CHAPTER-1:- INTRODUCTION TO STATISTICS AND DATA ANALYSIS

Ans 1.- (a) 15

(b) $\bar{x} = \frac{1}{15}(3.4+2.5+4.8+\ldots+4.8) = 3.787$

(c) Sample median is the 8^{th} value, after arranging the data in increasing order = 3.6.

(e) After trimming 20% of the highest data and 20% of the lowest data, the trimmed mean is,

$$\bar{x}_{20tr} = \frac{1}{9}(2.9+3.0+3.3+3.4+3.6+3.7+4.0+4.4+4.8)$$
$$= 3.678$$

Ans 2.- (a) Mean = 20.768, Median = 20.610

(b) $\bar{x}_{10tr} = 20.743$

Ans 4.- Mean for company A = 7.950   Median for company A = 8.250

Mean for company B = 10.260   Median for company B = 10.150

Ans 8.- $\bar{X} = 20.768$

$$S^2 = \frac{1}{20-1}\left[(18.71-20.768)^2+(21.41-20.768)^2+\ldots+(21.12-20.768)^2\right]$$
$$= 2.5345$$
$$S = \sqrt{2.5345} = 1.592$$

Ans 11.- $\bar{X}_C = 5.6$ $\bar{X}_T = 7.6$

$S_C^2 = 69.39 \Rightarrow S_C = 8.33$

$S_T^2 = 128.14 \Rightarrow S_T = 11.32$

Ans 19.- (a)

Stem	Leaf	Frequency
0	22233457	8
1	023558	6
2	035	3
3	0.3	2
4	057	3
5	0569	4
6	0005	4

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(b) Relative frequency distribution table

Class interval	Class midpoint	Frequency, f	Relative frequency
0.0 – 0.9	0.45	8	0.267
1.0 – 1.9	1.45	6	0.200
2.0 – 2.9	2.45	3	0.100
3.0 – 3.9	3.45	2	0.067
4.0 – 4.9	4.45	3	0.100
5.0 – 5.9	5.45	4	0.133
6.0 – 6.9	6.45	4	0.133

(c) $\bar{x} = 2.797$, $S^2 = 4.959$ (Variance)

$S = 2.227$ (Standard deviation)

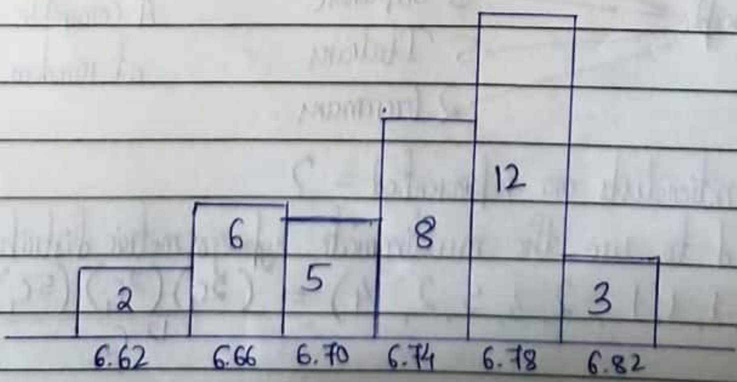
Sample range $= 6.5 - 0.2 = 6.3$

Ans 22.- (a) $\bar{x} = 6.7261$, $S^2 = 0.002872$

$S = 0.0536$

(b)

Class interval	Class midpoint	Frequency, f	Relative frequency
6.60 – 6.64	6.62	2	0.055
6.64 – 6.68	6.66	6	0.166
6.68 – 6.72	6.7	5	0.138
6.72 – 6.76	6.74	8	0.222
6.76 – 6.80	6.78	12	0.333
6.80 – 6.84	6.82	3	0.083



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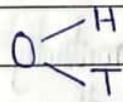
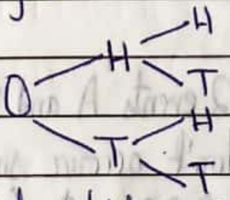
CHAPTER-02:- PROBABILITY

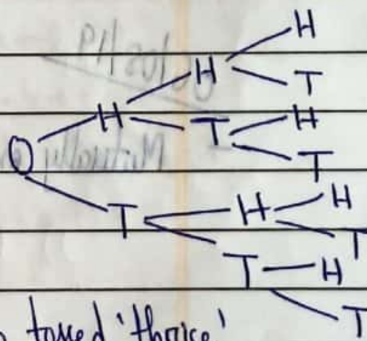
- Scientific experiment :- Any experiment is said to be scientific if the occurrence of the experiment can be predicted with certainity.

- Statistical experiment :- Any experiment is said to be statistical or random if it satisfies -
  (i) The set of all possible outcomes of this experiment is well known in advance.
  (ii) The exact outcome of the experiment cannot be predicted with certainity.
  (iii) The experiments must be performed under identical conditions.

- Sample space :- The set of all possible outcomes of a statistical experiment is called the sample space of that experiment.
  It is denoted by S.

S_1 - 
Coin tossed 'once'

S_2 - 
Coin tossed 'twice'

S_3 - 
Coin tossed 'thrice'

- Event :- Let S be a given sample space. Any ~~subject~~ subset of the sample space is called an event.
  If A and B are 2 events, then -
  A = Occours, B = Occours
  $A \cup B$ = Either A occours or B occours (at least one occours)
  $A \cap B$ = Both A and B occours
  ϕ = Impossible to occour
  $A \cap B^c$ = A occours but B doesn't occours.

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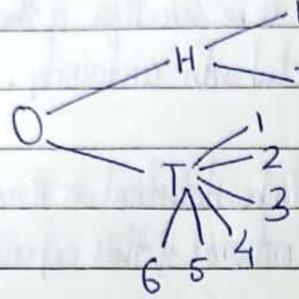
Q.- Example - 2.2, Page - 36

Ans.-



$S = \{HH, HT, T1, T2, T3, T4, T5, T6\}$

Q.- Example - 2.3, Page - 37

Ans.- $S = \{DDD, DDN, DND, NDD, DNN, NDN, NND, NNN\}$

06/08/19

- Mutually exclusive :- 2 events A and B are said to be mutually exclusive, if they don't occour simultaneously or in other words,
  $A \cap B = \phi$ (Both can't occour simultaneously)

- Mutually exhaustive :- 2 events are said to be mutually exhaustive if no event or events can be outside the occurrence of $A \cup B$ or in other words,
  $A \cup B = S$.

→ Exercises → P-42

7.- $S_1 = \{MMMM, MMMF, MMFM, MFMM, FMMM, MMFF, MFFM, FFMM, FMFM, MFMF, FMMF, FFFM, FFMF, FMFF, MFFF, FFFF\}$

$S_2 = \{0, 1, 2, 3, 4\}$

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5.- $S = \{2H, 2T, 4H, 4T, 6H, 6T, 1HH, 1HT, 1TH, 1TT, 3HH, 3HT, 3TH, 3TT, 5HH, 5HT, 5TH, 5TT\}$

9.- $S = \{2H, 2T, 1HH, 1HT, 1TH, 1TT\}$

11.- $S = \{M_1M_2, M_2M_1, M_1F_1, M_1F_2, F_1M_1, F_2M_1, M_2F_1, M_2F_2, F_1M_2, F_2M_2, F_1F_2, F_2F_1\}$

$A = \{M_1M_2, M_2M_1, M_1F_1, M_1F_2, M_2F_1, M_2F_2\}$

3.- $A = \{1, 3\}$, $B = \{x \mid x$ is a number on a die$\}$
$C = \{x \mid x^2 - 4x + 3 = 0\}$, $D = \{x \mid x$ is the number of heads when 6 coins are tossed$\}$

Here, $A = C$

$A = \{1, 3\}$
$B = \{1, 2, 3, 4, 5, 6\}$
$C = \{1, 3\}$
$D = \{1, 2, 3, 4, 5, 6\}$

$x^2 - 4x + 3 = 0$
$\Rightarrow x^2 - 3x - x + 3 = 0$
$\Rightarrow x(x-3) - 1(x-3) = 0$
$\Rightarrow (x-3)(x-1) = 0$
$\Rightarrow x = 3, 1$

10/08/19

- Probability of an event :- Let S be a given sample space, 'A' be any event ($A \subseteq S$), then we can define probability as a set function from S to R ($P: S \rightarrow R$) satisfying—
  (i) Probability of any event A is, $0 \leq P(A) \leq 1$.
  (ii) Probability of sample space is 1, $P(S) = 1$.
  (iii) If $A_1, A_2, \ldots, A_k$ are pairwise disjoint (mutually exclusive) events, then
  $$P(A_1 \cup A_2 \cup \ldots \cup A_k) = P(A_1) + P(A_2) + \ldots + P(A_k)$$

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- Consequences – (i) If ϕ is an empty set, then

$$P(\phi) = 0.$$

In (iii), if $A_1, A_2 = A_3 = \dots = A_k = \phi$,

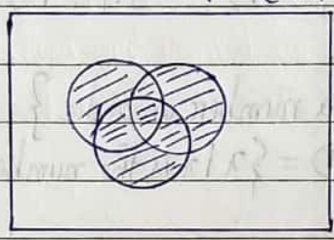
$$\Rightarrow P(A_1) = P(A_1) + (k-1) P(\phi)$$

$$\Rightarrow P(\phi) = 0.$$

(ii) Addition theorem,

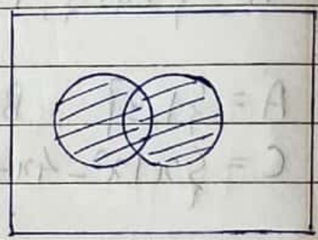
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$



$P(A \cup B \cup C)$

$P(A \cup B)$



(iii) If $A \subseteq B$, then,

$$P(A) \leqslant P(B)$$

$$P(A \cup B - A) = P(A) + P(B - A) \quad A \text{ and } B - A \text{ are mutually exclusive}$$

$$\geqslant P(A)$$

$$\Rightarrow P(B) \geqslant P(A)$$

(iv) $A \subseteq B \Rightarrow P(B - A) = P(B) - P(A)$

→ Exercises → P-59

53.- $P(S) = 0.7$, $P(B) = 0.4$, $P(S \cup B) = 0.8$

(i) $P(S \cap B) = P(S) + P(B) - P(S \cup B)$

$$= 0.7 + 0.4 - 0.8$$

$$= 0.3$$

(ii) $P(S \cup B)^c = 1 - P(S \cup B)$

$$= 1 - 0.8$$

$$= 0.2$$

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59.- (i) $P(3 \text{ Aces}) = \dfrac{{}^4C_3 \cdot {}^{48}C_2}{{}^{52}C_5}$

(ii) $P(4 \text{ Hearts and } 1 \text{ Club}) = \dfrac{{}^{13}C_4 \cdot {}^{13}C_1}{{}^{52}C_5}$

68.- X = Number of persons purchasing oven.

(i) $P(X \leq 2) = 0.4$

$\Rightarrow P(X \geq 3) = 1 - P(X \leq 2)$
$= 1 - 0.4$
$= 0.6$

(ii) $P(X = 6E) = 0.007$
$P(X = 6G) = 0.104$
$\Rightarrow P(6E \cup 6G) = 0.007 + 0.104$
$= 0.111$

So, the probability that atleast one of each type is purchased $= (1 - 0.111)$
$= 0.889$

50.- Assuming equal weights, $P(A) = 5/18$
$P(C) = 1/3$
$P(A \cap C) = 7/36$

58.- (i) Of the $6 \times 6 = 36$ elements in the sample space, only 5 elements (2,6), (3,5), (4,4), (5,3), and (6,2) add to 8. Hence the probability of obtaining a total of 8 is -

$P = 5/36$

(ii) 10 of the total elements in sample space add upto atmost 5 - (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2), (4,1). So, the probability of obtaining a total of at most 5 is -

$P = 10/36 = 5/18$.

64.- (i) $P(X > 6000) = 0.42$

$\Rightarrow P(X \leq 6000) = 1 - 0.42 = 0.58$

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(ii) $P(X \leqslant 4000) = 0.04$

$\Rightarrow P(X > 4000) = 1 - 0.04$

$= 0.96$

72:- (i) P(Meets specification) = 0.95

P(Too light) = 0.002

$\Rightarrow$ P(Too heavy) = 1 − 0.95 − 0.002 = 0.048

(ii) If all packages meet expectations, then total profit -

$(\$25 - \$20) \times 10{,}000 = \$50{,}000$

(iii) If all defective packages are rejected, the total profit reduced -

$(0.05) \times (10{,}000) \times \$5 + (0.05) \times (10{,}000) \times \$20 = \$12{,}500$

13/08/19

- Conditional probability :- Let 'S' be a given sample set and A and B are events $A, B \subseteq S$, we define conditional probability as

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

or, $$P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$$

- Independent events :- 2 events A and B are said to be independent if,

$P(A/B) = P(A)$

or, $P(B/A) = P(B)$

- Multiplication rule :- $P(A \cap B) = P(A/B) \cdot P(B)$

or, $P(A \cap B) = P(B/A) \cdot P(A)$

- Remark - If A and B are independent,

$P(A \cap B) = P(A) \cdot P(B)$ or,

$P(A \cap B) = P(B) \cdot P(A)$

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- General multiplication rule :- Let $(A_1, A_2, \ldots A_k)$ are k events of a given sample space, then,

$$P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1) . P(A_2/A_1) . P(A_3/A_1 \cap A_2) \ldots . P(A_k/A_1 \cap A_2 \cap \ldots \cap A_{k-1})$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) . P(A_2/A_1) . P(A_3/A_1 \cap A_2)$$
$$= P(A_1) . \frac{P(A_1 \cap A_2)}{P(A_1)} . \frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1 \cap A_2)}$$

Eg 37.-(66)

| 4 White, 3 Black | → | 3 White, 5 Black | → Black

ω_1 = White on first draw
ω_2 = White on second draw
Similarily, B_1 = Black on first draw
and, B_2 = Black on second draw

$$P(\omega_1 \cap B_2) + P(B_1 \cap B_2)$$
$$= P(B_2/\omega_1) P(\omega_1) + P(B_2/B_1) P(B_1)$$
$$= \left(\frac{5}{9} \times \frac{4}{7}\right) + \left(\frac{6}{9} \times \frac{3}{7}\right)$$
$$= \frac{20}{63} + \frac{18}{63}$$
$$= \frac{38}{63} = 0.6032$$

→ Exercises → P-69

75.-

Education	Male	Female	
Elementary	38	45	→ 83
Secondary	28	50	→ 78
College	22	17	→ 39
	↓ 88	↓ 112	

(i) $P(M/S) = \frac{P(M \cap S)}{P(S)} = \frac{28/200}{78/200} = \frac{14}{39}$

(ii) $P(C^c/F) = \frac{P(C^c \cap F)}{P(F)} = \frac{95/200}{112/200} = \frac{95}{112}$

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74.- No. of students — Grade A

Juniors – 10 — 3

Seniors – 30 — 10

Graduate – 10 — 5

$\downarrow$ 50 — $\downarrow$ 18

$P(S/A) = 10/18 = 5/9$

77.- Mathematics – 42

Psychology – 68

History – 54

Both Mathematics & History – 22

Both Mathematics & Psychology – 25

History, No Maths, No Psychology – 7

All 3 – 10

None of the 3 – 8

(i) $P(M \cap P \cap H) = 10/68 = 5/34$

(ii) $P(H \cap M / P^c) = \dfrac{P(H \cap M \cap P^c)}{P(P^c)} = \dfrac{22-10}{100-68} = \dfrac{12}{32} = \dfrac{3}{8}$

80.- $P(\text{Oil change}) = 0.25$

$P(\text{New oil filter}) = 0.40$

$P(\text{Both}) = 0.14$

(i) $P(\text{New oil filter}/\text{oil change}) = \dfrac{P(\text{New oil filter} \cap \text{Oil change})}{P(\text{oil change})} = \dfrac{0.14}{0.25} = 0.56$

(ii) $P(\text{oil change}/\text{New oil filter}) = \dfrac{P(\text{Oil change} \cap \text{New oil filter})}{P(\text{New oil filter})} = \dfrac{0.14}{0.40} = 0.35$

89.- Let A and B represent the availability of each fire engine. P(Available) = 0.96

(i) $P(A^c \cap B^c) = P(A^c)\,P(B^c)$

$= (0.04)(0.04) = 0.0016$

(ii) $P(A \cup B) = 1 - P(A^c \cap B^c)$

$= 1 - 0.0016$

$= 0.9984$

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93.- This is a parallel system of 2 series subsystems.

(i) $P = 1 - [1-(0.7)(0.7)][1-(0.8)(0.8)(0.8)]$

$= 0.75112$

(ii) $P = \frac{P(A^c \cap C \cap D \cap E)}{P(\text{System works})} = \frac{0.3 \times 0.8 \times 0.8 \times 0.8}{0.75112}$

$= 0.2045$

94.- Let $P(S)$ be the probability that the system works.

$\Rightarrow P(A^c/S^c) = \frac{P(A^c \cap S^c)}{P(S^c)}$

$= \frac{P(A^c)(1 - P(C \cap D \cap E))}{1 - P(S)}$

$= \frac{(0.3)[1-(0.8)(0.8)(0.8)]}{1 - 0.75112}$

$= 0.588$

14/08/19

- Total probability theorem :- If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the given sample space S such that

$P(B_i) \neq 0, \quad i = 1, 2, \ldots, k$

then for any event A

$P(A) = \sum_{i=1}^{k} P(A/B_i) P(B_i)$

By a partition, we mean,

(i) $B_i \cap B_j = \phi, \; i = j$ (Nothing in common, Mutually exclusive)

(ii) $\bigcup_{i=1}^{k} P_i = S$

- Proof :- Now, $A = A \cap S = A \cap (B_1 \cup B_2 \cup \ldots \cup B_k)$

$= \underbrace{(A \cap B_1)}_{A_1} \cup \underbrace{(A \cap B_2)}_{A_2} \cup \ldots \cup \underbrace{(A \cap B_k)}_{A_k}$

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It can be seen that A is also mutually exclusive

$$A_i \cap A_j = (A \cap B_i) \cap (A \cap B_j)$$
$$= A \cap (B_i \cap B_j)$$
$$= A \cap \phi$$
$$= \phi$$

$$P(A) = P(A_1 \cup A_2 \cup \dots \cup A_k)$$
$$= P(A_1) + P(A_2) + \dots + P(A_k)$$
$$= \sum_{j=1}^{k} P(A_j)$$

$$P(A) = \sum_{j=1}^{k} P(A \cap B_j)$$

$$\Rightarrow P(A) = \sum_{j=1}^{k} P(A/B_j)\, P(B_j)$$

Hence, proved.

17/08/19

- Baye's Theorem :- Let $B_1, B_2, \dots, B_k$ form a partition on a given sample space S, $P(B_i) \neq 0$, then for any event A, we have

$$P(B_i/A) = \frac{P(A/B_i) \cdot P(B_i)}{\sum_{j=1}^{k} P(A/B_j) \cdot P(B_j)}$$

- Proof :- We know that, $P(B_i/A) = \frac{P(A \cap B_i)}{P(A)}$

$$= \frac{P(A/B_i) \cdot P(B_i)}{P(A)}$$

$$= \frac{P(A/B_i)\, P(B_i)}{\sum_{j=1}^{k} P(A/B_j) \cdot P(B_j)}$$

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→ Exercises → Page – 76

95.- B_1 = A person above 40 has cancer. $P(B_1) = 0.05$
B_2 = A person above 40 doesn't have cancer. $P(B_2) = 0.95$
A = A person is ~~correctly~~ diagonised as having cancer $P(A/B_1) = 0.78$
B = A person is incorrectly diagonised as having cancer $P(A/B_2) = 0.06$

$$P(A) = P(A/B_1).P(B_1) + P(A/B_2).P(B_2)$$
$$= 0.78 \times 0.05 + 0.06 \times 0.95$$
$$= 0.096$$

97.- $$P(B_1/A) = \frac{P(A/B_1).P(B_1)}{P(A)}$$
$$= \frac{0.78 \times 0.05}{0.096}$$
$$= 0.406$$

96.-

L_1	L_2	L_3	L_4
40%	30%	20%	30%
0.2	0.1	0.5	0.2
$P(L_1)$	$P(L_2)$	$P(L_3)$	$P(L_4)$

A - Girl recieves a speeding ticket.
$$P(A) = P(A/L_1).P(L_1) + P(A/L_2).P(L_2) + P(A/L_3).P(L_3) + P(A/L_4).P(L_4)$$
$$= 0.4 \times 0.2 + 0.3 \times 0.1 + 0.2 \times 0.5 + 0.3 \times 0.2$$
$$= 0.27$$

98.- $$P(L_2/A) = \frac{P(A/L_2).P(L_2)}{P(A)}$$
$$= \frac{0.3 \times 0.1}{0.27}$$
$$= 0.11$$

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104.- Let A_i be the event that the ith patient is allergic to some type of weed. $P(A) = \frac{1}{2}$

(a) $P(A_1 \cap A_2 \cap A_3 \cap A_4^c) + P(A_1 \cap A_2 \cap A_3^c \cap A_4) + P(A_1 \cap A_2^c \cap A_3 \cap A_4) + P(A_1^c \cap A_2 \cap A_3 \cap A_4) =$

$P(A_1)P(A_2)P(A_3)P(A_4^c) + P(A_1)P(A_2)P(A_3^c)P(A_4) + P(A_1)P(A_2^c)P(A_3)P(A_4) + P(A_1^c)P(A_2)P(A_3)P(A_4)$

$= 4 \times \left(\frac{1}{2}\right)^4$

$= \frac{1}{4} = 0.25$

(b) $P(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c) = P(A_1^c)P(A_2^c)P(A_3^c)P(A_4^c)$

$= \left(\frac{1}{2}\right)^4$

$= \frac{1}{16} = 0.0625$

108.- M = Person makes a mistake $P(M) = 0.1$, $P(M^c) = 0.9$

(a) $P(M_1 \cap M_2 \cap M_3 \cap M_4) = (0.1)^4$

$= 0.0001$

(b) $P(J \cap C \cap R^c \cap W^c) = P(J)P(C)P(R^c)P(W^c)$

$= (0.1)(0.1)(0.9)(0.9)$

$= 0.0081$

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Ans 14.- $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$A = \{0, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7, 9\}$

$C = \{2, 3, 4, 5\}$, $D = \{1, 6, 7\}$

(i) $A \cup C = \{0, 2, 3, 4, 5, 6, 8\}$

(ii) $A \cap B = \{\phi\}$

(iii) $C^c = \{0, 1, 6, 7, 8, 9\}$

(iv) $(C^c \cap D) \cup B = \{1, 3, 5, 6, 7, 9\}$

(v) $(S \cap C)^c = \{0, 1, 6, 7, 8, 9\}$

(vi) $A \cap C \cap D^c = \{2, 4\}$

Ans 30.- With $n_1 = 2$ choices for the first question, $n_2 = 2$ choices for the second question and so on, the generalised multiplication rule yeilds -

No. of ways a true false test consisting of 9 questions can be answered

$= n_1 \cdot n_2 \cdot n_3 \cdots n_9$

$= 2^9 = 512$

Ans 37.- The first seat must be filled by a girl as there are 5 girls and 4 boys sitting in alternate and the other way round is not possible. So, the second seat is filled by a boy. Continuing in a similar manner, we get,

No. of ways 4 boys are 5 girls can sit in a row if the boys and girls must alternate.

