LECTURE-27

5 EIGENVALUES AND EIGENVECTORS

5.1 INTRODECTION

Eigenvalues and Eigenvectors are most important and interesting topic in Linear Algebra. Some basic concept of determinant and Matrices will be use to study the Eigenvalue and Eigenvector problems. One important thing is that, Eigenvalues and Eigenvectors we can find only from the square matrix.

Eigenvalues and Eigenvectors will help to solve many problems in Linear Algebra. However the basic concepts of Eigenvalues and Eigenvectors are useful throughout pure and applied mathematics. Eigenvalues are also used to study differential equations and continuous dynamics systems, they provide critical information in engineering design and they arise naturally in fields such as physics and chemistry.

Let us consider A be square matrix and x be the vector multiply with A as a in goes vector then outcomes vector is Ax (i.e. gradient of A). It is like a function, x is in goes then then $f(x)$ is outcomes function. Since Ax is a vector, so, question will be arise about the direction of Ax. This outcome vector may be goes different direction. But some particular vector of x, Ax will be parallel to x. Then we can write

$$Ax = \lambda x.$$

This special vector of x is called eigenvector of the matrix A and this scalar value of λ is called the eigenvalue of the matrix A.

Example 1. Let

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \qquad u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad and \quad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Are both u and v are eigenvectors of A?

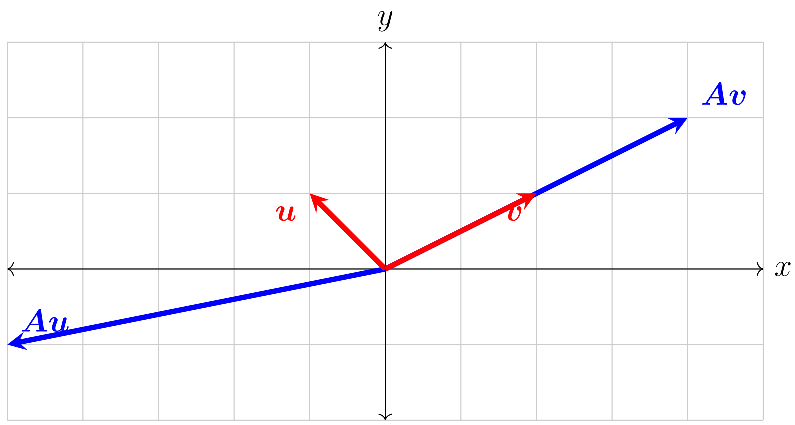
Solution:

$$Au = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} 5 \\ 1 \end{bmatrix} \neq \lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix} \neq \lambda u$$

where $\lambda = -1$. Therefore u is not an eigenvector of A, because Au is not a multiple of u. Again,

$$Av = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = (2) \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda v$$

Where $\lambda = 2$. Thus v is an eigenvector of A corresponding to an eigenvalue $\lambda = 2$, and Av is a multiple of v only.



NOTE: If λ is positive, then Ax and x are in same direction.

Example 2. Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, \qquad u = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \quad \textit{and} \quad v = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

Are both u and v are eigenvectors of A?

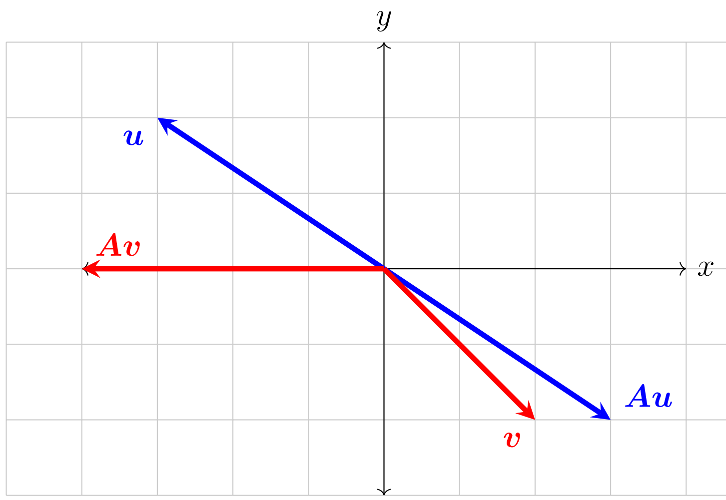
Solution:

$$Au = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} = (-1) \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \lambda u$$

where $\lambda = -1$. Therefore u is an eigenvector corresponding to an eigenvalue of (-1), and Au is a multiple of u only. Again,

$$Av = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ -2 \end{bmatrix} \neq \lambda v$$

Thus v is not an eigenvector of A, because Av is not a multiple of v.



NOTE: If λ is negative, then Ax and x are in opposite direction.

Above two examples, it is clear to see that some special vectors only the eigenvector corresponding to eigenvalue λ of the any matrix A. This scalar value λ may be zero or nonzero(i.e. positive, negative or imaginary). The numbers of eigenvalue (i.e. λ)and eigenvectors are depends on the order of the matrix. If the order of the matrix is $n \times n$ then we will get n eigenvalues and for each corresponding eigenvalues will get n linearly independent eigenvectors. But some times eigenvalue may repeat. In that case all eigenvalue are not linearly independent.

