Orthogonal Complement:

Two subspaces V and W of the space $\mathbb{R}^n$ are said to be orthogonal complement of each other if $V \perp W$ and $\dim V + \dim W = n$.

Orthogonal complement of V is denoted by $V^{\perp}$.

Ex: 1. x-axis is the orthogonal complement of y-axis in $\mathbb{R}^2$.

2. $y=x$ line is orthogonal complement of $y=-x$ line in $\mathbb{R}^2$.

3. x-axis is the orthogonal complement of yz-plane in $\mathbb{R}^3$.

Fundamental Theorem of Linear Algebra, Part-II:

The nullspace is the orthogonal complement of row space in $\mathbb{R}^n$ and the left nullspace is the orthogonal complement of column space in $\mathbb{R}^m$.

In symbol,

$$N(A) = (C(A^T))^{\perp} \text{ in } \mathbb{R}^n$$
$$N(A^T) = (C(A))^{\perp} \text{ in } \mathbb{R}^m$$

Problem Set 3.1

No. 1. $v_1 = (1,2,-2,1)$, $v_2 = (4,0,4,0)$, $v_3 = (1,-1,-1,-1)$,

$v_4 = (1,1,1,1)$.

$v_1 v_3$, $v_2 v_3$ are orthogonal pairs, since

$v_1^T v_3 = 0$ and $v_2^T v_3 = 0$.

No. 2.
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$$

(i) $x = (-2,1,0)$ is a vector orthogonal to the row space of A.

(ii) $y = (-1,-1,1)$ is a vector orthogonal to the column space of A.

(iii) $z = (1,2,1)$ is a vector orthogonal to the nullspace of A.

$$N(A) \perp C(A^T) \qquad z \in C(A^T).$$

So, $z \perp x$.

No. 7. $x = (1,4,0,2)$, $y = (2,-2,1,3)$

$\|x\| = \sqrt{1^2+4^2+0^2+2^2} = \sqrt{21}$, $\|y\| = \sqrt{2^2+(-2)^2+1^2+3^2}$
$= \sqrt{18} = 3\sqrt{2}$.

$x^T y = 2-8+0+6 = 0$.

$\Rightarrow x \perp y$.

No. 9.
$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$$

row space $= C(A^T)$

$(C(A^T))^\perp = N(A)$

Basis for the orthogonal complement of the row space of A is same as basis for nullspace $N(A)$.

$$Ax = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow u\begin{bmatrix} 1 \\ 1 \end{bmatrix} + v\begin{bmatrix} 0 \\ 1 \end{bmatrix} + w\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow u = -2,\ v = -2,\ w = 1$

Basis of $N(A) = \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$

i.e. Basis of $(C(A^T))^{\perp} = \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$

$$x = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = x_r + x_n$$

$$= x_r + \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

$$\Rightarrow x_r = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$$

No. 10. Plane (P): $x + 2y - z = 0$

A vector perpendicular to P is $(1, 2, -1)$.

$$A = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$$

$$N(A) = P$$

$$x + 2y - z = 0$$

$$\Rightarrow z = x + 2y$$

$$x = 1, y = 1 \Rightarrow z = 3$$

$$x = 0, y = 1 \Rightarrow z = 2$$

$$B = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$C(B^T) = P.$$

No. 11. Given: vectors $(1, 4, 4, 1)$, $(2, 9, 8, 2)$

Let $A = \begin{bmatrix} 1 & 4 & 4 & 1 \\ 2 & 9 & 8 & 2 \end{bmatrix}$

Row space $C(A^T) \perp$ nullspace $N(A)$

$$Ax = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 4 & 1 \\ 2 & 9 & 8 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 8 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \cdots\cdots \right\}$$

The vectors of the nullspace $N(A)$ are perpendicular to the two given vectors.

No. 12. Show that $x-y$ is orthogonal to $(x+y)$ if and only if $\|x\| = \|y\|$.

Proof: Let $(x-y)$ is orthogonal to $(x+y)$.

$$\Leftrightarrow (x-y) \perp (x+y)$$

$$\Leftrightarrow (x-y)^T (x+y) = 0.$$

$$\Leftrightarrow (x^T - y^T)(x+y) = 0$$

$$\Leftrightarrow x^T x + x^T y - y^T x - y^T y = 0$$

$$\Leftrightarrow \|x\|^2 - \|y\|^2 = 0 \quad (\because x^T y = y^T x)$$

$$\Leftrightarrow \|x\| = \|y\|$$

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