$= 5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1$

$= 2880$

Ans 43.- We know that, the number of permutations of n objects arranged in a circle is $(n-1)!$ So,

Number of ways in which 5 different trees can be planted in a circle is

$(5-1)! = 4! = 24$

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Ans 65.- $P(>6000) = 0.42$ $\quad$ $P(A) = 0.2$

$P(\leq 4000) = 0.04$ $\quad$ $P(B) = 0.35$

(a) $P(A^c) = 1 - 0.2 = 0.8$

(b) $P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - 0.2 - 0.35 = 0.45$

(c) $P(A \cup B) = 0.2 + 0.35 = 0.55$

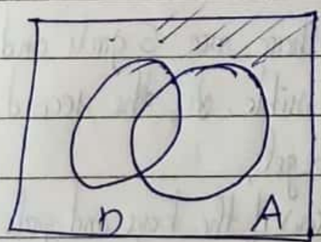
$A \rightarrow F$

$D \rightarrow D$ strain

$B = \bar{F} \cap D = (\bar{A} \cap D)$

$P(B) = P(\bar{F} \cap D)$

$=$



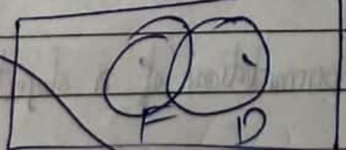
$P(\bar{D} \cap \bar{A}) = 1 - P(A \cup D)$

$= \cancel{1 - P(A) - P(D) + P(A \cap D)}$

$= \cancel{P} = 1 - P(A) - P(D) + P(A \cap D)$

$\cancel{P(B)} =$

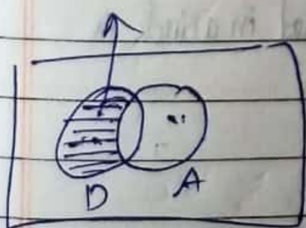
$=$



$P(\bar{A} \wedge \bar{D}) = 1 - P(A \cup D)$

$= \cancel{P(D) - P(D \cap \bar{A})}$

$D \cap \bar{A}$



$\cancel{P(A \cap B)}$

$= 1 - [P(D \cap \bar{A}) + P(A)]$

$= 1 - 0.35 - 0.20$

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CHAPTER-03:- RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

- Random Variable:- A random variable is a function that associates a real number with each element of the sample space.

$$X : S \to \mathbb{R}$$

i.e, for every $\omega \in S$, we have, $X(\omega)$ is a real number and the inverse images of X are called events.

Eg1.- A coin is tossed 3 times.

X = Number of heads $= \{0, 1, 2, 3\}$

So, $X(HTH) = 2$, $X(TTT) = 0$, $X(HHH) = 3$, $X(TTH) = 1$

$X^{-1}(1) = \{HTT, THT, TTH\} \subseteq S$

$X^{-1}(2) = \{HHT, HTH, THH\}$

$X^{-1}(3) = \{HHH\}$

$X^{-1}(0) = \{TTT\}$

Eg2.- A die is rolled 3 times

X = Sum of the faces $= \{3, 4, 5, \ldots, 18\}$

So, $X(115) = 7$, $X(551) = 11$

$X^{-1}(7) = \{115, 151, 511, 223, 232, 322, 142, 124, 214, 421, 412, 241\}$

→ Types of random variable

- (i) Discrete random variable:- A random variable X is said to be a discrete random variable if the sample space contains either finite or countably infinite number of points. In other words, the values of X are either finite or countably infinite.

- (ii) Continious random variable:- A random variable X is said to be a continious random variable if the sample space contains infinite number of possibilities that are uncountable or in other words, the value of X is uncountable.

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→ **Discrete Probability Distribution**

- Probability mass function (PMF) :- Let X be a discrete random variable defined on a sample space S. Any function $f(x)$ is said to be a PMF of X if

(i) $f(x) = P(X = x)$

(ii) $f(x) \geq 0$

(iii) $\sum_x f(x) = 1$

Eg - Number of heads in 3 tosses of a coin

X	0	1	2	3
$f(x) - P(X=x)$	1/8	3/8	3/8	1/8

$\leftarrow \sum_x f(x) = 1$

$f(x) > 0$

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- Cumulative distribution function (Cdf) :- Let X be a discrete random variable with PMF $f(x)$ $(P(X) = x)$, then the Cdf of 'X' is denoted as,

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$$

Eg1.-

X	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

$$F(x) = P(X \leq x) = \begin{cases} 0, & x < 0 \\ 1/8, & 0 \leq x < 1 \\ 4/8, & 1 \leq x < 2 \\ 7/8, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

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Eg2.-

x	1	2	3
$f(x)$	0.25	0.5	0.25

$$F(x) = P(x \leq x) = \begin{cases} 0 & ; x < 1 \\ 0.25 & ; 1 \leq x < 2 \\ 0.75 & ; 2 \leq x < 3 \\ 1 & ; x \geq 3 \end{cases}$$

- Probability density function (Pdf):- Let X be a continious random variable defined on a sample space S. Any continious function F(x) is said to be the Pdf of X if it satisfies-

(i) $F(x) \geq 0$
(ii) $\int_{-\infty}^{\infty} F(x)\,dx = 1$
(iii) $P(a < X < b) = \int_a^b f(x)\,dx$.

*Note :- If X is a continious random variable then,

$$\left.\begin{aligned} &P(X=a) = 0 \\ &P(a<x<b) = P(a \leq x < b) \\ &= P(a < x \leq b) = P(a \leq x \leq b) \\ &= \int_a^b f(x)\,dx \end{aligned}\right\}$$

This is only for continious random variable as probability at a point is 0 for continious, but not for discrete.

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- Cumulative distribution function (Cdf) of a contnious random variable X is denoted by F(x) and is given by,

$$F(x) = P(X < x) = \int_{-\infty}^{\infty} f(x)\,dx$$

where, f(x) is the pdf of X

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3.- W = Number of heads - Number of tails, when a coin is tossed thrice.

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$\omega(HHH) = 3-0 = 3 \qquad W = (TTT) = 0-3 = -3$

$\omega(THH) = 2-1 = 1 \qquad \omega(TTH) = 1-2 = -1$

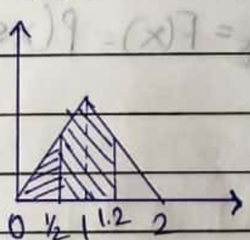
$W = \{-3, -1, 1, 3\}$

4.- X = Number of times a coin is tossed until 3 head occour in succession in 6 or less number of tosses.

$S = \{HHH, THHH, HTHHH, TTHHH, HHTHHH, THTHHH, HTTHHH, TTTHHH\}$

7.- X = Total time (in 100 hours) that a vaccum cleaner is used.

$$f(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1 \leq x < 2 \\ 0, & ew \end{cases}$$



(a) $P(X < 1.2) = \int_0^1 x\,dx + \int_1^{1.2} (2-x)\,dx$
$= 0.68$

(b) $P(½ < X < 1) = \int_{½}^1 x\,dx = 0.375$

(c) $Cdf = F(x) = P(X \leq x) = \begin{cases} 0, & x<0 \\ \int_0^x u\,du = \frac{x^2}{2}, & 0 \leq x < 1 \\ \int_0^1 x\,dx + \int_1^x (2-u)\,du, & 1 \leq x < 2 \\ 1, & x \geq 2. \end{cases}$

10.- X = Number obtained when a die is rolled once.

$S = \{1, 2, 3, 4, 5, 6\}$

$P(X = x) = \begin{cases} 1/6, & x = 1,2,3,4,5,6 \\ 0, & ew \end{cases}$

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11.- 7 TVs — 2 Defective, 5 Non-defective. A person purchases 3 TVs.

X = No. of defective TVs in the sample
= (0, 1, 2)

$$f(x) = P(X=x) = \frac{{}^2C_x \; {}^5C_{3-x}}{{}^7C_3}, \quad x = 0, 1, 2$$

x	0	1	2
$f(x)$	10/35	20/35	5/35

13.-

X	0	1	2	3	4
$f(x)$	0.41	0.37	0.16	0.05	0.01

$$\text{Cdf} = F(x) = P(X \leq x) = \begin{cases} 0, & x < 0 \\ 0.41, & 0 \leq x < 1 \\ 0.78, & 1 \leq x < 2 \\ 0.94, & 2 \leq x < 3 \\ 0.99, & 3 \leq x < 4 \\ 1, & x \geq 4 \end{cases}$$

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14.- X = The waiting time, in hours, between successive speeders spotted by a radar unit.

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-8x}, & x \geq 0 \end{cases}$$

(a) $P(X < 0.2) = F(0.2) = 1 - e^{-8 \times 0.2}$ ($\because$ 12 mins = 0.2 hour)

$= 1 - e^{-1.6} = 0.7981$

(b) $f(x) = F'(x) = -(-8e^{-8x}) = 8e^{-8x}$

$\therefore P(X < 0.2) = 8\int_0^{0.2} e^{-8x}\,dx = \left[-e^{-8x}\right]_0^{0.2}$

$= 0.7981$

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29.- $f(x) = \begin{cases} 3x^{-4}, & x > 1 \\ 0, & ew \end{cases}$

(a) $f(x) \geq 0$, and,
$\int_1^\infty 3x^{-4} = \left[-\frac{3x^{-3}}{3}\right]_1^\infty = 1$. So, this is a valid density function

(b) For $x \geq 1$, $F(x) = \int_1^x 3t^{-4}dt = 1 - x^{-3}$ So,
$F(x) = \begin{cases} 0, & x < 1 \\ 1-x^{-3}, & x \geq 1 \end{cases}$

(c) $P(X>4) = 1 - F(4)$
$= 1 - (1-4^{-3}) = 4^{-3} = 0.015$

30.- $f(x) = \begin{cases} k(3-x^2), & -1 \leq x \leq 1 \\ 0, & ew \end{cases}$

(a) We know that $\int_{-\infty}^{\infty} f(x)dx = 1$
$\Rightarrow 1 = k\int_{-1}^{1} 3-x^2 \, dx$
$= k\left(3x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = \frac{16}{3}k$

$\Rightarrow k = 3/16$
So, for $k = 3/16$, this is a valid density function

(b) For $-1 \leq x \leq 1$, $F(x) = \frac{3}{16}\int_{-1}^{x}(3-t)^2 dt$
$= \frac{3}{16}\left(3t - \frac{t^3}{3}\right)\Big|_{-1}^{x} = \frac{1}{2} + \frac{9}{16}x - \frac{x^3}{16}$

So, $P(X < 1/2) = \frac{1}{2} + \frac{9}{16}\times\frac{1}{2} - \frac{(1/2)^3}{16} = \frac{99}{128} = 0.773$

(c) $P(|X| < 0.8) = P(X < -0.8) + P(X > 0.8)$
$= F(-0.8) + 1 - F(0.8)$
$= 1 + \left(\frac{1}{2} - \frac{9}{16}\times 0.8 + \frac{1}{16}\times 0.8^3\right) - \left(\frac{1}{2} + \frac{9}{16}\times 0.8 - \frac{1}{16}\times 0.8^3\right)$
$= 0.164$

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35.- X = Number of cars that arrive at a specific intersection during a 20-second time period.

$P(X=x) = f(x) = \frac{e^{-6} 6^x}{x!}$, $x = 0, 1, 2, \ldots$

(a) $P(X>8) = \sum_{x=9}^{\infty} \frac{e^{-6} \cdot 6^x}{x!}$

or, $P(X<8) = 1 - P(X \leq 8)$

$= 1 - \sum_{x=0}^{8} \frac{e^{-6} \cdot 6^x}{x!}$

$= e^{-6}\left(\frac{6^0}{0!} + \frac{6^1}{1!} + \ldots + \frac{6^8}{8!}\right) = 0.1528$

(b) $P(X=2) = \frac{e^{-6} \cdot 6^2}{2!} = 0.0446$

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⇒ Joint Probability Distribution

- Let X and Y be 2 ~~students~~ discrete random variables defined on some sample space (2-Dimensional) (R^2).

We define $f(x,y)$ as the joint probability mass function of (x,y) if

(i) $f(x,y) = P(X=x, Y=y)$

(ii) $f(x,y) \geq 0$

(iii) $\sum_x \sum_y f(x,y) = 1$

(iv) $P((x,y) \in A) = \sum\sum_A f(x,y)$

Eg.- 3 Blue balls + 2 Red balls + 3 Green balls.

2 balls drawn at random.

X = No. of blue balls = (0, 1, 2)

Y = No. of green balls = (0, 1, 2)

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$Y\downarrow$ $X\rightarrow$	0	1	2	$P(Y=y)$
0	${}^3C_2/{}^8C_2 = 3/28$	$({}^3C_1 \cdot {}^3C_1)/{}^8C_2 = 9/28$	${}^3C_2/{}^8C_2 = 3/28$	$P(Y=0) = 15/28$
1	$({}^2C_1 \cdot {}^3C_1)/{}^8C_2 = 6/28$	$({}^3C_1 \cdot {}^2C_1)/{}^8C_2 = 6/28$	0	$P(Y=1) = 12/28$
2	${}^2C_2/{}^8C_2 = 1/28$	0	0	$P(Y=2) = 1/28$
$P(X=x)$	$P(X=0) = 10/28$	$P(X=1) = 15/28$	$P(X=2) = 3/28$	1

$P(X=0, Y=1) = 6/28$

$P(X\leq 1, Y=2) = P(X=0, Y=2) + P(X=1, Y=2) = \frac{1}{28} + 0 = 1/28$

$P(X\leq 1, Y\leq 2) = P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=2) + P(X=1, Y=0)$
$+ P(X=1, Y=1) + P(X=1, Y=2) = 25/28$

- Let X and Y be 2 continious random variables defined on some sample space R^2, then we define $f(x,y)$ as the joint Pdf of (x,y) if
  (i) $f(x,y) \geq 0$
  (ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$
  (iii) $P((x,y)\in A) = \iint_A f(x,y)\,dx\,dy$

Eg. - $f(x,y) = \begin{cases} 2/5\,(2x+3y), & 0\leq x\leq 1,\ 0\leq y\leq 1 \\ 0, & ew \end{cases}$

$= \frac{2}{5}\int_0^1\int_0^1 (2x+3y)\,dx\,dy$

$= \frac{2}{5}\int_0^1 (1+3y)\,dy$

$= \frac{2}{5}\left(1+3/2\right)$

$= \frac{5}{2}\times\frac{2}{5} = 1$

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$$P\left(0<x<\tfrac{1}{2},\ \tfrac{1}{4}<y<\tfrac{1}{2}\right)$$
$$= \frac{2}{5}\int_{1/4}^{1/2}\int_{0}^{1/2}(2x+3y)\,dx\,dy$$
$$= 13/160$$

- The marginal distributions of X and Y can be obtained from the joint distributions of X and Y as follows.

A) X and Y are both discrete, then,
$$g(x) = P(X=x) = \sum_y f(x,y)$$
and, $$h(y) = P(Y=y) = \sum_x f(x,y)$$

B) X and Y are both continious, then,
$$g(x) = \int_y f(x,y)\,dy$$
and, $$h(y) = \int_x f(x,y)\,dx$$

Eg.- $$g(x) = \begin{cases} \frac{2}{5}(2x + 3/2), & 0<x<1 \\ 0, & ew \end{cases}$$

$$h(y) = \begin{cases} \frac{2}{5}(1+3y), & 0<y<1 \\ 0, & ew \end{cases}$$

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- Let X and Y be 2 random variables (may be discrete or continious), then the conditional distribution of Y, given X = x, is given by
$$f(y\backslash x) = \frac{f(x,y)}{g(x)}, \quad g(x) \neq 0.$$

Similarity, the conditional distribution of X, given Y = y, is given by
$$f(x\backslash y) = \frac{f(x,y)}{h(y)}, \quad h(y) \neq 0$$

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- Eg.- From the table on Page-21 of this copy,

$$P(X=1 \setminus Y=0) = \frac{f(1,0)}{h(0)} = \frac{9/28}{15/28} = \frac{9}{15} = \frac{3}{5}$$

$$P(Y\leq 1 \setminus X=1) = \frac{f(1,0)+f(1,1)}{g(1)} = \frac{\frac{6}{28}+\frac{9}{28}}{15/28} = 1$$

$$P(0\leq X\leq 1 \setminus 0\leq Y\leq 1) = \frac{P(0\leq x\leq 1,\ 0\leq y\leq 1)}{P(0\leq y\leq 1)}$$

$$= \frac{P(0,0)+P(0,1)+P(1,0)+P(1,1)}{P(Y=0)+P(Y=1)}$$

$$= 24/27$$

- For Continious random variables -

$P(0\leq X\leq 1/2 \setminus Y=1)$

Step 1.- $f(x \setminus y) = \frac{f(x,y)}{h(y)} = \frac{\frac{2}{5}(2x+3y)}{\frac{2}{5}(1+3y)} = \frac{2x+3y}{1+3y}$, $0\leq x\leq 1$, $0\leq y\leq 1$.

Step 2.- $f(x \setminus y=1) = \frac{2x+3}{4}$, $0\leq x\leq 1$

Step 3.- So, $P(0\leq x\leq 1/2 \setminus y=1) = \int_0^{1/2} \frac{2x+3}{4}\, dx = \frac{1}{4}\left[\frac{1}{4}+\frac{3}{2}\right] = \frac{7}{16}$

$P(0\leq Y\leq 1/3 \setminus X=1)$

$$f(y \setminus x) = \frac{f(x,y)}{g(x)} = \frac{\frac{2}{5}(2x+3y)}{\frac{2}{5}(2x+\frac{3}{2})} = \frac{2x+3y}{2x+\frac{3}{2}}$$

$$f(y \setminus x=1) = \frac{2(2+3y)}{7}$$

$$f(0\leq y\leq 1/3 \setminus x=1) = \frac{2}{7}\int_0^{1/3} (2+3y)\, dy$$

$$P(0\leq Y\leq 1/3 \setminus 0\leq X\leq 1/2) = \frac{f(0\leq x\leq 1/2,\ 0\leq y\leq 1/3)}{f(0\leq x\leq 1/2)}$$

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→ **Statistically Independent Joint Probability Distribution**

- Let X and Y be 2 random variables with joint probability distribution $f(x,y)$ with marginal density of x and y as $g(x)$ and $h(y)$. Then X and Y are said to be statistically independent if $f(x,y) = g(x).h(y)$, i.e., joint p distribution of X and Y can be expressed as product of their marginal distributions.

Eg.- In previous problem (Example for continuous)
$f(x,y) \neq g(x).h(y)$. So, jpd is statistically dependent.

(Example for discrete)
$f(1,2) = 0$, $g(1) = \frac{15}{28}$, $h(2) = \frac{1}{28}$.

Since $f(1,2) \neq g(1).h(2)$. So, jpd is statistically dependent.

- If $f(x,y) = g(x).h(y)$ $\forall x, y$, then only jpd is statistically independent.

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37.- (a) $f(x,y) = Cxy$, $x = 1, 2, 3$, $y = 1, 2, 3$

$$\sum_{x=1,2,3} \sum_{y=1,2,3} f(x,y) = 1$$

$$\Rightarrow C \sum_{x=1,2,3} \sum_{y=1,2,3} xy = 1$$

$$\Rightarrow C[1+2+3+2+4+6+3+6+9] = 1$$
$$\Rightarrow C = 1/36$$

(b) $f(x,y) = C|x-y|$, $x = -2, 0, 2$, $y = -2, 3$

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$$\sum_{x=-2,0,2}\sum_{y=-2,3} f(x,y)=1$$
$$\Rightarrow C\sum_{x=-2,0,2}\sum_{y=-2,3}|x-y|=1$$
$$\Rightarrow C[0+5+2+3+4+1]=1$$
$$\Rightarrow C=1/15$$

38.- $f(x,y)=(x+y)/30$, $x=0,1,2,3$, $y=0,1,2$

(i) $P(X\leq 2, Y=1) = P(0\ 1)+P(1\ 1)+P(2\ 1)$
$$=\frac{1}{30}+\frac{2}{30}+\frac{3}{30}$$
$$=\frac{6}{30}=\frac{1}{5}$$

(ii) $P(X>Y) = P(1\ 0)+P(2\ 0)+P(3\ 0)+P(2\ 1)+P(3\ 1)+P(3\ 2)$
$$=\frac{1}{30}+\frac{2}{30}+\frac{3}{30}+\frac{3}{30}+\frac{4}{30}+\frac{5}{30}$$
$$=\frac{18}{30}=\frac{3}{5}$$

(iii) $P(X+Y=4)=P(2\ 2)+P(3\ 1)$
$$=\frac{4}{30}+\frac{4}{30}=\frac{8}{30}=\frac{4}{15}$$

42.- Let X, Y = Lifetime of 2 electric system $(X>0, Y>0)$

$$f(x,y)=\begin{cases} e^{-(x+y)}, & x>0, y>0 \\ 0, & ew\end{cases}$$

$P(0<X<1 \setminus Y=1)$

The conditional distribution of X and $Y=y$ can be written as,

$f(x\setminus y)=\dfrac{f(x,y)}{h(y)}$, $h(y)=\int_0^\infty e^{-(x+y)}$ $(y>0)$ $=\begin{cases} e^{-y}, & x>0 \\ 0, & ew\end{cases}$

So $f(x\setminus y)=\dfrac{e^{-(x+y)}}{e^{-y}}=\begin{cases} e^{-x}, & x>0 \\ 0, & ew\end{cases}$

$f(X\setminus Y=1)=\begin{cases} e^{-x}, & x>0 \\ 0, & ew\end{cases}$

Now, $\int_0^1 e^{-x}dx = 1-e^{-1}$

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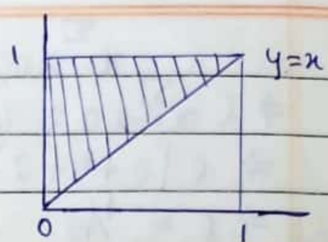
43.- $f(x,y) = \begin{cases} 4xy, & 0<x<1, \ 0<y<1 \\ 0, & ew \end{cases}$



(a) $P(0 < X < 1/2, \ 1/4 < Y < 1/2)$

$$4\int_0^{1/2}\int_{1/4}^{1/2} xy \, dy\, dx = \frac{3}{8}\int_0^{1/2} x \, dx$$

$$= 3/64$$

(b) $P(X < Y)$

$$4\int_0^1\int_x^1 xy\, dy\, dx \quad \text{OR} \quad 4\int_0^1\int_0^y xy\, dx\, dy$$

$$= 2\int_0^1 y^3\, dy$$

$$= 1/2$$

44.- Let X = Air pressure of the right tyre of aeroplane.
Y = Air pressure of the left tyre of aeroplane.