DEFINATION

An **eigenvector** of an $n \times n$ matrix A is anonzero vector x such that $Ax = \lambda x$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution x of $Ax = \lambda x$; such an x is called an eigenvector corresponding to λ.

Possible two cases of λ **of the Special equation** $Ax = \lambda x$

Case-1 The solution of $Ax = 0$,

If $\lambda = 0$, then the special equation becomes $Ax = 0$ and x is in the nullspace of A. Since x is a vector so A will be zero. i.e. A is a singular matrix. i.e. $det(A) = 0$

NOTE: If $\lambda = 0$, then A is a singular matrix and if A is a singular matrix then $\lambda = 0$.

Case-2 The solutions of $Ax = \lambda x$

If $\lambda \neq 0$ then the equation $Ax = \lambda x$ is becomes $(A - \lambda \cdot I)x = 0$. Again same condition, x is a nullspace of $(A - \lambda \cdot I)$. Since x is vector so $(A - \lambda \cdot I)$ will be zero. i.e. $(A - \lambda \cdot I)$ is a singular matrix. i.e. $det(A - \lambda \cdot I) = 0$. This is the key equation to find the eigenvalue, and is called **eigenvalue equation or**

characteristic equation.

EXAMPLES OF EIGENVALUES AND EIGENVECTORS

The first step is to understand how eigenvalues can be useful. One of their application is to ordinary differential equations.

Example 3. Let us consider the coupled pair of equations

$$\begin{aligned} \frac{dv}{dt} &= 4v - 5w, \quad v = 8 \quad t = 0 \\ \frac{dv}{dt} &= 2v - 3w, \quad w = 5 \quad t = 0. \end{aligned}$$

The system of matrix form is

$$\frac{du}{dt} = Au, \quad with \quad u = u(0)att = 0 \tag{1}$$

where

$$u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}, \qquad att = 0, \quad u(0) = \begin{bmatrix} 8) \\ 5 \end{bmatrix}, \qquad A = \begin{bmatrix} 4 & -5) \\ 2 & -3 \end{bmatrix}$$

The pure exponential solution of the above the equation (1) is $\quad u(t) = e^{at}u(0)$.

Now we are going to find the eigenvalue and eigenvector of the equation (1) with the help of coefficient matrix A.

The characteristic equation is $\quad det(A - \lambda I) = 0$.

$$i.e. \quad det(A - \lambda I) = 0$$

$$i.e. \quad \begin{vmatrix} 4 - \lambda & -5) \\ 2 & -3 - \lambda \end{vmatrix} = 0$$

$$i.e. \quad \lambda^2 - \lambda - 2 = 0$$

$$i.e. \quad \lambda = -1 \quad or \quad \lambda = 2$$

Therefor $\lambda = -1$ and $lambda = 2$ are the eigenvalue of the matrix A

NOTE: A matrix with zero determinant is singular, so there must be nonzero vector x in its nullspace. In fact the nullspace contains a whole line of eigenvectors. It is a subspace.

Let x_1 be the eigenvector of the corresponding eigenvalue $\lambda = -1$.

$$(A + I)x = 0$$

$$i.e. \quad \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution (the First eigenvalue)is any nonzero multiple of x_1

Eigenvaetor for $\lambda_1 = -1$,

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let x_2 be the eigenvector of the corresponding eigenvalue $\lambda_2 = 2$.

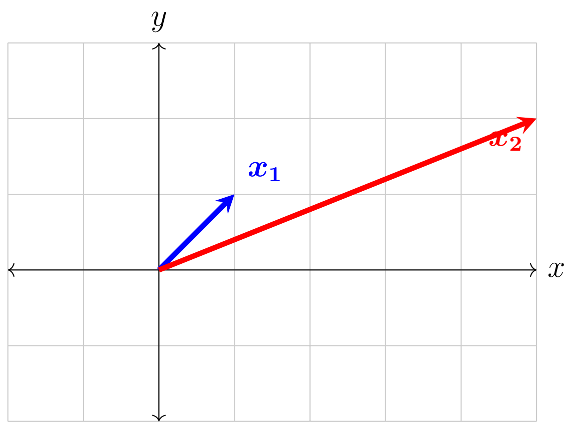
$$(A - 2I)x = 0$$

$$i.e. \quad \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution (the second eigenvector)is any nonzero multiple of x_2

Eigenvaetor for $\lambda_2 = 2$,

$$x_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$



The special solution of equation (1) is $u = e^{\lambda t}x$. Therefore the complete solution equation (1) is

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2. \tag{2}$$

Where c_1 and c_2 are two free parameter. The initial condition of the system is $u = u(0)$ at $t = 0$.

At $t = 0$, the equation (2) becomes

$$c_1 x_1 + c_2 x_2 = 0$$

$$i.e. \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

The constant are $c_1 = 3$ and $c_2 = 1$ Therefor the solution to the original equation is

$$u(t) = 3e^{-t} x_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 5 \\ 2 \end{bmatrix} \tag{3}$$

where $u(t) = 3e^{-1} + 5e^{2t}$, $w(t) = 3e^{-1} + 2e^{2t}$

Hence at $t = 0$, $v(0) = 8$ and $w(0) = 5$