Introduction to Algorithm Design

(Mathematical and Programming Background)

Iteration vs Recursion

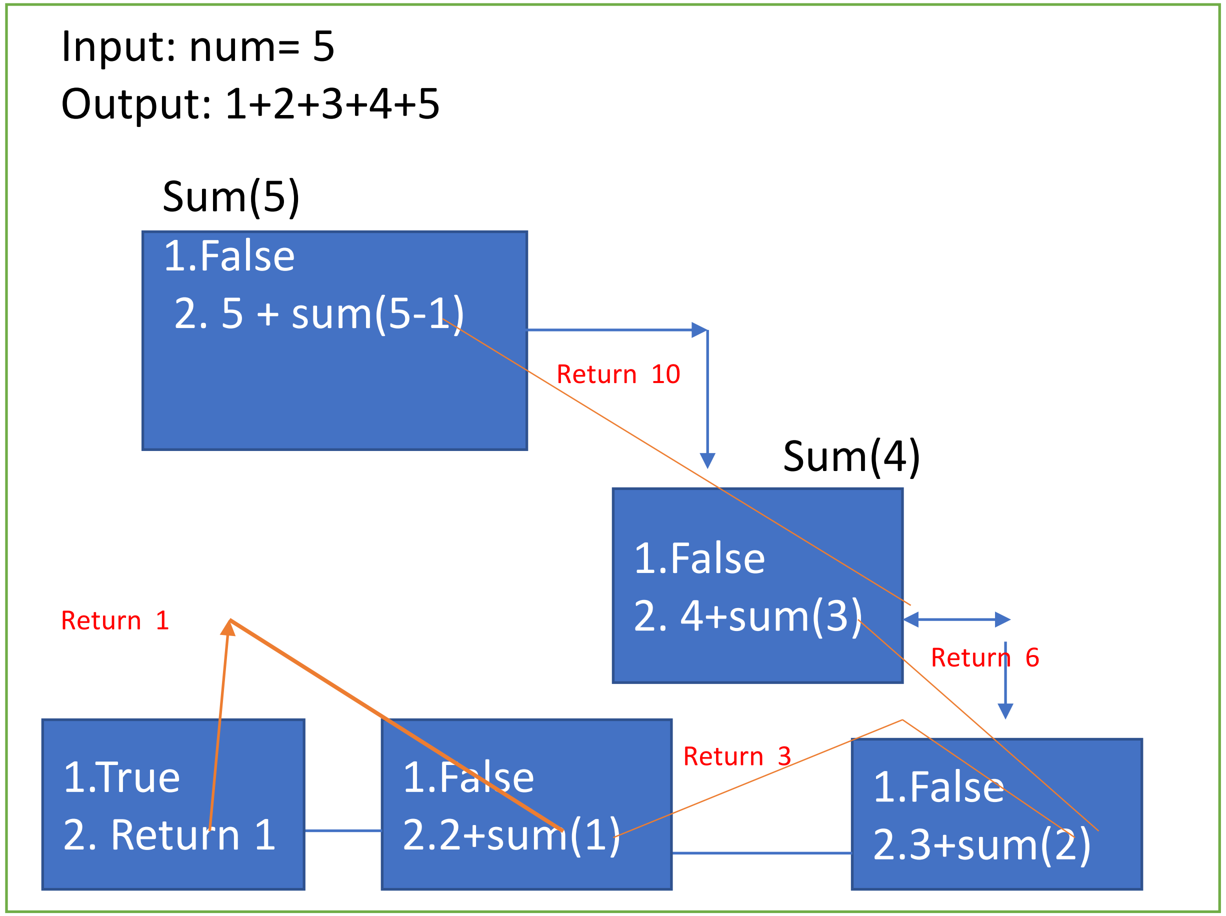
The process in which a function calls itself directly or indirectly is called recursion and the corresponding function is called as recursive function. Using recursive algorithm, certain problems can be solved quite easily.

Determine the sum of first n positive natural numbers?

<u>Pseudocode</u>

```
num: input "enter a number"
sum(num)
  {
        if(num <= 1)
              return  num
        else
              return ( num +sum(num -1))

}
```



Iterative method

Determine the sum of first n natural numbers?

```
i, num, s=0
input  num
i=0
do
{
        s=s+i
         i=i+1
}While( i<=num)
print  s
```

Factorial of number

The value of 5! is 120 as

$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

(5 distinct objects can be arranged into a sequence in 120 ways).

The value of 0! is 1

Iteration vs Recursion

```
input  n
factorial( n)
{
    if (n == 0)
        return 1
  else
    return (n * factorial(n - 1)
}
```

```
input  n
factorial( n)
{
    res = 1, i
    for (i := 2 to n)
        res = res*i;
    return res;
}
```

The Fibonacci numbers are a sequence of integers in which the first two elements are 0 & 1, and each following elements are the sum of the two preceding elements:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ..., 233

Iteration vs Recursion

Fibonacci numbers:

write a function to generate the nth Fibonacci number starting from 0,1

```
fiboRecursion(n)
{
      //use recursion
      if (n == 0)
       {
         return 0
      }
   else if (n == 1)
     {
         return 1
      }
 return fiboRecursion(n - 1) +fiboRecursion(n- 2)
  }
```

```
fibonacciLoop(number) {
     //use loop
     previouspreviousnumber
    previousnumber = 0
     currentnumber = 1

     for ( i := 1 to number ) {

        previousnumber = previousnumber

        previousnumber = currentnumber

      currentnumber = previouspreviousnumber +
previousnumber

     }
     return currentnumber;
  }
```

Mathematical Induction

Mathematical Induction is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number. A versatile proof technique applied to many types of problems.

