Lecture 2
Chapter 2 : Probability

2.1 Sample Space, 2.2 Events, 2.3 Counting Sample Points

The aim of this lecture is to explain the following concepts :

- Sample Space.
- Event.
- Counting Sample Points.

2.1 Sample Space :

Definition 1 *The set of all possible outcomes of a statistical experiment is called the **sample space** and is represented by the symbol S.*

Notes :

- Each outcome in a sample space is called an element or a member of the sample space, or simply a sample point.
- If the sample space has a finite number of elements, we may list the members separated by commas and enclosed in braces.
- Thus, the sample space S, of possible outcomes when a coin is flipped, may be written $S = \{H, T\}$, where H and T correspond to heads and tails, respectively.

Example 1 *: Consider the experiment of tossing a die. If we are interested in the number that shows on the top face, the sample space is* $S_1 = \{1, 2, 3, 4, 5, 6\}$.
If we are interested only in whether the number is even or odd, the sample space is simply $S_2 = \{even, odd\}$.

Example 2 : *An experiment consists of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a die is tossed once. To list the elements of the sample space providing the most information, we construct the tree diagram of Figure 2.1. The sample space is* $S = \{HH, HT, T1, T2, T3, T4, T5, T6\}$.

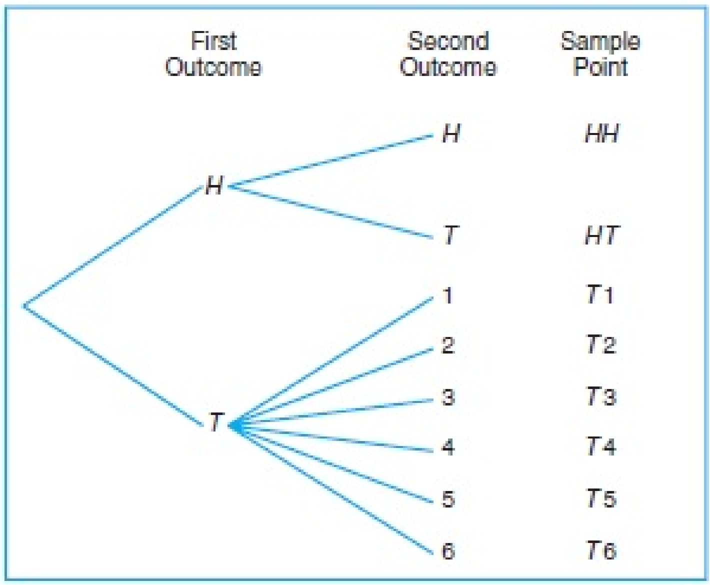

Figure 1: Tree diagram for Ex. 2

Example 3 : *Suppose that three items are selected at random from a manufacturing process. Each item is inspected and classified defective, D, or nondefective, N. To list the elements of the sample space providing the most information, we construct the tree diagram of Figure 2.2. The sample space is*

$$S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}$$

Suppose the experiment is to sample items randomly until one defective item is observed. The sample space for this case is $S = \{D, ND, NND, NNND, ...\}$

2.2 Events :

Definition 2 *An **event** is a subset of a sample space.*

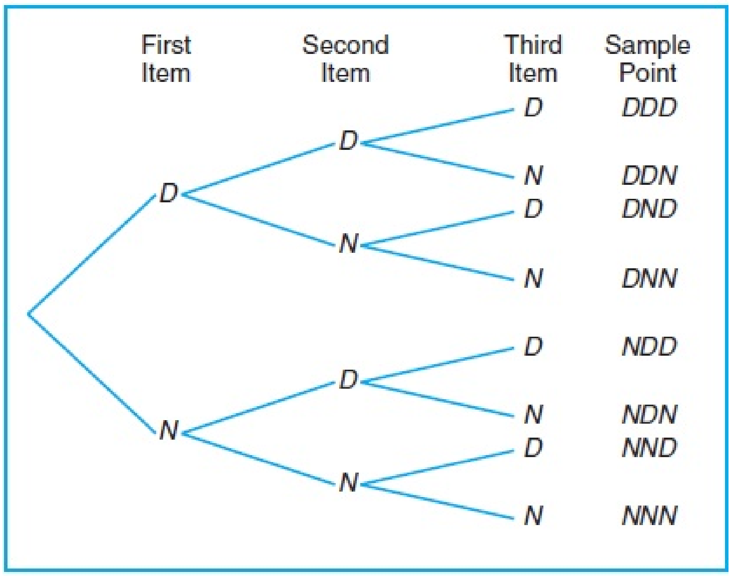

Figure 2: Tree diagram for Ex. 3

Example 4 ***:*** *The event A that the outcome when a die is tossed is divisible by 3. This will occur if the outcome is an element of the subset*$S_1 = \{3, 6\}$ *of the sample space* S_1 *in Example 1.*
In the event B that the number of defectives is greater than 1 in Example 3. This will occur if the outcome is an element of the subset $S = \{DDD, DDN, DND, NDD\}$ *of the sample space S.*

Definition 3 *The* ***complement*** *of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by the symbol* A'.

