(Lect. 36)
10.10 One and two sample test concerning variance

In this section we will discuss how to test population variance or standard deviation using the help of χ^2 distribution table and F distribution table.

ONE SAMPLE TEST VARIANCE

Here H_0: Null Hypothesis, the hypothesis, which refer to any hypothesis we wish to test.
H_1: Alternative hypothesis
Procedure for test for variance:

Step 1: Select a fixed significance level α
Step 2: State the Null Hypothesis H_0 and alternative hypothesis H_1 that is we have to test the Null Hypothesis $H_0 : \sigma^2 = {\sigma_0}^2$ against the alternative hypothesis H_1, which is one case among three following cases

$$H_1 : \sigma^2 < {\sigma_0}^2 (Case - I)$$
$$H_1 : \sigma^2 > {\sigma_0}^2 (Case - II)$$
$$H_1 : \sigma^2 \neq {\sigma_0}^2 (Case - III)$$

Step 3: Determine

$$\chi^2 = \frac{(n-1)s^2}{{\sigma_0}^2}$$

where n is the sample size, s^2 is the sample variance, and ${\sigma_0}^2$ is the value of ${\sigma_0}^2$ given by the null hypothesis.
If H_0 is true, χ^2 is a value of the chi-squared distribution with $v = n - 1$ degrees of freedom.
Step 4: Determine the critical region and fail to reject region based on α
Case-I

Critical region:
If the alternative hypothesis is $H_1 : \sigma^2 < {\sigma_0}^2$ (case-I) then the critical region is $\chi^2 < \chi^2_{1-\alpha}$.
Fail to reject region:
Fail to reject null hypothesis H_0 region is $\chi^2 \geq \chi^2_{1-\alpha}$.
Note: Here we have to determine $\chi^2_{1-\alpha}$ using the equation $P(\chi^2 > \chi^2_{1-\alpha}) = 1 - \alpha$ with n-1 degrees of freedom (use χ^2 distribution table)
(Case-II)
Critical Region
If the alternative hypothesis is $H_1 : \sigma^2 > {\sigma_0}^2$ (case-II) then the critical region is $\chi^2 > \chi^2_{\alpha}$
Fail to reject region:
Fail to reject null hypothesis H_0 region is $\chi^2 \leq \chi^2_{\alpha}$
Note: Here we have to determine χ^2_{α} using the equation $P(\chi^2 > \chi^2_{\alpha}) = \alpha$ with n-1 degrees of freedom (use χ^2 distribution table)

(Case-III)

Critical Region

If the alternative hypothesis is $H_1 : \sigma^2 \neq {\sigma_0}^2$ (case-III) then the critical region is $\chi^2 > \chi^2_{\alpha/2}$ or $\chi^2 < \chi^2_{1-\alpha/2}$

Fail to reject region

Fail to reject null hypothesis H_0 region is $\chi^2_{1-\alpha/2} \leq \chi^2_{\alpha/2}$

Note:

Here we have to determine $\chi^2_{\alpha/2}$ using the equation $P(\chi^2 > \chi^2_{\alpha/2}) = \alpha/2$ with n-1 degrees of freedom.

Determine $\chi^2_{1-\alpha/2}$ using the equation $P(\chi^2 > \chi^2_{1-\alpha/2}) = 1 - \alpha/2$ with n-1 degrees of freedom (use χ^2 distribution table) or using $\chi^2_{1-\alpha/2} = -\chi^2_{\alpha/2}$

