**LECTURE - 12 and 13**

**CHEPTER-4**

**4.3 Means and Variances of Linear Combinations of Random Variables**

**Theorem-4.5**
If a and b are constants, then E(aX + b) = aE(X) + b.
Substituting a=0 ,we get E(b)=b and b=0 we get E(aX)=aE(X)

**Example-4.17**
Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

| X | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|
| P(X=x) | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

Let g(X)=2X−1 represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.
**Solution** :
we can write E(2X-1) = 2E(X)-1.
Now
$\mu = E(X) = \sum_{x=4}^{9} xf(x)$

$$=(4)(\tfrac{1}{12}) + (5)(\tfrac{1}{12}) + (6)(\tfrac{1}{4}) + (7)(\tfrac{1}{4}) + (8)(\tfrac{1}{6}) + (9)(\tfrac{1}{6}) = \tfrac{41}{6}.$$

Therefore,
$\mu_{2X-1} = (2)(\frac{41}{6}) - 1 = \$12.67$

**Theorem-4.6**
The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,
E[g(X) ± h(X)] = E[g(X)] ± E[h(X)].

**Exercise-4.57**
Let X be a random variable with the following probability distribution:

| x | -3 | 6 | 9 |
|---|---|---|---|
| f(x) | $\frac{1}{6}$ | $\frac{1}{2}$ | $\frac{1}{3}$ |

Find E(X) and $E(X^2)$ and then, using these values, evaluate$E[(2X+1)^2]$.
**Solution** :
$E(X) = (-3)(\frac{1}{6}) + (6)(\frac{1}{2}) + (9)(\frac{1}{3}) = 5.5$
$E(X^2) = (9)(\frac{1}{6}) + (36)(\frac{1}{2}) + (81)(\frac{1}{3}) = 46.5$
$E[(2X+1)^2] = E(4(X^2) + 4(X) + 1) = 4E(X^2) + 4E(X) + 1$
$= 4(46.5) + 4(5.5) + 1 = 209$

**Theorem-4.7**
The expected value of the sum or difference of two or more functions of the random variables X and Y is the sum or difference of the expected values of the functions. That is,
E[g(X, Y ) ± h(X, Y )] = E[g(X, Y )] ± E[h(X, Y )].

**Theorem-4.8**
Let X and Y be two independent random variables.Then E(XY ) = E(X)E(Y).
Let X and Y be two independent random variables. Then $\sigma_{XY} = 0$.

**Theorem-4.9**
If X and Y are random variables with joint probability distribution f(x, y) and a, b, and c are constants, then $\sigma^2_{aX+bY+c} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}$
Setting b = 0, we see that $\sigma^2_{aX+c} = a^2\sigma^2_X = a^2\sigma^2$
Setting a = 1 and b = 0, we see that $\sigma^2_{X+c} = \sigma^2_X = \sigma^2$.
Setting b = 0 and c = 0, we see that $\sigma^2_{aX} = a^2\sigma^2_X = a^2\sigma^2$.
If X and Y are independent random variables, then $\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y$ and $\sigma^2_{aX-bY} = a^2\sigma^2_X + b^2\sigma^2_Y$

**Example-4.22**
If X and Y are random variables with variances $\sigma^2_X = 2 \text{and} \sigma^2_Y = 4$ and covariance $\sigma_{XY} = -2$, find the variance of the random variable Z = 3X-4Y+ 8.
**Solution** :
$\sigma^2_Z = \sigma^2_{3X-4Y+8} = \sigma^2_{3X-4Y}$

$$= 9\sigma^2_X + 16\sigma^2_Y - 24\sigma_{XY} = (9)(2) + (16)(4) - (24)(-2) = 130$$

**Example-4.23**
Let X and Y denote the amounts of two different types of impurities in a batch of a certain chemical product. Suppose that X and Y are independent random variables with variances $\sigma^2_X = 2$ and $\sigma^2_Y = 3$. Find the variance of the random variable Z = 3X-2Y+ 5.

