Boolean Algebra

Lecture-6

By
Bhagyalaxmi Behera
Asst. Professor (Dept. of ECE)

INTRODUCTION

- Because binary logic is used in all of today's digital computers and devices, the cost of the circuits that implement it is an important factor addressed by designers—be they computer engineers, electrical engineers, or computer scientists.

- Finding simpler and cheaper, but equivalent, realizations of a circuit can reap huge payoffs in reducing the overall cost of the design.

- Mathematical methods that simplify circuits rely primarily on Boolean algebra.

- Therefore, this topic provides a basic vocabulary and a brief foundation in Boolean algebra that will enable you to optimize simple circuits and to understand the purpose of algorithms used by software tools to optimize complex circuits involving millions of logic gates.

Boolean Algebra

Boolean algebra, like any other deductive mathematical system, may be defined

with

a set of elements, a set of operators, and

a number of unproved axioms or postulates.

BASIC DEFINITIONS

- The most common postulates used to formulate various algebraic structures are:

1. Closure. A set S is closed with respect to a binary operator if, for every pair of elements of S, the binary operator specifies a rule for obtaining a unique element of S.

For Example: N={1,2,3,4...}, is closed with respect to the binary operator + by the rules of arithmetic addition, since for any $a,b \in N$ we obtain a unique $c \in N$ by the operation $a+b=c$.

- The set of natural numbers is *not closed with respect to the binary operator - by the rules of arithmetic* subtraction, because $2-3=-1$ and $2,3 \in N$, while $(-1) \notin N$.

2. Associative law. A binary operator $*$ on a set S is said to be associative whenever

$(x * y) * z = x * (y * z)$ for all $x, y, z, \in S$

3. Commutative law. A binary operator * on a set *S is said to be commutative whenever*

$x * y = y * x$ for all $x, y \in S$

Identity element

4. Identity element. A set *S is said to have an identity element with respect to a binary* operation * on *S if there exists an element e* $\in$ *S with the property that*

*e * x = x * e = x for every x* $\in$ *S*

- *Example: The element 0 is an identity element with respect to the binary operator* + on the set of integers

  I = {. . ., -3, -2, -1, 0, 1, 2, 3,. . .},

  since x + 0 = 0 + x = x for any x $\in$ *I*

- The set of natural numbers, *N, has no identity element, since 0 is excluded from the set.*

Inverse

5. Inverse. A set *S having the identity element e with respect to a binary operator* * is said to have an inverse whenever, for every $x \in S$, *there exists an element* $y \in S$ such that

$$x * y = e$$

- *Example: In the set of integers, I, and the operator +, with e = 0, the inverse of* an element *a is (-a), since a + (-a) = 0.*

Distributive law

6. Distributive law. If $*$ and $\cdot$ are two binary operators on a set S, $*$ is said to be distributive over $\cdot$

$$x * (y \cdot z) = (x * y) \cdot (x * z)$$

Basic Definitions

- The operators and postulates have the following meanings:

The binary operator + defines addition. The additive identity is 0.

Ex. $a + 0 = 0 + a$

The additive inverse defines subtraction.

The binary operator $\cdot$ defines multiplication.

The multiplicative identity is 1. The multiplicative inverse of $a = 1/a$ defines division, i.e., $a \cdot 1/a = 1$

The only distributive law applicable is that of $\cdot$ over +:

$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Axiomatic Definition of Boolean Algebra

- Boolean algebra is defined by a set of elements, B, provided following postulates with two binary operators, + and $\cdot$, are satisfied:

1. (a) The structure is closed with respect to the operator +.
   (b) The structure is closed with respect to the operator . .

x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

The structure is *closed with respect to the two operators is obvious from the* tables, since the result of each operation is either 1 or 0 and 1, 0 $\in$ *B*.

2. (a) The element 0 is an identity element with respect to +;
   that is, $x + 0 = 0 + x = x$.
   (b) The element 1 is an identity element with respect to . ;
   that is, $x \cdot 1 = 1 \cdot x = x$.

3. (a) The structure is commutative with respect to +;
   that is, $x + y = y + x$.
   (b) The structure is commutative with respect to . ;
   that is, $x \cdot y = y \cdot x$.