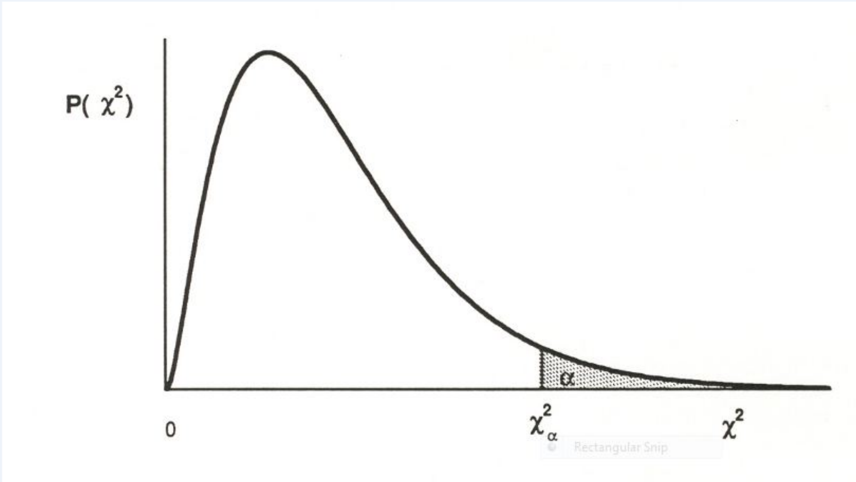

Figure J.1: The χ^2 distribution

Figure 1: **A**

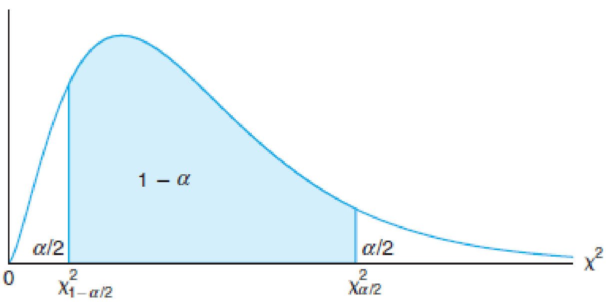

Figure 9.7: $P(\chi^2_{1-\alpha/2} < X^2 < \chi^2_{\alpha/2}) = 1 - \alpha$.