**Solution** :
$\sigma_Z^2 = \sigma_{3X-2Y+5}^2 = \sigma_{3X-2Y}^2 = 9\sigma_X^2 + 4\sigma_Y = (9)(2) + (4)(3) = 30$

**Exercise-4.58**
The total time, measured in units of 100 hours, that a teenager runs her hair dryer over a period of one year is a continuous random variable X that has the density function

$$\text{f(x)}=\begin{cases} x & 0 < x < 1 \\ 2 - x & 1 < x < 2 \\ 0 & elsewhere \end{cases}$$

Evaluate the mean of the random variable Y = $60X^2$ + 39X, where Y is equal to the number of kilowatt hours expended annually.
**Solution**:
E(Y)=E(60$X^2$ + 39X)=60E($X^2$)+39E(X)
$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 (x)(x)dx + \int_1^2 (x)(2-x)dx$

$$=\frac{x^3}{3}\Big|_0^1 + (2\frac{x^2}{2} - \frac{x^3}{3})\Big|_1^2 = 1$$
$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 (x^2)(x)dx + \int_1^2 (x^2)(2-x)dx$

$$=\frac{x^4}{4}\Big|_0^1 + (2\frac{x^3}{3} - \frac{x^4}{4})\Big|_1^2 = \frac{7}{6}$$

$E(Y) = (60)(\frac{7}{6}) + (39)(1) = 109$
Total time=(109)(100)=10900 hours.

**Exercise-4.60**
Suppose that X and Y are independent random variables having the joint probability distribution

| | | x | |
|---|---|---|---|
| | f(x,y) | 2 | 4 |
| | 1 | 0.10 | 0.15 |
| y | 3 | 0.20 | 0.30 |
| | 5 | 0.10 | 0.15 |

Find (a) E(2X-3Y ); (b) E(XY ).
**Solution**:

| | f(x,y) | x: 2 | 4 | Row total h(y) |
|---|---|---|---|---|
| | 1 | 0.10 | 0.15 | 0.25 |
| y | 3 | 0.20 | 0.30 | 0.50 |
| | 5 | 0.10 | 0.15 | 0.25 |
| | Column total g(x) | 0.40 | 0.60 | 1 |

E(2X-3Y)=2E(X)-3E(Y)=$2\sum_x xg(x) - 3\sum_y yh(y)$

=$2((2)(0.40) + (4)(0.60)) - 3((1)(0.25) + (3)(0.5) + (5)(0.25))$

$= 6.40 + 9 = 6.40 - 9 = -2.60$

$E(XY) = E(X)E(Y) = (\sum_x xg(x))(\sum_y yh(y)) = (3.20)(3) = 9.60$

**4.4 Chebyshev's Theorem**
The probability that any random variable X will assume a value within k standard deviations of the mean is at least $1 - 1/k^2$. That is,
$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - 1/k^2$.

**Example-4.23**
A random variable X has a mean $\mu = 8$, a variance $\sigma^2 = 9$, and an unknown probability distribution. Find (a)P$(-4 < X < 20)$ (b) P(| $(X - 8)$ |≥6).
**Solution**:
(a)$P(-4 < X < 20)$

=$P[8 - (4)(3) < X < 8 + (4)(3)] \geq \frac{15}{16}$.

(b) $P(\mid (X - 8) \mid \geq 6)$

=$1 - P(\mid (X - 8) \mid < 6) = 1 - P(-6 < X - 8 < 6)$

=$1 - P[8 - (2)(3) < X < 8 + (2)(3)] \leq \frac{1}{4}$

**Exercise-4.75**
An electrical firm manufactures a 100-watt light bulb, which, according to specifications written on the package, has a mean life of 900 hours with a standard deviation of 50 hours. At most, what percentage of the bulbs fail to last even 700 hours? Assume that the distribution is symmetric about the mean.
**Solution**:

X is the random variable define life of the 100-watt bulb
Here $\mu$=900 hours and $\sigma = 50$
To find the probability of$P(X \leq 700)$.
Given that the distribution is symmetric about the mean. According to Chebyshev's theorem
$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - 1/k^2$.

$$P(X \leq 700) = (0.5)(P(\mid X - 900 \mid \geq 200)) = (0.5)(1 - P(\mid X - 900 \mid \leq 200))$$

$$=(0.5)(1 - P(900 - (4)(50) < X < 900 + (4)(50))) \leq (0.5)(\tfrac{1}{4^2}) = 0.03215$$

Therefore the percentage of the bulbs fail to last even 700 hours is 3.215%.

