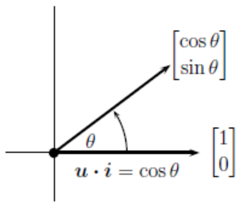
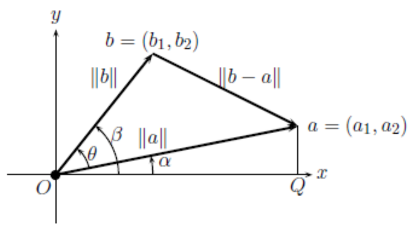
3.2 Cosines and Projections Onto Lines

Course outcome: The course outcome of this article is to know about the Cosine angle between two lines and also the projection matrix.

Inner Product and Cosines

The cosine of the angle is directly related to inner product. For this consider the triangle in two dimensional case. Suppose the vectors a and b make angles α and β with $X-$axis as in fig. The length $\| a \|$ is the hypotenuse in the triangle OaQ. So, the sine and cosine of α are $\sin\alpha = \dfrac{a_2}{\| a \|}, \quad \cos\alpha = \dfrac{a_1}{\| a \|}$.





The cosine of the angle $\theta = \beta - \alpha$ using inner products. For angle β, the sine is $\dfrac{b_2}{\| b \|}$ and the cosine is $\dfrac{b_1}{\| b \|}$.

The cosine formula

$$\begin{aligned}
\cos\theta &= \cos(\beta - \alpha)\\
&= \cos\beta\cos\alpha + \sin\beta\sin\alpha\\
&= \frac{a_1b_1 + a_2b_2}{\| a \|\| b \|}\\
&= \frac{a^Tb}{\| a \|\| b \|}\\
&= \frac{\langle a, b\rangle}{\| a \|\| b \|}
\end{aligned}$$

Law of Cosines: $\| b - a \|^2 = \| b \|^2 + \| a \|^2 - 2 \| b \|\| a \| \cos\theta$. When θ is a right angle, we have $\| b - a \|^2 = \| b \|^2 + \| a \|^2$, which is Pythagoras Theorem.

$$
\begin{aligned}
\| b-a \|^2 &= (b-a)^T(b-a) \\
&= (b^T-a^T)(b-a) \\
&= b^Tb - ab^T - a^Tb + a^Ta \\
&= b^Tb - a^Tb - a^Tb + a^Ta \\
&= \| b \|^2 - 2a^Tb + \| b \|^2 \\
&= \| b \|^2 - 2 \| a \| \| b \| \cos\theta + \| b \|^2 .
\end{aligned}
$$

When θ is a right angle, $\cos\theta = 0$. Thus $\| b-a \|^2 = \| a \|^2 + \| b \|^2$.

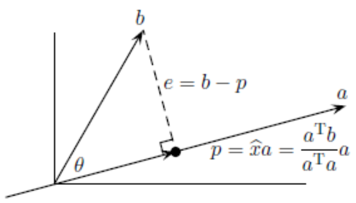
Projection Onto a Line

Suppose that we want to find the distance from a point b to the line in the direction of the vector a. We are looking also that instead of a line for the point p closest to b. The line connecting b to p is perpendicular to a. The situation is the same when we are given a plane or any subspace S instead of a line. Again, the problem is to find the point P on the subspace that is closed to b onto the subspace. Every point on the line is a multiple of a. So,

$$p = \widehat{x}a, \text{ where } \widehat{x} = \frac{a^Tb}{a^Ta}.$$

Hence, the projection of the vector b onto the line in the direction of a is

$$p = \widehat{x}a = \frac{a^Tb}{a^Ta}a = \frac{a^Tb}{\| a \|^2}a.$$



Example 1 Project $b = (1,2,3)$ onto the line through $a = (1,1,1)$ to get $\widehat{x}$ and p.
Solution: $\widehat{x} = \dfrac{a^Tb}{a^Ta} = \dfrac{1+1\times 2+1\times 3}{(\sqrt{1^2+1^2+1^2})^2} = \dfrac{6}{3} = 2$. The projection is $p = \widehat{x}a = 2(1,1,1) = (2,2,2)$. The angle between a and b is $\cos\theta = \dfrac{a^Tb}{\| a \| \| b \|} = \dfrac{6}{\sqrt{3}\sqrt{14}}$.