Figure 2: **B**

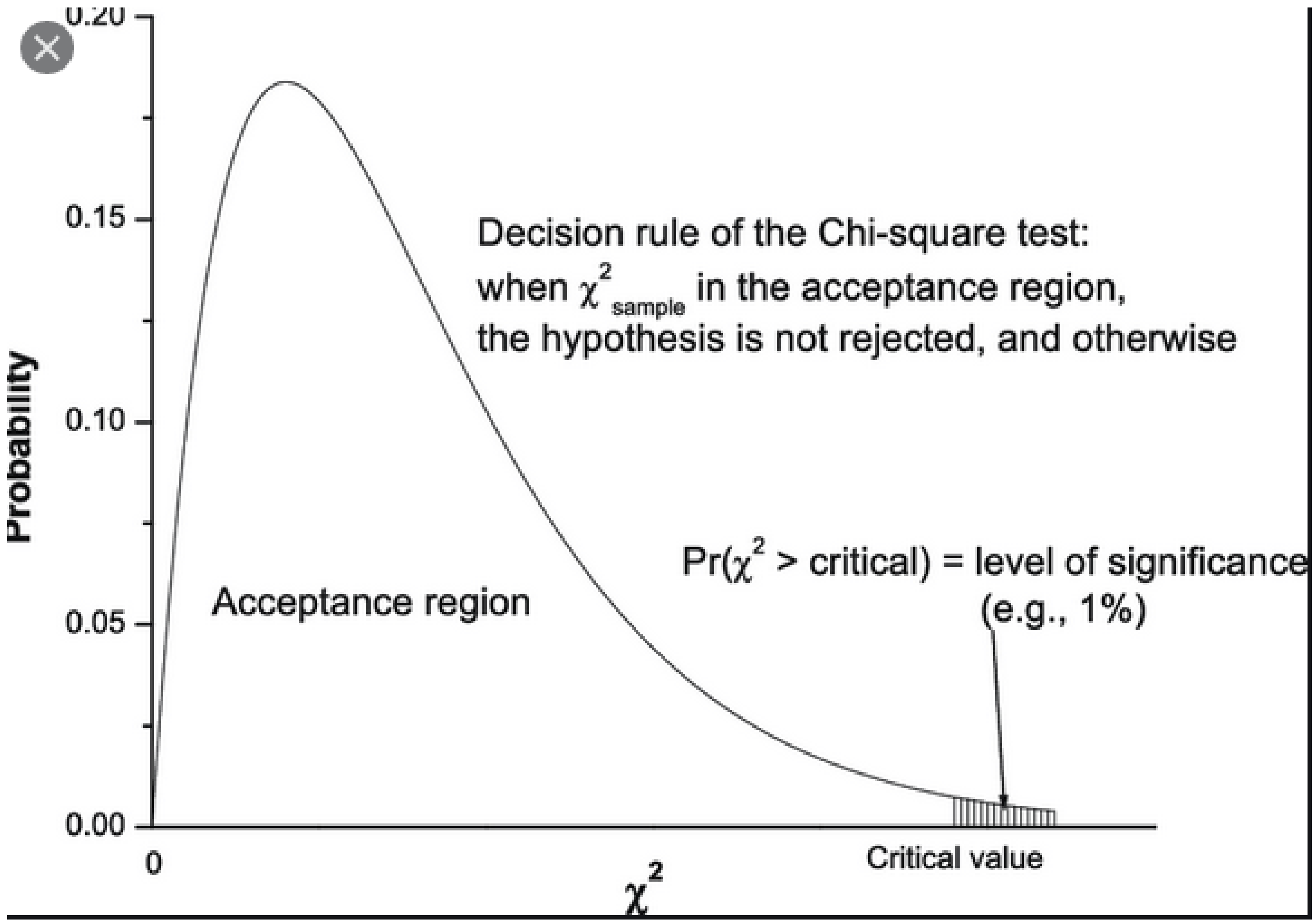

Figure 3: **C**

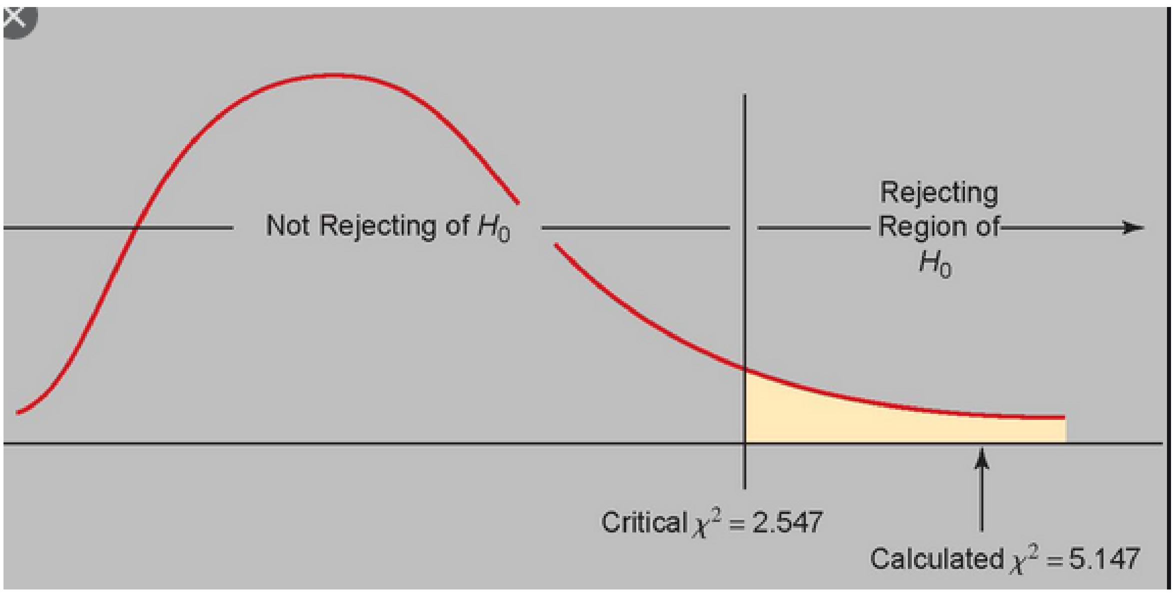

Figure 4: **D**

Question No 67:
The content of containers of a particular lubricant is known to be normally distributed with a variance of 0.03 litre. Test the hypothesis that $\sigma^2 = 0.03$ against the alternative that $\sigma^2 \neq 0.03$ for the random sample of 10 containers in Exercise 10.23 on page 398.
(Ex 10.23) Test the hypothesis that the average content of containers of a particular lubricant is 10 liters if the contents of a random sample of 10 containers are 10.2, 9.7, 10.1, 10.3, 10.1, 9.8, 9.9, 10.4, 10.3, and 9.8 liters. Use a 0.01 level of significance and assume that the distribution of contents is normal.)
Solution:
Step 1: It is given that α= 0.01
n =sample size= 10,
Step 2: we have to test the Null Hypothesis $H_0 : \sigma^2 = {\sigma_0}^2 = 0.03$ against the alternative hypothesis H_1, which is

$$H_1 : \sigma^2 \neq {\sigma_0}^2 = 0.03(case - III)$$

Step 3: Determine

$$\chi^2 = \frac{(n-1)s^2}{{\sigma_0}^2}$$

n =sample size= 10
Now using 10 sample points we have to determine sample mean, sample variance and sample standard deviation.
Sample mean =

$$\overline{x} = \frac{10.2 + 9.7 + 10.1 + 10.3 + 10.1 + 9.8 + 9.9 + 10.4 + 10.3 + 9.8}{10} = \frac{106.6}{10} = 10.06$$

Sample variance=s^2
Sample standard deviation =$s = +\sqrt{s^2}$