**Exercise-4.77**
A random variable X has a mean$\mu = 10$ and a variance$\sigma^2 = 4$. Using Chebyshev's theorem, find
$(a) P(\mid X - 10 \mid \geq 3)$;
$(b) P((\mid X - 10 \mid < 3)$;
$(c) P(5 < X < 15)$;
(d) the value of the constant c such that $P(\mid X - 10 \mid \geq c) \leq 0.04$.
**Solution**:
(a)$P(\mid X - 10 \mid \geq 3$

$$= 1 - P(\mid (X - 10) \mid < 3) = 1 - P(-3 < X - 10 < 3)$$

$$=1 - P[10 - (2)(\tfrac{3}{2}) < X < 10 + (2)(\tfrac{3}{2})] \leq \tfrac{4}{9}$$

(b) $P((\mid X - 10 \mid < 3)$

$$=P(-3 < X - 10 < 3) = P[10 - (2)(\tfrac{3}{2}) < X < 10 + (2)(\tfrac{3}{2})] \geq 1 - \frac{1}{(\frac{3}{2})^2} = \frac{5}{9}$$

(c)$P(5 < X < 15)$

$$=P[10 - (2)(\tfrac{5}{2}) < X < 10 + (2)(\tfrac{5}{2})] \geq 1 - \frac{1}{(\frac{5}{2})^2} = \frac{21}{25}$$

(d)$P(\mid X - 10 \mid \geq c)$

$$=1 - P(\mid X - 10 \mid \leq c) = 1 - P(10 - c < X < 10 + c)$$

$$= 1 - P(10 - 2(\tfrac{c}{2}) < X < 10 + 2(\tfrac{c}{2})) \leq \frac{1}{(\frac{c}{2})^2} = \frac{4}{c^2}$$

Given that $P(\mid X-10 \mid \geq c) \leq 0.04$.
Therefore $\frac{4}{c^2}=0.04 \implies c^2=100 \implies c=10$

**Example-4.78**
Compute$P(\mu-2\sigma<X<\mu+2\sigma)$, where X has the density function

$$\text{f(x)=}\begin{cases} 6x(1-x) & 0<x<1 \\ 0 & elsewhere \end{cases}$$

**Solution**:
E(X)=$\int_0^1 (x)(6x(1-x))dx = \int_0^1 (6x^2-6x^3)dx = \frac{1}{2}$

$E(X^2) = \int_0^1 (x^2)(6x(1-x))dx = \int_0^1 (6x^3-6x^4)dx = \frac{3}{10}$

$\sigma^2 = E(X^2) - (E(X))^2 = \frac{3}{10} - (\frac{1}{2})^2 = \frac{1}{20} \implies \sigma = \sqrt{(\frac{1}{20})} = 0.223$

$P(\mu-2\sigma<X<\mu+2\sigma) = P(0.053 < X < 0.9472) = \int_{0.053}^{0.947} 6x(1-x)dx = \int_{0.053}^{0.947}(6x-6x^2)dx = 6(\frac{x^2}{2}-\frac{x^3}{3})\Big|_{0.053}^{0.9472} = 0.9838$

By Chebyshev's theorem,
$P(\mu-2\sigma<X<\mu+2\sigma) \geq 1-\frac{1}{2^2} = 0.75$
$P(\mu-2\sigma<X<\mu+2\sigma) = 0.9838 \geq 0.75$
Hence Chebyshev's theorem is verified.

*****Completed*****