Schwarz Inequality:

$$| ab | \leq \| a \| \| b \| .$$

Exercise-3.2

1. (a) Given any two positive numbers x and y, choose the vector b equal to $(\sqrt{x}\ ,\sqrt{y})$, and choose $a = \left(\sqrt{y}\ \ ,\sqrt{x}\right))$. Apply the Schwarz inequality to compare the arithmetic mean $\frac{1}{2}(x+y)$ with the geometric mean $\sqrt{xy}$.

Solution: If $a = \begin{pmatrix} \sqrt{y} \\ \sqrt{x} \end{pmatrix}$, $b = \begin{pmatrix} \sqrt{x} \\ \sqrt{y} \end{pmatrix}$. Applying Schwarz inequality, we get

$$
\begin{aligned}
& a^T b \leq \| a \| \| b \| \\
& \Rightarrow \begin{pmatrix} \sqrt{y} & \sqrt{x} \end{pmatrix} \begin{pmatrix} \sqrt{x} \\ \sqrt{y} \end{pmatrix} \leq \| (\sqrt{y}, \sqrt{x}) \| \| (\sqrt{x}, \sqrt{y}) \| \\
& \Rightarrow \sqrt{xy} + \sqrt{xy} \leq \sqrt{(\sqrt{y})^2 + (\sqrt{x})^2} \sqrt{(\sqrt{x})^2 + (\sqrt{y})^2} \\
& \Rightarrow 2\sqrt{xy} \leq \sqrt{y+x}\sqrt{x+y} \\
& \Rightarrow \sqrt{xy} \leq \frac{x+y}{2} \quad (G.M. \leq A.M.)
\end{aligned}
$$

(b) Using the triangle inequality $\| x+y \| \leq \| x \| + \| y \|$, reduce to Schwarz inequality.

Solution: From the triangle inequality, we know

$$
\begin{aligned}
& \| x+y \| \leq \| x \| + \| y \| \\
& \Rightarrow \| x+y \|^2 \leq (\| x \| + \| y \|)^2 \\
& \Rightarrow (x+y)^T(x+y) \leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 \\
& \Rightarrow (x^T + y^T)(x+y) \leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 \\
& \Rightarrow x^T x + x^T y + y^T x + y^T y \leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 \\
& \Rightarrow \| x \|^2 + 2x^T y + \| y \|^2 \leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 \\
& \Rightarrow x^T y \leq \| x \| \| y \| \quad \text{(Schwarz inequality).}
\end{aligned}
$$

3. By using the correct b in the Schwarz Inequality, prove that

$$(a_1 + a_2 + \cdots + a_n)^2 \leq n(a_1^2 + a_2^2 + \cdots a_n^2).$$

When does equality hold?

Solution: Let $b = (1, 1, \cdots, 1)$ and $a = (a_1, a_2, \cdots, a_n)$. Using Schwarz inequality, we have

$$
\begin{aligned}
& a^T b \leq \| a \| \| b \| \\
& \Rightarrow \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \sqrt{1^2 + 1^2 + \cdots + 1^2} \\
& \Rightarrow a_1 + a_2 + \cdots a_n \leq \sqrt{n} \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \\
& \Rightarrow (a_1 + a_2 + \cdots a_n)^2 \leq n(a_1^2 + a_2^2 + \cdots + a_n^2).
\end{aligned}
$$

The inequality becomes equality, if $a_i = a_j$ for $i = j = 1, 2, \ldots, n$.

Assignments

Exercise-3.2, Q. 5,9.