$$= \sqrt{\frac{1}{9}\left(\begin{array}{l} (10.2 - 10.06)^2 + (9.7 - 10.06)^2 + (10.1 - 10.06)^2 \\ +10.3 - 10.06^2 + (10.1 - 10.06)^2 + (9.8 - 10.06)^2 \\ +(9.9 - 10.06)^2 + (10.4 - 10.06)^2 + (10.3 - 10.06)^2 + (9.8 - 10.06)^2 \end{array}\right)}$$

=0.246
So

$$\chi^2 = \frac{(n-1)s^2}{{\sigma_0}^2} = \frac{(10-1)(0.246)^2}{0.03} = 18.1548$$

Step 4: Determine the critical region and fail to reject region based on α for case-III
Now we have to determine $\chi^2_{\alpha/2}$ using the equation $P(\chi^2 > \chi^2_{\alpha/2}) = \alpha/2$ with n-1 degrees of freedom.
As $P(\chi^2 > \chi^2_{\alpha/2}) = 0.01/2 = 0.005$ with $n-1$=10-1=9 degrees of freedom.
Using χ^2 distribution table we get $\chi^2_{\alpha/2} = 23.589$
As $\chi^2_{1-\alpha/2} = -\chi^2_{\alpha/2}$

$$\Rightarrow \chi^2_{1-\alpha/2} = -23.589$$

As here $\chi^2_{1-\alpha/2} = -23.589 < \chi^2 = 18.1548 < \chi^2_{\alpha/2} = 23.589$
So it satisfy fail to reject region $\chi^2_{1-\alpha/2} \leq \chi^2_{\alpha/2}$
So our conclusion is Fail to reject $H_0 : \sigma^2 = 0.03$; that is the sample of 10 containers is not sufficient to show that σ^2 is not equal to 0.03.
Note: Critical Region:
If the alternative hypothesis is $H_1 : \sigma^2 \neq {\sigma_0}^2$ (case-III) then the critical region is $\chi^2 > \chi^2_{\alpha/2}$ or $\chi^2 < \chi^2_{1-\alpha/2}$
Fail to reject region:
Fail to reject null hypothesis H_0 region is $\chi^2_{1-\alpha/2} \leq \chi^2_{\alpha/2}$