$f(x,y) = \begin{cases} K(x^2+y^2), & 30 \leq X \leq 50, \ 30 \leq Y \leq 50 \\ 0, & ew \end{cases}$

(a) We know that,

$$K\int_{30}^{50}\int_{30}^{50} (x^2+y^2)\, dx\, dy = 1 \quad \Rightarrow K = \frac{3}{392} \times 10^{-4} = 7.65 \times 10^{-7}$$

(c) $P(\text{Both are underfilled}) = P(30 \leq X \leq 40, \ 30 \leq Y \leq 40)$

$$= \int_{30}^{40}\int_{30}^{40} K(x^2+y^2)$$

$$= \int_{30}^{40}\int_{30}^{40} \frac{3}{392}(x^2+y^2) \cdot 10^{-4}\, dx\, dy$$

$$= 2\,\frac{3}{392} \cdot 10^{-4}(40-30)\int_{30}^{40} x^2\, dx$$

$$= \frac{3}{196} \times 10^{-3}\; \frac{40^3 - 30^3}{3} = \frac{37}{196}$$

(b) $P(30 \leq X \leq 50, \ 30 \leq Y \leq 50) = \int_{30}^{50}\int_{30}^{50} K(x^2+y^2)\, dy\, dx = 10^{-3} \cdot \frac{3}{392}\left(\frac{40^3-30^3}{3} + \ldots\right.$

$$= \frac{49}{196}$$

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49.-

Y↓ X→	1	2	3	P(Y=y) h(y)↓
1	0.05	0.05	0.10	0.2
3	0.05	0.10	0.35	0.5
5	0.00	0.20	0.10	0.3
P(X=x) g(x)→	0.1	0.35	0.55	1

(c) $P(Y=3 \backslash X=2) = \frac{f(2,3)}{g(2)} = \frac{0.10}{0.35} = \frac{2}{7}$

(a)

x	1	2	3
$g(x)$	0.1	0.35	0.55

(b)

y	1	3	5
$h(y)$	0.2	0.5	0.3

50.-

Y↓ X→	2	4	P(Y=y) h(y)↓
1	0.10	0.15	0.25
3	0.20	0.30	0.50
5	0.10	0.15	0.25
P(X=x) g(x)→	0.40	0.60	1

(a)

x	2	4
$g(x)$	0.40	0.60

(b)

y	1	3	5
$h(y)$	0.25	0.50	0.25

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Ans 12.- $F(t) = \begin{cases} 0, & t<1 \\ \frac{1}{4}, & 1 \leq t < 3 \\ \frac{1}{2}, & 3 \leq t < 5 \\ \frac{3}{4}, & 5 \leq t < 7 \\ 1, & t \geq 7 \end{cases}$

(a) $P(T=5) = F(5) - F(4)$
$= \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

(b) $P(T>3) = 1 - F(3)$
$= 1 - \frac{1}{2} = \frac{1}{2}$

(c) $P(1.4 < T < 6) = F(6) - F(1.4)$
$= \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$

Ans 60.- $f(x,y,z) = \begin{cases} \frac{4xyz^2}{9}, & 0<x,y<1,\ 0<z<3 \\ 0, & ew \end{cases}$

(a) $g(y,z) = \frac{4}{9} \int_0^1 xyz^2\, dx = \frac{2}{9} yz^2$, for $0<y<1, 0<z<3$

(b) $h(y) = \frac{2}{9} \int_0^3 yz^2\, dz = 2y$, for $0<y<1$

(c) $P(\frac{1}{4}<X<\frac{1}{2}, Y>\frac{1}{3}, Z<2) = \frac{4}{9} \int_1^2 \int_{1/3}^1 \int_{1/4}^{1/2} xyz^2\, dx\, dy\, dz = \frac{7}{162}$

(d) $P(0<X<\frac{1}{2} \mid Y=\frac{1}{4}, Z=2) = 2 \int_0^{1/2} x\, dx = \frac{1}{4}$
$\left(\because f(x \mid (y,z)) = \frac{f(x,y,z)}{g(y,z)} = 2x \text{ for } 0<x<1\right)$

Ans 62.- $F(x) = \begin{cases} 0, & x<1 \\ 0.4, & 1 \leq x < 3 \\ 0.6, & 3 \leq x < 5 \\ 0.8, & 5 \leq x < 7 \\ 1.0, & x \geq 7 \end{cases}$

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56
76

(a) PMF →

X	1	3	5	7
$f(x)$	0.4	0.2	0.2	0.2

(b) $P(4 < X \leq 7) = P(X \leq 7) - P(X \leq 4)$
$= F(7) - F(4)$
$= 1 - 0.6 = 0.4$

$P(X > 0.3 \mid y = 0.5)$

$= \int_{0.3}^{0.5} 8x \, dx$

$= 8\left(\frac{x^2}{2}\right)_{0.3}^{0.5}$

$= 4\left((0.5)^2 - (0.3)^2\right)$

$= 4(0.8 \times 0.2)$

$= 0.64$

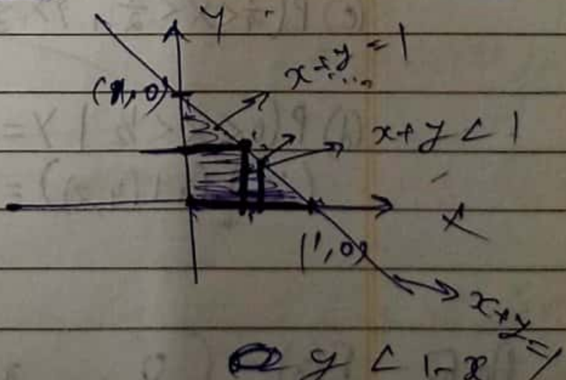
Ans 66:- $f(x,y) = \begin{cases} x+y, & 0 \leq x, y \leq 1 \\ 0, & ew \end{cases}$

(a) $g(x) = \int_0^1 (x+y)\,dy = x + \frac{1}{2}$, for $0 < x < 1$
$h(y) = \int_0^1 (x+y)\,dx = y + \frac{1}{2}$, for $0 < y < 1$

(b) $P(X > 0.5, Y > 0.5) = \int_{0.5}^1 \int_{0.5}^1 (x+y)\,dx\,dy = \int_{0.5}^1 \left[\frac{x^2}{2} + xy\right]_{0.5}^1 dy$
$= \int_{0.5}^1 \left(\frac{1}{2} + y\right) - \left(\frac{1}{8} + \frac{y}{2}\right) dy$
$= \frac{3}{8}$

$x + y = 1$
$x + 0.5 = 1$
$x = 0.5$

Ans 56:- (a) $g(x) = \int_{\cancel{x}} f(x,y)\,dy$



$= \int_{y=0}^{1-x} (6x)\,dy$

$8\left(\frac{x^2}{2}\right)_0^{0.5}$
$4(0.5)^2$
$4 \times \frac{1}{4} = 1$

$= 6x\,(y)_0^{1-x}$

$y < 1-x$

$= 6x(1-x), \quad 0 < x < 1$

$x + y < 1$
$x + y = 1.$

$0 < y < 1-x$

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$$h(y) = \int_x f(x,y)\,dx \qquad 0<y<1$$

$$= \int_{x=0}^{1-y} 6x\,dx$$

$$= 6\left(\frac{x^2}{2}\right)_0^{1-y}$$

$$= 3(1-y)^2, \qquad \underline{\underline{0<y<1}}.$$

$$f(x,y) = 6x$$

$$\neq 6x(1-x).\,3(1-y)^2$$

$$= g(x)\,h(y)$$

$\Rightarrow$ X and Y are not independent

(b) $P(\underline{X > 0.3} \mid Y = 0.5)$

$$f(x|y) = \frac{f(x,y)}{\underline{h(y)}}$$

$$= \frac{6x}{3(1-y)^2}$$

$$f(x|0.5) = \frac{2\cancel{6}x}{\cancel{3}(1-0.5)^2} = 8x, \quad \underline{0<x<0.5}$$

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CHAPTER-04:- MATHEMATICAL EXPECTATION

- Let X be a random variable with probability distribution f(x), then the mean of expected value of X is given by :-

$$\mu_X = E(X) = \sum_x x f(x) \quad — \text{ X is discrete.}$$

OR

$$= \int_{-\infty}^{\infty} x f(x)\, dx \quad — \text{ X is continuous.}$$

- Let X be a random variable with probability distribution f(x), and let g(x) be another function, then,

$$E = [g(X)] = \sum_x g(x) f(x) = \mu_{g(x)} \quad — \text{ X is discrete.}$$

OR

$$= \int_{-\infty}^{\infty} g(x) f(x)\, dx \quad — \text{ X is continuous}$$

Eg.- A coin is tossed thrice.

X	0	1	2	3
f(x)	1/8	3/8	3/8	1/8

$$E(X) = 0\times \tfrac{1}{8} + 1\times\tfrac{3}{8} + 2\times\tfrac{3}{8} + 3\times\tfrac{1}{8}$$
$$= 12/8$$

$$E(X^2) = 0^2\times\tfrac{1}{8} + 1^2\times\tfrac{3}{8} + 2^2\times\tfrac{3}{8} + 3^2\times\tfrac{1}{8} =$$

$$E(e^X) = e^0\times\tfrac{1}{8} + e^1\times\tfrac{3}{8} + e^2\times\tfrac{3}{8} + e^3\times\tfrac{1}{8} =$$

- If X and Y are 2 random variables with joint probability distribution f(x, y), then,

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) f(x,y) \quad — \text{ Discrete}$$

OR

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy \quad — \text{ Continuous}$$

Eg.- In Q10 of the exercise, $E(X,Y) = \sum_{x=1}^{3}\sum_{y=1}^{3} xy\, f(x,y)$

$$= 1\times1 f(1,1) + 1\times2 f(1,2) + 1\times3 f(1,3) + 2\times1 f(2,1) + 2\times2 f(2,2) + 2\times3 f(2,3) + 3\times1 f(3,1) + 3\times2 f(3,2) + 3\times3 f(3,3)$$

$$E(XY^0) = \sum_{x=1}^{3}\sum_{y=1}^{3} xy^0 f(x,y) = 1\times1^0 f(1,1) + 1\times2^0 f(1,2) + 1\times3^0 f(1,3) + 2\times1^0 f(2,1) + 2\times2^0 f(2,2) + 2\times3^0 f(2,3) + 3\times1^0 f(3,1) + 3\times2^0 f(3,2) + 3\times3^0 f(3,3).$$

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$$= 1\left(f(1,1)+f(1,2)+f(1,3)\right)+2\left(f(2,1)+f(2,2)+f(2,3)\right)+3\left(f(3,1)+f(3,2)+f(3,3)\right)$$

Eg 2.-Page 117- X = Comission

For 1st Appointment
$P(X=1000) = 0.7$
$P(X=0) = 0.3$

For 2nd Appointment
$P(X=1500) = 0.4$
$P(X=0) = 0.6$

Ans.-

X	0	1000	1500	2500
$P(X=x)$	$0.6\times0.3 = 0.18$	$0.7\times0.6 = 0.42$	$0.4\times0.3 = 0.12$	$0.7\times0.4 = 0.28$

$$E(x) = (0\times0.18)+(1000\times0.42)+(1500\times0.12)+(2500\times0.28)$$
$$= 1300$$

→ Exercises → Page - 121

7.- X = Profit
$P(X=4000) = 0.3$, $P(X=1000) = 0.7$

X	1000	4000
$f(X=x)$	0.7	0.3

$$E(x) = (0.7\times1000)+(0.3\times4000) = 1900$$

10.-

Y↓ X→	1	2	3	P(Y=y) h(y)
1	0.10	0.05	0.02	0.17
2	0.10	0.35	0.05	0.50
3	0.03	0.10	0.20	0.33
P(X=x) g(x)	0.23	0.5	0.27	1

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$\mu_y = E(Y) = \sum_y y h(y) = (1\times0.17)+(2\times0.5)+(3\times0.33)$
$= 2.16$

$\mu_x = E(X) = \sum_x x g(x) = (1\times0.23)+(2\times0.5)+(3\times0.27)$
$= 2.04$

12.- X = Profit (in \$5000)

$f(x) = \begin{cases} 2(1-x), & 0<x<1 \\ 0, & ew \end{cases}$

$E(X) = \int_0^1 x.\ 2(1-x)\,dx$
$= 2\int_0^1 x(1-x)\,dx$
$= 2\left[\frac{1}{2}-\frac{1}{3}\right] = \frac{1}{3}$

So, profit $= \frac{1}{3}\times5000 = \$1666.66$

15.- X = Hours (in 100 hours)

$f(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1\leq x<2 \\ 0, & ew \end{cases}$

$E(X) = \int_0^1 x.x\,dx + \int_1^2 x.(2-x)\,dx$
$= 1$

So, total number of hours $= 1\times100 = 100$ hours.

20.- $f(x) = \begin{cases} e^{-x}, & x>0 \\ 0, & ew \end{cases}$

$g(x) = e^{2X/3}$

$E[g(x)] = E(e^{2X/3}) = \int_0^\infty e^{2x/3}.e^{-x}\,dx$
$= \int_0^\infty e^{-x/3}\,dx$
$= 3$

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23.- (a) $E[g(X,Y)] = E(XY^2) = \sum_x \sum_y xy^2 f(x,y)$

$= [2\times1^2(0.10)] + [2\times3^2(0.2)] + [2\times5^2(0.1)] + [4\times1^2(0.15)] + [4\times3^2(0.3)]$

$+ [4\times5^2(0.15)]$

$= 35.2$

(b) $\mu_X = E(X) = (2\times0.40) + (4\times0.60)$

$= 3.20$

$\mu_Y = E(Y) = (1\times0.25) + (3\times0.50) + (5\times0.25)$

$= 3.00.$

	$f(x,y)$	x: 2	4	$P(Y=y)$ $h(y)$
y	1	0.1	0.15	0.25
	3	0.2	0.3	0.5
	5	0.1	0.15	0.25
$P(X=x)$ = $g(x)$		0.4	0.6	1

26.- $f(x,y) = \begin{cases} 4xy, & 0<x,\ y<1 \\ 0, & ew \end{cases}$

$Z = \sqrt{X^2+Y^2}$

$E(Z) = E(\sqrt{X^2+Y^2}) = \int_0^1\int_0^1 4xy\sqrt{x^2+y^2}\,dx\,dy$

$= \frac{4}{3}\int_0^1 [y(1+y^2)^{3/2} - y^4]\,dy$

$= 8(2^{3/2}-1)/15$

$= 0.9752$

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⇒ Variance and Co-variance of Random Variables

- Let X be a random variable with probability distribution $f(x)$, then we define,

$$\sigma^2 = E(X-\mu)^2$$
$$= \sum (x-\mu)^2 f(x)$$
OR
$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

Virat Kohli	Rohit Sharma
91	121
87	13
67	264
41	43
73	0
359	441

72 88

So, VK is more consistant that RS.

* Note:- $\sigma^2 = E(x^2) - \mu^2$

Proof -
$$E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$
$$= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f(x)\,dx$$
$$= \int_{-\infty}^{\infty} x^2 f(x) - \int_{-\infty}^{\infty} 2\mu x f(x)\,dx + \int_{-\infty}^{\infty} \mu^2 f(x)\,dx$$
$$= E(x^2) - 2\mu(\mu) + \mu^2 \times 1$$
$$= E(x^2) - 2\mu^2 + \mu^2$$
$$= E(x^2) - \mu^2 \text{ , Hence, proved.}$$

- Let X and Y be 2 random variables with probability distribution $f(x,y)$, then,

$$\sigma_{XY} = Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= \sum_x \sum_y (x-\mu_x)(y-\mu_y) f(x,y)$$
OR
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_y) f(x,y)\,dx\,dy$$

$$\sigma_{XY} = E(XY) - E(X).E(Y)$$
$$E[XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y]$$
$$= E[XY) - \mu_Y E(X) - \mu_X E(Y) + \mu_X\mu_Y]$$
$$= E(XY) - \mu_Y\mu_X - \cancel{\mu_X\mu_Y} + \cancel{\mu_X\mu_Y}$$
$$= E(XY) - \mu_X\mu_Y$$

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- Let X and Y be 2 random variables with co-variance σ_{XY} and standard deviations σ_X and σ_Y then we define the Correlation coefficient of (X,Y) as

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y}$$

*Note :- $-1 \leq \rho_{XY} \leq 1$

→ Exercises ⟶ Page - 131

34.-

x	-3	2	5
$f(x)$	0.3	0.2	0.5

$E(X^2) = ((-3)^2 \times 0.3) + (2^2 \times 0.2) + (5^2 \times 0.5) = 16$

$E(X) = (-3 \times 0.3) + (2 \times 0.2) + (5 \times 0.5) = 2$

$\sigma_X^2 = E(X^2) - \mu^2 = 16 - (2)^2 = 12 \quad (\mu = E(X))$

$\sigma_X = \sqrt{12} = 3.464$

50.- $f(x) = \begin{cases} 2(1-x), & 0<x<1 \\ 0, & ew \end{cases}$

$E(X^2) = \int_0^1 x^2 \cdot 2(1-x)\,dx = 1/6$

$E(X) = \int_0^1 x \cdot 2(1-x)\,dx = 1/3$

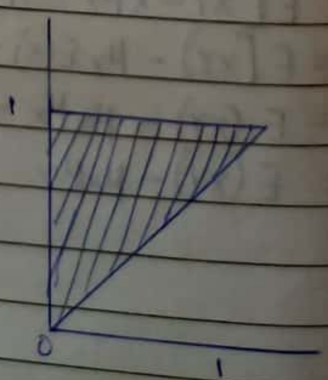
$\sigma_X^2 = E(X^2) - \mu^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18} \quad (\mu = E(x))$

$\sigma_X = \sqrt{1/18} = 0.235$

52.- $f(x,y) = \begin{cases} 2, & 0<x\leq y<1 \\ 0, & ew \end{cases}$

$g(x) = 2\int_x^1 dy = \begin{cases} 2(1-x), & 0<x<1 \\ 0, & ew \end{cases}$

$h(y) = 2\int_0^y dx = \begin{cases} 2y, & 0<y<1 \\ 0, & ew \end{cases}$



$E(Y^2) = 2\int_0^1 y^2 . y\,dy = 1/2$ $\quad$ $E(X^2) = 2\int_0^1 x^2.(1-x)\,dx = 1/3$

$E(Y) = 2\int_0^1 y.y\,dy = 2/3$ $\quad$ $E(X) = 2\int_0^1 x.(1-x)\,dx = 1/6$

$\sigma_Y^2 = E(Y^2) - (E(Y))^2 = 1/18$ $\quad$ $\sigma_X^2 = E(X^2) - (E(X))^2 = 1/18$

Now, $E(XY) = \int_0^1\int_0^y xy\,dx\,dy = \int_0^1 y^3\,dy$
$= 1/4.$

$\therefore$ $\rho_{XY} = \dfrac{\frac{1}{4} - \left(\frac{1}{3}\times\frac{2}{3}\right)}{\sqrt{1/18}\times\sqrt{1/18}} = \dfrac{1}{2} = 0.5$

⇒ **Means and Variances of Linear Combination of Random Variables**

- Theorem 1:- If a and b are constants then,
$$E(ax+b) = aE(x)+b$$
Proof:- $\int_{-\infty}^{\infty}(ax+b)f(x)\,dx = \int_{-\infty}^{\infty} ax f(x)\,dx + \int_{-\infty}^{\infty} b f(x)\,dx$
$= a\int_{-\infty}^{\infty} x f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$
$= a\times E(x) + b\times 1$
$= aE(x)+b.$

- Theorem 2:- $E(g(x)\pm h(x)) = E(g(x)) \pm E(h(x))$

- Theorem 3:- $E(g(x,y)\pm h(x,y)) = E(g(x,y)) \pm E(h(x,y))$

- Theorem 4:- If X and Y are 2 independent random variables, then,
$$E(X,Y) = E(X).E(Y)$$
Proof:- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$
$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, g(x)h(y)\,dx\,dy$
$= \int_{-\infty}^{\infty} x g(x)\,dx . \int_{-\infty}^{\infty} y h(y)\,dy$
$= E(X).E(Y)$

$f(x\backslash y) = g(x)$
$\Rightarrow \dfrac{f(x,y)}{h(y)} = g(x)$
$\Rightarrow f(x,y) = g(x)h(y)$

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- Corollary :- If X and Y are independent, then,
$\sigma_{XY} = 0$, and, $\rho_{XY} = 0$ (X and Y are uncorelated)
$E(g(x,y)) \pm E(h(x,y))$
$= E(XY) - E(X)E(Y)$
$= E(X)E(Y) - E(X)E(Y)$
$= 0$

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- Theorem-5:- If X and Y are 2 random variables with joint probability distribution $f(x,y)$ then,
$$\sigma^2_{(ax+by+c)} = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\sigma_{XY}$$

Proof:- $Z = aX + bY + c$
$\mu_Z = E(Z) = E(aX+bY+c) = aE(X) + b\,E(Y) + c$
$= a\mu_X + b\mu_Y + c$ --- (i)

$\sigma_Z^2 = E[Z - \mu_Z]^2$
$= E[aX + bY + \cancel{c} - a\mu_X - b\mu_Y - \cancel{c}]^2$
$= E[\underbrace{a(X-\mu_X)}_{A} + \underbrace{b(Y-\mu_Y)}_{B}]^2$
$= E[a^2(X-\mu_X)^2 + b^2(Y-\mu_Y)^2 + 2ab(X-\mu_X)(Y-\mu_Y)]$
$= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\sigma_{XY}$

- Theorem-6:- If X and Y are independent, then,
$\sigma^2_{aX \pm bY + c} = \sigma_X^2 a^2 + b^2\sigma^2_Y \pm 2ab\,\sigma_{XY}$ ($\because$ Independent, so $\sigma_{XY}=0$)
$= \sigma_X^2 a^2 + b^2\sigma^2_Y$

- Theorem-7:- If X and Y are independent, then,

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→ Chebyshev's Theorem

- The probability that the random variable X assumes a value within K standard deviation deviation of the means is atleast $(1 - 1/K^2)$, i.e.,

$$P(\mu - K\sigma < x < \mu + K\sigma) \geq 1 - 1/K^2 \quad \text{--- (i)}$$

$$P(x \leq \mu - K\sigma \text{ or, } x \geq \mu + K\sigma) \leq 1/K^2$$

$$P(x - \mu \leq -K\sigma \text{ or, } x - \mu \geq K\sigma) \leq 1/K^2$$

$$= P(|x - \mu| \geq K\sigma) \leq 1/K^2 \quad \text{--- (ii)}$$

P-141 Eg27 - $\mu = 8$, $\sigma^2 = 9$

Ans.- (i) $P(-4 < x < 20)$

According to Chebyshev's theorem,

$P(8 - (4\times3) < x < 8 + (4\times3))$

$\Rightarrow P(-4 < x < 20) \geq 1 - \frac{1}{K^2}$

$\Rightarrow P(-4 < x < 20) \geq 15/16$

$8 + K.3 = 20 \Rightarrow K = 4$

$1 - \frac{1}{K^2} = 1 - \frac{1}{4^2} = 1 - \frac{1}{16} = \frac{15}{16}$

(ii) $P(|x - \mu| \geq 6)$

$\Rightarrow P(|x - 8| \geq (3\times2))$

$\Rightarrow P(|x - 8| < \frac{1}{4})$

$k\sigma = 6 \Rightarrow k\times3 = 6 \Rightarrow k = 2$

$\frac{1}{k^2} = \frac{1}{2^2} = \frac{1}{4}$

→ Exercises → Page-143

77.- $\mu = 10$, $\sigma^2 = 4$ (i) $P(5 < x < 15)$, (ii) $P(|x - 10| < 3)$, (iii) $P(|x - 10| \geq 3)$

$\mu + k\sigma = 15 \Rightarrow 10 + k\times2 = 15 \Rightarrow k = 5/2$

(i) $1 - 1/k^2 = 1 - 1/(5/2)^2 = 1 - 4/25 = 21/25$

(ii) $P(|x - 10| < 3) = P(-3 < x - 10 < 3) = P(\underset{\mu - k\sigma}{7} < x < \underset{\mu + k\sigma}{13}) \geq 1 - 1/k^2$

$\mu + 2k = 13 \Rightarrow 2k = 10 \Rightarrow k = 3/2$

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(ii) $P(|x-\mu| \geq K\sigma) \leq 1/K^2$

$K\sigma = 3$, $\sigma = 2$

$\Rightarrow K = 3/2$, $K^2 = 9/4$, $\Rightarrow 1/K^2 = \frac{4}{9}$

$P(|X-10| \geq 3) \leq 4/9$

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57.-

X	-3	6	9
$f(x)$	$1/6$	$1/2$	$1/3$

$E(x) = (-3)(1/6) + (6 \times 1/2) + (9 \times 1/3) = 11/2 = 5.5$

$E(x^2) = (-3)^2(1/6) + (6^2 \times 1/2) + (9^2 \times 1/3) = 93/2 = 46.5$

$E[(2x+1)^2] = 4E(x^2) + 4E(x) + 1 = (4 \times 46.5) + (4 \times 5.5) + 1 = 209$

58.- $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 \leq x < 2 \\ 0, & ew \end{cases}$

$E(x) = \int_0^1 x.x\,dx + \int_1^2 x^2(2-x)\,dx = 1$

$E(x^2) = \int_0^1 x^2.x\,dx + \int_1^2 x^2(2-x)\,dx = 7/6$

$E(Y) = 60X^2 + 39X = (60 \times 7/6) + (39 \times 1)$

$= 109$

60.-

$Y\downarrow X\rightarrow$	2	4	$h(y)$
1	0.1	0.15	0.25
3	0.2	0.3	0.5
5	0.1	0.15	0.25
$g(x)$	0.4	0.6	1

(a) $E(2X-3Y) = 2E(x) - 3E(Y)$

$= (2 \times 3.2) - 3(3) = -2.60$

$E(x) = (2 \times 0.4) + (4 \times 0.6) = 3.2$

$E(Y) = (1 \times 0.25) + (3 \times 0.5) + (1 \times 0.25) = 3$

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(b) $E(XY) = \sum_x \sum_y xy f(x,y)$

