problems 19–37 are about the requirements for a basis.

19. Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbf{R}^3$. Then find a basis for the intersection of that plane with the xy-plane. Then find a basis for all vectors perpendicular to the plane.

20. If $v_1, \dots, v_n$ are linearly independent, the space they span has dimension ______. These vectors are a ______ for that space. If the vectors are the columns of an m by n matrix, then m is ______ than n.

21. Suppose $\mathbf{S}$ is a five-dimensional subspace of $\mathbf{R}^6$. True or false?

(a) Every basis for $\mathbf{S}$ can be extended to a basis for $\mathbf{R}^6$ by adding one more vector.

(b) Every basis for $\mathbf{R}^6$ can be reduced to a basis for $\mathbf{S}$ by removing one vector.

22. The columns of A are n vectors from $\mathbf{R}^m$. If they are linearly independent, what is the rank of A? If they span $\mathbf{R}^m$, what is the rank? If they are a basis for $\mathbf{R}^m$, what then?

23. U comes from A by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces.

24. Find three different bases for the column space of U above. Then find two different bases for the row space of U.

25. Find a basis for each of these subspaces of $\mathbf{R}^4$:

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

(c) All vectors that are perpendicular to $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$.

(d) The column space (in $\mathbf{R}^2$) and nullspace (in $\mathbf{R}^5$) of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

26. Suppose the columns of a 5 by 5 matrix A are a basis for $\mathbf{R}^5$.

(a) The equation $Ax = 0$ has only the solution $x = 0$ because ______.

(b) If b is in $\mathbf{R}^5$ then $Ax = b$ is solvable because ______.

Conclusion: A is invertible. Its rank is 5.

27. Suppose $v_1, v_2, \dots, v_6$ are six vectors in $\mathbf{R}^4$.

(a) Those vectors (do)(do not)(might not) span $\mathbf{R}^4$.

(b) Those vectors (are)(are not)(might be) linearly independent.

(c) Any four of those vectors (are)(are not)(might be) a basis for $\mathbf{R}^4$.

(d) If those vectors are the columns of A, then $Ax = b$ (has) (does not have) (might not have) a solution.

28. Find a counterexample to the following statement: If v_1, v_2, v_3, v_4 is a basis for the vector space $\mathbf{R}^4$, and if $\mathbf{W}$ is a subspace, then some subset of the v's is a basis for $\mathbf{W}$.

29. If A is a 64 by 17 matrix of rank 11, how many independent vectors satisfy $Ax = 0$? How many independent vectors satisfy $A^{\mathrm{T}}y = 0$?

30. Suppose $\mathbf{V}$ is known to have dimension k. Prove that

(a) any k independent vectors in $\mathbf{V}$ form a basis;
(b) any k vectors that span $\mathbf{V}$ form a basis.

In other words, if the number of vectors is known to be correct, either of the two properties of a basis implies the other.

31. For which numbers c and d do these matrices have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}.$$

32. *True or false*?

(a) If the columns of A are linearly independent, then $Ax = b$ has exactly one solution for every b. false, no sol^n.
(b) A 5 by 7 matrix never has linearly independent columns. True

33. Find a basis for each of these subspaces of 3 by 3 matrices:

(a) All diagonal matrices.
(b) All symmetric matrices ($A^{\mathrm{T}} = A$).
(c) All skew-symmetric matrices ($A^{\mathrm{T}} = -A$).

34. By locating the pivots, find a basis for the column space of

$$U = \begin{bmatrix} 0 & 5 & 4 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Express each column that is not in the basis as a combination of the basic columns. Find also a matrix A with this echelon form U, but a different column space.

35. Prove that if $\mathbf{V}$ and $\mathbf{W}$ are three-dimensional subspaces of $\mathbf{R}^5$, then $\mathbf{V}$ and $\mathbf{W}$ must have a nonzero vector in common. *Hint*: Start with bases for the two subspaces, making six vectors in all.

36. True or false (give a good reason)?

(a) If the columns of a matrix are dependent, so are the rows.
(b) The column space of a 2 by 2 matrix is the same as its row space.
(c) The column space of a 2 by 2 matrix has the same dimension as its row space.
(d) The columns of a matrix are a basis for the column space.

37. Find the dimensions of these vector spaces:

(a) The space of all vectors in $\mathbf{R}^4$ whose components add to zero.
(b) The nullspace of the 4 by 4 identity matrix.
(c) The space of all 4 by 4 matrices.

The next problems are about spaces in which the "vectors" are functions.

38. The cosine space $\mathbf{F}_3$ contains all combinations $y(x) = A\cos x + B\cos 2x + C\cos 3x$. Find a basis for the subspace that has $y(0) = 0$.

39. Write the 3 by 3 identity matrix as a combination of the other five permutation matrices! Then show that those five matrices are linearly independent. (Assume a combination gives zero, and check entries to prove each term is zero.) The five permutations are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

40. *Review*: Which of the following are bases for $\mathbf{R}^3$?

(a) $(1, 2, 0)$ and $(0, 1, -1)$.
(b) $(1, 1, -1), (2, 3, 4), (4, 1, -1), (0, 1, -1)$.
(c) $(1, 2, 2), (-1, 2, 1), (0, 8, 0)$.
(d) $(1, 2, 2), (-1, 2, 1), (0, 8, 6)$.

41. Find a basis for the space of polynomials $p(x)$ of degree ≤ 3. Find a basis for the subspace with $p(1) = 0$.

42. (a) Find all functions that satisfy $\frac{dy}{dx} = 0$.
(b) Choose a particular function that satisfies $\frac{dy}{dx} = 3$.
(c) Find all functions that satisfy $\frac{dy}{dx} = 3$.

43. Suppose $y_1(x)$, $y_2(x)$, $y_3(x)$ are three different functions of x. The vector space they span could have dimension 1, 2, or 3. Give an example of y_1, y_2, y_3 to show each possibility.