Question Number 68

Past experience indicates that the time required for high school seniors to complete a standardized test is a normal random variable with a standard deviation of 6 minutes. Test the hypothesis that $\sigma = 6$ against the alternative that $\sigma < 6$ if a random sample of the test times of 20 high school seniors has a standard deviation s = 4.51. Use a 0.05 level of significance.
Solution:
Step 1:
It is given that α= 0.05
Step 2:
Here we have to test the Null Hypothesis $H_0 : \sigma = \sigma_0 = 6$ against the alternative hypothesis H_1, which is $H_1 : \sigma < \sigma_0 = 6$ (case-I)
Also we have to test the Null Hypothesis $H_0 : \sigma^2 = {\sigma_0}^2 = 36$ against the alternative hypothesis H_1, which is $H_1 : \sigma^2 < {\sigma_0}^2 = 36$
Step 3:
Determine $\chi^2 = \frac{(n-1)s^2}{{\sigma_0}^2}$
It is given
n=sample size=20
${\sigma_0}^2 = 36$
sample standard deviation=$s = 4.51$
Now

$$\chi^2 = \frac{(n-1)s^2}{{\sigma_0}^2} == \frac{(20-1)4.51^2}{6^2} = 10.74$$

Step 4:
Determine the critical region and fail to reject region based on α for case-I

Here we have to determine $\chi^2_{1-\alpha}$ using the equation $P(\chi^2 > \chi^2_{1-\alpha}) = 1 - \alpha = 1 - 0.05 = 0.95$ with n-1=20-1=19 degrees of freedom (use χ^2 distribution table)
$\Rightarrow \chi^2_{1-\alpha} = 10.117$
$Here\ \chi^2 = 10.74 > \chi^2_{1-\alpha} = 10.117$
So it satisfy Fail to reject null hypothesis H_0, as $(\chi^2 \geq \chi^2_{1-\alpha})$.
Our conclusion is there was not sufficient evidence to conclude that the standard deviation is less then 6 at $\alpha = 0.05$ level of significance

Question No 71

A soft-drink dispensing machine is said to be out of control if the variance of the contents exceeds 1.15 decilitres. If a random sample of 25 drinks from this machine has a variance of 2.03 decilitres, does this indicate at the 0.05 level of significance that the machine is out of control? Assume that the contents are approximately normally distributed.

Solution:

Step 1:

It is given that $\alpha = 0.05$

Step 2:

Here we have to test the Null Hypothesis $H_0 : \sigma^2 = {\sigma_0}^2 = 1.15$ against the alternative hypothesis H_1, which is $H_1 : \sigma^2 > {\sigma_0}^2 = 1.15$

Step 3:

Determine

$$\chi^2 = \frac{(n-1)s^2}{{\sigma_0}^2}$$

It is given,

n=sample size=25

${\sigma_0}^2 = 1.15$

Sample variance=$s^2 = 2.03$

Now

$$\chi^2 = \frac{(n-1)s^2}{{\sigma_0}^2} = \frac{(25-1)2.03}{1.15} = 42.37$$

Step 4:

Determine the critical region and fail to reject region based on α for case-II.

Here we have to determine χ^2_α using the equation $P(\chi^2 > \chi^2_\alpha) = \alpha = 0.05 =$ with n-1=25-1=24 degrees of freedom (use χ^2 distribution table)

$$\Rightarrow \chi^2_\alpha = 36.415$$

$$\chi^2 = 42.37 > \chi^2_\alpha = 36.415$$

As it satisfy critical region $\chi^2 \geq \chi^2_\alpha$.

