Lecture 5
2.7 Bayes' Rule

The aim of this lecture is to explain the following concepts:

- Total Probability.
- Bayes' Rule

Theorem of total probability or the rule of elimination:

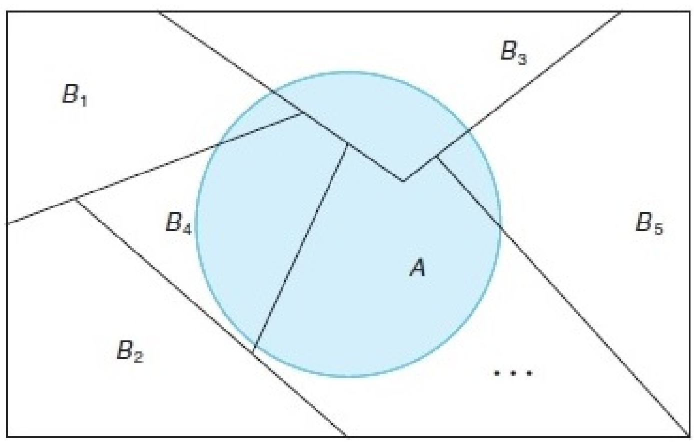

Figure 1: Partitioning the sample space S.

Theorem 0.1 *If the events $B_1, B_2, ..., B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for i = 1,2,.....,k, then for any event A of S,*

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i)$$

Theorem 0.2 *If the events $B_1, B_2, ..., B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for i = 1,2,.....,k, then for any event A of S,*

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)} \quad for\ r = 1, 2, ..., k.$$

Exercises:

95. In a certain region of the country it is known from past experience that the probability of selecting an adult over 40 years of age with cancer is 0.05. If the probability of a doctor correctly diagnosing a person with cancer as having the disease is 0.78 and the probability of incorrectly diagnosing a person without cancer as having the disease is 0.06, what is the probability that an adult over 40 years of age is diagnosed as having cancer?
Solution :
Consider the events:
C: an adult selected has cancer,
D: the adult is diagnosed as having cancer.
P(C) = 0.05,
$P(D|C) = 0.78$,
$P(C^{'}) = 0.95$
and $P(D|C^{'}) = 0.06$.
So, $P(D) = P(C \cap D) + P(C' \cap D)$
= (0.05)(0.78) + (0.95)(0.06)
= 0.096.

96. Police plan to enforce speed limits by using radar traps at four different locations within the city limits. The radar traps at each of the locations L1, L2, L3, and L4 will be operated 40%, 30%, 20% ,and 30% of the time. If a person who is speeding on her way to work has probabilities of 0.2, 0.1, 0.5, and 0.2, respectively, of passing through these locations, what is the probability that she will receive a speeding ticket?

Solution : Let $S_1, S_2, S_3,\ and\ S_4$ represent the events that a person is speeding as he passes through the respective locations and
let R represent the event that the radar traps is operating resulting in a speeding ticket.
Then the probability that he receives a speeding ticket:
$P(R) = \sum_{i=1}^{4} P(R|S_i)P(S_i)$
$= (0.4)(0.2) + (0.3)(0.1) + (0.2)(0.5) + (0.3)(0.2)$
$= 0.27.$

97. Referring to Exercise 2.95, what is the probability that a person diagnosed as having cancer actually has the disease?

Solution : $P(C|D) = \dfrac{P(C \cap D)}{P(D)} = \dfrac{0.039}{0.096} = 0.40625.$

98. If the person in Exercise 2.96 received a speeding ticket on her way to work, what is the probability that she passed through the radar trap located at L2?

Solution : $P(S_2|R) = \dfrac{P(R \cap S_2)}{P(R)} = \dfrac{0.03}{0.27} = 1/9.$