$= E(X).E(Y) = (3.2)(3) = 9.6$

64.- $g(x) = \begin{cases} 8/x^3, & x>2 \\ 0, & ew \end{cases}$

$h(y) = \begin{cases} 2y, & 0<y<1 \\ 0, & ew \end{cases}$

$Z = XY$

$E(Z) = E(XY) = E(X).E(Y)$

$= \int_2^\infty \frac{8}{x^3}.x\,dx \cdot \int_0^1 2y\,dy$

$= 4 \times \frac{2}{3}$

$= 8/3$

OR

$f(x,y) = g(x).h(y) = \begin{cases} 16y/x^3, & 0<y<1, x>0 \\ 0, & ew \end{cases}$

$F(XY) = 16\int_0^1\int_2^\infty \frac{y^2}{x^2}\,dx\,dy$

$= 16 \times \frac{1}{2} \int_0^1 y^2\,dy$

$= 8/3$

67.- $f(x,y) = \begin{cases} 2/7\,(x+2y), & 0<x<1, 1<y<2 \\ 0 & ew \end{cases}$

$E[g(X,Y)] = E\left[\frac{X}{Y^3} + X^2Y\right] = E\left(\frac{X}{Y^3} + X^2Y\right)$

$= \frac{2}{7}\int_1^2\int_0^1 \left(\frac{x}{y^3} + x^2y\right)(x+2y)\,dx\,dy$

$= \frac{2}{7}\int_1^2\int_0^1 \left(\frac{x^2}{y^3} + \frac{2x}{y^2} + x^3y + 2x^2y^2\right)dx\,dy$

$= \frac{2}{7}\int_1^2 \left(\frac{1}{3y^3} + \frac{1}{y^2} + \frac{y}{4} + \frac{2y^2}{3}\right)dy$

$= \frac{46}{63}$

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78.- $f(x)=\begin{cases} 6x(1-x), & 0<x<1 \\ 0, & ew \end{cases}$

$\left\{\begin{array}{l} P(\mu-K\sigma < X < \mu+K\sigma) \geq 1-1/K^2 \\ \text{Here } K=2 \\ \text{So, } 1-1/K^2 = 1-1/2^2 = 3/4 \end{array}\right.$

$P(\mu-2\sigma < X < \mu+2\sigma) \geq 4$

$E(X) = 6\int_0^1 x^2(1-x)\,dx = 1/2$

$E(X^2) = 6\int_0^1 x^3(1-x)\,dx = 6/20 = 3/10$

$\sigma^2 = E(X^2) - (E(X))^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$

$\sigma = \sqrt{1/20} = 1/2\sqrt{5}$

So, $P\left(\frac{1}{2} - 2\times\frac{1}{\sqrt{20}} < X < \frac{1}{2} + \frac{2}{\sqrt{20}}\right)$

$= P\left(\frac{1}{2} - \frac{1}{\sqrt{5}} < X < \frac{1}{2} + \frac{1}{\sqrt{5}}\right)$

$= 6\int_{\frac{1}{2}-\frac{1}{\sqrt{5}}}^{\frac{1}{2}-\frac{1}{\sqrt{5}}} x(1-x)\,dx = 6\int_{0.05}^{0.95}(x-x^2)\,dx$

$= 0.985$

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Ans 4.- Assigning the values of $3w$ and w to heads and tails respectively, we get,
$P(H) = 3/4 \quad P(T) = 1/4$

Now the sample space for the given experiment is $S = \{HH, HT, TH, TT\}$

Let X represent the number of tails that occour in 2 tosses of the coin, we have,

$$P(X=0) = P(HH) = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$$
$$P(X=1) = P(HT) + P(TH) = \frac{3}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{3}{4} = \frac{3}{8}$$
$$P(X=2) = P(TT) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

The probability distribution for X is then,

X	0	1	2
f(x)	9/16	3/8	1/16

for which we get, $\mu = E(x)$
$$= 0 \times \frac{9}{16} + 1 \times \frac{3}{8} + 2 \times \frac{1}{16}$$
$$= \frac{1}{2}$$

Ans 35.-

X	2	3	4	5	6
f(x)	0.01	0.25	0.4	0.3	0.04

$\mu = (2 \times 0.01) + (3 \times 0.25) + (4 \times 0.4) + (5 \times 0.3) + (6 \times 0.4) = 1$

$E(X^2) = (2^2 \times 0.01) + (3^2 \times 0.25) + (4^2 \times 0.4) + (5^2 \times 0.3) + (6^2 \times 0.4) = 2$

So, $\sigma^2 = 2 - 1^2 = 1$

Ans ~~74~~.-

CHAPTER-5:- SOME DISCRETE PROBABILITY DISTRIBUTIONS

⇒ Bernoulli Distribution

- If x is a Bernoulli random variable, which takes the values of 0 or 1 depending upon the experiment which may result either in a failure or success.
- Let the probability of a success be p and failure be q $(p+q=1)$. Then the probability distribution of the Bernoulli random variable is given by,—

$$P(X=x) = f(x) = p^x q^{1-x} \quad , \quad \text{for } x=0,1$$

$x=0$, $f(x)=q$
$x=1$, $f(x)=p$

X	0	1
$P(X=x)$	q	p

$X=$ No. of successes.

→ Bernoulli process

- It possesses the following properties
  (i) The experiment consists of repeated trials.
  (ii) Each trial results in an outcome that may be classified as success or failure.
  (iii) The probability of success is denoted by p and failure with q. $(p+q=1)$
  (iv) The repeated trials are independent.

$$E(x) = \sum_{x=0}^{1} x f(x) = p$$

$$E(x^2) = \sum_{x=0}^{1} x^2 f(x) = p$$

$$\sigma^2 = E(x^2) - (E(x))^2 = p - p^2 \qquad [q=(1-p)]$$
$$= p(1-p) = pq$$

⇒ Binomial Distribution

- The binomial distribution is a special case of Bernoulli process.
- A Bernoulli trial can result in a success with probability p and failure with probability q $(p+q=1)$, then the probability distribution of the binomial random variable, $X=$ Number of successes in n Bernoulli trials, is given by —

$$B(x,n,p) = {}^nC_x\, p^x q^{n-x} \quad , \quad x=0,1,2,\ldots,n$$

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x = No. of successes in n trials
n = No. of trials
p = Probability of success in one trial

$$ {}^nC_x = \frac{n!}{x!\,(n-x)!} \qquad {}^nP_x = \frac{n!}{(n-x)!} $$

$X = I_1 + I_2 + \dots + I_n$

$E(x) = p + p + p + \dots + p = np$

$\sigma^2 = npq$

→ Multinomial distribution

- If a given trial can result in k outcomes namely $E_1, E_2, \dots E_k$ with probabilities $p_1, p_2, \dots p_k$, then the probability distribution of random variables $x_1, x_2, \dots, x_k$ ~~respectively~~ representing the number of occurrences of $E_1, E_2, \dots, E_k$ is given by

$$f(x_1, \dots, x_k, n, p_1, \dots, p_k)$$

$$= \frac{n!}{x_1! \times x_2! \times \dots \times x_k!}\, p_1^{x_1} \times p_2^{x_2} \times \dots \times p_k^{x_k}$$

$\Sigma p_i = 1, \quad \Sigma x_i = n$

17/09/19

→ Exercises → Page - 157

9.- X = Number of trucks with blowouts.

P (Fail to complete without blowouts) = 1/4

P (Fail to complete with blowouts) = 3/4

$n = 15, \quad X \sim B(x, 15, 3/4) = {}^{15}C_x \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{15-x}$
↳ (Follows)

(a) $P(3 \leq X \leq 6) = \sum_{x=3}^{6} {}^{15}C_x \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{15-x}$

(b) $P(X < 4) = P(X \leq 3) = \sum_{x=0}^{3} {}^{15}C_x \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{15-x}$

(c) $P(X > 5) = P(X \geq 6) = \sum_{x=6}^{15} {}^{15}C_x \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{15-x}$

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11.- $x=5$, $n=7$.

$P(x=5) = {}^7C_x (0.9)^x (0.1)^{7-x}$

$= {}^7C_5 (0.9)^5 (0.1)^{7-5}$

$= {}^7C_5 (0.9)^5 (0.1)^2$

15.- X = Contract the disease , OR, X = Survive the disease

$n=5$, $x=2$

$P(x=2) = {}^5C_x (0.4)^x (0.6)^{5-x}$

$= {}^5C_2 (0.4)^2 (0.6)^3$

$n=5$, $x=3$

$P(x=3) = {}^5C_x (0.6)^x (0.4)^{5-x}$

$= {}^5C_3 (0.6)^3 (0.4)^2$

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⇒ Hypergeometric Distribution

- The probability distribution of the hypergeometric random variable X, where X = Number of successes, in a random sample of size 'n' selected from 'N' items out of which 'K' are labelled as successes ~~are~~ and 'N−K' as failure is given by.

$$H(x, N, n, K) = \frac{{}^KC_x \; {}^{N-K}C_{n-x}}{{}^NC_n}, \quad x = 0, 1, 2, \ldots$$

Mean, $\mu = \frac{nK}{N}$

Variance, $\sigma^2 = \frac{nK}{N}\left(1 - \frac{K}{N}\right)$

Eg.- Let N = Total no. of TVs at a shop

K = No. of TVs at shop functioning

n = No. of TVs you will buy

X = No. of functioning TVs in lot of 'n'.

Selecting n from N = ${}^NC_n$

Selecting x from K functioning TVs = ${}^KC_x$

" n−x " N−K non functioning TVs = ${}^{N-K}C_{n-x}$

$\therefore P(X=x) = \frac{{}^{N-K}C_{n-x} \; {}^KC_x}{{}^NC_n}$

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→ **Multivariate Hypergeometric Distribution**

- If N items can be partitioned into, say $A_1, A_2, \ldots, A_k$ with $a_1, a_2, \ldots, a_k$ elements respectively, then the probability distribution on the random variables, $x_1, x_2, \ldots, x_k$ representing the number of elements selected from $A_1, A_2, \ldots, A_k$ in a random sample of size n is given by -

$$f(x_1, x_2, \ldots, x_k, a_1, a_2, \ldots, a_k, n) = \frac{({}^{a_1}C_{x_1})({}^{a_2}C_{x_2})\ldots({}^{a_k}C_{x_k})}{{}^{N}C_n}$$

$\Sigma x_i = n, \quad \Sigma a_i = N$

Eg.- 20 Students
- Excellent (A_1) : 3 (a_1)
- Good (A_2) : 10 (a_2)
- Average (A_3) : 5 (a_3)
- Bad (A_4) : 2 (a_4)

10 students required : (x_1) 1 Excellent x_1
(x_2) 3 Good x_2
(x_3) 4 Average x_3
(x_4) 2 Bad x_4

$$f(1, 3, 4, 2, 3, 10, 5, 2, 10) = \frac{({}^{3}C_1)({}^{10}C_3)({}^{5}C_4)({}^{2}C_2)}{{}^{20}C_{10}}$$

→ **Exercises → Page - 165**

31.- N(6) — Doctors (4), Nurses (2)

$n = 3$

X = Number of doctors

$$P(X = x) = \frac{{}^{4}C_x \; {}^{2}C_{3-x}}{{}^{6}C_3}, \quad x = 0, 1, 2, 3$$

$$P(2 \le X \le 3) = \frac{{}^{4}C_2 \; {}^{2}C_1}{{}^{6}C_3} + \frac{{}^{4}C_3 \; {}^{2}C_0}{{}^{6}C_3}$$

$$= \frac{4}{5}$$

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32.- N(10) — Defective (3)
— Non defective (7)

$n = 4.$

X = Number of defectives

$$P(X=x) = \frac{{}^3C_x \, {}^7C_{4-x}}{{}^{10}C_4}$$

$$P(X \leq 2) = \sum_{x=0}^{2} \frac{{}^3C_x \, {}^7C_{4-x}}{{}^{10}C_4}$$

⇒ Negative Binomial Distribution

- If repeated independent trials can result in a success with probability p and failure with probability (q) $(p+q=1)$, then the probability distribution of the random variable x, X = The number of trial required to get the k^{th} success is given by,

$$P(X=x) = B^*(x, k, p) = {}^{x-1}C_{k-1}\, p^k q^{x-k}, \quad x = k, k+1, \ldots \quad k = 1, 2, 3, \ldots$$

Example 14.- (P-165)
(a) ${}^5C_3\,(0.55)^4\,(0.45)^2$
(b) $B^*(4, 4, 0.55) + B^*(5, 4, 0.55) + B^*(6, 4, 0.55) + B^*(7, 4, 0.55)$

14/10/19

→ Geometric Distribution

- If repeated independent trials can result in a success with probability p and failure with probability q $(p+q=1)$, then the probability distribution of the random variable x, X = The no. of trials in which first success occurs, is given by -

$$g(x, p) = pq^{x-1}, \quad x = 1, 2, 3, \ldots$$

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P-166 Eg15.- $P(\text{item is defective}) = 0.01$

$g(5, 0.01) = 0.01 \times (0.99)^{5-1} = 0.0096$

Eg16.- $P(\text{connected}) = 0.05$

$g(5, 0.05) = 0.05 \times (0.95)^{5-1} = 0.04$

→ Poission's distribution

- Let X be a binomial random variable with probability distribution $b(x, n, p)$. When $n \to \infty$, $p \to 0$, $\mu = np$ fixed, we have,

$b(x, n, p) \xrightarrow[\substack{p \to 0 \\ \mu = np}]{n \to \infty} P(x, \mu)$

$\mu = E(X) = np$ (Mean), $P(x, \mu) = \dfrac{e^{-\mu} \mu^x}{x!}$, $x = 0, 1, 2, \ldots, \infty$

$\sigma^2 = np$ (Variance)

P-170 Eg19.- $P(\text{commiting an accident of a given day}) = 0.005$

$n = 400$, $p = 0.005$

$\mu = np = 400 \times 0.005 = 2$

$P(X=1) = \dfrac{e^{-2} \cdot 2^1}{1!} = 0.27$

Eg20.- $P(\text{defective glass item}) = \dfrac{1}{1000} = 0.001$

$n = 8000$, $p = 0.001$

$\mu = np = 8000 \times 0.001 = 8$

$P(X < 7) = \sum_{x=0}^{6} \dfrac{e^{-8} \cdot 8^x}{x!} = 0.313$

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→ Exercises → Pages – 189

49.- This problem requires the use of negative binomial distribution, where,
$x = 10 \quad k = 5, \quad p = 0.3$
$B^*(10, 5, 0.3) = {}^9C_4 \ (0.3)^5 \ (0.7)^5$
$= 0.0515$

50.- (a) This problem requires the use of negative binomial distribution, where,
$x = 7, \quad k = 3, \quad p = 0.5$
$B^*(7, 3, 0.5) = {}^6C_2 \ (0.5)^3 \ (0.5)^4$
$= 0.1172$

(b) This problem requires the use of geometric distribution, where,
$x = 4, \quad p = 0.5, \Rightarrow q = 0.5$
$g(4, 0.5) = 0.5 \times (0.5)^{4-1}$
$= 0.0625$

51.- The probability that all coins turn up the same is $2/8$.
This problem requires the use of geometric distribution, where,
$p = \frac{6}{8} = \frac{3}{4}, \quad q = \frac{2}{8} = \frac{1}{4}$
$P(X<4) = \sum_{x=1}^{3} g(x, 3/4) = \sum_{x=1}^{3} \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{x-1} = \frac{63}{64}$
$= 0.984$

60.- (a) This problem requires the use of poission distribution, where,
$\mu = 12 \quad X \sim P(x, 12)$ & $f(x) = \frac{e^{-12} \cdot 12^x}{x!}$
$P(X<7) = P(X \leq 6) = \sum_{x=0}^{6} \frac{e^{-12} \, 12^x}{x!} = 0.0458$
$= p$

(b) This problem requires the use of binomial distribution, where,
$p = 0.0458, \quad q = 0.9542, \quad n = 3 \quad x = 2$
$B(2, 3, 0.0458) = {}^3C_2 \, (0.0458)^2 \, (0.9542)^{3-2} = 0.0060$

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69.- Here, given that, $p = 0.001$
and, $n = 4000$

Mean number of people who will die $= \mu = np$
$= 4000 \times 0.001 = 4$

70.- (a) $n = 100$, $p = 1\% = \frac{1}{100} = 0.01$
Mean, $\mu = np = 100 \times 0.01 = 1$

(b) $\sigma^2 = np = 1$

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Ans16.- P(Engine fails) = 0.4

P(Engine doesn't fail) = 1 − 0.4 = 0.6

Probability of 2 or more of 4 engines operating when $p = 0.4$ is

$$P(X \geq 2) = 1 - P(X \leq 1) \qquad (n=4,\ r=2)$$
$$= 0.8208.$$

and, the probability of 1 or more of 2 engines operating when $p = 0.4$ is

$$P(X \geq 1) = 1 - P(X = 0) \qquad (n=2,\ r=1)$$
$$= 0.8400$$

So, the 2 engine plane has a slightly higher probability for a succesful flight when $p = 0.6$.

Ans19.- Let X_1 = No. of times encountered with green light with P(Green) = 0.35

X_2 = No. of times encountered with yellow light with P(Yellow) = 0.05, and,

X_3 = No. of times encountered with red light with P(Red) = 0.60. Then,

$$f(x_1, x_2, x_3) = \binom{n}{x_1, x_2, x_3} (0.35)^{x_1} (0.05)^{x_2} (0.60)^{x_3}$$

Ans22.- Red : Black : White = 8 : 4 : 4

$$\left\{ \begin{array}{l} P(Red) = \frac{8}{16} \\ P(Black) = \frac{4}{16} \\ P(White) = \frac{4}{16} \end{array} \right.$$

Among 8 offsprings, we need, 5 red, 2 black and 1 white.

So, by using the multinomial distribution, we have,

$$\binom{8}{5,2,1} \left(\frac{1}{2}\right)^5 \left(\frac{1}{4}\right)^2 \left(\frac{1}{4}\right)^1 = \frac{21}{256}$$

Ans 43.-

12 people → 2 Canadians, 3 Japanese, 5 Italians, 2 Germans

A committe of 4 people is selected at random

(a) P (All nationalists are represented = ?

We need to use the multivariate hypergeometric distribution.

So, $f(1,1,1,1,\ 2,3,5,2,\ 4) = \dfrac{({}^2C_1)({}^3C_1)({}^5C_1)({}^2C_1)}{{}^{12}C_4} = \dfrac{4}{33}$

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(b) P(All nationalists except Italians are represented) = ?

Using multivariate hypergeometric distribution again, we get,

$f(1,1,2,\ 2,3,2,\ 4) + f(1,2,1,\ 2,3,2,4) + f(2,1,1,2,3,2,4)$

$$= \frac{({}^2C_1)({}^3C_1)({}^2C_2)}{{}^{12}C_4} + \frac{({}^2C_1)({}^3C_2)({}^2C_1)}{{}^{12}C_4} + \frac{({}^2C_2)({}^3C_1)({}^2C_1)}{{}^{12}C_4}$$

$$= \frac{8}{165}$$

Ans 44.- ~~3 Green~~, 2 Blue, 4 Red, 3 Green ⇒ Total = 9 Balls

In ~~and~~ a random sample,

P(Both blue balls and atleast one red ball) = ?

Using multivariate hypergeometric distribution, we get,

$f(2,1,2,\ 2,4,3,5) + f(2,2,1,2,4,3,5) + f(2,3,0,2,4,3,5)$

$$= \frac{({}^2C_2)({}^4C_1)({}^5C_2)}{{}^9C_5} + \frac{({}^2C_2)({}^4C_2)({}^5C_1)}{{}^9C_5} + \frac{({}^2C_2)({}^4C_3)({}^5C_0)}{{}^9C_5}$$

$$= \frac{17}{63}$$

Ans 47.-

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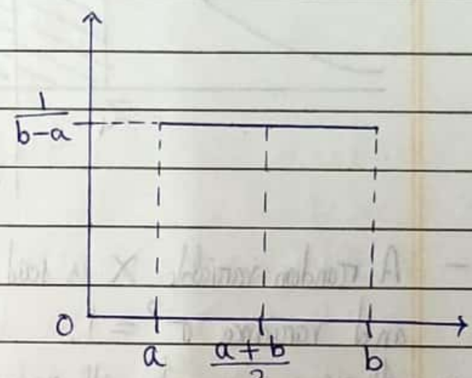
CHAPTER-6:- SOME CONTINIOUS PROBABILITY DISTRIBUTIONS

⇒ Uniform Distribution

- If $X \sim U(a,b)$, then it has the probability distribution function,

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & ew \end{cases}$$



$$\mu = E(x) = \frac{1}{b-a} \int_a^b x\, dx$$
$$= \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$
$$E(x^2) = \frac{1}{b-a} \int_a^b x^2\, dx$$
$$= \frac{b^3-a^3}{3(b-a)} = \frac{a^2+b^2+ab}{3}$$
$$\sigma^2 = E(x^2) - E^2(x)$$
$$= \frac{a^2+b^2+ab}{3} - \frac{(a+b)^2}{4} = \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12}$$

⇒ Normal Distribution

- If $X \sim N(x, \mu, \sigma)$, $E(x) = \mu$, $Va(x) = \sigma^2$, then it has the probability distribution function, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}$, $\quad -\infty < x < \infty$, $-\infty < \mu < \infty$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}\, dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}\, dx = \sigma\sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} e^{-\frac{(x+1)^2}{2}}\, dx = \sqrt{2\pi}, \quad \because \text{Here, } \sigma = 1, \ \mu = -1$$

So, $\sigma\sqrt{2\pi} = \sqrt{2\pi}$ is the total integration

↓

This integral is independent of μ.

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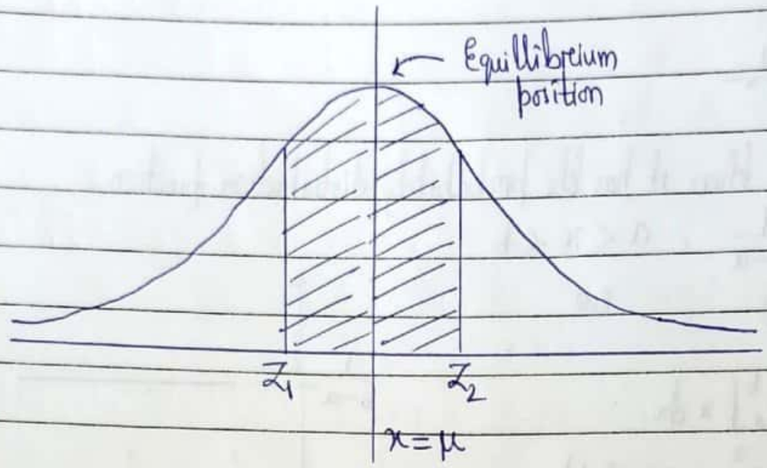
2, 4, 7, 8, 10, 15, 22
24, 26, 29, 34
41, 47, 54



- A random variable X is said to be a standard random variable it it has mean, $\mu=0$ and variance, $\sigma^2=1$.
- If X is a normal with mean μ and variance σ^2 then,

$$Z = \frac{X-\mu}{\sigma} \text{ is a standard normal.}$$

$$\begin{cases} Y = ax+b \\ E(X)=\mu,\ Va(X)=\sigma^2 \\ E(Y) = a\mu+b \\ Va(Y) = a^2\sigma^2 \end{cases}$$

Then the probability distribution function of a standard normal random variable is.