Our conclusion is Reject H_0

So there is sufficient evidence to conclude, at level $\alpha = 0.05$, that the soft drink machine is out of control

TWO SAMPLE TEST VARIANCE

Step 1: Select a fixed significance level α

Step 2: State the Null Hypothesis H_0 and alternative hypothesis H_1 that is we have to test the Null Hypothesis

$$H_0 : {\sigma_1}^2 = {\sigma_2}^2$$

against the alternative hypothesis H_1, which is one case among three following cases

$$H_1 : {\sigma_1}^2 < {\sigma_2}^2 (case - I)$$

$$H_1 : {\sigma_1}^2 > {\sigma_2}^2 (case - II)$$

$$H_1 : {\sigma_1}^2 \neq {\sigma_2}^2 (case - III)$$

Step 3: Determine f value of testing ${\sigma_1}^2 = {\sigma_2}^2$ which is the ratio $f = \frac{{s_1}^2}{{s_2}^2}$
where ${s_1}^2$ and ${s_2}^2$ are the variance computed from the two samples of size n_1 and n_2
If the two populations are approximately normally distributed and the null hypothesis is true, then the ratio $f = \frac{{s_1}^2}{{s_2}^2}$ is a value of the F-distribution with $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$ degrees of freedom.

Step 4: Determine the critical region and fail to reject region based on α, using F-distribution (variance ratio distribution) table with $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$ degrees of freedom.
Note:The F-distribution is used in two sample situations to draw inference about the population variance.

Case-I

Critical region:

If the alternative hypothesis is $H_1 : {\sigma_1}^2 < {\sigma_2}^2$ (case-I) then the critical region is $f < f_{(1-\alpha)}(v_1, v_2)$.

Fail to reject region:

Fail to reject null hypothesis H_0 region is $f \geq f_{(1-\alpha)}(v_1, v_2)$.
NOTE: Here we have to determine $\chi^2_{1-\alpha}$ using the equation $P(\chi^2 > \chi^2_{1-\alpha}) = 1 - \alpha$ with n-1 degrees of freedom (use χ^2 distribution table)

(Case-II)

Critical region:

If the alternative hypothesis is $H_1 : {\sigma_1}^2 > {\sigma_2}^2$ (case-I) then the critical region is $f > f_{(\alpha)}(v_1, v_2)$.
Fail to reject region:

Fail to reject null hypothesis H_0 region is $f \leq f_{(\alpha)}(v_1, v_2)$.
NOTE: Here we have to determine χ^2_{α} using the equation $P(\chi^2 > \chi^2_{\alpha}) = \alpha$ with n-1 degrees of freedom (use χ^2 distribution table)

(Case-III)
Critical region:

If the alternative hypothesis is $H_1 : {\sigma_1}^2 \neq {\sigma_2}^2$ (case-III) then the critical region is $f < f_{(1-\alpha/2)}(v_1, v_2)$ or $f > f_{(\alpha/2)}(v_1, v_2)$

Fail to reject region:
Fail to reject null hypothesis H_0 region is $f_{(1-\alpha/2)}(v_1, v_2) \leq f \leq f_{(\alpha/2)}(v_1, v_2)$.
NOTE: Here we have to determine $\chi^2_{\alpha/2}$ using the equation $P(\chi^2 > \chi^2_{\alpha/2}) = \alpha/2$ with n-1 degrees of freedom.
Determine $\chi^2_{1-\alpha/2}$ using the equation $P(\chi^2 > \chi^2_{1-\alpha/2}) = 1 - \alpha/2$ with n-1 degrees of freedom (use χ^2 distribution table) or using $\chi^2_{1-\alpha/2} = -\chi^2_{\alpha/2}$

Question No 73
A study is conducted to compare the lengths of time required by men and women to assemble a certain product. Past experience indicates that the distribution of times for both men and women is approximately normal but the variance of the times for women is less than that for men. A random sample of times for 11 men and 14 women produced the following data:

$$\begin{array}{cc} Men & Women \\ \hline n_1 = 11 & n_2 = 14 \\ s_1 = 6.1 & s_2 = 5.3 \end{array} \tag{0.1}$$

Test the hypothesis that ${\sigma_1}^2 = {\sigma_2}^2$ against the alternative that ${\sigma_1}^2 = {\sigma_2}^2$.