44. *Review*: Suppose A is 5 by 4 with rank 4. Show that $Ax = b$ has no solution when the 5 by 5 matrix $[A \;\; b]$ is invertible. Show that $Ax = b$ is solvable when $[A \;\; b]$ is singular.

45. Find a basis for the space of functions that satisfy

(a) $$\frac{dy}{dx} - 2y = 0.$$

(b) $$\frac{dy}{dx} - \frac{y}{x} = 0.$$

2.4 THE FOUR FUNDAMENTAL SUBSPACES

The previous section dealt with definitions rather than constructions. We know what a basis is, but not how to find one. Now, starting from an explicit description of a subspace, we would like to compute an explicit basis.

Subspaces can be described in two ways. First, we may be given a set of vectors that span the space. (*Example*: The columns span the column space.) Second, we may be told which conditions the vectors in the space must satisfy. (*Example*: The nullspace consists of all vectors that satisfy $Ax = 0$.)

The first description may include useless vectors (dependent columns). The second description may include repeated conditions (dependent rows). We can't write a basis by inspection, and a systematic procedure is necessary.

The reader can guess what that procedure will be. When elimination on A produces an echelon matrix U or a reduced R, we will find a basis for each of the subspaces to show how it can be broken into simple pieces. For linear algebra, the simple pieces are matrices of ***rank* 1**:

$$\textbf{Rank 1} \qquad A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} \quad \text{has } r = 1.$$

Every row is a multiple of the first row, so the row space is one-dimensional. In fact, we can write the whole matrix as *the product of a column vector and a row vector*:

$$A = \textbf{(column)(row)} \qquad \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$$

The product of a 4 by 1 matrix and a 1 by 3 matrix is a 4 by 3 matrix. *This product has rank* 1. At the same time, the columns are all multiples of the same column vector; the column space shares the dimension $r = 1$ and reduces to a line.

Every matrix of rank* 1 *has the simple form* $A = uv^{\mathrm{T}}$ = *column times row.

The rows are all multiples of the same vector v^{T}, and the columns are all multiples of u. The row space and column space are lines—the easiest case.

Problem Set 2.4

1. Describe the four subspaces in three-dimensional space associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

2. Find the dimension and a basis for the four fundamental subspaces for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. Find the dimension and construct a basis for the four subspaces associated with each of the matrices

$$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

4. If the product AB is the zero matrix, $AB = 0$, show that the column space of B is contained in the nullspace of A. (Also the row space of A is in the left nullspace of B, since each row of A multiplies B to give a zero row.)

5. True or false: If $m = n$, then the row space of A equals the column space. If $m < n$, then the nullspace has a larger dimension than ______.

6. Find the rank of A and write the matrix as $A = uv^{\mathrm{T}}$:

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix}.$$

7. If the columns of A are linearly independent (A is m by n), then the rank is ______, the nullspace is ______, the row space is ______, and there exists a ______-inverse.

8. If $Ax = b$ always has at least one solution, show that the only solution to $A^{\mathrm{T}}y = 0$ is $y = 0$. *Hint*: What is the rank?

9. Find a left-inverse and/or a right-inverse (when they exist) for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}.$$

10. Suppose A is an m by n matrix of rank r. Under what conditions on those numbers does

(a) A have a two-sided inverse: $AA^{-1} = A^{-1}A = I$?
(b) $Ax = b$ have *infinitely many* solutions for *every* b?

11. Find a matrix A that has $\mathbf{V}$ as its row space, and a matrix B that has $\mathbf{V}$ as its nullspace, if $\mathbf{V}$ is the subspace spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}.$$

12. Why is there no matrix whose row space and nullspace both contain $(1, 1, 1)$?

13. Find a basis for each of the four subspaces of

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

14. If a, b, c are given with $a \neq 0$, choose d so that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = uv^{\mathrm{T}}$$

has rank 1. What are the pivots?

15. (*A paradox*) Suppose A has a right-inverse B. Then $AB = I$ leads to $A^{\mathrm{T}}AB = A^{\mathrm{T}}$ or $B = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. But that satisfies $BA = I$; it is a *left*-inverse. Which step is not justified?

16. If $Ax = 0$ has a nonzero solution, show that $A^{\mathrm{T}}y = f$ fails to be solvable for some right-hand sides f. Construct an example of A and f.

17. Suppose the only solution to $Ax = 0$ (m equations in n unknowns) is $x = 0$. What is the rank and why? The columns of A are linearly ______.

18. Find a 1 by 3 matrix whose nullspace consists of all vectors in $\mathbf{R}^3$ such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace.

19. Construct a matrix with $(1, 0, 1)$ and $(1, 2, 0)$ as a basis for its row space and its column space. Why can't this be a basis for the row space and nullspace?

20. Suppose the 3 by 3 matrix A is invertible. Write bases for the four subspaces for A, and also for the 3 by 6 matrix $B = [A \quad A]$.

21. If A has the same four fundamental subspaces as B, does $A = cB$?

22. If the entries of a 3 by 3 matrix are chosen randomly between 0 and 1, what are the most likely dimensions of the four subspaces? What if the matrix is 3 by 5?

23. (Important) A is an m by n matrix of rank r. Suppose there are right-hand sides b for which $Ax = b$ has *no solution*.

(a) What inequalities ($<$ or $\leq$) must be true between m, n, and r?
(b) How do you know that $A^{\mathrm{T}}y = 0$ has a nonzero solution?

24. Construct a matrix with the required property, or explain why you can't.

(a) Column space contains $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, row space contains $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}2\\5\end{bmatrix}$.

(b) Column space has basis $\begin{bmatrix}1\\2\\3\end{bmatrix}$, nullspace has basis $\begin{bmatrix}3\\2\\1\end{bmatrix}$.