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$$

- Observation :- $X \sim n(\mu, \sigma^2)$

$$P(a<X<b) = P\left(\frac{a-\mu}{\sigma} < \underbrace{\frac{X-\mu}{\sigma}}_{Z} < \frac{b-\mu}{\sigma}\right)$$

$$= P\left(\underbrace{\frac{a-\mu}{\sigma}}_{Z_1} < Z < \underbrace{\frac{b-\mu}{\sigma}}_{Z_2}\right)$$

$$= P(Z<Z_2) - P(Z<Z_1)$$

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2.- $X \sim U(1,5)$

$$f(x) = \begin{cases} \frac{1}{5-1} = \frac{1}{4}, & 1<x<5 \\ 0, & ew \end{cases}$$

$$P(X > 2.5 \mid X \leq 4) = \frac{P(X>2.5, X\leq 4)}{P(X\leq 4)}$$

$$= \frac{P(2.5 < X \leq 4)}{P(X\leq 4)}$$

$$P(2.5 < X \leq 4) = \frac{1}{4}\int_{2.5}^{4} dx = 3/8$$

$$P(X \leq 4) = \frac{1}{4}\int_{1}^{4} dx = 3/4$$

So, $P(X > 2.5 \mid X \leq 4) = \dfrac{3/8}{3/4} = \dfrac{1}{2}$

4.- $$f(x) = \begin{cases} 1/10, & 0<x<10 \\ 0, & ew \end{cases}$$

(a) $P(X>7) = \frac{1}{10}\int_{7}^{10} dx = 3/10$

(b) $P(2<X<7) = \frac{1}{10}\int_{2}^{7} dx = 1/2$

7.- (b) $P(Z>k) = 0.2946$

$\Rightarrow P(Z<k) = 1-0.2946 = 0.7054$

$\Rightarrow k = 0.54$ (From normal table)

(c) $P(-0.93 < Z < k) = 0.7235$

$\Rightarrow P(Z<k) - P(Z<-0.93) = 0.7235$

$\Rightarrow P(Z<k) = 0.7235 + P(Z<-0.93)$

$= 0.7235 + 0.1762$

$= 0.8997$

$\Rightarrow k = 1.28$ (From normal table)

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(a) $P(z < k) = 0.0427$

$\Rightarrow k = -1.72$ (From normal table)

8.- Given, $\mu = 30$, $\sigma = 6$, $X \sim N(30, 6)$

(a) $P(X > 17) = 1 - P(X \leq 17)$

$= 1 - P\left(z < \frac{17-30}{6}\right)$

$= 1 - P(z < -2.16)$

$= 1 - 0.0154 = 0.9846$ (From table)

(e) $P(x_1 < X < x_2) = 0.75$

$P(X < x_1) = 0.1250$ (12.5%)

$\Rightarrow P\left(z < \frac{x_1 - 30}{6}\right) = 0.1250$

$\Rightarrow \frac{x_1 - 30}{6} = -1.15$ (From table)

$\Rightarrow x_1 = 23.1$

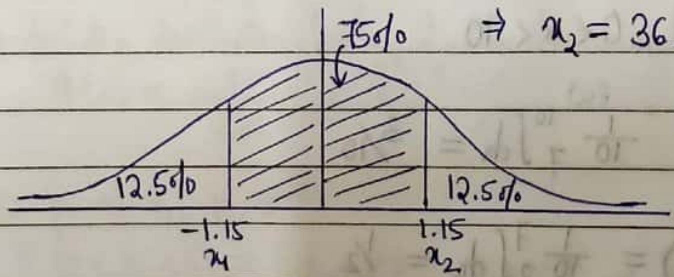
$P(X > x_2) = 0.1250$ (12.5%)

$\Rightarrow P(X < x_2) = 1 - 0.1250 = 0.8750$

$\Rightarrow P\left(z < \frac{x_2 - 30}{6}\right) = 0.8750$

$\Rightarrow \frac{x_2 - 30}{6} = 1.15$ (From table)

$\Rightarrow x_2 = 36.9$



(b) $P(X < 22) = P\left(z < \frac{22-30}{6}\right)$

$= P(z < -1.33) = 0.0918$ (From table)

(c) $P(32 < X < 41) = P(X < 41) - P(X < 32)$

$= P\left(z < \frac{41-30}{6}\right) - P\left(z < \frac{32-30}{6}\right)$

$= P(z < 1.83) - P(z < 0.66)$

$= 0.9664 - 0.7454 = 0.221$ (From table)

(d) $P(X = x) = 80\% = 0.80$

$\Rightarrow P\left(z = \frac{x-30}{6}\right) = 0.80$

$\Rightarrow \frac{x-30}{6} = 0.84$ (From table)

$\Rightarrow x = 35.04$

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11.- $\mu = 200$ ml, $\sigma = 15$ ml

(a) $P(x > 224) = 1 - P(x < 224)$
$= 1 - P\left(z < \frac{224-200}{15}\right)$
$= 1 - P(z < 1.6)$
$= 1 - 0.9452 = 0.0548$ (From table)

(c) $P(x > 230) = 1 - P(x < 230)$
$= 1 - P\left(z < \frac{230-200}{15}\right)$
$= 1 - P(z < 2)$
$= 1 - 0.9772 = 0.0228$ (From table) {Probability of overflow of 1 cup}

Number of overflow in 1000 cups $= 0.0228 \times 1000$
$= 22.8 \approx 23$ cups

(b) $P(191 < x < 209) = P(x < 209) - P(x < 191)$
$= P\left(z < \frac{209-200}{15}\right) - P\left(z < \frac{191-200}{15}\right)$
$= P(z < 0.6) - P(z < -0.6)$
$= 0.7257 - 0.2743 = 0.4514$ (From table)

(d) $P(x < x) = 25\% = 0.25$
$\Rightarrow P\left(z < \frac{x-200}{15}\right) = 0.25$
$\Rightarrow \frac{x-200}{15} = -0.67$ (From table)
$\Rightarrow x = 189.95$ ml

Ans 15.- (a) $P(x > 30) = 1 - P(x < 30)$
$= 1 - P\left(z < \frac{30-24}{3.8}\right)$
$= 1 - P(z < 1.57)$
$= 1 - 0.9418 = 0.0582 = p$ (From table) {Probability of atleast half an hour}

(e) ${}^3C_2\, p^2 (1-p)^{3-2} = {}^3C_2 (0.0582)^2 (0.9418)^1$
$= 9.57 \times 10^{-3}$
$= 0.00957$

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(b) $P\left(Z < \frac{15-24}{3.8}\right) = P(Z < -2.36)$

$\Rightarrow Z = 0.0091$ (From table)

$P(X > 15) = P(Z > -2.36)$

$= 1 - P(Z < -2.36)$

$= 1 - 0.0091 = 0.9909$

So, He is late 99.09% of the time

(c) $P\left(Z < \frac{25-24}{3.8}\right) = P(Z < 0.26)$

$\Rightarrow Z = 0.6026$ (From table)

$P(X > 25) = P(Z > 0.26)$

$= 1 - P(Z < 0.26)$

$= 1 - 0.6026 = 0.3074$

(d)

Ans 10.- $P(\underbrace{\mu - 3\sigma}_{x_1} < X < \underbrace{\mu + 3\sigma}_{x_2})$

$Z_1 = \frac{(\mu - 3\sigma) - \mu}{\sigma} = -3$, $Z_2 = \frac{(\mu + 3\sigma) - \mu}{\sigma} = 3$

$\Rightarrow P(\mu - 3\sigma < X < \mu + 3\sigma) = P(-3 < Z < 3)$

$= P(Z < 3) - P(Z < -3)$

$= 0.9987 - 0.0013$ (From table)

$= 0.9974$

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Ans 22.- (a) $x_1 = \mu + 1.3\sigma$, and, $x_2 = \mu - 1.3\sigma$

$\Rightarrow z_1 = 1.3$, and, $\Rightarrow z_2 = -1.3$.

Now, $P(X > \mu + 1.3\sigma) + P(X < \mu - 1.3\sigma)$

$= P(Z > 1.3) + P(Z < -1.3)$

$= 1 - P(Z < 1.3) + P(Z < -1.3)$

$= 1 - 0.9032 + 0.0968$

$= 0.1936 = \cancel{13}.\ 19.36\%$

(b) $x_1 = \mu + 0.52\sigma$, and, $x_2 = \mu - 0.52\sigma$

$\Rightarrow z_1 = 0.52$, and, $\Rightarrow z_2 = -0.52$

Now, $P(\mu - 0.52\sigma < X < \mu + 0.52\sigma)$

$= P(-0.52 < Z < 0.52)$

$= P(Z < 0.52) - P(Z < -0.52)$

$= 0.6985 - 0.3015$

$= 0.3970 = 39.70\%$

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⇒ Normal Approximation to Binomial Distribution

- Theorem 1:- If X is a binomial random variable with mean, $\mu = np$ and variance, $\sigma^2 = npq$, where $q = 1-p$, then the limiting form of the distribution of $Z = \frac{X - np}{\sqrt{npq}}$ as $n \to \infty$, is the standard normal distribution. $n(z, 0, 1)$.

- Theorem 2:- Let x be a binomial random variable with parameters n and p. For large n, X has a approximately normal distribution with mean, $\mu = np$ and variance, $\sigma^2 = npq$, then $P(X \leq x)$ can be written as,

$P(X \leq x) = P\left(Z < \frac{x + 0.5 - np}{\sqrt{npq}}\right)$

$P(a \leq X \leq b) = P(X \leq b) - P(X \leq a)$

$= P\left(Z \leq \frac{b + 0.5 - np}{\sqrt{npq}}\right) - P\left(Z \leq \frac{a + 0.5 - np}{\sqrt{npq}}\right)$

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24.- $n = 400$, $p = 0.5$

(a) $P(185 < X \leq 210) = P\left(Z < \frac{210.5 - 200}{\sqrt{400 \times 0.5 \times 0.5}}\right) - P\left(Z < \frac{184.5 - 200}{\sqrt{400 \times 0.5 \times 0.5}}\right)$

$= P(Z < 1.05) - P(Z < -1.55)$

$= 0.8531 - 0.0606$

$= 0.7925$

(b) $P(X = 205) = P\left(Z < \frac{205.5 - 200}{10}\right) - P\left(Z < \frac{204.5 - 200}{10}\right)$

$= P(Z < 0.55) - P(Z < 0.45)$

$= 0.7088 - 0.6736$

$= 0.0352$

(c) $P(X \leq 175) + P(X > 227) = P\left(Z < \frac{175.5 - 200}{10}\right) + \left\{1 - P\left(Z < \frac{227.5 - 200}{10}\right)\right\}$

$= P(Z < -2.45) + 1 - P(Z < 2.75)$

$= 0.0071 + 1 - 0.9970$

$= 0.0101$

26.- $p = 10\% = \cancel{0.1}\ 0.1$, $n = 100$ $\Rightarrow np = 10$, $\sqrt{npq} = 3$

(a) $P(X > 13) = 1 - P(X \leq 13)$

$= 1 - P\left(Z \leq \frac{13.5 - 10}{3}\right)$

$= 1 - P(Z \leq 1.16)$

$= 1 - 0.8770 = 0.1230$

(b) $P(X < 8) = P(X \leq 7)$

$= P\left(Z \leq \frac{7.5 - 10}{3}\right)$

$= P(Z \leq -0.83)$

$= 0.2033$

29.-

34.-

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29.- $p = 0.2$, $n = 1000$ $\Rightarrow np = 200$, $\sqrt{npq} = 12.64$

(a) $P(170 \leq X \leq 185) = P(X \leq 185) - P(X \leq 170)$
$= P\left(Z \leq \frac{185.5-200}{12.64}\right) - P\left(Z \leq \frac{170.5-200}{12.64}\right)$
$= P(Z \leq -1.14) - P(Z \leq -2.33)$
$= 0.1271 - 0.0099$
$= 0.1172$

(b) $P(210 < X \leq 225) = P(X \leq 225) - P(X \leq 209)$
$= P\left(Z \leq \frac{225.5-200}{12.64}\right) - P\left(Z \leq \frac{209.5-200}{12.64}\right)$
$= P(Z \leq 2.01) - P(Z \leq 0.75)$
$= 0.9778 - 0.7734$
$= 0.2044$

34.- $n = 180$, $p = 1/6$ $\Rightarrow np = 30$, $\sqrt{npq} = 5$

(a) $P(X > 25) = 1 - P(X \leq 24)$
$= 1 - P\left(Z \leq \frac{24.5-30}{5}\right)$
$= 1 - P(Z \leq -1.1)$
$= 1 - 0.1357 = 0.8643$

(b) $P(33 < X \leq 41) = P(X \leq 41) - P(X \leq 32)$
$= P\left(Z \leq \frac{41.5-30}{5}\right) - P\left(Z \leq \frac{32.5-30}{5}\right)$
$= P(Z \leq 2.3) - P(Z \leq 0.5)$
$= 0.9893 - 0.6915 = 0.2978$

(c) $P(X = 30) = P\left(Z \leq \frac{30.5-30}{5}\right) - P\left(Z \leq \frac{29.5-30}{5}\right)$
$= \cancel{0.1}\ P(Z \leq 0.1) - P(Z \leq -0.1)$
$= 0.5398 - 0.4602$
$= 0.0796$

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Eg 15.- $n = 100, \; p = 0.4$

$$P(X \leq 29) = P\left(Z < \frac{29.5 - (10 \times 0.4)}{\sqrt{10 \times 0.4 \times 0.6}}\right)$$

$$= P(Z < -2.14)$$

$$= 0.0162$$

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→ Gamma Functions

$$\int_0^\infty x^{\alpha-1} e^{-x} \, dx \quad (\alpha > 0) = \Gamma(x)$$

Properties :- (i) If α is an integer, then, $\Gamma(\alpha) = (\alpha - 1)!$

(ii) If $\alpha > 0$, $\Gamma(\alpha) = (\alpha - 1)\Gamma(1/2)$

(iii) $\Gamma(1/2) = \sqrt{\pi}$

$$\int_0^\infty x^{\alpha-1} e^{-x/\beta} \, dx = \Gamma(\alpha)\beta^\alpha$$

Proof :- Let $\frac{x}{\beta} = z$

$$\Rightarrow dx = \beta \, dz$$

$$= \int_0^\infty (\beta z)^{\alpha-1} e^{-z} . \beta \, dz$$

$$= \beta \int_0^\infty (\beta z)^{\alpha-1} e^{-z} \, dz = \beta^\alpha \int_0^\infty z^{\alpha-1} e^{-z} \, dz$$

$$= \beta^\alpha \, \Gamma(\alpha)$$

⇒ Gamma Distribution

- The continious random variable X has a gamma distribution with parameter α and β, if its density function is given by -

$$f(x, \alpha, \beta) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & x > 0, \alpha > 0, \beta > 0 \\ 0, & ew \end{cases}$$

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$$\mu = E(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x \cdot x^{\alpha-1} e^{-x/\beta}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(\alpha+1) \cdot \beta^{(\alpha+1)}$$

$$= \frac{\alpha \Gamma(\alpha)\, \beta^{(\alpha+1)}}{\Gamma(\alpha)\, \beta^\alpha} = \alpha\beta.$$

$$\sigma^2 = E(x^2) - E(x)^2$$

$$E(x^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^2 \cdot x^{\alpha-1} e^{-x/\beta}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha+1} e^{-x/\beta}\, dx \qquad \Gamma\alpha+2 = (\alpha+1)\alpha\,\Gamma\alpha$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(\alpha+2)\beta^{\alpha+2}$$

$$= \frac{\alpha(\alpha+1)\Gamma(\alpha)\beta^{\alpha+2}}{\Gamma(\alpha)\beta^\alpha} = \alpha(\alpha+1)\beta^2$$

So, $\sigma^2 = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2$

$$= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2$$

$$= \alpha\beta^2$$

→ Cumulative Distribution Function of a Gamma Random Variable

- Let us define, $F(\alpha, x) = \frac{1}{\Gamma\alpha} \int_0^x y^{\alpha-1} e^{-y}\, dy$

So, $P(X \le x) = \frac{1}{\Gamma\alpha\, \beta^\alpha} \int_0^x t^{\alpha-1} e^{-t/\beta}\, dt$

Let $t/\beta = y \Rightarrow dt = \beta\, dy$

Then, $P(X \le x) = \frac{\beta}{\Gamma\alpha\, \beta^\alpha} \int_0^{x/\beta} y^{\alpha-1} e^{-y}\, dy$

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$\Rightarrow P(X \leq x) = F(\alpha, \frac{x}{\beta})$

$\Rightarrow$ Exponential Distribution

- The continious random variable x has an exponential distribution with parameter β if its density is given by.

$$f(x, \beta) = \begin{cases} 1/\beta \; e^{-x/\beta} & , x>0, \beta>0 \\ 0 & , ew \end{cases}$$

$x \sim G(x, 1, \beta)$.
Here, $\alpha = 1$, so,
$\mu = E(X) = 1.\beta = \beta$
$\sigma^2 = E(X^2) - E(X)^2 = 1.\beta^2 = \beta^2$

$$P(X \leq x) = \begin{cases} 0, & x<0 \\ 1-e^{-x/\beta}, & x>0 \end{cases}$$

$$\frac{1}{\beta}\int_0^x e^{-t/\beta} dt = 1 - e^{-x/\beta}$$

P-213 Eg17.- T = Lifetime of a component (Years)
Given that $\mu = \beta = 5$.
$T \sim f(x, 5) = \frac{1}{5} e^{-x/5}, x \geq 0$

$$P(T>8) = 1 - P(T \leq 8) = 1 - \frac{1}{5}\int_0^8 e^{-x/5} dx = 1 - \frac{1}{5} e^{-8/5} = 0.2$$

X = Number of components surviving more than 8 years.
$X \sim b(x, 5, 0.2)$

$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - \sum_{x=0}^{1} {}^5C_x \, 0.2^x \, 0.8^{5-x} = 0.2627$$

P-215 Eg19.- X = Survival time (Weeks)

$X \sim G(x, 5, 10)$, $\alpha = 5$, $\beta = 10$

$$P(X \leq 60) = \frac{1}{\beta^5 \Gamma 5} \int_0^{60} x^{\alpha-1} e^{-x/\beta}$$

$$= \frac{1}{10^5 \times 24} \int_0^{60} x^4 e^{-x/10}$$

$$= 0.714$$

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41.- Given: X has a Gamma ~~function~~ distribution with,

$\alpha = 2$, $\beta = 1$.

Pdf - $f(x) = \dfrac{x^{\alpha-1} e^{-x/\beta}}{\Gamma\alpha \; \beta^\alpha}$

$$\Rightarrow f(x) = \frac{x^{2-1} e^{-x/1}}{\Gamma 1 \; 1^2} = \frac{x e^{-x}}{1 \times 1^2} = x e^{-x} \, dx$$

$$P(1.8 < X < 2.4) = \int_{1.8}^{2.4} x e^{-x} \, dx = 0.1543$$

46.- X = Number of years that switch survived.

$\beta = 2$.

$X \sim f(x, 2) = \frac{1}{2} e^{-x/2}$, $x > 0$

$$P(X \leq x) = 1 - e^{-(x/\beta)}$$
$$= 1 - e^{-x/2}, \quad x > 0.$$

$$P(X \leq x) = 1 - e^{-1/2} = 1 - 0.6065 = 0.3935$$

$n = 100$, $p = 0.3935$, $q = 0.6065$.

As the sample size is large, so, here, we will use Normal approximation to binomial distribution

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$$P(x \leq 30) = P\left(z \leq \frac{30+0.5-(100 \times 0.3935)}{\sqrt{100 \times 0.3935 \times 0.6065}}\right)$$

$$= P(z \leq -1.81)$$

$$= 0.0351$$

54.- X = Lifetime of 'a' resistor.

$\mu = 10, \quad \sigma = \sqrt{50}$

$$\left.\begin{array}{l} \mu = \alpha\beta = 10 \\ \sigma^2 = \alpha\beta^2 = 50 \end{array}\right\} \Rightarrow \begin{array}{l} \beta = 5 \\ \alpha = 2 \end{array}$$

$$f(x, \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma\alpha} x^{\alpha-1} e^{-x/\beta}, & x>0, \alpha>0, \beta>0 \\ 0, & ew \end{cases}$$

(a) $$P(X \leq 50) = \int_0^{50} \frac{1}{\beta^\alpha \Gamma\alpha} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \frac{1}{5^2 \Gamma 2} \int_0^{50} x^{2-1} e^{-x/5} dx$$

$$= \frac{1}{25 \times 1!} \int_0^{50} x e^{-x/5} dx$$

$$= \frac{1}{25} \int_0^{50} x e^{-x/5} dx$$

$$= 0.9995$$

(b) $$P(X \leq 10) = \frac{1}{25} \int_0^{10} x e^{-x/5} dx$$

$$= 0.5940$$

44.- X = Consumption of electric power (millions of kW hour)

$X \sim G(\alpha, \beta)$

$$\left.\begin{array}{l} \mu = \alpha\beta = 6 \\ \sigma^2 = \alpha\beta^2 = 12 \end{array}\right\} \Rightarrow \beta = 2, \ \alpha = 3.$$

(a) $\beta = 2, \ \alpha = 3$

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(b) $P(X > 12) = 1 - P(X \leq 12)$
$= 1 - F(3, 6)$
$= 1 - 0.0840 = 0.916$

Some Other Distributions (Supplementary)

⇒ Chi-Squared Distribution (χ^2)

- The continious random variable x has a chi-squared distribution with n degrees of freedom, if its density is given by –

$$f(x) = \frac{1}{2^{n/2}\,\Gamma(n/2)}\, x^{\frac{n}{2}-1} e^{-x/2}, \quad x > 0, \; n > 0$$

$X \sim \chi^2_n \sim G\left(\frac{n}{2}, 2\right)$

α = Shape Parameter
β = Scale Parameter.

