LECTURE-24 and 25

Moments and Moment-Generating Functions

Chapter-7.3:

Moments:

The **rth moment** about the origin of the random variable X is given by

$$\mu'_r = E(X^r) = \begin{cases} \sum_x x^r f(x), & if \;\; \text{X} \;\; is\ discrete \\ \int_{-\infty}^{\infty} x^r f(x)dx, & if \;\; \text{X} \;\; is\ continuous. \end{cases}$$

Moment-Generating Functions:

The **Moment-Generating Function** of the random variable X is given by $E(e^{tX})$

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx} f(x), & if \;\; \text{X} \;\; is\ discrete \\ \int_{-\infty}^{\infty} e^{tx} f(x)dx, & if \;\; \text{X} \;\; is\ continuous. \end{cases}$$

Questions:

7.17: A random variable X has the discrete uniform distribution

$$f(x,k) = \begin{cases} \frac{1}{k}, & i = 1, 2, 3, \ldots, k, \\ 0, & elsewhere. \end{cases}$$

Show that the moment-generating function of X is

$$M_X(t) = \frac{e^t(1 - e^{kt})}{k(1 - e^t)}$$

Ans: Given

$$f(x,k) = \begin{cases} \frac{1}{k}, & i = 1, 2, 3, \ldots, k, \\ 0, & elsewhere. \end{cases}$$

Moment-Generating Function of r.v. X is

$$M_X(t) = E(e^{tX})$$
$$= \sum_0^\infty e^{tx} f(x) = \sum_0^k e^{tx} \tfrac{1}{k}$$
$$= \tfrac{1}{k}(e^t + e^{2t} + \ldots + e^{kt})$$
$$= \tfrac{e^t}{k}(1 + e^t + \ldots + e^{(k-1)t})$$
$$= \tfrac{e^t}{k}\left(\tfrac{(e^t)^{k-1+1}-1}{e^t-1}\right)$$
$$= \tfrac{e^t(1-e^{kt})}{k(1-e^t)} \tag{1}$$

Note: From Binomial expansion $1 + x + x^2 + x^3 + \ldots + x^n = \frac{x^{n+1}-1}{x-1}$

<u>**7.19**:</u> A random variable X has the Poisson distribution

$$p(x, \mu) = \frac{e^{-\mu}\mu^x}{x!} \ for x = 0, 1, 2, ...$$

Show that the moment-generating function of X is

$$M_X(t) = e^{\mu(e^t-1)}$$

Using $M_X(t)$, find the mean and variance of the Poisson distribution.

Ans: Given X is a random variable having Poisson distribution

$$p(x, \mu) = \frac{e^{-\mu}\mu^x}{x!} \ for \ \ x = 0, 1, 2, ...$$

So moment-generating function of X is given by

$$M_X(t) = E(e^{tX})$$
$$= \sum_0^\infty e^{tx} f(x)$$
$$= \sum_0^\infty e^{tx} \tfrac{e^{-\mu}\mu^x}{x!}$$
$$= e^{-\mu} \sum_0^\infty \tfrac{(\mu e^t)^x}{x!}$$
$$= e^{-\mu}\left(1 + \tfrac{(\mu e^t)}{1!} + \tfrac{(\mu e^t)^2}{2!} + \tfrac{(\mu e^t)^3}{3!} + \ldots\right)$$
$$= e^{-\mu} e^{\mu e^t}$$
$$= e^{\mu(e^t-1)} \tag{2}$$

$$M_X(t) = e^{\mu(e^t-1)}$$

$$\frac{dM_X(t)}{dt} = e^{\mu(e^t-1)}\mu e^t$$

$$\frac{d^2M_X(t)}{dt^2} = e^{\mu(e^t-1)}(\mu e^t)^2 + e^{\mu(e^t-1)}\mu e^t$$

Mean of r.v. X is

$$E(X) = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \left.e^{\mu(e^t-1)}\mu e^t\right|_{t=0} = \mu$$

$$E(X^2) = \left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} = \mu^2 + \mu$$