4. (a) The operator . is distributive over +;
   that is, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.
   (b) The operator + is distributive over . ;
   that is, $x + (y \cdot z) = (x + y) \cdot (x + z)$.

x	y	z	$y + z$	$x \cdot (y + z)$	$x \cdot y$	$x \cdot z$	$(x \cdot y) + (x \cdot z)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

5. For every element $x \in B$, *there exists an element* $x \in B$ *(called the complement of x)* such that

(a) $x + x' = 1$

since 0 + 0’ = 0 + 1 = 1 and 1 + 1’ = 1 + 0 = 1.

(b) $x \, . \, x' = 0.$

since 0 . 0’ = 0 . 1 = 0 and 1 . 1’ = 1 . 0 = 0.

6. There exist at least two elements $x, y \in B$ *such that* $x \neq y$.

Duality principle

- *Duality principle states that every algebraic expression deducible from* the postulates of Boolean algebra remains valid if the operators and identity elements are interchanged.

- In a two-valued Boolean algebra, the identity elements and the elements of the set *B are the same: 1 and 0.*

- *The duality principle has many applications. If* the *dual of an algebraic expression is desired, we simply interchange OR and AND* operators and replace 1's by 0's and 0's by 1's.

Postulates and Theorems

Postulate 2	(a)	$x + 0 = x$	(b)	$x \cdot 1 = x$
Postulate 5	(a)	$x + x' = 1$	(b)	$x \cdot x' = 0$
Theorem 1	(a)	$x + x = x$	(b)	$x \cdot x = x$
Theorem 2	(a)	$x + 1 = 1$	(b)	$x \cdot 0 = 0$
Theorem 3, involution		$(x')' = x$		
Postulate 3, commutative	(a)	$x + y = y + x$	(b)	$xy = yx$
Theorem 4, associative	(a)	$x + (y + z) = (x + y) + z$	(b)	$x(yz) = (xy)z$
Postulate 4, distributive	(a)	$x(y + z) = xy + xz$	(b)	$x + yz = (x + y)(x + z)$
Theorem 5, DeMorgan	(a)	$(x + y)' = x'y'$	(b)	$(xy)' = x' + y'$
Theorem 6, absorption	(a)	$x + xy = x$	(b)	$x(x + y) = x$

Basic Theorems

Theorem 1(a): $x + x = x$

$= (x + x) \cdot 1$	by postulate 2(b)
$= (x + x) \cdot (x + x')$	5(a)
$= x + xx'$	4(b)
$= x + 0$	5(b)
$= x$	2(a)

Dual

Dual

back

Theorem 1(b): $x \cdot x = x$

$= x \cdot x + 0$	by postulate 2(a)
$= xx + xx'$	5(b)
$= x\,(x + x')$	4(a)
$= x \cdot 1$	5(a)
$= x$	2(b)

THEOREM 2(a): $x + 1 = 1.$

Statement	Justification
$x + 1 = 1 . (x + 1)$	*postulate 2(b)*
$= (x + x')(x + 1)$	*5(a)*
$= x + x' . 1$	*4(b)*
$= x + x'$	*2(b)*
$= 1$	5(a)

THEOREM 2(b): $x . 0 = 0$

by applying duality property.

THEOREM 6(a): $x + x\,y = x.$

(Method-1)

Statement	Justification
$x + x\,y = x\,.\,1 + x\,y$	*postulate 2(b)*
$= x(1 + y)$	*4(a)*
$= x(y + 1)$	*3(a)*
$= x\,.\,1$	*2(a)*
$= x$	*2(b)*

(Method-2)

x	y	xy	$x + xy$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

THEOREM 6(b): $x\,(x + y) = x$

by applying duality property.

DeMorgan's theorem

$$(x + y)' = x'y'$$

x	y	$x + y$	$(x + y)'$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

x'	y'	$x'y'$
1	1	1
1	0	0
0	1	0
0	0	0

Operator Precedence

- The operator precedence for evaluating Boolean expressions is

(1) parentheses

(2) NOT

(3) AND

(4) OR

In other words, expressions inside parentheses must be evaluated before all other operations. The next operation that holds precedence is the complement, and then follows the AND and, finally, the OR.