$\mu = E(X) = \alpha\beta = \frac{n}{2} \times 2 = n$
$\sigma^2 = E(X^2) - E(X)^2 = \alpha\beta^2 = \frac{n}{2} \times 4 = 2n$

⇒ Beta Distribution

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1}\, dx = \frac{\Gamma(\alpha) \cdot \Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Eg.- $\int_0^1 x^5 (1-x)^{19} = B(6, 20) = \frac{\Gamma(6) \cdot \Gamma(20)}{\Gamma(6+20)}$

$= \frac{5! \cdot 19!}{25!} =$

$X \sim B(\alpha, \beta)$

$$f(x, \alpha, \beta) = \begin{cases} \frac{1}{B(\alpha, \beta)}\, x^{\alpha-1} (1-x)^{\beta-1}, & x > 0,\ \alpha > 0,\ \beta > 0 \\ 0, & ew \end{cases}$$

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$$\mu = E(x) = \frac{1}{B(\alpha,\beta)} \int_0^1 x \cdot x^{\alpha-1} (1-x)^{\beta-1}\, dx$$

$$= \frac{1}{B(\alpha,\beta)} B(\alpha+1, \beta)$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}$$

$$= \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}\cancel{\Gamma(\beta)}} \times \frac{\alpha \cancel{\Gamma(\alpha)}\cancel{\Gamma(\beta)}}{(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}}$$

$$= \frac{\alpha}{\alpha+\beta}$$

$$E(x^2) = \frac{1}{B(\alpha,\beta)} \int_0^1 x^2 \cdot x^{\alpha-1} (1-x)^{\beta-1}\, dx$$

$$= \frac{1}{B(\alpha,\beta)} B(\alpha+2, \beta)$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}$$

$$= \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}\cancel{\Gamma(\beta)}} \times \frac{\alpha(\alpha+1)\cancel{\Gamma(\alpha)}\cancel{\Gamma(\beta)}}{(\alpha+\beta+1)(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

So, $\sigma^2 = E(x^2) - E(x)^2$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

CHAPTER - 7:- FUNCTIONS OF RANDOM VARIABLES (OPTIONAL)

→ Moments and Moment Generating Functions

- Let 'X' be a random variable with probability distribution function $f(x)$. Then, we define

$$\mu_r' = r^{th} \text{ moment of } X = E(X^r)$$
$$= \sum_x x^r f(x)\, dx, \quad \text{if } X \text{ is discrete}$$
$$= \int_{-\infty}^{\infty} x^r f(x)\, dx, \quad \text{if } X \text{ is continious}$$

→ Moment Generating Functions

- We define $M_X(t) = E(e^{tx}) = \sum_x e^{tx} f(x)\, dx, \quad \text{if } X \text{ is discrete}$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx, \quad \text{if } X \text{ is continious}$$

- Theorem:- Let X be a random variable with moment generating function $M_X(t)$, then,

$$\frac{d^r}{dt^r} M_X(t) \Big|_{t=0} = E[X^r]$$

Proof:- $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

$$\Rightarrow M_X'(t) = \int_{-\infty}^{\infty} x e^{tx} f(x)\, dx$$
$$\Rightarrow M_X'(t) \big|_{t=0} = \int_{-\infty}^{\infty} x f(x)\, dx = E(X).$$

So, in general,

$$M_X^{(r)}(t) = \int_{-\infty}^{\infty} x^r e^{tx} f(x)\, dx$$
$$\Rightarrow M_X^{(r)}(t) \big|_{t=0} = \int_{-\infty}^{\infty} x^r f(x)\, dx = E[X^r]$$

- Let $X \sim b(x, n, p)$ $f(x) = {}^nC_x\, p^x q^{n-x}$, $x = 0$ to n.
(X follows binomial distribution)

Then, $M_X(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx}\, {}^nC_x\, p^x q^{n-x}$

$$= \sum_{x=0}^{n} {}^nC_x\, (pe^t)^x q^{n-x}$$

$$\Rightarrow M_X(t) = (pe^t + q)^n = (q + pe^t)^n$$

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($\because$ General binomial distribution of $(a+b)^n = \sum_{x=0}^{n} {}^nC_x \, a^x b^{n-x}$)

Now,

$E[X^r] = M_X^{(r)}(t) \,|\, t=0$

$\mu = E(X) = M_X'(t) \,|\, t=0$

$M_X'(t) = n(q+pe^t)^{n-1} \times pe^t$

$\Rightarrow M_X'(0) = np = \mu$ $\{\because p+q=1\}$

$M_X''(t) = n(n-1)(q+pe^t)^{n-2}(pe^t)^2 + M_X'(t)$

$E(X^2) = M_X''(0) = n(n-1)p^2 + np$

$\therefore \sigma^2 = E(X^2) = E(X)^2$

$= n(n-1)p^2 + np - n^2 - p^2$

$= np - np^2$

$= np(1-p)$ $\{\because 1-p=q\}$

$= npq$

– Let $X \sim P(\mu)$, $f(x) = \dfrac{e^{-\mu}\mu^x}{x!}$, $x = 0, 1, \ldots, \infty$

(X follows Poission's distribution)

So, $M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \dfrac{e^{-\mu}\mu^x}{x!}$ (It is a discrete distribution)

$= e^{-\mu} \sum_{x=0}^{\infty} \dfrac{(\mu e^t)^x}{x!}$

$= e^{-\mu} e^{\mu e t}$

$= e^{\mu(e^t - 1)}$

($\because$ General distribution of $e^r = \sum_{x=0}^{\infty} \dfrac{r^x}{x!}$) $(r = \mu e^t)$

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$$M_X(t) = e^{\mu(e^t-1)}$$
$$\Rightarrow M_X'(t) = e^{\mu(e^t-1)} \cdot \mu e^t$$
$$\Rightarrow M_X'(0) = \mu = E(x)$$

$$M_X''(t) = M_X'(t)\mu e^t + M_X(t)\mu e^t$$
$$= M_X'(t)\mu e^t + M_X'(t)$$
$$\Rightarrow M_X''(0) = \mu^2 + \mu = E(x^2)$$

So, now,

$$\sigma^2 = E(x^2) - E(x)^2$$
$$= \mu^2 + \mu - \mu^2 = \mu$$

→ Exercises → Page – 250

17.- $f(x) = \begin{cases} \frac{1}{K}, & x = 1, 2, \dots, K \\ 0, & ew \end{cases}$

$$M_X(t) = E(e^{tx}) = \sum_{x=1}^{k} \frac{e^{tx}}{k}$$
$$= \frac{1}{k}\sum_{x=1}^{k} e^{tx}$$
$$= \frac{1}{k}\left[e^t + e^{2t} + \dots + e^{kt}\right] \longrightarrow \text{Geometric progression}$$
$$= \frac{1}{k}\left[\frac{e^t(e^{kt}-1)}{e^t-1}\right] \longrightarrow \text{Sum of k terms of a GP}$$

19.- $M_X(t) = e^{\mu(e^t-1)}$

Poisson

20.- $M_X(t) = e^{4(e^t-1)}$

$X \sim P(4)$

$M_X(t)$ for poisson distribution $= e^{\mu(e^t-1)}$

$\Rightarrow$ Here, $\mu = \sigma^2 = 4$

$\Rightarrow \sigma = 2$

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So, $P(\mu - 2\sigma < X < \mu + 2\sigma)$

$= P(0 < X < 8)$

$= \sum_{x=0}^{8} \frac{e^{-\mu} 4^{x}}{x!}$

$= 0.9786$ (Poission table)

04/11/19

⇒ Function of a Random Variable

- Theorem 1:- Suppose that X is a discrete random variable with probability distribution $f(x)$. Let $Y = U(x)$ define a one-to-one transformation between the values of X and Y so that the equation $y = U(x)$ can be uniquely solved for x in terms of y. Say $x = \omega(y)$, then the probability distribution of Y is given by $g(y) = f(\omega(y))$.

Eg $f(x) = \frac{3}{4}\left(\frac{1}{4}\right)^{x-1}$, $x = 1, 2, \ldots$, $Y = X^2$.

Ans- This is a geometric distribution,

$Y = X^2 \Rightarrow X = \sqrt{Y}$ (No negative values)

$= \omega(y)$.

$g(y) = f(\sqrt{y}) = \frac{3}{4}\left(\frac{1}{4}\right)^{\sqrt{y}-1}$, $y = 1, 4, 9, 16, 25, \ldots$

- Theorem 2:- Suppose that X is a continious random variable with probability distribution $f(x)$. Let $Y = U(x)$ define a one-to-one transformation between the values of X and Y so that the equation $y = U(x)$ can be uniquely solved for x in terms of y. say $x = \omega(y)$, then the probability distribution of Y is given by

$g(y) = f(\omega(y))\,|J|$, where, $J = \omega'(y)$ is the Jacobian.

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Eg $f(x) = \begin{cases} x/12 , & 1<x<5 \\ 0 , & ew \end{cases}$. $Y = 2X-3$

Ans.- $Y = 2X - 3 \Rightarrow X = \frac{Y+3}{2}$
$= \omega(y)$
$\Rightarrow \omega'(y) = \frac{1}{2}$

$\left\{ \begin{array}{l} 1 < x < 5 \\ \Rightarrow 2 < 2x < 10 \\ \Rightarrow -1 < 2x-3 < 7 \end{array} \right\}$

Now, $g(y) = f\left(\frac{y+3}{2}\right) \times \left|\frac{1}{2}\right|$
$= \frac{(y+3)/2}{12} \times \frac{1}{2}$
$= \begin{cases} \frac{y+3}{48} , & -1<x<7 \\ 0 , & ew \end{cases}$

2.- $f(x) = {}^3C_x \left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{3-x}$, $x = 0, 1, 2, 3, \ldots$
$Y = X^2$
$\Rightarrow X = \sqrt{Y} = \omega(y)$
Now,
$g(y) = f(\sqrt{y}) = {}^3C_{\sqrt{y}} \left(\frac{2}{5}\right)^{\sqrt{y}} \left(\frac{3}{5}\right)^{3-\sqrt{y}}$, $x = 0, 1, 4, 9, \ldots$

5.- $f(x) = \begin{cases} 1 , & 0<x<1 \\ 0 , & ew \end{cases}$ $\rightarrow$ Uniform distribution, $X \sim U(0,1)$
$Y = -2\ln X$
$\Rightarrow X = e^{-Y/2}$, $y>0$ $= \omega(y)$
$g(y) = f(\omega(y))\,|\omega'(y)|$
$= f(e^{-y/2})\,|(e^{-y/2})'|$
$= 1 \times \frac{1}{2}e^{-y/2}$
$= \begin{cases} \frac{1}{2}e^{-y/2} , & y>0 \\ 0 , & ew \end{cases}$

- Theorem 3 :- Suppose X_1 and X_2 are 2 discrete variables with joint probability mass function $f(x_1, x_2)$ such that $(X_1, X_2) \sim f(x_1, x_2)$. Let $Y_1 = U_1(X_1, X_2)$ and $Y_2 = U_2(X_1, X_2)$ define a one-to-one correspondance between the pairs (x_1, x_2) and (y_1, y_2) so that the equations $y_1 = U_1(x_1, x_2)$ and $y_2 = U_2(x_1, x_2)$ can be uniquel solved for x_1 and x_2 in terms of y_1 and y_2. Say, $x_1 = w_1(y_1, y_2)$ and $x_2 = w_2(y_1, y_2)$, then the joint probabilit distribution of y_1 and y_2 is given by,

$$g(y_1, y_2) = f(w_1(y_1, y_2), w_2(y_1, y_2))$$

- Theorem 4 :- Suppose X_1 and X_2 are 2 continious variables with joint probability mass function $f(x_1, x_2)$ such that $(X_1, X_2) \sim f(x_1, x_2)$. Let $Y_1 = U_1(X_1, X_2)$ and $Y_2 = U_2(X_1, X_2)$ define a one-to-one correspondance between the pairs (x_1, x_2) and (y_1, y_2) so that the equations $y_1 = U_1(x_1, x_2)$ and $y_2 = U_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2. Say $x_1 = w_1(y_1, y_2)$ and, $x_2 = w_2(y_1, y_2)$, then the joint probability distribution of y_1 and y_2 is given by,

$$g(y_1, y_2) = f(w_1(x_1, x_2), w_2(x_1, x_2)) \, |J|$$

where, $J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$, is the Jacobian

- $X_1 \sim P_1(\mu_1)$, $X_2 \sim P(\mu_2)$

X_1 and X_2 are independent $\quad x_1 = 0, 1, \ldots, \infty$

$x_2 = 0, 1, \ldots, \infty$

$$f(x_1) = \frac{e^{-\mu_1} \mu_1^{x_1}}{x_1!}$$

$$f(x_2) = \frac{e^{-\mu_2} \mu_2^{x_2}}{x_2!}$$

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Then, $f(x_1, x_2) = \dfrac{e^{-(\mu_1+\mu_2)} \cdot \mu_1^{x_1} \cdot \mu_2^{x_2}}{x_1! \cdot x_2!}$

$Y_1 = X_1 + X_2, \quad Y_2 = X_2, \quad Y_1 > Y_2$

On solving, we get,

$x_1 = y_1 - y_2, \quad y_1 > y_2, \quad x_2 = y_2$

The joint probability distribution of y_1, y_2 is given by,

$$g(y_1, y_2) = \frac{e^{-(\mu_1+\mu_2)} \cdot \mu_1^{y_1-y_2} \cdot \mu_2^{y_2}}{(y_1-y_2)!\, y_2!}$$

$$g(y_1) = \sum_{y_2=0}^{y_1} g(y_1, y_2)$$

$$= \sum_{y_2=0}^{y_1} \frac{e^{-(\mu_1+\mu_2)} \mu_1^{y_1-y_2} \cdot \mu_2^{y_2}}{(y_1-y_2)!\, y_2!}$$

$$= \frac{e^{-(\mu_1+\mu_2)}}{y!} \sum_{y_2=0}^{y_1} \frac{y_1!}{(y_1-y_2)!\, y_2!} \mu_1^{y_1-y_2} \cdot \mu_2^{y_2}$$

$$= \frac{e^{-(\mu_1+\mu_2)}}{y!} \sum_{y_2=0}^{y_1} {}^{y_1}C_{y_2} \cdot \mu_1^{y_1-y_2}\, \mu_2^{y_2}$$

$$\Rightarrow g(y_1) = \frac{e^{-(\mu_1+\mu_2)}}{y_1!} (\mu_1+\mu_2)^{y_1}, \quad y_1 = 0, 1, \ldots, \infty$$

$Y_1 \sim P(\mu_1+\mu_2), \quad Y_1 = X_1 + X_2$

4.- $f(x_1, x_2) = \dfrac{x_1 x_2}{18}, \quad x_1 = 1, 2; \; x_2 = 1, 2, 3$

$Y = X_1 X_2, \quad Y = 1, 2, 3, 4, 6$

$P(Y=1) = P(X_1X_2 = 1) = P(X_1 = 1, X_2 = 1) = \frac{1}{18}$

$P(Y=2) = P(X_1 X_2 = 2) = P(X_1 = 2, X_2 = 1) + P(X_1 = 1, X_2 = 2) = \frac{2}{9}$

$P(Y=3) = P(X_1 X_2 = 3) = P(X_1 = 1, X_2 = 3) = \frac{3}{18}$

$P(Y=4) = P(X_1 X_2 = 4) = P(X_1 = 2, X_2 = 2) = 4/18$

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$P(Y=6) = P(X_1X_2=6) = P(X_1=2, X_2=3) = 6/18$

Y	1	2	3	4	6
$g(y)$	$1/18$	$4/18$	$3/18$	$4/18$	$6/18$

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3.- $f(x_1, x_2) = \dfrac{2!}{x_1!\, x_2!\, (2-x_1-x_2)!} \left(\frac{1}{4}\right)^{x_1} \left(\frac{1}{3}\right)^{x_2} \left(\frac{5}{12}\right)^{2-x_1-x_2}$

$Y_1 = X_1 + X_2$ $\quad x_1 = 0, 1, 2 \quad x_2 = 0, 1, 2$

$Y_2 = X_1 - X_2$ $\quad 0 \leq x_1 + x_2 \leq 2$

$\Rightarrow X_1 = \dfrac{Y_1+Y_2}{2}, \quad X_2 = \dfrac{Y_1 - Y_2}{2}$

$Y_1 = 0, 1, 2, \quad Y_2 = -2, -1, 0, 1, 2$

According to the given condition,

$Y_1 + Y_2 = 0, 2, 4, \quad Y_1 \geq Y_2$

$g(y_1, y_2) = f(w_1(y_1, y_2), w_2(y_1, y_2))$

$= \dfrac{2!}{\left(\frac{y_1+y_2}{2}\right)! \left(\frac{y_1-y_2}{2}\right)! (2-y_1)!} \times \left(\frac{1}{4}\right)^{\frac{y_1+y_2}{2}} \left(\frac{1}{3}\right)^{\frac{y_1-y_2}{2}} \times \left(\frac{5}{12}\right)^{2-y_1}$

8.- (i) Y = Profit (in \$5000)

$Y = X^2, \quad f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & ew \end{cases}$

$\Rightarrow w(y) = x = \sqrt{y}, \quad 0 < y < 1$

$g(y) = f(w(y))\,|w'(y)| = f(\sqrt{y})\, \dfrac{1}{2\sqrt{y}}$

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$$= \begin{cases} 2(1-\sqrt{y})\dfrac{1}{2\sqrt{y}} &, 0<y<1 \\ 0 &, ew \end{cases}$$

$$= \begin{cases} y^{-1/2} - 1 &, 0<y<1 \\ 0 &, ew \end{cases}$$

(ii) P(Profit < \$500)

$$\Rightarrow P\left(Y < \frac{1}{10}\right) = \int_0^{1/10} \frac{1}{\sqrt{y}} - 1 \, dy$$

$$=$$

12.- $f(x) = \begin{cases} e^{-(x_1+x_2)} &, x_1>0, x_2>0 \\ 0 &, ew \end{cases}$

$Y_1 = X_1 + X_2, \quad Y_2 = \dfrac{X_1}{X_1+X_2}$

So, $X_1 = Y_1 Y_2, \quad X_2 = Y_1 - Y_1 Y_2; \quad Y_1 > 0, \quad 0 < Y_2 < 1$

$g(y_1, y_2) = f[w_1(y_1; y_2), w_2(y_1, y_2)] \, [J]$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix}$$

$$= -y_1 y_2 - y_1(1-y_2)$$

$$= -y_1$$

$g(y_1, y_2) = f[y_1 y_2, y_1 - y_1 y_2] \, y_1$

$$= \begin{cases} y_1 e^{-y_1} &, y_1 > 0, \quad 0<y_2<1 \\ 0 &, ew \end{cases}$$

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$$g(y_1) = \int_0^1 y_1 e^{-y_1}\, dy_2 = \begin{cases} y_1 e^{-y_1}, & y_1 > 0 \\ 0, & ew \end{cases}$$

$$g(y_2) = \int_0^\infty y_1 e^{-y_1}\, dy_1 = \begin{cases} 1, & 0 < y_2 < 1 \\ 0, & ew \end{cases}$$

$$g(y_1) \cdot g(y_2) = g(y_1, y_2)$$

So, y_1 and y_2 are independent. Hence, proved.

10.- $f(x,y) = \begin{cases} 24xy, & 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad x+y \leq 1 \\ 0, & ew \end{cases}$

$Z = X + Y$

Let $U = X$, $Z = X + Y$. $\qquad 0 < U < 1, \quad Z > U$

$\Rightarrow X = U$, $Y = Z - U$ $\qquad 0 < Z < 1$

$$g(u,z) = f(w_1(u,z), w_2(u,z))\, |J|$$

$$|J| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

So, $g(u,z) = \begin{cases} 24u(z-u), & 0 < u < 1, \ 0 < z < 1, \ u < z \\ 0, & ew \end{cases}$

$$g(z) = \begin{cases} \int_0^z 24u(z-u)\, du, & 0 < z < 1 \\ 0, & ew \end{cases}$$

$$= \begin{cases} 4z^3, & 0 < z < 1 \\ 0, & ew \end{cases}$$

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SOME OTHER THEOREMS (SUPPLEMENTARY)

→ **Uniqueness Theorem**

- Let X and Y be 2 random variables with moment generating functions $X \to M_X(t)$ and $Y \to M_Y(t)$ respectively. If $M_X(t) = M_Y(t)$ for all t, then X and Y have the same probability distributions.

- $M_{X+a}(t) = e^{at} M_X(t)$

$$M_{X+a}(t) = E\left[e^{(x-a)t}\right] = E\left[e^{at} \cdot e^{xt}\right]$$
$$= e^{at} M_X(t)$$

$$\left\{ \because M_X(t) = E\left[e^{tx}\right] \right\}$$

→ **Linear Combinations of Random Variables**

- If $X_1, X_2, \ldots, X_n$ are independent variables with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, then, the random variable Y
$Y = a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$ has the
mean, $\mu_Y = a_1 \mu_1 + a_2 \mu_2 + \ldots + a_n \mu_n$, and
variance, $\sigma_Y^2 = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \ldots + a_n^2 \sigma_n^2$

Eg 7.- (Left Out) Show that moment generating function of random variable X having a normal probability distribution with μ and variance σ^2 is given by:-

$$M(x)(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

Ans.- $M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

Given: X follows normal distribution i.e.

$$X \sim N(x, \mu, \sigma) \text{ such that } f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

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$$\therefore M_x(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-x^2+2x\mu-\mu^2+2\sigma^2 tx}{2\sigma^2}\right\} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-(x+(\mu+\sigma^2 t))^2 - 2\mu\sigma^2 t - \sigma^4 t^2}{2\sigma^2}\right\} dx$$

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-[x+(\mu+\sigma^2 t)]^2}{2\sigma^2}\right\} dx$$

Let $w = \frac{x-(\mu+\sigma^2 t)}{\sigma} \Rightarrow dx = \sigma\, dw$

$$\therefore M_x(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \times \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-w^2/2} \sigma\, dw$$

$$= \exp\left(\mu t - \frac{\sigma^2 t^2}{2}\right) \times \int_{-\infty}^{\infty} \frac{e^{-w^2/2}}{\sqrt{2\pi}} dw$$

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

Hence, proved.

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CHAPTER - 8 :- FUNDAMENTAL SAMPLING DISTRIBUTIONS AND DATA DESCRIPTIONS

⇒ Population and Sample

- A population consists of totality of the observations with which we are concerned, i.e. we want to study certain parameters of the population, which may not be known to us.
- If the population is small enough, then we can study the entire population or else, it would be very difficult to study a population.

- A sample is a subset of the population.
- Let $X_1, X_2, \ldots, X_n$ be n independent random variables, each having the same probability distribution $f(x)$. Define $X_1, X_2, \ldots, X_n$ to be a random sample of size n from the population $f(x)$ and let us write the joint probability distribution as,
  $f(x_1, x_2, \ldots, x_n) = f(x_1) \cdot f(x_2) \ldots f(x_n)$
  ↳ Likelihood function
  ↳ IID (Independent and Identically Distributed)

⇒ Statistic

- Any function of the random variables constituting a random sample is called a statistic.

$$T = g(x_1, x_2, \ldots, x_n)$$

Eg.- $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $S^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$

- The probability distribution of a statistic is known as sampling distribution.

→ Sampling distribution of $\bar{X}$ and S^2

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

$\mu_{\bar{X}} = \frac{n\mu}{n} = \mu$, $\sigma_{\bar{X}} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$ } Mean and Variance of Mean

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→ Central Limit Theorem

- If $\bar{x}$ is the mean of a random sample of size n taken from a population with mean μ and variance σ^2, then the limiting form of the distribution of Z,
$$Z = \left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right) \text{ as } n \to \infty$$ is the standard normal distribution, $N(0,1)$.

11/09/19

P-260 Eg 4.- X = Lifetime of a bulb

$\mu = 800$ hours, $\sigma = 40$ hours, $n = 16$, $\bar{x} < 775$.

$P(\bar{x} < 775) = P\left(Z < \left(\frac{775-800}{40}\right)\sqrt{16}\right)$

$= P(Z < -2.5)$

$= 0.0062$

Eg 5.- X = Time taken by bus

$\mu = 28$ mins, $\sigma = 5$ mins, $n = 40$, $\bar{x} > 30$.

$P(\bar{x} > 30) = P(\bar{x} \geq 30.5)$

$= 1 - P(\bar{x} < 30.5)$

$= 1 - P(Z < 3.16)$

$= 1 - 0.9992 = 0.0006$

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17.- $\mu = 50$, $\sigma = 5$, $n = 16$

$P(\mu_{\bar{x}} - 1.9\sigma_{\bar{x}} < \bar{x} < \mu_{\bar{x}} - 0.4\sigma_{\bar{x}})$

$= P(50 - 1.9 \times \frac{5}{4} < \bar{x} < 50 - 0.4 \times \frac{5}{4})$

$= P(47.62 < \bar{x} < 49.5)$

$= P(\bar{x} < 49.5) - P(\bar{x} < 47.62)$

$= P(Z < -0.4) - P(Z < -1.9)$

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$= 0.3446 - 0.0287$
$= 0.3159$

19.- $\mu = 78.3$, $\sigma = 5.6$, n changes from 64 to 196.

$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}$

At $n = 64$, $\sigma_{\bar{x}}^2 = 0.49$
At $n = 196$, $\sigma_{\bar{x}}^2 = 0.16$.
Change in value $= 0.49 - 0.16 = 0.33$

20.- $f(x) = \begin{cases} 1/3, & x = 2, 4, 6 \\ 0, & ew \end{cases}$

$n = 54$.
$\mu = E(x) = \sum x f(x) = 2\times\frac{1}{3} + 4\times\frac{1}{3} + 6\times\frac{1}{3} = 4.$

$E(x^2) = 4\times\frac{1}{3} + 16\times\frac{1}{3} + 36\times\frac{1}{3} = 18.7$
$\sigma^2 = E(x^2) - E(x)^2 = 18.7 - 16$
$= 2.7$

$\Rightarrow \sigma = 1.64$
Now, $P(4.1 < \bar{x} < 4.4) = P(\bar{x} < 4.4) - P(\bar{x} < 4.1)$
$= P(z < 1.79) - P(z < 0.44)$
$= 0.9633 - 0.6700$
$= 0.2933$

23.- X = Number of cherries

X	4	5	6	7
$f(x)$	0.2	0.4	0.3	0.1

(a) $\mu = E(x) = 4\times0.2 + 5\times0.4 + 6\times0.3 + 7\times0.1 = 5.3$
$E(x^2) = 16\times0.2 + 25\times0.4 + 36\times0.3 + 49\times0.1 = 28.9$
$\sigma^2 = E(x^2) - E(x)^2 = 0.81 \Rightarrow \sigma = 0.9$

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(b) $\mu_{\bar{x}} = \mu = 5.3$

$\sigma^2_{\bar{x}} = \frac{\sigma^2}{n} = 0.02$

(c) $P(\bar{x} < 5.5) = P\left(z < \frac{5.5 - 5.3}{\cancel{36}\ 0.9}\sqrt{36}\right)$

$= P(z < 1.33)$

$= 0.9082$

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24.- $\mu = 40$ ohms, $\sigma = 2$ ohms, $n = 36$

$\bar{x} = \frac{1458}{36} = 40.5$

$P(\bar{x} > 40.5) = 1 - P(\bar{x} \leq 40.5)$

$= 1 - P\left(z < \frac{40.5 - 40}{2} \cdot \sqrt{36}\right)$

$= 1 - P(z < 1.5)$

$= 1 - 0.9332 = 0.0668$

26.- $\mu = 3.2$ mins, $\sigma = 1.6$ mins, $n = 64$

(a) $P(\bar{x} < 2.7) = P\left(z < \frac{2.7 - 3.2}{1.6}\sqrt{64}\right)$

$= P(z < -2.5)$

$= 0.0062$

(b) $P(\bar{x} > 3.5) = 1 - P(\bar{x} < 3.5)$

$= 1 - P\left(z < \frac{3.5 - 3.2}{1.6}\sqrt{64}\right)$

$= 1 - P(z < 1.5)$

$= 1 - 0.9332$

$= 0.0668$

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(c) $P(3.2 < \bar{x} < 3.4) = P(\bar{x} < 3.4) - P(\bar{x} < 3.2)$

$= P\left(z < \frac{3.4 - 3.2}{1.6}\sqrt{64}\right) - P\left(z < \frac{3.2 - 3.2}{1.6}\right)$

$= P(z < 1) - P(z < 0)$

$= 0.8413 - 0.5000$

$= 0.3413$

30.- $\mu = 540$, $\sigma = 50$

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⇒ Sampling Distribution of S^2

- Theorem :- If S^2 is the variance of a random sample of size n taken from normal population with mean μ and variance σ^2, then the statistic
$$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$$ has a Chi-Squared distribution with $n-1$ degrees of freedom.

