1.6 Inverses and Transposes

Course Outcome: Students will have understanding about existence of inverse, Gauss-Jordan method to find the inverse of square matrices and transpose of matrices.

Existence of Inverse:

Inverse of a square matrix 'A' exist if it is nonsingular i.e. $|A| \neq 0$. It is denoted by A^{-1}.
If a square matrix has full set of pivots, then it is also nonsingular. So, inverse of a square matrix exist if it has full set of pivots.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$|A| = 4 - 6 = -2 \neq 0$

$\Rightarrow A^{-1}$ will exist.

Minor of $A = M = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

Cofactor of $A = C = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

Adjoint of A (Adj. A) $= C^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$.

$$A^{-1} = \frac{\text{Adj. } A}{|A|} = \frac{C^T}{|A|} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}.$$

Ex: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A| = ad - bc \neq 0$

$\Rightarrow A^{-1}$ will exist.

Minor of $A = M = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$

Cofactor of $A = C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Adj. $A = C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A^{-1} = \frac{\text{Adj. } A}{|A|} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

Ex: $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$|A| = 2(4-1) + 1(-2) = 6 - 2 = 4 \neq 0$

$\Rightarrow A^{-1}$ will exist.

Minor of $A = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}$

Cofactor of $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Adj. A = Transpose of cofactor $= \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$A^{-1} = \dfrac{\text{Adj. } A}{|A|} = \dfrac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Ex: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$

Points to remember:

1. The inverse of a square matrix, if exist, is unique and is known as both sided inverse.
2. $AA^{-1} = I = A^{-1}A$.
3. $(AB)^{-1} = B^{-1}A^{-1}$, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.
4. $Ax = b \Rightarrow x = A^{-1}b$.

Gauss-Jordan Method: (The calculation of A^{-1})

Ex: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$|A| = 4 - 6 = -2 \neq 0$

$\Rightarrow A^{-1}$ will exist.

$[A \mid I]$

$$= \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 - 3R_1$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \quad R_1 \leftarrow R_1 + R_2$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right] \quad R_2 \leftarrow -\frac{1}{2} R_2$$

$= [I \mid A^{-1}]$,

where $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Ex: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$|A| = 1(6-4) - 1(3-2) + 1(2-2) = 2 - 1 = 1 \neq 0$

$\Rightarrow A^{-1}$ will exist.

$[A \mid I]$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad R_3 \leftarrow R_3 - R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow R_1 - R_3 \\ R_2 \leftarrow R_2 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad R_1 \leftarrow R_1 - R_2$$

$= [I \mid A^{-1}]$,

where $A^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

Transpose :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

Symmetric matrix: A square matrix 'A' is said to be symmetric if $A^T = A$.

Skew-symmetric matrix: A square matrix 'A' is said to be skew-symmetric if $A^T = -A$.

If A is any real square matrix, then $A + A^T$ is always symmetric and $A - A^T$ is always skew-symmetric.

Let $B = A + A^T$. Then

$$B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = B$$

$\Rightarrow B$ is symmetric.

Let $C = A - A^T$. Then

$$C^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -C$$

$\Rightarrow C$ is skew-symmetric.

$$A = \underbrace{\frac{A + A^T}{2}}_{\text{symmetric}} + \underbrace{\frac{A - A^T}{2}}_{\text{skew-symmetric}}$$

$\Rightarrow$ Any real square matrix can be expressed as a sum of symmetric and skew-symmetric matrix.

Ex: Express the matrix $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ as sum of symmetric and skew-symmetric matrix.

Soln: $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $A^T=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$.

$$A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$$

$$\Rightarrow \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 5/2\\ 5/2 & 4\end{bmatrix}}_{\text{symmetric}} + \underbrace{\begin{bmatrix}0 & -1/2\\ 1/2 & 0\end{bmatrix}}_{\text{skew-symmetric}}$$

Ex: $A=\begin{bmatrix}1 & 3 & 5\\ 3 & 12 & 18\\ 5 & 18 & 30\end{bmatrix}$

$$= \begin{bmatrix}1 & 3 & 5\\ 0 & 3 & 3\\ 0 & 3 & 5\end{bmatrix} \quad \begin{matrix}R_2 \leftarrow R_2 - 3R_1 \ (3)\\ R_3 \leftarrow R_3 - 5R_1 \ (5)\end{matrix}$$

$$= \begin{bmatrix}1 & 3 & 5\\ 0 & 3 & 3\\ 0 & 0 & 2\end{bmatrix} \quad R_3 \leftarrow R_3 - R_2 \ (1)$$

$= U$

LU-factorization:

$$L=\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ 5 & 1 & 1\end{bmatrix}, \quad U=\begin{bmatrix}1 & 3 & 5\\ 0 & 3 & 3\\ 0 & 0 & 2\end{bmatrix}$$

$LU = A$

LDU-factorization:

$$L=\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ 5 & 1 & 1\end{bmatrix}, \quad D=\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{bmatrix}, \quad U=\begin{bmatrix}1 & 3 & 5\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$$

$LDU = A$

Here $U = L^T$. So, $LDL^T = A$.