(c) Dimension of nullspace $= 1 +$ dimension of left nullspace.

(d) Left nullspace contains $\begin{bmatrix}1\\3\end{bmatrix}$, row space contains $\begin{bmatrix}3\\1\end{bmatrix}$.

(e) Row space $=$ column space, nullspace $\neq$ left nullspace.

25. Which subspaces are the same for these matrices of different sizes?

(a) $[A]$ and $\begin{bmatrix}A\\A\end{bmatrix}$. (b) $\begin{bmatrix}A\\A\end{bmatrix}$ and $\begin{bmatrix}A & A\\A & A\end{bmatrix}$.

Prove that all three matrices have the same rank r.

26. What are the dimensions of the four subspaces for A, B, and C, if I is the 3 by 3 identity matrix and 0 is the 3 by 2 zero matrix?

$$A = \begin{bmatrix}I & 0\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}I & I\\0^{\mathrm{T}} & 0^{\mathrm{T}}\end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix}0\end{bmatrix}.$$

27. (a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?
(b) If a 3 by 4 matrix has rank 3, what are its column space and left nullspace?

28. Without computing A, find bases for the four fundamental subspaces:

$$A = \begin{bmatrix}1 & 0 & 0\\6 & 1 & 0\\9 & 8 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 2 & 3\\0 & 0 & 1 & 2\end{bmatrix}.$$

29. Without elimination, find dimensions and bases for the four subspaces for

$$A = \begin{bmatrix}0 & 3 & 3 & 3\\0 & 0 & 0 & 0\\0 & 1 & 0 & 1\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}1 & 1\\4 & 4\\5 & 5\end{bmatrix}.$$

30. (Left nullspace) Add the extra column b and reduce A to echelon form:

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}.$$

A combination of the rows of A has produced the zero row. What combination is it? (Look at $b_3 - 2b_2 + b_1$ on the right-hand side.) Which vectors are in the nullspace of A^{T} and which are in the nullspace of A?

31. Following the method of Problem 30, reduce A to echelon form and look at zero rows. The b column tells which combinations you have taken of the rows:

(a) $\begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix}$.

From the b column after elimination, read off $m - r$ basis vectors in the left nullspace of A (combinations of rows that give zero).

32. True or false (with a reason or a counterexample)?

(a) A and A^{T} have the same number of pivots.
(b) A and A^{T} have the same left nullspace.
(c) If the row space equals the column space then $A^{\mathrm{T}} = A$.
(d) If $A^{\mathrm{T}} = -A$ then the row space of A equals the column space.

33. If you exchange the first two rows of a matrix A, which of the four subspaces stay the same? If $y = (1, 2, 3, 4)$ is in the left nullspace of A, write down a vector in the left nullspace of the new matrix.

34. Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that A is not invertible?

35. Explain why $v = (1, 0, -1)$ cannot be a row of A and also be in the nullspace.

36. Describe the four subspaces of $\mathbf{R}^3$ associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

37. Suppose A is the sum of two matrices of rank one: $A = uv^{\mathrm{T}} + wz^{\mathrm{T}}$.

(a) Which vectors span the column space of A?
(b) Which vectors span the row space of A?
(c) The rank is less than 2 if ______ or if ______.
(d) Compute A and its rank if $u = z = (1, 0, 0)$ and $v = w = (0, 0, 1)$.

Problem Set 2.6

1. The solutions to the linear differential equation $d^2u/dt^2 = u$ form a vector space (since combinations of solutions are still solutions). Find two independent solutions, to give a basis for that solution space.

2. On the space $\mathbf{P}_3$ of cubic polynomials, what matrix represents d^2/dt^2? Construct the 4 by 4 matrix from the standard basis $1, t, t^2, t^3$. Find its nullspace and column space. What do they mean in terms of polynomials?

3. With initial values $u = x$ and $du/dt = y$ at $t = 0$, what combination of basis vectors in Problem 1 solves $u'' = u$? This transformation from initial values to solution is linear. What is its 2 by 2 matrix (using $x = 1, y = 0$ and $x = 0, y = 1$ as basis for $\mathbf{V}$, and your basis for $\mathbf{W}$)?

4. Suppose A is a linear transformation from the x-y plane to itself. Why does $A^{-1}(x + y) = A^{-1}x + A^{-1}y$? If A is represented by the matrix M, explain why A^{-1} is represented by M^{-1}.

5. What 3 by 3 matrices represent the transformations that
   (a) project every vector onto the x-y plane?
   (b) reflect every vector through the x-y plane?
   (c) rotate the x-y plane through 90°, leaving the z-axis alone?
   (d) rotate the x-y plane, then x-z, then y-z, through 90°?
   (e) carry out the same three rotations, but each one through 180°?

6. The matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ produces a ***stretching*** in the x-direction. Draw the circle $x^2 + y^2 = 1$ and sketch around it the points $(2x, y)$ that result from multiplication by A. What shape is that curve?

7. Verify directly from $c^2 + s^2 = 1$ that reflection matrices satisfy $H^2 = I$.

8. Every straight line remains straight after a linear transformation. If z is halfway between x and y, show that Az is halfway between Ax and Ay.

9. What matrix has the effect of rotating every vector through 90° and then projecting the result onto the x-axis? What matrix represents projection onto the x-axis followed by projection onto the y-axis?

10. Does the product of 5 reflections and 8 rotations of the x-y plane produce a rotation or a reflection?

11. From the cubics $\mathbf{P}_3$ to the fourth-degree polynomials $\mathbf{P}_4$, what matrix represents multiplication by $2 + 3t$? The columns of the 5 by 4 matrix A come from applying the transformation to $1, t, t^2, t^3$.

