2.4: The Four Fundamental Subspaces

Course Outcomes: Students will have understanding about four fundamental subspaces of matrices and one sided inverse of rectangular matrices.

The four fundamental subspaces of a matrix are

1. The column space $C(A)$
2. The nullspace $N(A)$
3. The row space $C(A^T)$
4. The left nullspace $N(A^T)$

The Row Space: The row space of a matrix A of order $m \times n$ contains all the linear combinations of the rows of A or columns of A^T. It is denoted by $C(A^T)$. It is a subspace of $\mathbb{R}^n$.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$A^T y = b$

$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b$

$\Rightarrow y_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = b$

$y_1 = 0, y_2 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y_1 = 1, y_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$y_1 = 0, y_2 = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$y_1 = 1, y_2 = 1 \Rightarrow b = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$\vdots$

$C(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \dots \right\}$

$= \mathbb{R}^2.$

<u>Ex</u>: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$A^T y = b$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b$$

$$\Rightarrow y_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = b$$

$y_1 = 0, \; y_2 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y_1 = 1, \; y_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$y_1 = 0, \; y_2 = 1 \Rightarrow b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$y_1 = 1, \; y_2 = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$\vdots$

$$C(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \ldots \right\}$$

i.e. $y = 2x$ line.

<u>Ex</u>: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$A^T y = b$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b$$

$$\Rightarrow y_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = b$$

$$\Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$C(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ i.e. origin of $\mathbb{R}^2$.

<u>The Left Nullspace</u>: The left nullspace of a matrix A of order $m \times n$ contains all the vectors y such that $A^T y = 0$. It is denoted by $N(A^T)$. It is a subspace of $\mathbb{R}^m$.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$A^T y = 0$

$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow y_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow y_1 = 0, \; y_2 = 0$

$N(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ i.e. origin of $\mathbb{R}^2$.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$A^T y = 0$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow y_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow y_1 = 0, \; y_2 = 0$

$y_1 = 2, \; y_2 = -1$

$y_1 = 4, \; y_2 = -2$

$y_1 = 6, \; y_2 = -3$

$\vdots$

$N(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \end{bmatrix}, \ldots \right\}$,

which is the line $y = -\frac{x}{2}$ in $\mathbb{R}^2$.

Ex: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A^T y = 0$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow y_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ Any value of y_1 and y_2 can satisfy.

$\Rightarrow N(A^T) = \mathbb{R}^2$

Points to remember:

1. If A is a nonsingular matrix order n, then $C(A^T) = \mathbb{R}^n$ and $N(A^T) =$ origin of $\mathbb{R}^n$.

2. If A is a zero matrix of order n, then $C(A^T) =$ origin of $\mathbb{R}^n$ and $N(A^T) = \mathbb{R}^n$.

3. If A is a nonzero singular matrix of order 2, then its row space as well as left nullspace is a line passing through origin in $\mathbb{R}^2$.

4. If A is a nonzero singular matrix of order 3, then its row space as well as left nullspace is either a line passing through origin or a plane passing through origin in $\mathbb{R}^3$.

Fundamental Theorem of Linear Algebra, Part-I:

Let A be a matrix of order $m \times n$ with rank r.

Then
$$\dim . C(A) = r$$
$$\dim . C(A^T) = r$$
$$\dim . N(A) = n - r$$
$$\dim . N(A^T) = m - r .$$

Ex: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}_{2\times 2}$.

$$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad R_2 \leftarrow R_2 - 3R_1 \text{, echelon form}$$

Rank of $A = r = 1$.

$\dim . C(A) = r = 1$, $\dim . N(A) = n - r = 2 - 1 = 1$

$\dim C(A^T) = r = 1$, $\dim N(A^T) = m - r = 2 - 1 = 1$.

The column space:

$$Ax = b$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$$

$$\Rightarrow x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 6 \end{bmatrix} = b$$

$x_1 = 0,\ x_1 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = 1,\ x_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$x_1 = 0,\ x_2 = 1 \Rightarrow b = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$x_1 = 1,\ x_2 = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

$\vdots$

$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \ldots \right\}$,

which is the line $y = 3x$ in $\mathbb{R}^2$.

Basis of $C(A) = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$.

The row space:

$$A^T y = b$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b$$

$$\Rightarrow y_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = b$$

$y_1 = 0,\ y_2 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y_1 = 1,\ y_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$y_1 = 0,\ y_2 = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$y_1 = 1,\ y_2 = 1 \Rightarrow b = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$\vdots$

$C(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \ldots \right\}$,

which is the line $y = 2x$ in $\mathbb{R}^2$.

Basis of $C(A^T) = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

The nullspace:

$Ax = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = 0,\ x_2 = 0$

$x_1 = 2,\ x_2 = -1$

$x_1 = 4,\ x_2 = -2$

$x_1 = 6,\ x_2 = -3$

$\vdots$

$$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \end{bmatrix}, \ldots \right\},$$

which is the line $y = -\frac{x}{2}$ in $\mathbb{R}^2$.

Basis of $N(A) = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$.

The left nullspace:

$A^T y = 0$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y_2\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow y_1 = 0,\ y_2 = 0$

$y_1 = 3,\ y_2 = -1$

$y_1 = 6,\ y_2 = -2$

$y_1 = 9,\ y_2 = -3$

$\vdots$

$$N(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 9 \\ -3 \end{bmatrix}, \ldots \right\},$$

which is the line $y = -\frac{x}{3}$ in $\mathbb{R}^2$.

Basis of $N(A^T) = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$.

Note: 1. The nullspace is called the kernel of A and its dimension $n-r$ is the nullity.

2. The no. of independent columns = the no. of independent rows.