4.2 Properties of the Determinant

1. $det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

2. $det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

3. The determinant of the identity matrix is 1, eg-$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ and $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

4. The determinant changes sign when two rows are interchanged, i.e.,$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$.

5. The determinant depends linearly on the first row, i.e.,$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$ and $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

6. If any two rows of A are equal then $det(A) = 0$, eg, $\begin{vmatrix} a & c \\ a & c \end{vmatrix} = 0.$

7. Substracting a multiple of one row from another row leaves the same determinant, i.e., $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a - tc & b - td \\ c & d \end{vmatrix}$.

8. If A is singular then $det(A) = 0$ and if A is nonsingular then $det(A) \neq 0$.

9. $det(AB) = det(A)det(B)$

10. $det(A) = det(A^T)$

Exercise-4.2

4. By applying row operations to produce an upper triangular U, compute

(i) $det\begin{pmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{pmatrix}.$

Solution:

$$\begin{vmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 2 & 5 & 3 \end{vmatrix} \quad R_2 \leftarrow R_2 - 2R_1,\ R_3 \leftarrow R_3 + R_1$$

$$= \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & 5 \end{vmatrix} \quad R_4 \leftarrow R_4 + 2R_2$$

$$= \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{vmatrix} \quad R_4 \leftarrow R_4 + \frac{5}{2}R_3$$

$$= 1 \times (-1) \times (-2) \times 10 = 20.$$

(ii) $det\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{pmatrix}.$

Solution:

$$\begin{aligned}
\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{vmatrix} &= \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{vmatrix} \quad R_2 \leftarrow R_2 + \frac{1}{2}R_1 \\
&= \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & -2 \end{vmatrix} \quad R_3 \leftarrow R_3 + \frac{2}{3}R_2 \\
&= \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{vmatrix} \quad R_4 \leftarrow R_4 + \frac{3}{4}R_3 \\
&= 2 \times \frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} = 5.
\end{aligned}$$

5. Find the determinants of:

   (a) a rank one matrix $A = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \end{pmatrix}$

   (b) the upper triangular matrix $U = \begin{pmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

   (c) the lower triangular matrix U^T.

   (d) the inverse matrix U^{-1}.

   (e) the reverse-triangular matrix that results from row exchanges, $M = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{pmatrix}$

Solution:

(a) $A = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 \\ 8 & -4 & 8 \\ 4 & -2 & 4 \end{pmatrix}$. Hence, $det(A) = \begin{vmatrix} 2 & -1 & 2 \\ 8 & -4 & 8 \\ 4 & -2 & 4 \end{vmatrix} =$ $\begin{vmatrix} 2 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$

(b) $\mid U \mid = 4 \times 1 \times 2 \times 2 = 16$

(c) $\mid U^T \mid = \mid U \mid = 16.$

(d) $\mid U^{-1} \mid = \dfrac{1}{\mid U \mid} = \dfrac{1}{16}.$

(e) $\mid M \mid = \begin{vmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{vmatrix} = 4 \times 1 \times 2 \times 2 = 16.$

Assignments

Exercise-4.2, Q. 2,6,13.