12. The matrix $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ yields a ***shearing*** transformation, which leaves the y-axis unchanged. Sketch its effect on the x-axis, by indicating what happens to $(1, 0)$ and $(2, 0)$ and $(-1, 0)$—and how the whole axis is transformed.

13. The product $(AB)C$ of linear transformations starts with a vector x and produces $u = Cx$. Then rule 2V applies AB to u and reaches $(AB)Cx$.

(a) Is this result the same as separately applying C then B then A?

(b) Is the result the same as applying BC followed by A? Parentheses are unnecessary and the associative law $(AB)C = A(BC)$ holds for linear transformations. This is the best proof of the same law for matrices.

14. In the vector space P_3 of all $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, let **S** be the subset of polynomials with $\int_0^1 p(x)\,dx = 0$. Verify that **S** is a subspace and find a basis.

15. What is the axis and the rotation angle for the transformation that takes (x_1, x_2, x_3) into (x_2, x_3, x_1)?

16. If S and T are linear with $S(v) = T(v) = v$, then $S(T(v)) = v$ or v^2?

17. Which of these transformations satisfy $T(v+w) = T(v)+T(w)$, and which satisfy $T(cv) = cT(v)$?

(a) $T(v) = v/\|v\|$. (b) $T(v) = v_1 + v_2 + v_3$.
(c) $T(v) = (v_1, 2v_2, 3v_3)$. (d) $T(v) =$ largest component of v.

18. The space of all 2 by 2 matrices has the four basis "vectors"

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}.$$

For the linear transformation of *transposing*, find its matrix A with respect to this basis. Why is $A^2 = I$?

19. A linear transformation must leave the zero vector fixed: $T(0) = 0$. Prove this from $T(v + w) = T(v) + T(w)$ by choosing $w =$ ______. Prove it also from the requirement $T(cv) = cT(v)$ by choosing $c =$ ______.

20. Which of these transformations is not linear? The input is $v = (v_1, v_2)$.

(a) $T(v) = (v_2, v_1)$. (b) $T(v) = (v_1, v_1)$.
(c) $T(v) = (0, v_1)$. (d) $T(v) = (0, 1)$.

21. Find the 4 by 4 cyclic permutation matrix: (x_1, x_2, x_3, x_4) is transformed to $Ax = (x_2, x_3, x_4, x_1)$. What is the effect of A^2? Show that $A^3 = A^{-1}$.

22. Suppose $T(v) = v$, except that $T(0, v_2) = (0, 0)$. Show that this transformation satisfies $T(cv) = cT(v)$ but not $T(v + w) = T(v) + T(w)$.

23. A *nonlinear* transformation is invertible if $T(x) = b$ has exactly one solution for every b. The example $T(x) = x^2$ is not invertible because $x^2 = b$ has two solutions for positive b and no solution for negative b. Which of the following transformations (from the real numbers $\mathbf{R}^1$ to the real numbers $\mathbf{R}^1$) are invertible? None are linear, not even (c).

(a) $T(x) = x^3$. (b) $T(x) = e^x$.
(c) $T(x) = x + 11$. (d) $T(x) = \cos x$.

24. Find the 4 by 3 matrix A that represents a *right shift*: (x_1, x_2, x_3) is transformed to $(0, x_1, x_2, x_3)$. Find also the *left shift* matrix B from $\mathbf{R}^4$ back to $\mathbf{R}^3$, transforming (x_1, x_2, x_3, x_4) to (x_2, x_3, x_4). What are the products AB and BA?

25. Prove that T^2 is a linear transformation if T is linear (from $\mathbf{R}^3$ to $\mathbf{R}^3$).

26. The "cyclic" transformation T is defined by $T(v_1, v_2, v_3) = (v_2, v_3, v_1)$. What is $T(T(T(v)))$? What is $T^{100}(v)$?

27. Suppose a linear T transforms $(1, 1)$ to $(2, 2)$ and $(2, 0)$ to $(0, 0)$. Find $T(v)$ when

(a) $v = (2, 2)$. (b) $v = (3, 1)$. (c) $v = (-1, 1)$. (d) $v = (a, b)$.

28. A linear transformation from $\mathbf{V}$ to $\mathbf{W}$ has an *inverse* from $\mathbf{W}$ to $\mathbf{V}$ when the range is all of $\mathbf{W}$ and the kernel contains only $v = 0$. Why are these transformations not invertible?

(a) $T(v_1, v_2) = (v_2, v_2)$ $\quad \mathbf{W} = \mathbf{R}^2$.
(b) $T(v_1, v_2) = (v_1, v_2, v_1 + v_2)$ $\quad \mathbf{W} = \mathbf{R}^3$.
(c) $T(v_1, v_2) = v_1$ $\quad \mathbf{W} = \mathbf{R}^1$.

29. For these transformations of $\mathbf{V} = \mathbf{R}^2$ to $\mathbf{W} = \mathbf{R}^2$, find $T(T(v))$.

(a) $T(v) = -v$.
(b) $T(v) = v + (1, 1)$.
(c) $T(v) = 90^\circ$ rotation $= (-v_2, v_1)$.
(d) $T(v) =$ projection $= \left(\frac{v_1 + v_2}{2}, \frac{v_1 + v_2}{2}\right)$.

30. Find the *range* and *kernel* (those are new words for the column space and nullspace) of T.

(a) $T(v_1, v_2) = (v_2, v_1)$. (b) $T(v_1, v_2, v_3) = (v_1, v_2)$.
(c) $T(v_1, v_2) = (0, 0)$. (d) $T(v_1, v_2) = (v_1, v_1)$.

Problems 31–35 may be harder. The input space V contains all 2 by 2 matrices M.

31. Suppose T transposes every matrix M. Try to find a matrix A that gives $AM = M^{\mathrm{T}}$ for every M. Show that no matrix A will do it. *To professors:* Is this a linear transformation that doesn't come from a matrix?

