Lecture 27
Sampling Distributions (Contd.)

1 Sampling Distributions of S^2

In the last class, we have studied about the sampling distribution of $\overline{X}$. In this Section we will get to know about the sampling distribution of S^2. First let us study two important continuous distributions such as χ^2 and Student's t-distribution.

Definition 1 *Let a continuous random variable X has Chi-squared distribution with n degrees of freedom(DF), then it has the density*

$$f(x) = \frac{1}{\Gamma(1/2)2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, \ x > 0.$$

Remark 1.1 *We say $X \sim \chi_n$, then $X \sim G(n/2, 2)$ distribution. See figure 1.*

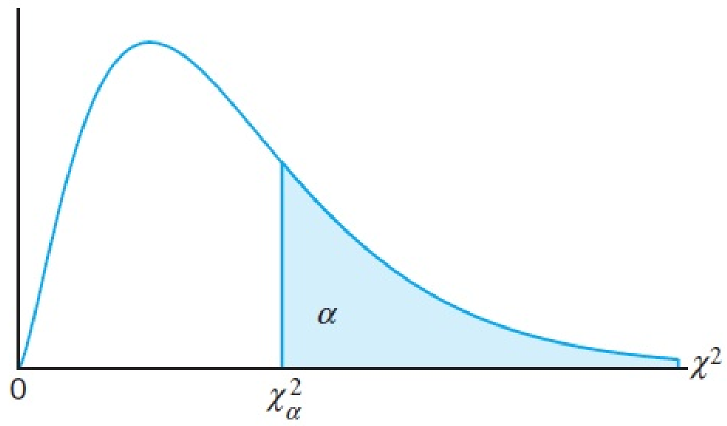

Figure 1: The chi-squared distribution

Define χ^2_α such that $P(\chi^2 > \chi^2_\alpha) = \alpha$, which is also called $100(1-\alpha)\%$ critical point. It is equal to the area under to the curve to the right of this value. Table of critical values of Chi-Squared distribution have been tabulated. students are advised to find out these critical points from the table for various values of α and v. For example if $v = 7$ and $\alpha = 0.05$ yields the value of $\chi^2_{0.05} = 14.067$.

Theorem 1.1 *If S^2 is the variance of a random sample of size n taken from a normal population having the variance σ^2, then the statistic*

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{\sigma^2}$$

has a chi-squared distribution with $v = n - 1$ degrees of freedom.

Theorem 1.2 *Let Z be a standard random variable and V a chi-squared random varibale with v DF. If Z and V are independent, then the distribution of the random varibale T, where*

$$T = \frac{Z}{\sqrt{V/v}}$$

is given by the density function

$$h(t) = \frac{\Gamma[(v+1)/2]}{\Gamma(v/2)\sqrt{\pi v}} \left(1 + \frac{t^2}{v}\right)^{-\frac{v+1}{2}}, \quad -\infty < t < \infty.$$

This is known as the **t-distribution** with v DF. As a consequence of the above result, we may state the following corollary.

Corrolary 1.1 *Let $X_1, X_2, \ldots, X_n$ be n be independent normal random variables with mean μ and variance σ^2. Then the random variable $T = \sqrt{n}\left(\frac{\overline{X}-\mu}{S}\right)$ has a t-distribibition with $v = n-1$ DF.*

The distribution of T is similar to the distribution of Z in that both are symmetric about a mean of 0. However for large values of n, both the distributions behave alike. we can conclude the following theorem.

Theorem 1.3 *Let T follows a t-distribution with n DF. As $n \to \infty$, the density of T converges to a $N(0,1)$ distribution.*

Usually, when $n \geq 30$, both the distribution behave same. Students can find out the value of t_α, where t_α such that $P(T > t_\alpha) = \alpha = P(T < -t_\alpha)$, see figure. For example,

Example 1 *The t-value with $v = 14$ DF that leaves an area of 0.025 to the left, and therefore an area of 0.975 to the right, is*

$$t_{0.975} = -t_{0.025} = -2.145.$$

Example 2 *Find $P(-t_{0.025} < T < t_{0.05})$.*
Answer:- *Since $t_{0.05}$ leaves an area 0.05 to the right, and $-t_{0.025}$ leaves an area of 0.025 to the left, we find a total area of*

$$1 - 0.05 - 0.025 = 0.925$$

between $-t_{0.025}$ and $t_{0.05}$. Hence

$$P(-t_{0.025} < T < t_{0.05}) = 0.925.$$

Question-37:-For a chi-squared distribution, find
(a) $\chi^2_{0.025}$ when $v = 15$,
(b)$\chi^2_{0.01}$ when $v = 7$;
(c)$\chi^2_{0.05}$ when $v = 24$
. **Answer:-**From the table it can be seen that a)27.488,
b) 18.475 and c) 26.217.

Question-41:-Assume the sample variances to be continuous measurements. Find the probability that a random sample of 25 observations, from a normal population with variance $\sigma^2 = 6$, will have a sample variance S^2

(a) greater than 9.1;

(b) between 3.462 and 10.745.

Answer:-a) From the question $P(S^2 > 9.1) = P\Big(\frac{(n-1)S^2}{\sigma^2} > 36.4\Big) = P\Big(\chi^2 > 36.4\Big) = 0.05.$

b)$P\Big(3.462 < S^2 < 10.745\Big) = P\Big(13.848 < \frac{(n-1)S^2}{\sigma^2} < 42.980\Big) = 0.95 - 0.01 = 0.94.$

Assignments- Q- 40, 45 of page 285.