Problem Set - 6.1

2) Given, $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, where $a>0$, $ac>b^2$.

$|A-\lambda I| = 0$

$\Rightarrow \begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = 0$

$\Rightarrow (a-\lambda)(c-\lambda) - b^2 = 0$

$\Rightarrow \lambda^2 - (a+c)\lambda + (ac-b^2) = 0$

$\Rightarrow \lambda = \dfrac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2}$

$\Rightarrow \lambda = \dfrac{(a+c) \pm \sqrt{a^2+2ac+c^2-4ac+4b^2}}{2}$

$\Rightarrow \lambda = \dfrac{(a+c) \pm \sqrt{(a-c)^2+4b^2}}{2}$

$a>0$,
$b^2>0$
$\therefore ac>0 \Rightarrow c>0$.

$\therefore \lambda_1 = \dfrac{(a+c) + \sqrt{(a-c)^2+4b^2}}{2}$

$\lambda_2 = \dfrac{(a+c) - \sqrt{(a-c)^2+4b^2}}{2}$

$\lambda_1 > 0$ because sum of positive numbers.

$\lambda_2 > 0$ because $\lambda_1\lambda_2 = ac-b^2 > 0$ & $\lambda_1 > 0$.

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5) $f = x^2+4xy+2y^2$ —— (1)

(0,0) is the stationary point.

Comparing (1) with $f = ax^2+2bxy+cy^2$, we got,

$a=1$, $2b=4 \Rightarrow b=2$; $c=2$

$\therefore ac-b^2 = 1.2 - 2^2 = 2-4 = -2 < 0$

So, f has a saddle point at the origin. (Proved)

$f = x^2+4xy+2y^2$

$\Rightarrow f = x^2+4xy+4y^2-2y^2$

$\Rightarrow f = (x+2y)^2 - 2y^2$ (Ans:)

8)

a) $A = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$

A is symmetric.

Here, ~~$ac-b^2=$~~

$a=1,\ b=3,\ c=5.$

$\therefore\ ac-b^2 = 5-3^2 = 5-9 = -4 < 0$

$\therefore$ A is not positive definite.

$f = x^T A x$

$\Rightarrow f = [x\ y]\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow f = [x\ y]\begin{bmatrix} x+3y \\ 3x+5y \end{bmatrix}$

$\Rightarrow f = x^2+3xy+3xy+5y^2$

$\Rightarrow f = x^2+6xy+5y^2$ (Ans)

b) $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

A is symmetric.

Here.

$a=1,\ b=-1,\ c=1$

$\therefore\ ac-b^2 = 1-(-1)^2$

$= 0$

$\therefore$ A is not +'ve definite.

$f = x^T A x$

$\Rightarrow f = x^2-2xy+y^2$ (Ans)

$f(x,y)=0$

$\Rightarrow (x-y)^2 = 0$

$\Rightarrow x-y=0$

$\Rightarrow \boxed{x=y}$ → This is the required line.

c) $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

A is symmetric,

Here, $a=2,\ b=3,\ c=5.$

$\therefore\ ac-b^2 = 2.5-3^2 = 10-9 = 1 > 0$

and $a=2>0$

$\therefore$ A is '+'ve definite.

$f = x^T A x$

$\Rightarrow f = 2x^2+6xy+5y^2$ (Ans)

d) $A = \begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$

A is symmetric.

Here,

$a=-1,\ b=2,\ c=-8$

$\therefore\ ac-b^2 = 8-2^2 = 4 > 0$

but $a=-1<0$

$\therefore$ A is not +'ve definite.

$f = x^T A x$

$\Rightarrow f = -x^2+4xy-8y^2$ (Ans)

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9) a) Given,

$F = -1+4(e^x-x)-5x\sin y+6y^2$

$F_x = 4(e^x-1)-5\sin y$

$F_{xx} = 4e^x$

$\therefore\ F_{xx}\big/_{(0,0)} = 4$

$F_y = -5x\cos y+12y$

$F_{yy} = 5x\sin y+12$

$\therefore\ F_{yy}\big/_{(0,0)} = 12$

$\therefore\ F_{xy} = F_{yx} = -5\cos y$

$F_{xy}\big/_{(0,0)} = -5$

$F_{xx}.F_{yy}-(F_{xy})^2\big/_{(0,0)} = 4.12-(-5)^2 = 48-25 = 23 > 0$

& $F_{xx}\big/_{(0,0)} = 4 > 0$

So, F has a minima at the point $x=0, y=0$.

b) Given,

$$F = (x^2-2x)\cos y$$

$F_x = (2x-2)\cos y$

$F_{xx} = 2\cos y$

$\therefore F_{xx}/_{(1,\pi)} = -2$

$F_y = (2x-x^2)\sin y$

$F_{yy} = (2x-x^2)\cos y$

$\therefore F_{yy}/_{(1,\pi)} = 1.\cos\pi = -1$

$$F_{xy} = F_{yx} = (2-2x)\sin y$$

$$\therefore F_{xy}/_{(1,\pi)} = 0$$

$\therefore F_{xx}.F_{yy} - (F_{xy})^2/_{(1,\pi)} = (-2).(-1) - 0^2 = 2 > 0$

but, $F_{xx}/_{(1,\pi)} = -2 < 0$

So, F has a maxima at the point $x=1;\ y=\pi$

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17) $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$\therefore A^TA = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$

We know that A^TA is symmetric for any A.

Here, 1st order principal submatrix: $A_1 = [1]$

$\therefore |A_1| = 1 > 0$

2nd " " " : $A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$

$\therefore |A_2| = 13-4 = 9 > 0$

$\therefore A^TA$ is positive definite.

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$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\therefore A^T A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} \longrightarrow \text{symmetric}$$

Here, 1^{st} order principal submatrix: $A_1 = [6]$

$\therefore |A_1| = 6 > 0$

2^{nd} " " " : $A_2 = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$

$\therefore |A_2| = 36 - 25$
$= 11 > 0$

$\therefore A^T A$ is '+'ve definite.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\therefore A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

Here, 1^{st} order principal submatrix: $A_1 = [2]$

$\therefore |A_1| = 2 > 0$

2^{nd} " " " : $A_2 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

$\therefore |A_2| = 10 - 9 = 1 > 0$

3^{rd} " " " : $A_3 = A^T A$

$\therefore |A_3| = 2(25-16) - 3(15-12) + 3(12-15)$

$= 18 - 9 - 9$

$= 0$

$\therefore A^T A$ is not '+'ve definite.