32. Suppose $T(M) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Find a matrix with $T(M) \neq 0$. Describe all matrices with $T(M) = 0$ (the kernel of T) and all output matrices $T(M)$ (the range of T).

33. M is any 2 by 2 matrix and $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. The linear transformation T is defined by $T(M) = AM$. What rules of matrix multiplication show that T is linear?

34. Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. Show that the identity matrix I is not in the range of T. Find a nonzero matrix M such that $T(M) = AM$ is zero.

35. The transformation T that transposes every matrix is definitely linear. Which of these extra properties are true?

(a) $T^2 =$ identity transformation.
(b) The kernel of T is the zero matrix.
(c) Every matrix is in the range of T.
(d) $T(M) = -M$ is impossible.

The next problems are about changing the basis.

36. (a) What matrix transforms $(1, 0)$ into $(2, 5)$ and transforms $(0, 1)$ to $(1, 3)$?

Proof Every b in the column space is a combination Ax of the columns. In fact, b is Ax_r, with x_r in the row space, since the nullspace component gives $Ax_n = 0$. If another vector x_r' in the row space gives $Ax_r' = b$, then $A(x_r - x_r') = b - b = 0$. This puts $x_r - x_r'$ in the nullspace and the row space, which makes it orthogonal to itself. Therefore it is zero, and $x_r = x_r'$. Exactly one vector in the row space is carried to b. ∎

> ***Every matrix transforms its row space onto its column space.***

On those r-dimensional spaces A is invertible. On its nullspace A is zero. When A is diagonal, you see the invertible submatrix holding the r nonzeros.

A^T goes in the opposite direction, from $\mathbf{R}^m$ to $\mathbf{R}^n$ and from $C(A)$ back to $C(A^T)$. Of course the transpose is not the inverse! A^T moves the spaces correctly, but not the individual vectors. That honor belongs to A^{-1} if it exists—and it only exists if $r = m = n$. We cannot ask A^{-1} to bring back a whole nullspace out of the zero vector.

When A^{-1} fails to exist, the best substitute is the ***pseudoinverse*** A^+. This inverts A where that is possible: $A^+Ax = x$ for x in the row space. On the left nullspace, nothing can be done: $A^+y = 0$. Thus A^+ inverts A where it is invertible, and has the same rank r. One formula for A^+ depends on the ***singular value decomposition***—for which we first need to know about eigenvalues.

Problem Set 3.1

1. Which pairs are orthogonal among the vectors v_1, v_2, v_3, v_4?

$$v_1 = \begin{bmatrix}1\\2\\-2\\1\end{bmatrix}, \quad v_2 = \begin{bmatrix}4\\0\\4\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}, \quad v_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

2. Find a vector x orthogonal to the row space of A, and a vector y orthogonal to the column space, and a vector z orthogonal to the nullspace:

$$A = \begin{bmatrix}1 & 2 & 1\\2 & 4 & 3\\3 & 6 & 4\end{bmatrix}.$$

3. Find all vectors in $\mathbf{R}^3$ that are orthogonal to $(1, 1, 1)$ and $(1, -1, 0)$. Produce an orthonormal basis from these vectors (mutually orthogonal unit vectors).

4. Two lines in the plane are perpendicular when the product of their slopes is -1. Apply this to the vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$, whose slopes are x_2/x_1 and y_2/y_1, to derive again the orthogonality condition $x^Ty = 0$.

5. Give an example in $\mathbf{R}^2$ of linearly independent vectors that are not orthogonal. Also, give an example of orthogonal vectors that are not independent.

6. How do we know that the ith row of an invertible matrix B is orthogonal to the jth column of B^{-1}, if $i \neq j$?

7. Find the lengths and the inner product of $x = (1, 4, 0, 2)$ and $y = (2, -2, 1, 3)$.

8. *Why are these statements false?*

(a) If $\mathbf{V}$ is orthogonal to $\mathbf{W}$, then $\mathbf{V}^{\perp}$ is orthogonal to $\mathbf{W}^{\perp}$.

(b) $\mathbf{V}$ orthogonal to $\mathbf{W}$ and $\mathbf{W}$ orthogonal to $\mathbf{Z}$ makes $\mathbf{V}$ orthogonal to $\mathbf{Z}$.

9. Find a basis for the orthogonal complement of the row space of A:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

Split $x = (3, 3, 3)$ into a row space component x_r and a nullspace component x_n.

10. Let $\mathbf{P}$ be the plane in $\mathbf{R}^2$ with equation $x + 2y - z = 0$. Find a vector perpendicular to $\mathbf{P}$. What matrix has the plane $\mathbf{P}$ as its nullspace, and what matrix has $\mathbf{P}$ as its row space?

11. Find all vectors that are perpendicular to $(1, 4, 4, 1)$ and $(2, 9, 8, 2)$.

12. Show that $x - y$ is orthogonal to $x + y$ if and only if $\|x\| = \|y\|$.

13. Let $\mathbf{S}$ be the subspace of $\mathbf{R}^4$ containing all vectors with $x_1 + x_2 + x_3 + x_4 = 0$. Find a basis for the space $\mathbf{S}^{\perp}$, containing all vectors orthogonal to $\mathbf{S}$.

14. Find the orthogonal complement of the plane spanned by the vectors $(1, 1, 2)$ and $(1, 2, 3)$, by taking these to be the rows of A and solving $Ax = 0$. Remember that the complement is a whole line.

15. Let $\mathbf{S}$ be a subspace of $\mathbf{R}^n$. Explain what $(\mathbf{S}^{\perp})^{\perp} = \mathbf{S}$ means and why it is true.

16. Illustrate the action of A^{T} by a picture corresponding to Figure 3.4, sending $\boldsymbol{C}(A)$ back to the row space and the left nullspace to zero.