Pdf - $$\frac{1}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} x^{\frac{n-1}{2}-1} e^{-\frac{x}{2}}$$

- Theorem :- Let $X_1, X_2, \ldots, X_n$ be independent random variables that are all normal $(\sim N(\mu, \sigma^2))$.

$$T = \sqrt{n}\left(\frac{\bar{x}-\mu}{s}\right) \sim t \text{ distribution with } n-1 \text{ degrees of freedom}$$

- For large sample sizes $(n \geq 30)$, $t \approx N(0,1)$.

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CHAPTER- 9:- ONE- AND TWO - SAMPLE ESTIMATION PROBLEMS

⇒ Statistical Inferences

- Let X comes from a population with probability distribution $X \sim f_{\theta}(x)$, where 'θ' is unknown. Statistical inference is that study of statistics in which we can infer (estimate) the unknown parameter θ by means of a random sample of size n from the distribution of $f_{\theta}(x)$.
- Generally, there are 3 important methods in infering about an unknown parameter :-
(i) Point estimation, (ii) Interval estimation, (iii) Hypothesis testing.

→ Point estimation

- In this method, we try to find out the unknown value of θ pointwise based on a random sample of size n taken from the population. It is of the following types :-
(i) Maximum Likelihood Estimator (MLE).
(ii) Bayesian Estimator
(iii) Method of Moment Estimator
(iv) Equivariant Estimator
(v) Unbiased Estimator
(vi) Least Square Estimator.

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⇒ Maximum Likelihood Estimator

- Let the random variable X comes from a population with probability distribution $f_{\theta}(x)$, where θ may be unknown. In order to estimate the value of θ pointwise we take a random sample $X_1, X_2, \ldots, X_n$ from the population. Since a random sample is always independent and identically distributed random variables then, we write the likelihood function (joint distribution) given by,

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$$l = f_\theta(x_1, \ldots, x_n) = f_\theta(x_1) \cdot f_\theta(x_2) \ldots f_\theta(x_n)$$

- To find out the MLE of θ, we will maximise the likelihood function with respect to θ.
To avoid complexity, log function is given by,

$$L = \ln l = \sum_{i=1}^{n} \ln f_{(\theta)}(x_i)$$

Now, taking $\frac{dL}{d\theta} = 0 \Rightarrow$

The value of θ for which $\frac{dL}{d\theta} = 0$, given the MLE of θ
i.e. $\theta = \hat{\theta}_{MLE}$

- Let $X \sim P(\mu)$ μ is unknown

$$f(x) = \frac{e^{-\mu}\mu^x}{x!}, x = 0, 1, 2, \ldots \infty, \quad \text{--- (i)}$$

Let $X_1, X_2, \ldots, X_n$ be a random sample from equation (i)
The likelihood function thus is given by,

$$l = \frac{e^{-\mu}\mu^{x_1}}{x_1!} \cdot \frac{e^{-\mu}\mu^{x_2}}{x_2!} \cdots \frac{e^{-\mu}\mu^{x_n}}{x_n!}$$

$$\Rightarrow l = \frac{e^{-n\mu}\mu^{\sum_{i=1}^{n} x_i}}{(x_1! \ldots x_n!)}$$

Thus the log likelihood function can be written as $L = \ln l$

$$\Rightarrow L = \log(e^{-n\mu}) + \log(\mu^{\sum_{i=1}^{n} x_i}) - \log(x_1! \ldots x_n!)$$

$$= -n\mu + \sum_{i=1}^{n} x_i \log\mu - \log(x_1! \ldots x_n!)$$

Now, $\frac{dL}{d\mu} = 0 \Rightarrow -n + \frac{\sum_{i=1}^{n} x_i}{\mu} - 0 = 0$

$$\Rightarrow \frac{1}{\mu} = \frac{n}{\sum_{i=1}^{n} x_i}$$

$$\Rightarrow \hat{\mu}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{X}$$

- $X \sim N(\mu, \sigma^2)$

(i) μ is unknown but σ^2 is known.

(ii) μ is known but σ^2 is unknown.

(iii) Both μ and σ^2 are unknown

$X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$l = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(x_1-\mu)^2} \cdot \ldots \ldots \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(x_n-\mu)^2}$$

$$= \sigma^{-n} (2\pi)^{-n/2} \cdot e^{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$\Rightarrow \log l = -n\log\sigma \times \left(\frac{-n}{2}\right)\log 2\pi \times \frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

(i) $\frac{\delta L}{\partial\mu} = 0 \Rightarrow \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0$

$$\Rightarrow \Sigma(x_i-\mu) = 0$$

$$\Rightarrow \Sigma x_i - n\mu = 0$$

$$\Rightarrow \hat{\mu}_{MLE} = \frac{\Sigma x_i}{n} = \bar{X}$$

(ii) $\frac{\partial L}{\partial\sigma} = 0 \Rightarrow \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2 = 0$

$$\Rightarrow \frac{1}{\sigma}\left(-n + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right) = 0$$

$$\Rightarrow \sigma^2_{MLE} = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}$$

(iii) Here, $\hat{\mu}_{MLE} = \bar{x}$

and, $\sigma^2_{MLE} = \frac{\sum_{1}^{n}(x_i-\bar{x})^2}{n} \Rightarrow \hat{\sigma}^2_{MLE} = \frac{(n-1)s^2}{n}$

$\left\{ \text{Sample variance, } s^2 = \frac{\Sigma(x_i-\bar{x})^2}{n-1} \Rightarrow \Sigma(x_i-\bar{x})^2 = s^2(n-1) \right\}$

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Eg.- $f(x) = \begin{cases} \frac{1}{b-a} , & a<x<b \\ 0 , & ew \end{cases}$

$l = \frac{1}{b-a}$ l is maximum only when $b-a$ is minimum

$b = X$ max x_i

$a = X$ ~~max~~ min x_i

$a < x_1, x_2, \dots, x_n < b$

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85.- $f(\theta) = \begin{cases} \frac{1}{\theta} , & 0 < x < \theta \\ 0 , & ew \end{cases}$

'θ' is unknown.

$l = \frac{1}{\theta^n}$. l is maximum only when 'θ' is minimum.

$0 < x_1, x_2, \dots, x_n < \theta$

Let $0 < x_{(1)} \leq x_{(2)} \leq x_{(3)} \dots \leq x_{(n)} < \theta$

So, $\hat{\theta}_{MLE} = X_{(1)} \min(X_1, X_2, \dots, X_n)$

→ Interval Estimation

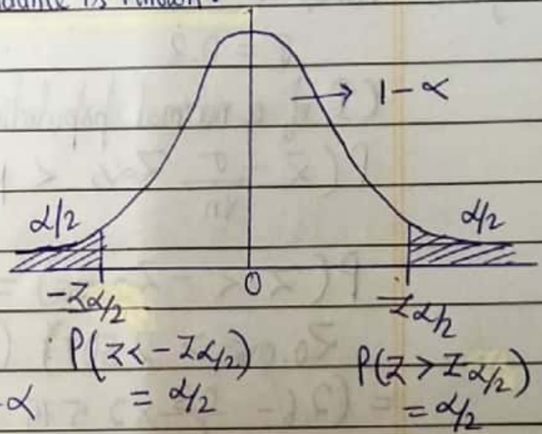
- In this type of estimation, we are interested to find out 2 statistics T_1 and T_2 such that

$$P(T_1 \leq \theta \leq T_2) = 1-\alpha$$

where, 'θ' is the unknown, 'α' is preassigned level of significance.
Then, $T_1 \leq \theta \leq T_2$ is valued $100(1-\alpha)\%$ Confidence Interval (CI) for estimating θ.

- Note that, the statistics T_1 and T_2 are functions of a random sample

- Confidence Interval of a Normal Population when Variance is Known :-



$$Z = \sqrt{n}\left(\frac{\bar{x}-\mu}{\sigma}\right) \sim N(0,1)$$

$$P(-Z_{\alpha/2} < Z < Z_{\alpha/2}) = 1-\alpha$$

$$\Rightarrow P\left(-Z_{\alpha/2} < \frac{\sqrt{n}(\bar{x}-\mu)}{\sigma} < Z_{\alpha/2}\right) = 1-\alpha$$

$$\Rightarrow P\left(\frac{-\sigma}{\sqrt{n}} Z_{\alpha/2} < \bar{x}-\mu < \frac{\sigma}{\sqrt{n}} Z_{\alpha/2}\right) = 1-\alpha$$

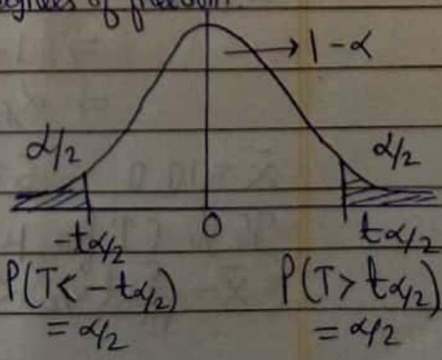
$$\Rightarrow P\left(\frac{-\sigma}{\sqrt{n}} Z_{\alpha/2} - \bar{x} < -\mu < \frac{\sigma}{\sqrt{n}} Z_{\alpha/2} - \bar{x}\right) = 1-\alpha$$

$$\Rightarrow P\left(\bar{x} - \frac{\sigma}{\sqrt{n}} Z_{\alpha/2} < \mu < \bar{x} + \frac{\sigma}{\sqrt{n}} Z_{\alpha/2}\right) = 1-\alpha$$

So, $100(1-\alpha)\%$ CI for μ.

- Confidence Interval for μ of a Normal Population when Variance is Unknown :-

$$Z = \sqrt{n}\left(\frac{\bar{x}-\mu}{s}\right) \sim t \text{ distribution with } (n-1) \text{ degrees of freedom.}$$



$$P(-t_{\alpha/2} < T < t_{\alpha/2}) = 1-\alpha$$

$$\Rightarrow P\left(-t_{\alpha/2} < \frac{\sqrt{n}(\bar{x}-\mu)}{s} < t_{\alpha/2}\right) = 1-\alpha$$

$$\Rightarrow P\left(\frac{-s}{\sqrt{n}} t_{\alpha/2} < \bar{x}-\mu < \frac{s}{\sqrt{n}} t_{\alpha/2}\right) = 1-\alpha$$

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$$\Rightarrow P\left(\frac{-S}{\sqrt{n}} t_{\alpha/2} - \bar{x} < -\mu < \frac{S}{\sqrt{n}} t_{\alpha/2} - \bar{x}\right) = 1-\alpha$$

$$\Rightarrow P\left(\bar{x} - \frac{S}{\sqrt{n}} t_{\alpha/2} < \mu < \bar{x} + \frac{S}{\sqrt{n}} t_{\alpha/2}\right) = 1-\alpha$$

So, $100(1-\alpha)\%$ CI for μ.

$$\left\{ \bar{x} < \frac{S}{\sqrt{n}} t_{\alpha/2} < \mu < \bar{x} + \frac{S}{\sqrt{n}} t_{\alpha/2} \right\}$$

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Eg 2.- Here, $n=36$, $\bar{x}=2.6$, $1-\alpha=0.99$.
$\sigma=0.3$ $\Rightarrow \alpha=0.01$

CI of a normal population when σ is known.

$$P\left(\bar{x} - \frac{\sigma}{\sqrt{n}} Z_{\alpha/2} < \mu < \bar{x} + \frac{\sigma}{\sqrt{n}} Z_{\alpha/2}\right) = 1-\alpha.$$

$P(Z < -Z_{0.005}) = 0.005$

$Z_{0.005} > 2.507$ (From the table)

$$= \left(2.6 - \frac{0.3}{6} \times 2.575 < \mu < 2.6 + \frac{0.3}{6} \times 2.575\right)$$

$$= (2.471 < \mu < 2.728)$$

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P-309
Eg 5.- $n=7$ 9.8, 10.2, 10.4, 9.8, 10.0, 10.2, 9.6

95% CI $\Rightarrow$ $95\% = 1-\alpha$
$\Rightarrow 1-\alpha = 0.95$
$\Rightarrow \alpha/2 = 0.025$

$$S^2 = \frac{\Sigma (x_i - \bar{x})^2}{n-1}$$

$$t_{\alpha/2} = t_{0.025} = 2.447$$

$\bar{x} = 10.0$, $S = 0.283$

95% CI for μ

$$\left(\bar{x} - \frac{S}{\sqrt{n}} t_{\alpha/2} < \mu < \bar{x} + \frac{S}{\sqrt{n}} t_{\alpha/2}\right)$$

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$$= \left(10 - \frac{0.283}{\sqrt{7}} \times 2.447 < \mu < 10 + \frac{0.283}{\sqrt{7}} \times 2.447\right)$$

$$= (9.73 < \mu < 10.26)$$

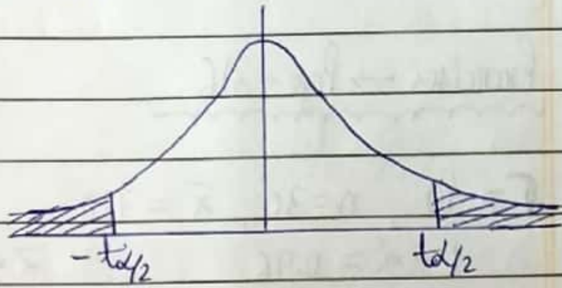
$$T = \frac{\sqrt{n}(\bar{x}-\mu)}{s} \sim t_{(n-1)} \text{ degrees of freedom.}$$

$$P(T > t_{\alpha/2}) = \alpha/2$$

$$t_{\alpha/2} = 2.447$$

$$(n-1) = \sigma = \nu \cdot$$

$$\alpha = 0.025$$



– <u>Confidence Interval for σ^2 when Mean may or may not be Known :–</u>

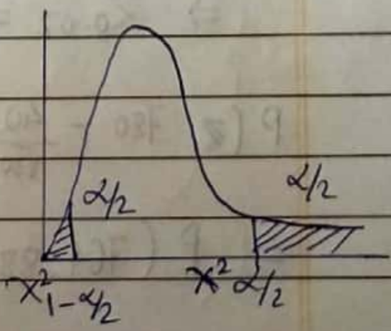
$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)} \text{ degrees of freedom}$$

$$\Rightarrow P\left(\chi^2_{1-\alpha/2} < \chi^2 < \chi^2_{\alpha/2}\right) = 1-\alpha$$

$$\Rightarrow P\left(\chi^2_{\alpha/2} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{\alpha/2}\right) = 1-\alpha$$

$$\Rightarrow P\left(\frac{\chi^2_{\alpha/2}}{(n-1)s^2} < \frac{1}{\sigma^2} < \frac{\chi^2_{\alpha/2}}{(n-1)s^2}\right) = 1-\alpha$$

$$\Rightarrow P\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{\alpha/2}}\right) = 1-\alpha.$$



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Eg17.- $n = 10$, $S = 46.4, 46.1, 45.8, 47.0, 46.1, 45.9, 45.8, 46.9, 45.2, 46.0$

95% CI for σ $\Rightarrow$ $1-\alpha = 0.95$

$\Rightarrow \alpha/2 = 0.025$ at $n-1$ degrees of freedom

$$P(\chi^2 > \chi^2_{\alpha/2}) = \alpha/2$$

$$\chi^2_{0.025} = 19.023 \qquad n = 9$$

$$P(\chi^2 > \chi^2_{0.975}) = 0.975$$

$$\{1 - \alpha/2 = 0.975 \quad \chi^2_{0.975} = 2.7, \ \alpha = 0.975 \text{ at } n = 9\}$$

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$S^2 = 0.286$

CI for σ^2 $P\left(\frac{9\times 0.286}{19.023} < \sigma^2 < \frac{9\times 0.286}{2.7}\right)$

$= P(0.135 < \sigma^2 < 0.95)$

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2.- $\sigma = 40$, $n = 30$, $\bar{x} = 780$.

$1-\alpha = 0.96$ $Z_{\alpha/2} = Z_{0.02} = 0.50$

$\alpha/2 = 0.02$

CI of $\mu = P\left(\bar{x} - \frac{\sigma}{\sqrt{n}} Z_{\alpha/2} < \mu < \bar{x} + \frac{\sigma}{\sqrt{n}} Z_{\alpha/2}\right) = 1-\alpha$

$P(Z < -Z_{0.02}) = 0.02$

$-Z_{0.02} = \cancel{-20.} -2.07$

$\Rightarrow Z_{0.02} = 2.07$

$P\left(\cancel{z}\ 780 - \frac{40}{\sqrt{30}} \times 2.07 < \mu < 780 + \frac{40}{\sqrt{30}} \times 2.07\right)$

$= P(764.883 < \mu < 795.117)$

4.- $n = 50$, $\bar{x} = 174.5$, $S = 6.9$

(a) 98 % CI - $1-\alpha = 0.98$ If $n > 30$ → Normal distribution

$\alpha/2 = 0.01$

CI for $\mu = P\left(\bar{x} - \frac{S}{\sqrt{n}} Z_{0.01} < \mu < \bar{x} + \frac{S}{\sqrt{n}} Z_{0.01}\right) = 1-\alpha$

$P(Z < -Z_{0.01}) = 0.01$

$\Rightarrow Z_{0.01} = 2.32$ (from table)

$P\left(174.5 - \frac{6.9}{\sqrt{50}} \times 2.32 < \mu < 174.5 + \frac{6.9}{\sqrt{50}} \times 2.32\right)$

$= P(172.237 < \mu < 176.763)$

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7.- Here, μ is known.

$$P\left(\bar{x} - \frac{\sigma}{\sqrt{n}} Z_{\alpha/2} < \mu < \bar{x} + \frac{\sigma}{\sqrt{n}} Z_{\alpha/2}\right) = 1-\alpha$$

$$= P\left(\frac{-\sigma}{\sqrt{n}} Z_{\alpha/2} < \bar{x} - \mu < \frac{\sigma}{\sqrt{n}} Z_{\alpha/2}\right) = 1-\alpha$$

$-a < x < a$
$\Rightarrow |x| < a.$

Error, $e = |\bar{x} - \mu| < \frac{\sigma}{\sqrt{n}} Z_{\alpha/2}$

or, $n < \left(\frac{\sigma}{e} Z_{\alpha/2}\right)^2$

if μ is known, $\sigma \to S$.

Given, $S = 0.0015$

95% CI - $1-\alpha = 0.95 \Rightarrow \alpha/2 = 0.025$

$e = 0.0005$

So, $n < \left(\frac{S}{e} Z_{0.025}\right)$

$Z_{\alpha/2} > +Z_{0.025} > 1.96$

$\Rightarrow n < \left(\frac{0.0015}{0.0005} \times 1.96\right)$

$\Rightarrow n < 5.88$, i.e. $n \leq 5$.

4.-

3.4	2.5	4.8	2.9	3.6
2.8	3.3	5.6	3.7	2.8
4.4	4.0	5.2	3.0	4.8

Here, $n = 15$.
$\bar{x} = 3.78$.

95% CI - $1-\alpha = 0.95$
$\Rightarrow \alpha/2 = 0.025$

CI for $\mu = \left(\bar{x} - \frac{S}{\sqrt{n}} t_{\alpha/2} < \mu < \bar{x} + \frac{S}{\sqrt{n}} t_{\alpha/2}\right)$

$=$

$$CI \text{ for } \sigma^2 = \left(\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{\alpha/2}} \right) = 1-\alpha$$

$=$

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⇒ Two - Sample Problems

- $X \sim f_{(\theta_1)}(x)$, $Y \sim f_{(\theta_2)}(y)$
- Confidence Interval for 2 Means of 2 Different Populations (σ_1^2 and σ_2^2 are known):-

$P(T_1 < \mu_1 - \mu_2 < T_2) = 1-\alpha$

$$Z = \frac{\bar{X} - \bar{Y} - \mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$\Rightarrow P(-Z_{\alpha/2} < Z < Z_{\alpha/2}) = 1-\alpha$

$$\Rightarrow P\left(-Z_{\alpha/2} < \frac{\bar{X} - \bar{Y} - \mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} < Z_{\alpha/2} \right) = 1-\alpha$$

$$\Rightarrow P\left(\bar{X}_1 - \bar{X}_2 - Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < \bar{X}_1 - \bar{X}_2 + Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \right) = 1-\alpha$$

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67, 68 → Not included in syllabus.

71.- $\mu = 3$, $\sigma^2 = 1$, $n = 5$ 1.9, 2.4, 3.0, 3.5, 4.2

$s^2 = 0.815 \Rightarrow s = 0.90$

95% CI for σ^2 → $1-\alpha = 0.95 \Rightarrow \alpha/2 = 0.025$

CI for σ^2 is - $\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{\alpha/2}} \right) = 1-\alpha$

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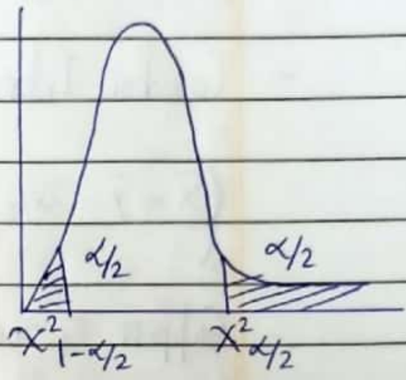
$P(\chi^2 > \chi^2_{0.025}) = 0.025$

From table, at $n=4$, $\alpha = 0.025$, $\chi^2_{0.025} = 11.143$

$P(\chi^2 > \chi^2_{0.975}) = 0.975$

From table, $\chi^2_{0.975} = 0.484$



$$CI = \left(\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}} \right)$$

$$= \left(\frac{4 \times s^2}{11.143} < \sigma^2 < 6.73 \right)$$

$$= (0.292 < \sigma^2 < 6.73)$$ So, the claim is true.

If S is unknown, So,

$$\left(\bar{x} - \frac{s}{\sqrt{n}} t_{\alpha/2} < \mu < \bar{x} + \frac{s}{\sqrt{n}} t_{\alpha/2} \right)$$

$P(T > t_{\alpha/2}) = \alpha/2$

$\Rightarrow P(T > t_{0.025}) = 0.025$, $\bar{x} = 3$.

With $n=4$, $\alpha = 0.025$,

$t_{0.025} = 2.776$

So, $\left(3 - \frac{0.9}{\sqrt{5}} \times 2.776 < \mu < 3 + \frac{0.9}{\sqrt{5}} \times 2.776 \right)$

$$= (1.88 < \mu < 4.11)$$ So, the claim is true.

