Lecture 4
2.6 Conditional Probability, Independence, and the Product Rule

The aim of this lecture is to explain the following concepts :

- Conditional Probability.
- Independence.
- The Product Rule

Definition 1 *The* ***conditional probability*** *of* B*, given* A*, denoted by* $P(B|A)$*, is defined by*

$$(B|A) = \frac{P(A \cap B)}{P(A)} \quad , \textit{ provided } P(A) > 0$$

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Definition 2 *Two events* A *and* B *are* ***independent*** *if and only if*

$$P(B|A) = P(B) \textit{ or } P(A|B) = P(A),$$

assuming the existences of the conditional probabilities. Otherwise, A *and* B *are dependent.*

The Product Rule, or the Multiplicative Rule :

Theorem 0.1 *If in an experiment the events* A *and* B *can both occur, then*

$$P(A \cap B) = P(A)P(B|A), \quad \textit{provided } P(A) > 0$$

.

Theorem 0.2 *Two events A and B are independent if and only if*

$$P(A \cap B) = P(A)P(B)$$

. *Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.*

Theorem 0.3 *If, in an experiment, the events $A_1, A_2, ..., A_k$ can occur, then*

$$P(A_1 \cap A_2 \cap .... \cap A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2).....P(A_k|A_1 \cap A_2 \cap ..... \cap A_{k-1}).$$

If the events $A_1, A_2, ..., A_k$ are independent, then

$$P(A_1 \cap A_2 \cap .... \cap A_k) = P(A_1)P(A_2).....P(A_k).$$

Definition 3 *A collection of events $A = \{A_1, ..., A_n\}$ are* ***mutually independent*** *if for any subset of A, $A_{i1}, ..., A_{ik}$, for $k \leq n$, we have*

$$P(A_{i1} \cap ..... \cap A_{ik}) = P(A_{i1})......P(A_{ik}).$$

Exercises :

74. A class in advanced physics is composed of 10 juniors, 30 seniors, and 10 graduate students. The final grades show that 3 of the juniors, 10 of the seniors, and 5 of the graduate students received an A for the course. If a student is chosen at random from this class and is found to have earned an A, what is the probability that he or she is a senior?
Solution :
$P(S|A) = 10/18 = 5/9$.

75. A random sample of 200 adults are classified below by sex and their level of education attained.

Education	*Male*	*Female*
Elementary	38	45
Secondary	28	50
College	22	17

If a person is picked at random from this group, find the probability that

(a) the person is a male, given that the person has a secondary education;

(b) the person does not have a college degree, given that the person is a female.

Solution : Consider the events:
M: a person is a male;
S: a person has a secondary education;
C: a person has a college degree.

(a) $P(M|S) = \frac{28}{78} = \frac{14}{39}$.

(b) $P(C'|M') = \frac{95}{112}$.

77. In the senior year of a high school graduating class of 100 students, 42 studied mathematics, 68 studied psychology, 54 studied history, 22 studied both mathematics and history, 25 studied both mathematics and psychology, 7 studied history but neither mathematics nor psychology, 10 studied all three subjects, and 8 did not take any of the three. Randomly select a student from the class and find the probabilities of the following events.

(a) A person enrolled in psychology takes all three subjects.

(b) A person not taking psychology is taking both history and mathematics.

Solution :

(a) $P(M \cap P \cap H) = \frac{10}{68} = \frac{5}{34}$.

(b) $P(H \cap M|P') = \frac{P(H \cap M \cap P')}{P(P')} = \frac{22-10}{100-68} = \frac{12}{32} = \frac{3}{8}$.

80. The probability that an automobile being filled with gasoline also needs an oil change is 0.25; the probability that it needs a new oil filter is 0.40; and the probability that both the oil and the filter need changing is 0.14.

(a) If the oil has to be changed, what is the probability that a new oil filter is needed?

(b) If a new oil filter is needed, what is the probability that the oil has to be changed?

Solution : Consider the events:
C: an oil change is needed,
F: an oil filter is needed.

(a) $P(F|C) = \dfrac{P(F \cap C)}{P(C)} = \dfrac{0.14}{0.25} = 0.56.$

(b) $P(C|F) = \dfrac{P(C \cap F)}{P(F)} = \dfrac{0.14}{0.40} = 0.35.$

89. A town has two fire engines operating independently. The probability that a specific engine is available when needed is 0.96.

(a) What is the probability that neither is available when needed?

(b) What is the probability that a fire engine is available when needed?

Solution : Let A and B represent the availability of each fire engine.

(a) $P(A' \cap B') = P(A')P(B') = (0.04)(0.04) = 0.0016.$

(b) $P(A \cup B) = 1 - P(A' \cap B') = 1 - 0.0016 = 0.9984.$

91. Find the probability of randomly selecting 4 good quarts of milk in succession from a cooler containing 20 quarts of which 5 have spoiled, by using

(a) the first formula of Theorem 2.12 on page 68

(b) the formulas of Theorem 2.6 and Rule 2.3 on pages 50 and 54, respectively.

Solution :

(a) $P(Q_1 \cap Q_2 \cap Q_3 \cap Q_4) = P(Q_1)P(Q_2|Q_1)P(Q_3|Q_1 \cap Q_2)P(Q_4|Q_1 \cap Q_2 \cap Q_3) = (15/20)(14/19)(13/18)(12/17) = 91/323.$

(b) Let A be the event that 4 good quarts of milk are selected. Then $P(A) = \dfrac{\binom{15}{4}}{\binom{20}{4}} = \dfrac{91}{323}.$