17. If $\mathbf{S} = \{0\}$ is the subspace of $\mathbf{R}^4$ containing only the zero vector, what is $\mathbf{S}^{\perp}$? If $\mathbf{S}$ is spanned by $(0, 0, 0, 1)$, what is $\mathbf{S}^{\perp}$? What is $(\mathbf{S}^{\perp})^{\perp}$?

18. If $\mathbf{V}$ and $\mathbf{W}$ are orthogonal subspaces, show that the only vector they have in common is the zero vector: $\mathbf{V} \cap \mathbf{W} = \{0\}$.

19. The fundamental theorem is often stated in the form of *Fredholm's alternative*: For any A and b, one and only one of the following systems has a solution:

(i) $Ax = b$.

(ii) $A^{\mathrm{T}}y = 0$, $y^{\mathrm{T}}b \neq 0$.

Either b is in the column space $\boldsymbol{C}(A)$ or there is a y in $\boldsymbol{N}(A^{\mathrm{T}})$ such that $y^{\mathrm{T}}b \neq 0$. Show that it is contradictory for (i) and (ii) both to have solutions.

20. If $\mathbf{V}$ is the orthogonal complement of $\mathbf{W}$ in $\mathbf{R}^n$, is there a matrix with row space $\mathbf{V}$ and nullspace $\mathbf{W}$? Starting with a basis for $\mathbf{V}$, construct such a matrix.

21. Find a matrix whose row space contains $(1, 2, 1)$ and whose nullspace contains $(1, -2, 1)$, or prove that there is no such matrix.

22. Construct a homogeneous equation in three unknowns whose solutions are the linear combinations of the vectors $(1, 1, 2)$ and $(1, 2, 3)$. This is the reverse of the previous exercise, but the two problems are really the same.

23. If $AB = 0$ then the columns of B are in the ______ of A. The rows of A are in the ______ of B. Why can't A and B be 3 by 3 matrices of rank 2?

24. In Figure 3.4, how do we know that Ax_r is equal to Ax? How do we know that this vector is in the column space? If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, what is x_r?

25. (a) If $Ax = b$ has a solution and $A^{\mathrm{T}}y = 0$, then y is perpendicular to ______.
(b) If $A^{\mathrm{T}}y = c$ has a solution and $Ax = 0$, then x is perpendicular to ______.

26. If Ax is in the nullspace of A^{T} then $Ax = 0$. Reason: Ax is also in the ______ of A and the spaces are ______. *Conclusion: $A^{\mathrm{T}}A$ has the same nullspace as A.*

27. (Recommended) Draw Figure 3.4 to show each subspace for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

28. Redraw Figure 3.4 for a 3 by 2 matrix of rank $r = 2$. Which subspace is $\mathbf{Z}$ (zero vector only)? The nullspace part of any vector x in $\mathbf{R}^2$ is $x_n =$ ______.

29. This is a system of equations $Ax = b$ with *no solution*:

$$\begin{aligned} x + 2y + 2z &= 5 \\ 2x + 2y + 3z &= 5 \\ 3x + 4y + 5z &= 9. \end{aligned}$$

Find numbers y_1, y_2, y_3 to multiply the equations so they add to $0 = 1$. You have found a vector y in which subspace? The inner product $y^{\mathrm{T}}b$ is 1.

30. Construct an unsymmetric 2 by 2 matrix of rank 1. Copy Figure 3.4 and put one vector in each subspace. Which vectors are orthogonal?

31. Find the pieces x_r and x_n, and draw Figure 3.4 properly, if

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

32. Construct a matrix with the required property or say why that is impossible.

(a) Column space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) Row space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution and $A^{\mathrm{T}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(d) Every row is orthogonal to every column (A is not the zero matrix).

(e) The columns add up to a column of 0s, the rows add to a row of 1s.

33. Suppose A is a symmetric matrix ($A^{\mathrm{T}} = A$).

(a) Why is its column space perpendicular to its nullspace?

(b) If $Ax = 0$ and $Az = 5z$, which subspaces contain these "eigenvectors" x and z? **Symmetric matrices have perpendicular eigenvectors** (see Section 5.5).

Example 3 Project onto the "θ-direction" in the x-y plane. The line goes through $a = (\cos\theta, \sin\theta)$ and the matrix is symmetric with $P^2 = P$:

$$P = \frac{aa^{\mathrm{T}}}{a^{\mathrm{T}}a} = \frac{\begin{bmatrix} c \\ s \end{bmatrix}\begin{bmatrix} c & s \end{bmatrix}}{\begin{bmatrix} c & s \end{bmatrix}\begin{bmatrix} c \\ s \end{bmatrix}} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}.$$

Here c is $\cos\theta$, s is $\sin\theta$, and $c^2 + s^2 = 1$ in the denominator. This matrix P was discovered in Section 2.6 on linear transformations. Now we know P in any number of dimensions. We emphasize that it produces the projection p:

To project b onto a, multiply by the projection matrix P: $p = Pb$.

Transposes from Inner Products

Finally we connect inner products to A^{T}. Up to now, A^{T} is simply the reflection of A across its main diagonal; the rows of A become the columns of A^{T}, and vice versa. The entry in row i, column j of A^{T} is the (j, i) entry of A:

$$\textbf{Transpose by reflection} \qquad (A^{\mathrm{T}})_{ij} = (A)_{ji}.$$

There is a deeper significance to A^{T}. Its close connection to inner products gives a new and much more "abstract" definition of the transpose:

3J The transpose A^{T} can be defined by the following property: The inner product of Ax with y equals the inner product of x with $A^{\mathrm{T}}y$. Formally, this simply means that

$$(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}A^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y). \tag{8}$$

This definition gives us another (better) way to verify the formula $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$. Use equation (8) twice:

$$\textbf{Move } \boldsymbol{A} \textbf{ then move } \boldsymbol{B} \qquad (ABx)^{\mathrm{T}}y = (Bx)^{\mathrm{T}}(A^{\mathrm{T}}y) = x^{\mathrm{T}}(B^{\mathrm{T}}A^{\mathrm{T}}y).$$

The transposes turn up in reverse order on the right side, just as the inverses do in the formula $(AB)^{-1} = B^{-1}A^{-1}$. We mention again that these two formulas meet to give the remarkable combination $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$.

