Sample Induction Proofs

Below are model solutions to some of the practice problems on the induction worksheets. The solutions given illustrate all of the main types of induction situations that you may encounter and that you should be able to handle. Use these solutions as models for your writing up your own solutions in exams and homework. For additional examples, see the following examples and exercises in the Rosen text: Section 4.1, Examples 1–10, Exercises 3, 5, 7, 13, 15, 19, 21, 23, 25, 45. Section 4.3, Example 6, Exercises 13, 15.

1. Prove by induction that, for all $n \in \mathbb{Z}_+$, $\sum_{i=1}^n (-1)^i i^2 = (-1)^n n(n+1)/2$.

   Proof: We will prove by induction that, for all $n \in \mathbb{Z}_+$,

   $$(1) \qquad \sum_{i=1}^n (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}.$$

   Base case: When $n = 1$, the left side of (1) is $(-1)1^2 = -1$, and the right side is $(-1)1(1+1)/2 = -1$, so both sides are equal and (1) is true for $n = 1$.

   Induction step: Let $k \in \mathbb{Z}_+$ be given and suppose (1) is true for $n = k$. Then

   $$\begin{aligned}
   \sum_{i=1}^{k+1} (-1)^i i^2 &= \sum_{i=1}^k (-1)^i i^2 + (-1)^{k+1}(k+1)^2 \\
   &= \frac{(-1)^k k(k+1)}{2} + (-1)^{k+1}(k+1)^2 \quad \text{(by induction hypothesis)} \\
   &= \frac{(-1)^k (k+1)}{2} (k - 2(k+1)) \\
   &= \frac{(-1)^k (k+1)}{2} (-k - 2) \\
   &= \frac{(-1)^{k+1}(k+2)}{2}.
   \end{aligned}$$

   Thus, (1) holds for $n = k + 1$, and the proof of the induction step is complete.

   Conclusion: By the principle of induction, (1) is true for all $n \in \mathbb{Z}_+$.

2. Find and prove by induction a formula for $\sum_{i=1}^n \frac{1}{i(i+1)}$, where $n \in \mathbb{Z}_+$.

   Proof: We will prove by induction that, for all $n \in \mathbb{Z}_+$,

   $$(1) \qquad \sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

   Base case: When $n = 1$, the left side of (1) is $1/(1 \cdot 2) = 1/2$, and the right side is $1/2$, so both sides are equal and (1) is true for $n = 1$.

Induction step: *Let $k \in \mathbb{Z}_+$ be given and suppose (1) is true for $n = k$. Then*

$$
\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \text{(by induction hypothesis)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \quad \text{(by algebra)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \quad \text{(by algebra)} \\
&= \frac{k+1}{k+2}.
\end{aligned}
$$

Thus, (1) holds for $n = k+1$, and the proof of the induction step is complete.

Conclusion: *By the principle of induction, (1) is true for all $n \in \mathbb{Z}_+$.*

3. Find and prove by induction a formula for $\sum_{i=1}^{n}(2i-1)$ (i.e., the sum of the first n odd numbers), where $n \in \mathbb{Z}_+$.

Proof: *We will prove by induction that, for all $n \in \mathbb{Z}_+$,*

$$
\text{(1)} \qquad\qquad \sum_{i=1}^{n}(2i-1) = n^2.
$$

Base case: *When $n = 1$, the left side of (1) is 1, and the right side is $1^2 = 1$, so both sides are equal and (1) is true for $n = 1$.*

Induction step: *Let $k \in \mathbb{Z}_+$ be given and suppose (1) is true for $n = k$. Then*

$$
\begin{aligned}
\sum_{i=1}^{k+1}(2i-1) &= \sum_{i=1}^{k}(2i-1) + (2(k+1)-1) \\
&= k^2 + 2(k+1) - 1 \quad \text{(by induction hypothesis)} \\
&= k^2 + 2k + 1 = (k+1)^2.
\end{aligned}
$$

Thus, (1) holds for $n = k+1$, and the proof of the induction step is complete.

Conclusion: *By the principle of induction, (1) is true for all $n \geq 2$.*

4. Find and prove by induction a formula for $\prod_{i=2}^{n}(1 - \frac{1}{i^2})$, where $n \in \mathbb{Z}_+$ and $n \geq 2$.

Proof: *We will prove by induction that, for all integers $n \geq 2$,*

$$
\text{(1)} \qquad\qquad \prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}.
$$

Base case: *When $n = 2$, the left side of (1) is $1 - 1/2^2 = 3/4$, and the right side is $(2+1)/4 = 3/4$, so both sides are equal and (1) is true for $n = 2$.*

Induction step: *Let $k \geq 2$ be given and suppose (1) is true for $n = k$. Then*

$$\prod_{i=2}^{k+1}\left(1 - \frac{1}{i^2}\right) = \prod_{i=2}^{k}\left(1 - \frac{1}{i^2}\right)\left(1 - \frac{1}{(k+1)^2}\right)$$

$$= \frac{k+1}{2k}\left(1 - \frac{1}{(k+1)^2}\right) \quad \text{(by induction hypothesis)}$$

$$= \frac{k+1}{2k} \cdot \frac{(k+1)^2 - 1}{(k+1)^2}$$

$$= \frac{k^2 + 2k}{2k(k+1)} = \frac{k+2}{2(k+1)}.$$

Thus, (1) holds for $n = k + 1$, and the proof of the induction step is complete.
Conclusion: *By the principle of induction, (1) is true for all $n \in \mathbb{Z}_+$ with $n \geq 2$.*