If μ is unknown, So,

$P(Z < -Z_{0.025}) = 0.025$

$-Z_{0.025} = -1.96 \Rightarrow Z_{0.025} = 1.96$

Now, $CI = \left(\bar{x} - \frac{1}{\sqrt{5}} \times 1.96 < \mu < \bar{x} + \frac{1}{\sqrt{5}} (1.96) \right)$

$$= (2.12 < \mu < 3.87)$$ So, the claim is true.

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⇒ Two-Sample Estimation

- Confidence Interval, where σ_1 and σ_2 are

$$\left(\bar{X}-\bar{Y}-Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} < \mu_1-\mu_2 < \bar{X}-\bar{Y}+Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}\right)$$

Suppose σ_1 and σ_2 are unknown but equal

$$\left(\bar{X}-\bar{Y}-t_{\alpha/2}\,S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} < \mu_1-\mu_2 < \bar{X}-\bar{Y}+t_{\alpha/2}\,S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\right)$$

follows a t-distribution with (n_1+n_2-2) degrees of freedom.

Where, $S_p^2 = \dfrac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}$

Eg - $n_1 = 50$, $n_2 = 75$ $\quad\quad 1-\alpha = 0.96$

A $\quad\quad$ B. $\quad\quad \Rightarrow \alpha/2 = 0.02$

$\bar{X} = 36$, $\bar{Y} = 42$ $\quad\quad \Rightarrow Z_{\alpha/2} = 2.05$

Given, $\sigma_2 = 8$, $\sigma_1 = 6$.

$$\left(\bar{Y}-\bar{X}-Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} < \mu_B-\mu_A < \bar{Y}-\bar{X}+Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}\right)$$

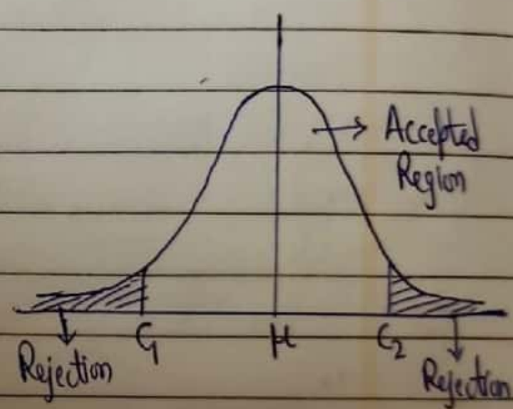
$$= \left(42-36-2.05\sqrt{\frac{64}{75}+\frac{36}{50}} < \mu_B-\mu_A < 42-36+2.05\sqrt{\frac{64}{75}+\frac{36}{50}}\right)$$

$$= (3.43 < \mu_B-\mu_A < 8.57)$$

CHAPTER-10 :- ONE- AND TWO- SAMPLE TESTS OF HYPOTHESES

⇒ Testing of Hypothesis

- A statistical hypothesis is an assertion or conjucture concerning one or two populations.
- The statistical hypothesis ~~ge~~ is generally divided into 2 categories:-
  (i) The null hypothesis (H_0), and (ii) The alternate hypothesis (H_1)
- Based on the size of the random sample of size 'n', we either reject the null hypothesis, (H_1 is accepted) or we fail to reject the null hypothesis.
- There are 3 types of tests :-
  (i) The right sided test - $H_0: \mu = 4\text{cm}$ {Diameter, e.g.}
  $H_1: \mu > 4\text{cm}$
  (ii) The left sided test - $H_0: \mu = 4\text{cm}$
  $H_1: \mu < 4\text{cm}$
  (iii) The both sided test - $H_0: \mu = 4\text{cm}$
  $H_1: \mu \neq 4\text{cm}$



- Critical region :- It is the region in which either we reject H_0 or fail to reject H_0.

→ Types of Errors

- While testing a statistical hypothesis, we generally come across 2 types of errors:-
- (i) Type I Error ($P(\text{Type I Error}) = \alpha$) :-
  Rejection of the null hypothesis when it is true is called type-I error.
  $P(H_0 \text{ is rejected} \mid H_0 \text{ is true}) = \alpha$
- (ii) Type II Error ($P(\text{Type II Error}) = \beta$) :-
  Non-rejection of the null hypothesis when it is false is called type-II error.
  $P(H_0 \text{ is not rejected} \mid H_0 \text{ is false}) = \beta$

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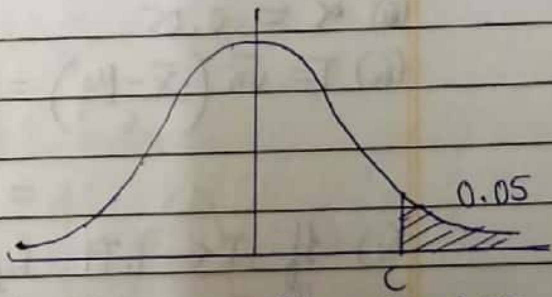
- $(1-\beta)$ is called the power of the test.
- It is very difficult to minimise both the errors simultaneously. So we keep one of the errors, α, fixed and try to minimise the other, β.

- Steps:- (0) Identify the problem
  (i) Write $H_0 : \mu = \mu_0$
  (ii) Write $H_1 : \mu \neq \mu_0, > \mu_0, < \mu_0$.
  (iii) Choose α
  (iv) Most powerful test.
  (v) Find the critical condition
  (vi) Conclude.

Eg :- $n = 100$, $\bar{x} = 71.8$, $\sigma = 8.9$, $\alpha = 5\%$



$P(z>C) = 0.05$
$\Rightarrow C = 1.65$

(i) $H_0 : \mu = 70$
(ii) $H_1 : \mu > 70$ (Right sided test)
(iii) $\alpha = 5\% = 0.05$
(iv) $z = \left(\frac{\bar{x} - \mu_0}{\sigma}\right)\sqrt{n} = \frac{71.8 - 70}{8.9} \times 10$

$= 2.02$

(v) $P(z>C) = 0.05$, $P(z<C) = 0.95$
$\Rightarrow C = 1.65$
Reject H_0; if $z > 1.65$.
Fail to reject H_0, if $z < 1.65$.
(vi) As $2.02 > 1.65$, so, reject H_0.

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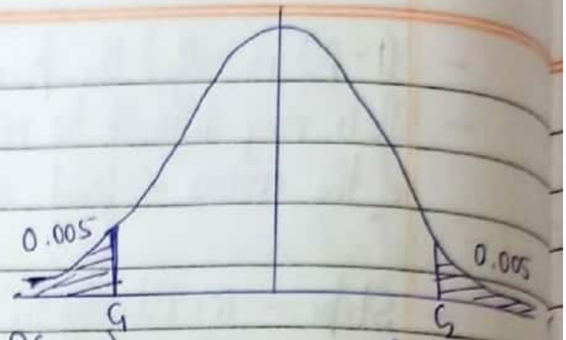
Eg :- $\sigma = 0.5$ Kg, $\mu = 8$ Kg, $\alpha = 0.01$, $n = 50$, $\bar{x} = 7.8$
(i) $H_0 : \mu = 8$
(ii) $H_1 : \mu \neq 8$ (Both sided test)
(iii) $\alpha = 0.01$

(iv) $Z = \sqrt{n}\left(\frac{\bar{X}-\mu_0}{\sigma}\right) = \sqrt{50}\left(\frac{7.8-8}{0.5}\right)$

$= -2.83$

(v) If $Z > 2.57$, or $Z < -2.57$, Reject H_0.
Otherwise fail to reject H_0.

(vi) $Z = -2.83 < -2.57$
So, Reject H_0.



$P(Z<C_1)=0.005$
$\Rightarrow C_1 = -2.57$

$P(Z>C_2)=0.005$
$\Rightarrow C_2 = 2.57$

P-382

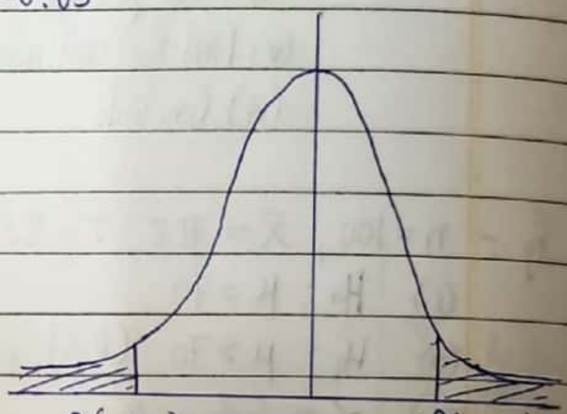
Eg 5.- $n=12$, $\bar{X} = 42$ kW/yr, $S = 11.9$, $\alpha = 0.05$

(i) $H_0: \mu = 46$
(ii) $H_1: \mu < 46$ (Left sided test)
(iii) $\alpha = 0.05$
(iv) $T = \sqrt{n}\left(\frac{\bar{X}-\mu_0}{S}\right) = \sqrt{12}\left(\frac{42-46}{11.9}\right)$

$= -1.16$

(v) If $T < -1.79$, Reject H_0.
If $T > -1.79$, fail to reject H_0.

(vi) As $-1.16 > -1.79$, so, fail to reject H_0.



$P(T<C)=0.005$
$= -1.796$

$P(T>C)$
$=0.005$
$\Rightarrow T = 1.796$
Negative

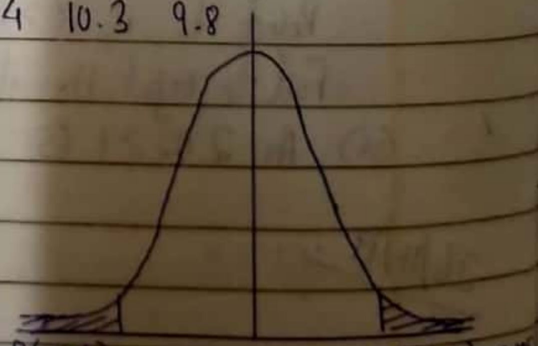
→ Exercise → Page - 398

23.- 10.2 9.7 10.1 10.3 10.1 9.8 9.9 10.4 10.3 9.8

$n = 10$, $\bar{X} = 10.06$, $S = 0.24$

(i) $H_0: \mu = 10$
(ii) $H_1: \mu \neq 10$ (Both sided test)
(iii) $\alpha = 0.01$
(iv) $T = \sqrt{n}\left(\frac{\bar{X}-\mu_0}{S}\right) = \sqrt{10}\left(\frac{10.06-10}{0.24}\right)$

$= 0.79$.



$P(T<C) = 0.005$
$\Rightarrow C = -3.250$

$P(T>C) = 0.005$
$\Rightarrow C = 3.25$

(v) If $T > 3.25$ or $T < -3.25$, Reject H_0.
Otherwise, fail to reject H_0.

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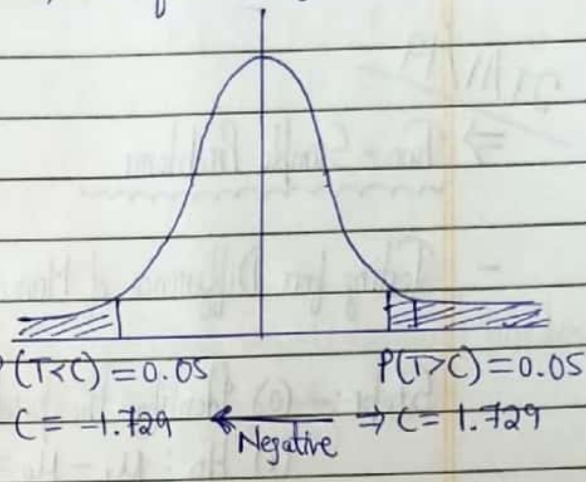
(vi) As $0.79 > -3.25$, and, $0.79 < 3.25$, So fail to reject H_0.



29.- $n=20$, $\bar{x}=33.1$, $S=4.3$, $\alpha=0.05$

(i) $H_0: \mu=35$

(ii) $H_1: \mu<35$ (Left sided test)

(iii) $\alpha=0.05$

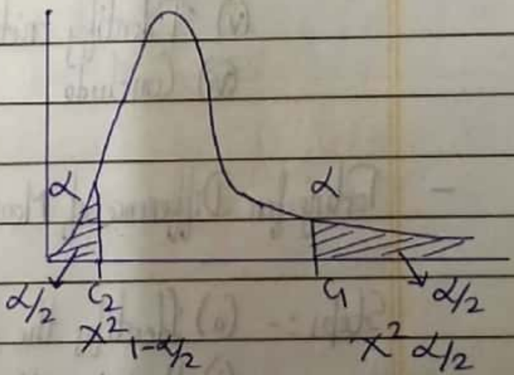
(iv) $T=\sqrt{n}\left(\frac{\bar{x}-\mu_0}{S}\right)=\sqrt{20}\left(\frac{33.1-35}{4.3}\right)$

$= -1.97$

(v) If $T<-1.729$, Reject H_0, otherwise, fail to reject H_0.

(vi) As $-1.97 < -1.729$, So, Reject H_0

– Testing for variance when mean may or may not be known :-



(0) Identify the problem

(i) $H_0: \sigma^2=\sigma_0^2$

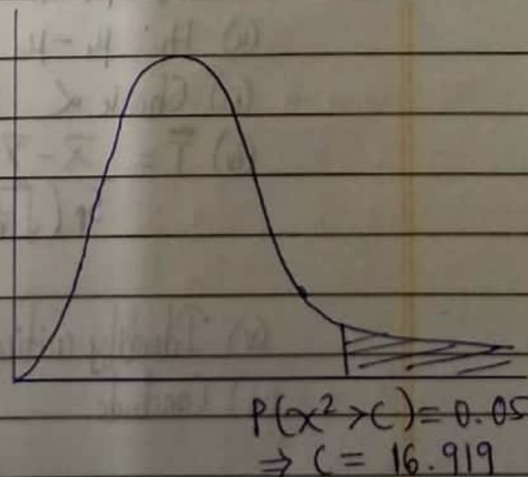
(ii) $H_1: \sigma^2 \neq \sigma_0^2$, $<\sigma_0^2$, $>\sigma_0^2$

(iii) Choose α

(iv) $\chi^2=\frac{(n-1)S^2}{\sigma^2}$, $(n-1)$ degrees of freedom.

(v) Find the critical condition.

(vi) Conclude.



P-408 Eg 12.- $\sigma=0.9 \Rightarrow \sigma^2=0.81$, $n=10$, $\alpha=5\%$.

$S=1.2 \Rightarrow S^2=1.44$

(i) $H_0: \sigma^2=0.81$

(ii) $H_1: \sigma^2>0.81$ (Right sided test)

(iii) $\alpha=0.05$

(iv) $\chi^2=\frac{9\times1.44}{0.81}=16$

(v) If $\chi^2 > 16.919$, Reject H_0, otherwise fail to reject H_0.

(vi) As $16 < 16.919$, so, fail to reject H_0.

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⇒ Two-Sample Problems

- Testing for Difference of Means when σ_1 and σ_2 are known:—

Steps:- (0) Identify the problems

(i) $H_0 : \mu_1 - \mu_0 = d_0$

(ii) $H_1 : \mu_1 - \mu_0 \neq d_0, < d_0, > d_0$

(iii) Choose α.

(iv) $$Z = \frac{\bar{X} - \bar{Y} - d_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

(v) Identify critical region.

(vi) Conclude.

- Testing for Difference of Means when σ_1 and σ_2 are unknown but equal:-

Steps:- (0) Identify the problem.

(i) $H_0 : \mu_1 - \mu_0 = d_0$

(ii) $H_1 : \mu_1 - \mu_0 \neq d_0, < d_0, > d_0$

(iii) Choose α

(iv) $$T = \frac{\bar{X} - \bar{Y} - d_0}{S_p\left(\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right)}$$ ~ t distribution with (n_1+n_2-2) degrees of freedom

where, $$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}$$

(v) Identify critical region

(vi) Conclude

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- Testing for 2 Variances when μ_1, μ_2 may or may not be known :-

Steps :- (0) Identify the problem

(i) $H_0 : \sigma_1^2 = \sigma_2^2$

(ii) $H_1 : \sigma_1^2 \neq \sigma_2^2, < \sigma_2^2, > \sigma_2^2$

(iii) Choose α

(iv) $f = \dfrac{S_1^2}{S_2^2} \sim f$ distribution with (n_1-1) and (n_2-1) degrees of freedom

and, $f_{1-\alpha}(n_1, n_2) = \dfrac{1}{f_\alpha(n_2, n_1)}$

(v) Identify critical region

(vi) Conclude.

→ Exercises → Page - 398

35.-

Treatment	2.1	5.3	1.4	4.6	0.9	$\Rightarrow \bar{x} = 2.86$
No Treatment	1.9	0.5	2.8	3.1		$\Rightarrow \bar{y} = 2.075$

(i) $H_0 : \mu_1 - \mu_2 = 0$

(ii) $H_1 : \mu_1 - \mu_2 > 0$

(iii) $\alpha = 0.05$

(iv) $T = \dfrac{\bar{x} - \bar{y} - d_0}{S_p\sqrt{\frac{1}{n_1} - \frac{1}{n_2}}} = 0.7$

(v) If $T > 1.895$, reject H_0

Otherwise, fail to reject H_0

(vi) As $T < 1.895$ so, fail to reject H_0.

→ Exercise → Page - 411

71.- $\sigma^2 > 1.15$, $n = 25$, $s^2 = \cancel{0.2m}\ 2.03$

$\alpha = 0.05$ $\quad P(T > C) = 0.05$, 24 degrees of freedom

$\Rightarrow C = 36.45$

(i) $H_0: \sigma^2 = 1.15$

(ii) $H_1: \sigma^2 > 1.15$

(iii) $\alpha = 0.05$

(iv) $\chi^2 = \dfrac{(n-1)s^2}{\sigma^2} = \dfrac{24 \times 2.03}{1.15} = 42.36$

(v) If $\chi^2 > C$, reject H_0.
otherwise, fail to reject H_0.

(vi) As $\chi^2 > 36.45$, so, reject H_0.

73.- $n_1 = 11$, $n_2 = 14$, $S_1 = 6.1$, $S_2 = 5.3$
$n_1 - 1 = 10$, $n_2 - 1 = 13$ degrees of freedom.

(i) $H_0: \sigma_1^2 = \sigma_2^2$

(ii) $H_1: \sigma_1^2 > \sigma_2^2$

(iii) $\alpha = 0.05$

(iv) $f = \dfrac{S_1^2}{S_2^2} = \dfrac{(6.1)^2}{(5.3)^2} = 1.324$

(v) If $f > C$ ($C = 2.67$), reject H_0.
otherwise fail to reject H_0.

(vi) As $f < 2.67$, so fail to reject H_0.

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⇒ Goodness of Fit Chi-Squared Test

Cell	1 2 3 . . . K	
O_i	O_1 O_2 O_3 . . . O_k	
e_i	e_1 e_2 e_3 . . . e_k	
$(O_i - e_i)^2$		$\Sigma (O_i - e_i)^2$
$\dfrac{(O_i - e_i)^2}{e_i}$		$\Sigma \dfrac{(O_i - e_i)^2}{e_i}$

$\Sigma O_i = \Sigma e_i$

$e_i = n p_i$

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Eg - A die is rolled 120 times.

Cell	1	2	3	4	5	6
O_i	20	22	17	18	19	24
e_i	20	20	20	20	20	20
$(O_i-e_i)^2$	0	4	9	4	1	6
$\frac{(O_i-e_i)^2}{e_i}$	0	0.2	0.45	0.2	0.05	0.8

$e_i = n(P_i)$, $n=120$
$= 120 \times \frac{1}{6} = 20$

$$\chi^2 = \sum_{i=1}^{6} \frac{(O_i-e_i)^2}{e_i} = 1.7$$

At $\alpha = 5\% = 0.05$, $6-1=5$ dF,
$C = 11.070$

If $\chi^2 > C$, Reject H_0, Distribution is not proper.
otherwise, fail to reject H_0, distribution is proper.
As $1.7 < 11.070$, so, fail to reject H_0, i.e, distribution is proper.

- Steps :- (i) H_0 : The distribution is proper
(ii) H_1 : The distribution is not proper.
(iii) Choose α.
(iv) $\chi^2 = \sum_{i=1}^{k} \frac{(O_i-e_i)^2}{e_i}$ with $(k-1)$ degrees of freedom.
(v) Identify critical region C.
(vi) Conclude.

→ Exercises → Page-

80.-

Grade	A	B	C	D	F
O_i	14	18	32	20	16
e_i	20	20	20	20	20
$(O_i-e_i)^2$	36	4	144	0	16
$\frac{(O_i-e_i)^2}{2}$	1.8	0.2	7.2	0	0.8

$e_i = n(P_i)$, $n = 100$
$= 100 \times \frac{1}{5} = 20$

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$$\chi^2 = \sum_{i=1}^{6} \frac{(O_i - e_i)^2}{e_i} = 10$$

At $\alpha = 5\% = 0.05$, $5-1 = 4$ dF,
$C = 9.488$

If $\chi^2 > C$, ~~fail to~~ reject H_0, distribution is not proper.
Otherwise, fail to reject H_0, distribution is proper.
As $10 > 9.488$, so, reject H_0, i.e. distribution is not proper.

83.-

Cell	1	2	3	4	5	6	7	8
O_i	136	60	34	12	9	1	3	1
e_i	32	32	32	32	32	32	32	32
$(O_i - e_i)^2$	10816	788	4	400	529	961	841	961
$\frac{(O_i - e_i)^2}{2}$	338	24.5	0.125	12.5	16.53	30.03	26.28	30.03

$e_i = nP_i, n=256$
$= 256 \times \frac{1}{8} = 32$

$$\chi^2 = \sum_{i=1}^{\infty} \frac{(O_i - e_i)^2}{e_i} = 477.995$$

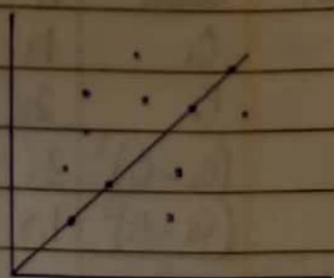
At $\alpha = 5\% = 0.05$, $8-1 = 7$ dF,
$C = 14.067$

If $\chi^2 > C$ ~~fa~~ reject H_0, distribution is not proper.
Otherwise, fail to reject H_0, distribution is proper.
As $477.995 > 14.067$, so, reject H_0, i.e, distribution is not proper.

⇒ Line of Regression

- $y = b_0 + b_1 x$ → Equation of a line
- Let $(x_1, y_1), (x, y_2), \ldots, (x_n, y_n)$ be two random samples taken from 2 populations. Our aim is to fit a straight line such that maximum number of points will lie on this ~~line~~ straight line.

→ Least Square Method



- In this method, we try to minimise the sum of the squares of these points taken horizontally.

$$D = e_1^2 + e_2^2 + \cdots + e_n^2$$

$$\frac{\partial D}{\partial b_0} = 0 \quad \text{and,} \quad \frac{\partial D}{\partial b_1} = 0$$

where, $b_1 = \dfrac{n \sum_i x_i y_i - \sum_i x_i \sum_i y_i}{n \sum_i x_i^2 - (\sum x_i)^2} = \dfrac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2}$

$$b_0 = \frac{\sum_i y_i - b_1 \sum_i x_i}{n}$$

→ Co-relation Coefficient

- Population co-relation coefficient -

$$\rho_{xy} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{E(XY) - E(X)E(Y)}{\sigma_x \sigma_y}, \quad -1 \leq \rho \leq 1$$

- Sample co-relation coefficient -

$$r = \frac{S_{xy}}{\sqrt{S_{xx} S_{xy}}}, \quad -1 \leq r < 1$$

where, $S_{xy} = \sum_i (x_i - \bar{x})(y_i - \bar{y})$

$S_{xx} = \sum_i (x_i - \bar{x})^2$

$S_{xy} = \sum_i (y_i - \bar{y})^2$

P-450

Eg 1.- $\sum x_i = 1104$, $\sum_i y_i = 1124$, $\sum x_i y_i = 41355$, $\sum x_i^2 = 41086$
Solve for b_1, b_0 when $n = 33$.

$$b_1 = \frac{33 \times 41355 - (1104 \times 1124)}{33 \times 41086 - (1104)^2} = \frac{123819}{137022} = 0.903$$

$$b_0 = \frac{1124 - (0.903 \times 1104)}{33} = 3.85$$

All the Best
2/12/19.