Problem Set 3.2

1. (a) Given any two positive numbers x and y, choose the vector b equal to $(\sqrt{x}, \sqrt{y})$, and choose $a = (\sqrt{y}, \sqrt{x})$. Apply the Schwarz inequality to compare the arithmetic mean $\frac{1}{2}(x + y)$ with the geometric mean $\sqrt{xy}$.
   (b) Suppose we start with a vector from the origin to the point x, and then add a vector of length $\|y\|$ connecting x to $x + y$. The third side of the triangle goes from the origin to $x + y$. *The triangle inequality asserts that this distance*

cannot be greater than the sum of the first two:

$$\|x + y\| \le \|x\| + \|y\|.$$

After squaring both sides, and expanding $(x + y)^{\mathrm{T}}(x + y)$, reduce this to the Schwarz inequality.

2. Square the matrix $P = aa^{\mathrm{T}}/a^{\mathrm{T}}a$, which projects onto a line, and show that $P^2 = P$. (Note the number $a^{\mathrm{T}}a$ in the middle of the matrix $aa^{\mathrm{T}}aa^{\mathrm{T}}$!)

3. By choosing the correct vector b in the Schwarz inequality, prove that

$$(a_1 + \cdots + a_n)^2 \le n(a_1^2 + \cdots + a_n^2).$$

When does equality hold?

4. Verify that the length of the projection in Figure 3.7 is $\|p\| = \|b\| \cos\theta$, using formula (5).

5. (a) Find the projection matrix P_1 onto the line through $a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and also the matrix P_2 that projects onto the line perpendicular to a.
(b) Compute $P_1 + P_2$ and $P_1 P_2$ and explain.

6. The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$—note that all six edges have length $\sqrt{2}$, so the tetrahedron is regular—what is the cosine of the angle between the rays going from the center $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ to the vertices? (The bond angle itself is about 109.5°, an old friend of chemists.)

7. Explain why the Schwarz inequality becomes an equality in the case that a and b lie on the same line through the origin, and only in that case. What if they lie on opposite sides of the origin?

8. Prove that the *trace* of $P = aa^{\mathrm{T}}/a^{\mathrm{T}}a$—which is the sum of its diagonal entries—always equals 1.

9. Find the matrix that projects every point in the plane onto the line $x + 2y = 0$.

10. In n dimensions, what angle does the vector $(1, 1, \ldots, 1)$ make with the coordinate axes? What is the projection matrix P onto that vector?

11. What multiple of $a = (1, 1, 1)$ is closest to the point $b = (2, 4, 4)$? Find also the point closest to a on the line through b.

12. Is the projection matrix P invertible? Why or why not?

13. The Schwarz inequality has a one-line proof if a and b are normalized ahead of time to be unit vectors:

$$|a^{\mathrm{T}}b| = \left|\sum a_j b_j\right| \le \sum |a_j||b_j| \le \sum \frac{|a_j|^2 + |b_j|^2}{2} = \frac{1}{2} + \frac{1}{2} = \|a\|\,\|b\|.$$

Which previous problem justifies the middle step?

14. Show that the length of Ax equals the length of $A^{\mathrm{T}}x$ if $AA^{\mathrm{T}} = A^{\mathrm{T}}A$.

15. Suppose P is the projection matrix onto the line through a.

(a) Why is the inner product of x with Py equal to the inner product of Px with y?

(b) Are the two angles the same? Find their cosines if $a = (1, 1, -1)$, $x = (2, 0, 1)$, $y = (2, 1, 2)$.

(c) Why is the inner product of Px with Py again the same? What is the angle between those two?

16. What matrix P projects every point in $\mathbf{R}^3$ onto the line of intersection of the planes $x + y + t = 0$ and $x - t = 0$?

Problems 17–26 ask for projections onto lines. Also errors $e = b - p$ and matrices P.

17. *Draw* the projection of b onto a and also compute it from $p = \widehat{x}a$:

(a) $b = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. (b) $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

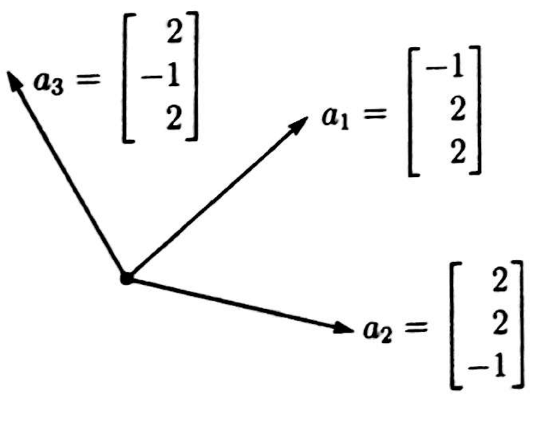
18. Construct the projection matrices P_1 and P_2 onto the lines through the a's in Problem 17. Is it true that $(P_1 + P_2)^2 = P_1 + P_2$? This *would* be true if $P_1P_2 = 0$.

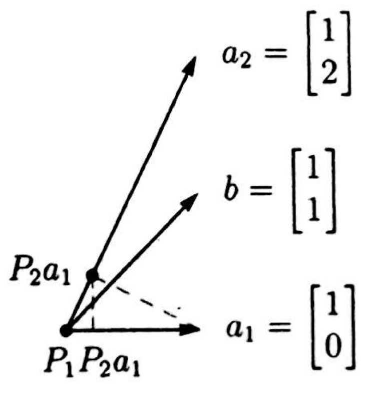
19. Project the vector b onto the line through a. Check that e is perpendicular to a:

(a) $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. (b) $b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $a = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$.