In this example since $A^T = A$, so $LDU = A$ is $LDL^T = A$.

Note: If $A = A^T$ can be factored into $A = LDU$ without row exchanges, then U is the transpose of L and $A = LDL^T$.

Points to rember:

1. $(A+B)^T = A^T + B^T$
2. $(AB)^T = B^T A^T$, $(ABC)^T = C^T B^T A^T$.
3. $(A^T)^T = A$
4. $(A^{-1})^T = (A^T)^{-1}$

Problem Set 1.6

No. 6. (a) If A is invertible and $AB = AC$, prove that $B = C$.

Proof: Let A be invertible and $AB = AC$.

$$AB = AC$$

$$\Rightarrow A^{-1}(AB) = A^{-1}(AC)$$

$$\Rightarrow (A^{-1}A)B = (A^{-1}A)C \quad \text{(by associative law)}$$

$$\Rightarrow IB = IC$$

$$\Rightarrow B = C.$$

(b) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, find an example with $AB = AC$ but $B \neq C$.

Sol^n: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}.$$

$AB = AC$ but $B \neq C$.

No. 10. $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$[A_1 | I]$

$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 - R_1$

$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 - R_3$

$= [I | A^{-1}]$

$\therefore A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$[A_3 | I]$

$= \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_3$

$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow R_1 - R_3 \\ R_2 \leftarrow R_2 - R_3 \end{array}$

$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \quad R_1 \leftarrow R_1 - R_2$

$= [I | A^{-1}]$

$\therefore A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$[A_2 | I]$

$= \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$

$= \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 + \frac{1}{2}R_1$

$= \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & 2/3 & 1 \end{array}\right] \quad R_3 \leftarrow R_3 + \frac{2}{3}R_2$

$= \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & 3/2 & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & 2/3 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 + \frac{3}{4}R_3$

$= \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & 3/2 & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & 2/3 & 1 \end{array}\right] \quad R_1 \leftarrow R_1 + \frac{2}{3}R_2$

$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & 1/2 & \frac{3}{4} \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow \frac{1}{2}R_1 \\ R_2 \leftarrow \frac{2}{3}R_2 \\ R_3 \leftarrow \frac{3}{4}R_3 \end{array}$

$= [I | A^{-1}]$

$\therefore A^{-1} = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}$

No. 11. Let B be a square matrix.

Let $A = B + B^T$ and $K = B - B^T$.

$A^T = (B+B^T)^T = B^T + (B^T)^T = B^T + B = A$

$\Rightarrow$ A is symmetric.

Again, $K^T = (B-B^T)^T = B^T - (B^T)^T = B^T - B = -(B-B^T)$
$= -K$

$\Rightarrow$ K is skew-symmetric.

$A = B + B^T = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

$K = B - B^T = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

$B = \dfrac{B+B^T}{2} + \dfrac{B-B^T}{2}$

$\Rightarrow \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{\text{symmetric}} + \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\text{skew-symmetric}}$

No. 15. $A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix}$

$|A| = c(0 - ef) = -cef$

A is singular if $|A| \neq 0$

$\Rightarrow -cef \neq 0$

$\Rightarrow cef \neq 0$.

The required conditions for A to be invertible are $\boxed{a, b, c, d, e, f \in \mathbb{R} \text{ such that } cef \neq 0}$

$B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$, $|B| = a(de - 0) - b(ce) = e(ad - bc)$

B is invertible $\Rightarrow |B| \neq 0$

$\Rightarrow \boxed{e(ad-bc) \neq 0}$, $a, b, c, d, e \in \mathbb{R}$.

No. 17 (a) $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $B=\begin{bmatrix}0&1\\1&0\end{bmatrix}$

$$A+B=\begin{bmatrix}1&1\\1&1\end{bmatrix}$$

$A+B$ is not invertible although A and B are invertible.

(b) $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $B=\begin{bmatrix}0&0\\0&1\end{bmatrix}$

$$A+B=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$A+B$ is invertible although A and B are not invertible.

(c) $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $B=\begin{bmatrix}2&0\\0&2\end{bmatrix}$, $C=\begin{bmatrix}3&0\\0&3\end{bmatrix}$

All of A, B and $A+B$ are invertible.

No. 42. $A=\begin{bmatrix}2&c&c\\c&c&c\\8&7&c\end{bmatrix}$

For $c=0, 2, 7$, the matrix is not invertible, as for these three values of c the determinant of the matrix is zero.

$c=0 \Rightarrow$ zero column (or zero row).

$c=2 \Rightarrow$ identical rows

$c=7 \Rightarrow$ identical columns.