The technique involves two steps to prove a statement, as stated below

- **Step 1:Base case step: Inductive base**: the property holds for n=1
- **Inductive hypothesis**: assume for all natural number n >=1,the property holds for n
- **Step 2(Inductive step)** − It proves that if the statement is true for the n^{th} iteration (or number *n*), then it is also true for $(n+1)^{th}$ iteration (or number *n+1*).

Problem 1:

Using the principle of mathematical induction ,prove that

$1^2 + 2^2 + 3^2 + \ldots.. + n^2 = (1/6)\{n(n + 1)(2n + 1\}$ for all $n \in N$.

Mathematical Induction

Base: for n=1 both sides of equation are equal ,hence base is true.

Putting n =1 in the given statement, we get

LHS = $1^2 = 1$ and RHS = $(1/6) \times 1 \times 2 \times (2 \times 1 + 1) = 1$.

Therefore LHS = RHS.

Thus, P(1) is true.

Assume : it is true for all n

$P(k): 1^2 + 2^2 + 3^2 + \ldots\ldots + k^2 = (1/6)\{k(k + 1)(2k + 1)\}$.

Required to prove: $P(k+1)=1^2 + 2^2 + 3^2 + \ldots\ldots + k^2+(k+1)^2 = (1/6)\{(k+1)(k + 2)(2(k+1) + 1)\}$

now $1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + k^2 + (k+1)^2$

$= (1/6) \{k(k + 1)(2k + 1)\} + (k+ 1)^2$

$= (1/6)\{(k+ 1).(k(2k+ 1)+6(k + 1))\}$

$= (1/6)\{(k+ 1)(2k^2 + 7k + 6) \})$ 2k2+4k+3k+6= 2k(k+2) +3(k+2) =(k+2)(2k+3)

$= (1/6)\{(k + 1)(k + 2)(2k + 3)\}$ 2k+2+1

$= 1/6\{(k + 1)(k + 2)[2(k + 1) + 1]\}$= P(k+1) is true

Hence, by the principle of mathematical induction, P(n) is true for all $n \in N$.

Mathematical Induction

Problem2

Claim: for all n , if 1+x > 0 then $(1+x)^n$>=1+nx

Base: for n=1 both sides of the equation are equal to 1+x hence base case is true

Assume: $(1+x)^n$ >= 1+nx p(n) is true

Required to prove: $(1+x)^{n+1}$>= 1+(n+1)x

$(1+x)^{n+1}$=(1+x) $(1+x)^n$ 2^3 =2^2 * 2

>= (1+x)(1+nx) 1+nx+x+n x^2 =1+x(n+1)+nx2

= 1+(n+1)x + nx^2

>= 1+(n+1)x

Mathematical Induction

Problem3

Claim: for all n>=1, $\sum_{i=1}^{n} 1/2^{\wedge}i <= 1$ (2 to the power i)

Base: the claim is clearly trye for n=1 since 1/2 <1 P(1) is true

Assume : the claim is true for n $\sum_{i=1}^{n} 1/2^{\wedge}i <= 1$ P(n) is true

Required to prove : The claim for n+1

$\sum_{i=1}^{n+1} \frac{1}{2i}$

$=1/2+1/4+1/8+\dots\dots\dots+1/2^{n+1}$

$=1/2+1/2(1/2+1/4+1/8\dots.+1/2^{n})$

$=1/2+1/2 \sum_{i=1}^{n} \frac{1}{2i}$

$<=1/2+1/2 .1 =1$ proved

Mathematical Induction

Problem4

Claim: for n any positive integer 6^{n}-1 is divisible by 5

Base: the statement p1 says that 6^{1}-1 is divisible by 5

Assume : for n>= 1and suppose pk holds 6^{k}-1 is divisible by 5

Required to prove: 6^{k+1}-1 is divisible by 5

6^{k+1}-1 = 6* 6^{k} -1

= 6(6^{k}-1)-1+6   6* 6^{k} - 6 - 1 +6

= 6(6^{k}-1) +5

First term is divisible by 5 and 2nd term is also divisible by 5 therefore 6^{k+1}-1 is divisible by 5

Mathematical Induction

Question: for n >= 7 ,n! >3^n,proves it holds for all natural number.

For any n>=7 let Pn be the statement that n! >3^n

Base : the statement P7 says that 7!= 5040 > 3^7 =2187 which is true

Assume: for k >=7 and suppose that pk holds that is k! >3^k

Required to prove: Pk+1 holds that is (k+1)! > 3^{k+1}

(k+1)! = (k+1)k!

> (k+1) 3^k

>= (7+1) 3^k

>= 8 * 3^k

> = 3* 3^k

=3^{k+1}

(k+1)!=1*2….*k*(k+1)

4!= 1*2*3*4=3!*4