20. In Problem 19, find the projection matrix $P = aa^{\mathrm{T}}/a^{\mathrm{T}}a$ onto the line through each vector a. Verify in both cases that $P^2 = P$. Multiply Pb in each case to compute the projection p.

For Problems 21–26, consult the accompanying figures.





21. Compute the projection matrices $aa^{\mathrm{T}}/a^{\mathrm{T}}a$ onto the lines through $a_1 = (-1, 2, 2)$ and $a_2 = (2, 2, -1)$. Multiply those projection matrices and explain why their product P_1P_2 is what it is.

The two guesses are dependent, because they are based on the same bidding—but not identical, because they are looking at different hands. Say the chance that they are both too high or both too low is zero, but the chance of opposite errors is $\frac{1}{3}$. Then $E(e_1e_2) = \frac{1}{3}(-1)$, and the inverse of the covariance matrix is W^TW:

$$\begin{bmatrix} E(e_1^2) & E(e_1e_2) \\ E(e_1e_2) & E(e_2^2) \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = C = W^TW.$$

This matrix goes into the middle of the weighted normal equations.

Problem Set 3.3

1. Solve $Ax = b$ by least squares, and find $p = A\hat{x}$ if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Verify that the error $b - p$ is perpendicular to the columns of A.

2. Write out $E^2 = \|Ax - b\|^2$ and set to zero its derivatives with respect to u and v, if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} u \\ v \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

Compare the resulting equations with $A^TA\hat{x} = A^Tb$, confirming that calculus as well as geometry gives the normal equations. Find the solution $\hat{x}$ and the projection $p = A\hat{x}$. Why is $p = b$?

3. Suppose the values $b_1 = 1$ and $b_2 = 7$ at times $t_1 = 1$ and $t_2 = 2$ are fitted by a line $b = Dt$ *through the origin*. Solve $D = 1$ and $2D = 7$ by least squares, and sketch the best line.

4. The following system has no solution:

$$Ax = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix} = b.$$

Sketch and solve a straight-line fit that leads to the minimization of the quadratic $(C - D - 4)^2 + (C - 5)^2 + (C + D - 9)^2$. What is the projection of b onto the column space of A?

5. Find the best least-squares solution $\hat{x}$ to $3x = 10$, $4x = 5$. What error E^2 is minimized? Check that the error vector $(10 - 3\hat{x}, 5 - 4\hat{x})$ is perpendicular to the column $(3, 4)$.

6. Find the projection of b onto the column space of A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

Split b into $p + q$, with p in the column space and q perpendicular to that space. Which of the four subspaces contains q?

7. If the vectors a_1, a_2, and b are orthogonal, what are $A^{\mathrm{T}}A$ and $A^{\mathrm{T}}b$? What is the projection of b onto the plane of a_1 and a_2?

8. If P is the projection matrix onto a line in the x-y plane, draw a figure to describe the effect of the "reflection matrix" $H = I - 2P$. Explain both geometrically and algebraically why $H^2 = I$.

9. Find the best straight-line fit (least squares) to the measurements

$$\begin{array}{llllll} b=4 & \text{at} & t=-2, & b=3 & \text{at} & t=-1, \\ b=1 & \text{at} & t=0, & b=0 & \text{at} & t=2. \end{array}$$

Then find the projection of $b = (4, 3, 1, 0)$ onto the column space of

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

10. (a) If $P = P^{\mathrm{T}}P$, show that P is a projection matrix.
(b) What subspace does the matrix $P = 0$ project onto?

11. The vectors $a_1 = (1, 1, 0)$ and $a_2 = (1, 1, 1)$ span a plane in $\mathbf{R}^3$. Find the projection matrix P onto the plane, and find a nonzero vector b that is projected to zero.

12. Find the projection matrix P onto the space spanned by $a_1 = (1, 0, 1)$ and $a_2 = (1, 1, -1)$.

13. What 2 by 2 matrix projects the x-y plane onto the $-45°$ line $x + y = 0$?

14. Show that if u has unit length, then the rank-1 matrix $P = uu^{\mathrm{T}}$ is a projection matrix: It has properties (i) and (ii) in 3N. By choosing $u = a/\|a\|$, P becomes the projection onto the line through a, and Pb is the point $p = \hat{x}a$. Rank-1 projections correspond exactly to least-squares problems in one unknown.

15. If $\mathbf{V}$ is the subspace spanned by $(1, 1, 0, 1)$ and $(0, 0, 1, 0)$, find
(a) a basis for the orthogonal complement $\mathbf{V}^{\perp}$.
(b) the projection matrix P onto $\mathbf{V}$.
(c) the vector in $\mathbf{V}$ closest to the vector $b = (0, 1, 0, -1)$ in $\mathbf{V}^{\perp}$.

16. If P is the projection matrix onto a k-dimensional subspace $\mathbf{S}$ of the whole space $\mathbf{R}^n$, what is the column space of P and what is its rank?

17. Suppose P is the projection matrix onto the subspace $\mathbf{S}$ and Q is the projection onto the orthogonal complement $\mathbf{S}^{\perp}$. What are $P + Q$ and PQ? Show that $P - Q$ is its own inverse.

18. We want to fit a plane $y = C + Dt + Ez$ to the four points

$$\begin{array}{llllll} y=3 & \text{at} & t=1,\ z=1 & y=6 & \text{at} & t=0,\ z=3 \\ y=5 & \text{at} & t=2,\ z=1 & y=0 & \text{at} & t=0,\ z=0. \end{array}$$