Variance of r.v. X is

$$V(X) = E(X^2) - (E(X))^2 = \mu^2 + \mu - \mu^2 = \mu$$

<u>7.20</u>: The moment-generating function of a certain Poisson random variable X is given by

$$M_X(t) = e^{4(e^t-1)}$$

Find

$$P(\mu - 2\sigma < X < \mu + 2\sigma)$$

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Ans: Given moment-generating function of a certain Poisson random variable X is

$$M_X(t) = e^{4(e^t-1)}$$

So mean $\mu = 4$ and variance $\sigma^2 = 4 \Rightarrow \sigma = 2$

Now

$$\begin{aligned} &P(\mu - 2\sigma < X < \mu + 2\sigma) \\ &= P(0 < X < 8) \\ &= \textstyle\sum_1^7 f(x) \\ &= \textstyle\sum_1^7 \frac{e^{-4}4^x}{x!} \\ &= .9489 \end{aligned} \tag{3}$$

Note: From book, page 810 (Poisson distribution table for $r = 7$ and $\mu = 4$)

Relation between moment-generating function $M_X(t)$ and rth moment μ'_r:

Let X be a random variable with moment-generating function $M_X(t)$. Then

$$\left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0} = \mu'_r$$

Example 7.6: Find the moment-generating function of the binomial random variable X and then use it to verify that $\mu = np$ and $\sigma^2 = npq$.

Ans: We know from binomial distribution, probability density function is

$$f(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & if \;\; \text{x} = 0, 1, \dots n \\ 0, \quad elsewhere. \end{cases}$$

$$M_X(t) = E(e^{tX}) = \sum_x e^{tx} f(x) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x} = (pe^t + q)^n$$

1^{st} moment or Mean of binomial distribution is

$$\mu'_1 = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \left.\frac{d(pe^t + q)^n}{dt}\right|_{t=0} = \left. n(pe^t + q)^{n-1} pe^t \right|_{t=0} = np$$

2^{nd} moment of binomial distribution is

$$\begin{aligned} E(X^2) = \mu'_2 &= \left.\tfrac{d^2 M_X(t)}{dt^2}\right|_{t=0} \\ &= \left.\tfrac{d^2 (pe^t+q)^n}{dt^2}\right|_{t=0} \\ &= \left.\tfrac{d}{dt}\left(\tfrac{d(pe^t+q)^n}{dt}\right)\right|_{t=0} \\ &= \left.\tfrac{d}{dt}\left(n(pe^t + q)^{n-1} pe^t\right)\right|_{t=0} \\ &= \left. n(n-1)(pe^t + q)^{n-2}(pe^t)^2 + n(pe^t + q)^{n-1} pe^t \right|_{t=0} \\ &= n(n-1)p^2 + np = n^2p^2 + np - np^2 \end{aligned} \tag{4}$$

Variance of binomial distribution is

$\sigma^2 = \mu'_2 - {\mu'_1}^2 = \mu'_2 - \mu^2 = n^2p^2 + np - np^2 - (np)^2 = np(1-p) = npq.$

Let X and Y be two random variables with moment-generating functions $M_X(t)$ and $M_Y(t)$ respectively. If $M_X(t) = M_Y(t)$ for all values of t, then X and Y have the same probability distribution.

$$M_{X+a}(t) = e^{at} M_X(t)$$

$$M_{aX}(t) = M_X(at)$$

If $X_1, X_2, X_3, \ldots, X_n$ are independent random variables with moment-generating functions $M_{X_1}(t), M_{X_2}(t), M_{X_3}(t), \ldots, M_{X_n}(t)$ respectively, and $Y = X_1 + X_2 + \ldots + X_n$, then

$$M_Y(t) = M_{X_1}(t) M_{X_2}(t) M_{X_3}(t) \ldots M_{X_n}(t)$$

Assignment Question: Show that the moment-generating function of the random variable X having a normal probability distribution with mean μ and variance σ^2 is given by

$$M_X(t) = exp(\mu t + \frac{1}{2}\sigma^2 t^2)$$