1.4 Matrix Notation and Matrix Multiplication

Course Outcome: Students will have understanding how to write system of equations using matrix notation, about matrix multiplication and how to compute elimination matrices and their use to convert a matrix into upper triangular form.

Ex:
$$2u + v + w = 5$$
$$4u - 6v = -2$$
$$-2u + 7v + 2w = 9$$

$\Rightarrow Ax = b$,

where $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$ (coefficient matrix)

$x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ (unknown matrix)

and $b = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$ (right hand side column vector)

Let A be a matrix with order $m \times n$ and B be a matrix with order $p \times q$. Then multiplication of A with B is possible if $n = p$ and order of AB is $m \times q$.

Ex: $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$, $B = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}_{1\times 3}$

$AB = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$ (matrix product which is column times row)

$BA = [4 + 10 + 18] = [32]$ (inner product which is row times column)

Ex: $A = \begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix}$, $x = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

Ax by rows: $\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 1\times 2 + 1\times 5 + 6\times 0 \\ 3\times 2 + 0\times 5 + 1\times 0 \\ 1\times 2 + 1\times 5 + 4\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}$$

Ax by columns: $\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

$$= 2\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}$$

Points to remember:

1. Matrix multiplication is associative:

$$(AB)C = A(BC)$$

2. Matrix multiplication is distributive:

$$A(B+C) = AB + AC \text{ and } (B+C)D = BD + CD.$$

3. Matrix multiplication is not commutative:

Usually $AB \neq BA$

Elimination Matrices:

Ex: Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$ $\quad R_2 \leftarrow R_2 - 2R_1 \ (2)$
$\quad R_3 \leftarrow R_3 + R_1 \ (-1)$

$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad R_3 \leftarrow R_3 + R_2 \ (-1)$

$= U$

The elimination matrices are

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$$E_{32} E_{31} E_{21} A = U$$

In this example 2, -1 and -1 are the multipliers of the 2nd row 1st column, 3rd row 1st column and 3rd row 2nd column places respectively. We are writing the opposite value of the multipliers in the respective places of a 3rd order identity matrix to get the elimination matrices.

Note: If a row operation is $R_i \leftarrow R_i - l R_j$, then l is the multiplier for the i-th row and j-th column place.

Problem Set 1.4

No. 1 $[A]_{2\times 2}$, where $a_{ij} = i+j$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

$[B]_{2\times 2}$, where $b_{ij} = (-1)^{i+j}$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad BA = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

No. 4 (a) Diagonal matrix:

A square matrix is said to be a diagonal matrix if all the off diagonal elements are zero i.e. $a_{ij} = 0$ for $i \neq j$.

Ex: $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(b) Symmetric matrix:

A square matrix A is said to be a symmetric matrix if $A^T = A$ i.e. $a_{ij} = a_{ji}$ for all i and j.

Ex: $A = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

(c) Upper triangular matrix:

A square matrix is said to be an upper triangular matrix if all its lower diagonal elements are zero i.e. $a_{ij} = 0$ if $i > j$.

Ex: $A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

(d) Skew-symmetric matrix:

A square matrix 'A' is said to be a skew-symmetric matrix if $A^T = -A$ i.e. $a_{ij} = -a_{ji}$ for all i and j.

Ex: $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$

No. 5. (i)

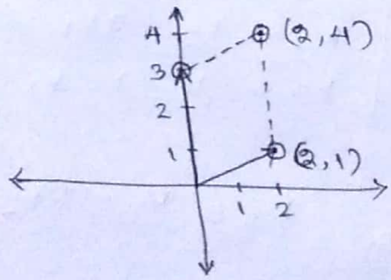
$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 4\times3 + 0\times4 + 1\times5 \\ 0\times3 + 1\times4 + 0\times5 \\ 4\times3 + 0\times4 + 1\times5 \end{bmatrix} = \begin{bmatrix} 17 \\ 4 \\ 17 \end{bmatrix}$$

(ii)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\times5 + 0\times(-2) + 0\times3 \\ 0\times5 + 1\times(-2) + 0\times3 \\ 0\times5 + 0\times(-2) + 1\times3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

(iii)

$$\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\times1 + 0\times1 \\ 1\times1 + 3\times1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



No. 19 (i) $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = A$$

$$A^3 = A$$

$$\vdots$$

$$A^k = A$$

(ii) $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & (-1)^2 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 1 & 0 \\ 0 & (-1)^3 \end{bmatrix}$$

$\vdots$

$$B^k = \begin{bmatrix} 1 & 0 \\ 0 & (-1)^k \end{bmatrix}$$

(iii) $C = AB = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$$C^2 = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C^3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\vdots$

$$C^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

<u>No. 27</u>

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 4R_1 \ (4) \\ R_3 \leftarrow R_3 + 2R_1 \ (-2) \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \quad R_3 \leftarrow R_3 - 2R_2 \ (2)$$

$= U$

The multipliers are $4, -2$ and 2.

The elimination matrices are

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

$E_{32}E_{31}E_{21}A = U$.

$\Rightarrow MA = U$,

where $M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$.

No. 21. Given : $A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$A(\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \quad A(\theta_2) = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$A(\theta_1)A(\theta_2) = \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\cdot\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$= A(\theta_1+\theta_2)$$

$$A(-\theta) = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A(\theta)A(-\theta) = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta.\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

No. 28.

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad R_2 \leftarrow R_2 + \tfrac{1}{2}R_1 \quad (-\tfrac{1}{2})$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad R_3 \leftarrow R_3 + \tfrac{2}{3}R_2 \quad (-\tfrac{2}{3})$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/3 \end{bmatrix} \quad R_4 \leftarrow R_4 + \frac{3}{4}R_3 \; \left(-\frac{3}{4}\right)$$

The multipliers are $-\frac{1}{2}$, $-\frac{2}{3}$ and $-\frac{3}{4}$.

The required elimination matrices are

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{bmatrix}$$