Definition 4 *The* ***intersection*** *of two events A and B, denoted by the symbol* $A \cap B$, *is the event containing all elements that are common to A and B.*

Definition 5 *Two events A and B are* ***mutually exclusive****, or* ***disjoint****, if* $A \cap B = \phi$, *that is, if A and B have no elements in common.*

Definition 6 *The* ***union*** *of the two events A and B, denoted by the symbol* $A \cup B$, *is the event containing all the elements that belong to A or B or both.*

Exercises :

3. Which of the following events are equal?

(a) $A = \{1, 3\}$

(b) $B = \{x \mid x \textit{ is a number on a die}\}$

(c) $C = \{x \mid x^2 - 4x + 3 = 0\}$

(d) $D = \{x \mid x \textit{ is the number of heads when six coins are tossed}\}$

Solution :

(a) $A = \{1, 3\}$

(b) $B = \{1, 2, 3, 4, 5, 6\}$

(c) $C = \{x \mid x^2 - 4x + 3 = 0\} = \{x \mid (x-1)(x-3) \;=\; 0\} = \{1, 3\}$

(d) $D = \{0, 1, 2, 3, 4, 5, 6\}$

Clearly, A = C.

7. Four students are selected at random from a chemistry class and classified as male or female. List the elements of the sample space S_1, using the letter M for male and F for female. Define a second sample space S_2 where the elements represent the number of females selected.
Solution:

$$S_1 = \{MMMM, MMMF, MMFM, MFMM, FMMM, MMFF, MFMF, MFFM,\\ FMFM, FFMM, FMMF, MFFF, FMFF, FFMF, FFFM, FFFF\}$$
$$S_2 = \{0, 1, 2, 3, 4\}$$

2.3 Counting Sample Points :
Multiplication Rule : If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

Example 5 *: How many sample points are there in the sample space when a pair of dice is thrown once?*
Solution : *The first die can land face-up in any one of* $n_1 = 6$ *ways. For each of these 6 ways, the second die can also land face-up in* $n_2 = 6$ *ways. Therefore, the pair of dice can land in* $n_1 n_2 = (6)(6) = 36$ *possible ways.*

Generalized Multiplication Rule : If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 ...... n_k$ ways.

Example 6 *: Sam is going to assemble a computer by himself. He has the choice of chips from two brands, a hard drive from four, memory from three, and an accessory bundle from five local stores. How many different ways can Sam order the parts?*
Solution : *Since* $n_1 = 2, n_2 = 4, n_3 = 3, and n_4 = 5$, *there are* $n_l \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5 = 120$ *different ways to order the parts.*

Definition 7 *A* ***permutation*** *is an arrangement of all or part of a set of objects.*

Notes :

- For any non-negative integer n, $n!$, called **n factorial**, is defined as $n! = n(n-1)(n-2).......(2)(1)$, with special case $0! = 1$.
- The number of permutations of n objects is $n!$.
- The number of permutations of n distinct objects taken r at a time is

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

.

Example 7 *: In one year, three awards (research, teaching, and service) will be given to a class of 25 graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?*
Solution : *Since the awards are distinguishable, it is a permutation problem. The total number of sample points is*

$$^{25}P_{3} = \frac{25!}{(25-3)!} = \frac{25!}{22!} = (25)(24)(23) = 13,800$$

.

Notes :

- The number of permutations of n objects arranged in a circle is $(n-1)!$.
- The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind,....., n_k of a kth kind is
$$\frac{n!}{n_1!\ n_2!\ ....\ n_k!}$$
.

Example 8 ***:*** *In a college football training session, the defensive coordinator needs to have 10 players standing in a row. Among these 10 players, there are 1 freshman, 2 sophomores, 4 juniors, and 3 seniors. How many different ways can they be arranged in a row if only their class level will be distinguished?*
Solution : *We find that the total number of arrangements is*

$$\frac{10!}{1!\ 2!\ 4!\ 3!} = 12,600$$

.

Notes :

- The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second, and so forth, is
$$\binom{n}{n_1,\ n_2, ...,n_r} = \frac{n!}{n_1!\ n_2!\ ....\ n_r!},\ \textit{where}\ n_1 + n_2 + ... + n_r = n$$
.
- The number of combinations of n distinct objects taken r at a time is
$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$