Column Space and Null Space of a Matrix

Column Space of a matrix is all linear combination of the columns of A. and denoted by C(A).
Column Space of Matrix Let $C_1, C_2, ..., C_n$ be 1st column, 2nd column,....,nth column of the matrix $A_{m \times n}$
then $C(A) = \{a_1C_1 + a_2C_2 + ... + a_nC_n / a_1, a_2, ...a_n \in R\}$
where R is set of real numbers.

Steps for finding C($A_{m \times n}$)
Given: Suppose we are given a matrix A
Output: C(A)
Step 1: find Echelon form of A ,say U is echelon form of A
step 2: find the pivot column in U
step 3: then C(A) is linear combination of those column of A which are corresponding to pivot column of U.
For Ex Let 1st and 5th are only pivot column in U,then $C(A) = \{a_1C_1 + a_5C_5/a_1, a_5 \in R\}$

Note : The system $Ax = b$ is solvable iff the vector b can be expressed as a combination of the columns of A . then b is in the column cpace of A
Note : C(A) is a subspace of R^m

Null Space of Matrix
let A be $m \times n$ matrices.then
Null Space of A consists of all vectors x such that Ax=0
and denoted by N(A).
i.e. $N(A) = \{x \in R^n / Ax = 0\}$
Note : N(A) is subspace of R^n

Exercise 2.1.5 : find the column space and null space of the matrices
(a):

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Sol: Since echelon form of A is itself A. i.e. $U = A$
and first column of U is pivot. so

$$C(A) = \left\{ a \begin{bmatrix} 1 \\ 0 \end{bmatrix} / a \in R \right\}$$

for the null space solve $Ax = 0$

$Aug.matrix = \begin{bmatrix} A & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

since since x_2 is free variable , so assume $x_2 = k$ where k is real number.so $x_1 - k = 0$, $x_1 k$

$N(A) = \left\{ \begin{bmatrix} k \\ k \end{bmatrix} / k \in R \right\} = \left\{ k \begin{bmatrix} 1 \\ 1 \end{bmatrix} / k \in R \right\}$

(b)

$$B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} = U$$

Since first and third columns are pivot in U.

So $C(B) = \left\{ a \begin{bmatrix} 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ 3 \end{bmatrix} \mid a, b \in R \right\}$

for null space solve $BX = 0$

Aug. matrix= $[B \mid 0] = \left[\begin{array}{ccc|c} 0 & 0 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$ Since x_2 is free variable so $x_2 = k, K \in R$

$$0x_1 + 0x_2 + 3x_3 = 0 \Rightarrow x_3 = 0$$
$$x_1 + 2k + 3x_3 = 0$$
$$x_1 + 2k + 3 \times 0 = 0$$
$$x_1 = -2k$$

$N(A) = \left\{ \begin{bmatrix} -2k \\ k \\ 0 \end{bmatrix} \mid k \in R \right\} = \left\{ k \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \mid k \in R \right\}$

(c)

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$C(A) = \left\{ a \begin{bmatrix} 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mid a, b, c \in R \right\}$$
$$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

Since x_1, x_2, x_3 are free variable.

So $x_1 = k_1,\ x_2 = k_2,\ x_3 = k_3,\ k_1, k_2, k_3 \in R.$

So $N(A) = \left\{ \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \mid k_1, k_2, k_3 \in R \right\} = R^3$

Exercise 2.1.24: For which Right hand side(find a condition on b_1, b_2, b_3) are these systems solvable?

(a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Aug. matrix $= [A|b] = \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_1 + R_3]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_1 + b_3 \end{array}\right]$

i.e

$$1x_1 + 4x_2 + 2x_3 = b_1$$

$$0x_1 + 0x_2 + 0x_3 = b_2 - 2b_1$$

$$0x_1 + 0x_2 + 0x_3 = b_1 + b_3$$

solution exist only if $b_2 - 2b_1 = 0$, $b_1 + b_3 = 0$

$\Rightarrow b_2 = 2b_1$ and $b_3 = -b_1$

(b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Aug. matrix $= [A|b] = \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_1 + b_3 \end{array}\right]$

i.e

$$1x_1 + 4x_2 = b_1$$

$$0x_1 + 1x_2 = b_2 - 2b_1$$

$$0x_1 + 0x_2 = b_1 + b_3$$

solution exist only if $b_3 + b_1 = 0$