Solution:

Step 1: Let us take significance level $\alpha = 0.05$
Step 2: State the Null Hypothesis H_0 and alternative hypothesis H_1 that is we have to test the Null Hypothesis $H_0 : {\sigma_1}^2 = {\sigma_2}^2$ against the alternative hypothesis H_1, which is $H_1 : {\sigma_1}^2 > {\sigma_2}^2$ (case-II)
Step 3: Given $n_1 = 11$ and $n_2 = 14$,
$s_1 = 6.1$ and $s_2 = 5.3$
Determine f value of testing ${\sigma_1}^2 = {\sigma_2}^2$ which is the ratio

$$f = \frac{{s_1}^2}{{s_2}^2}$$

where ${s_1}^2$ and ${s_2}^2$ are the variance computed from the two samples of size n_1 and n_2
Now

$$f = \frac{{s_1}^2}{{s_2}^2} = \frac{6.1^2}{5.3^2} = 1.33$$

Step 4: Determine the critical region and fail to reject region based on α, using F-distribution table with $v_1 = n_1 - 1 = 11 - 1 = 10$ and $v_2 = n_2 - 1 = 14 - 1 = 13$ degrees of freedom.
As here alternative hypothesis is $H_1 : {\sigma_1}^2 > {\sigma_2}^2$ (case-II)
then the critical region is $f > f_{(\alpha)}(v_1, v_2)$.

Fail to reject null hypothesis H_0 region is $f \leq f_{(\alpha)}(v_1, v_2)$.
Now $f_{(\alpha)}(v_1, v_2) = f_{(0.05)}(10, 13) = 2.67$ (refer to page no 820, F distribution table)
As here $f = 1.33 \leq f_{(\alpha)}(v_1, v_2) = f_{(0.05)}(10, 13) = 2.67$.
So it satisfy Fail to reject null hypothesis H_0 region.
So, our conclusion is " the variability of the time to assemble the product is not significantly greater for men".

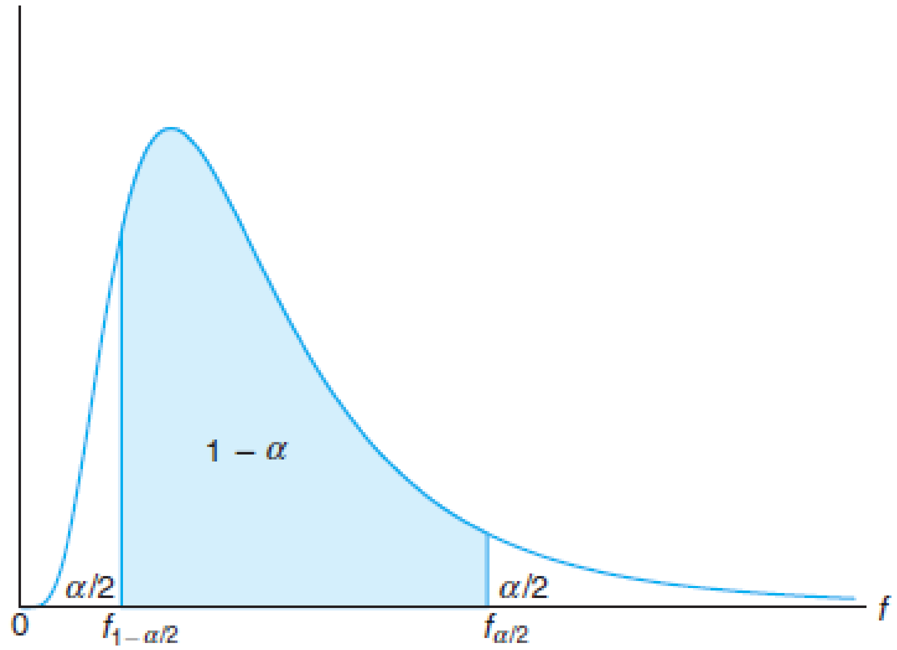

Figure 5: **E**