5. Prove that $n! > 2^n$ for $n \geq 4$.

Proof: *We will prove by induction that*

$(*)$ $$n! > 2^n$$

holds for all $n \geq 4$.
Base case: *Our base case here is the first n-value for which $(*)$ is claimed, i.e., $n = 4$. For $n = 4$, the left and right sides of $(*)$ are 24 and 16, respectively, so $(*)$ is true in this case.*
Induction step: *Let $k \geq 4$ be given and suppose $(*)$ is true for $n = k$. Then*

$$(k+1)! = k!(k+1)$$

$$> 2^k(k+1) \quad \text{(by induction hypothesis)}$$

$$\geq 2^k \cdot 2 \quad \text{(since } k \geq 4 \text{ and so } k + 1 \geq 2))$$

$$= 2^{k+1}.$$

Thus, $(*)$ holds for $n = k + 1$, and the proof of the induction step is complete.
Conclusion: *By the principle of induction, it follows that $(*)$ is true for all $n \geq 4$.*

6. Prove that for any real number $x > -1$ and any positive integer x, $(1 + x)^n \geq 1 + nx$.

Proof: *Let x be a real number in the range given, namely $x > -1$. We will prove by induction that for any positive integer n,*

$(*)$ $$(1 + x)^n \geq 1 + nx.$$

holds for any $n \in \mathbb{Z}_+$.
Base case: *For $n = 1$, the left and right sides of $(*)$ are both $1 + x$, so $(*)$ holds.*
Induction step: *Let $k \in \mathbb{Z}_+$ be given and suppose $(*)$ is true for $n = k$. We have*

$$(1+x)^{k+1} = (1+x)^k(1+x)$$

$$\geq (1 + kx)(1 + x) \quad \text{(by ind. hyp. and since } x > -1 \text{ and thus } (1 + x) > 0)$$

$$= 1 + (k+1)x + kx^2 \quad \text{(by algebra)}$$

$$\geq 1 + (k+1)x \quad \text{(since } kx^2 \geq 0).$$

Hence $(*)$ holds for $n = k + 1$, and the proof of the induction step is complete.
Conclusion: *By the principle of induction, it follows that $(*)$ holds for all $n \in \mathbb{Z}_+$.*

7. Prove that $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$ for all $n \in \mathbb{Z}_+$.

   Proof: We seek to show that, for all $n \in \mathbb{Z}_+$,

   $(*)$ $$\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}.$$

   Base case: When $n = 1$, the left side of $(*)$ is $f_1^2 = 1$, and the right side is $f_1 f_2 = 1 \cdot 1 = 1$, so both sides are equal and $(*)$ is true for $n = 1$.

   Induction step: Let $k \in \mathbb{Z}_+$ be given and suppose $(*)$ is true for $n = k$. Then

   $$\sum_{i=1}^{k+1} f_i^2 = \sum_{i=1}^{k} f_i^2 + f_{k+1}^2$$
   $$= f_k f_{k+1} + f_{k+1}^2 \quad \text{(by ind. hyp. } (*) \text{ with } n = k)$$
   $$= f_{k+1}(f_k + f_{k+1}) \quad \text{(by algebra)}$$
   $$= f_{k+1} f_{k+2} \quad \text{(by recurrence for } f_n).$$

   Thus, $(*)$ holds for $n = k + 1$, and the proof of the induction step is complete.

   Conclusion: By the principle of induction, it follows that $(*)$ is true for all $n \in \mathbb{Z}_+$.

8. Prove that $f_n \geq (3/2)^{n-2}$ for all $n \in \mathbb{Z}_+$.

   Proof: We will show that for all $n \in \mathbb{Z}_+$,

   $(*)$ $$f_n \geq (3/2)^{n-2}$$

   Base cases: When $n = 1$, the left side of $(*)$ is $f_1 = 1$, and the right side is $(3/2)^{-1} = 2/3$, so $(*)$ holds for $n = 1$. When $n = 2$, the left side of $(*)$ is $f_2 = 1$, and the right side is $(3/2)^0 = 1$, so both sides are equal and $(*)$ is true for $n = 2$.

   Thus, $(*)$ holds for $n = 1$ and $n = 2$.

   Induction step: Let $k \geq 2$ be given and suppose $(*)$ is true for all $n = 1, 2, \ldots, k$. Then

   $$f_{k+1} = f_k + f_{k-1} \quad \text{(by recurrence for } f_n)$$
   $$\geq (3/2)^{k-2} + (3/2)^{k-3} \quad \text{(by induction hypothesis with } n = k \text{ and } n = k - 1)$$
   $$= (3/2)^{k-1} \left((3/2)^{-1} + (3/2)^{-2}\right) \quad \text{(by algebra)}$$
   $$= (3/2)^{k-1} \left(\frac{2}{3} + \frac{4}{9}\right)$$
   $$= (3/2)^{k-1} \frac{10}{9} > (3/2)^{k-1}.$$

   Thus, $(*)$ holds for $n = k + 1$, and the proof of the induction step is complete.

   Conclusion: By the principle of strong induction, it follows that $(*)$ is true for all $n \in \mathbb{Z}_+$.

   Remarks: Number of base cases: Since the induction step involves the cases $n = k$ and $n = k - 1$, we can carry out this step only for values $k \geq 2$ (for $k = 1$, $k - 1$ would be 0 and out of range). This in turn forces us to include the cases $n = 1$ and $n = 2$ in the base step. Such multiple bases cases are typical in proofs involving recurrence sequences. For a three term recurrence we would need to check three initial cases, $n = 1, 2, 3$, and in the induction step restrict k to values 3 or greater.

9. Prove that $\sum_{i=1}^{n} f_i = f_{n+2} - 1$ for all $n \in \mathbb{Z}_+$.

Proof: *We will prove by induction that, for all $n \in \mathbb{Z}_+$,*

$$(*) \qquad \sum_{i=1}^{n} f_i = f_{n+2} - 1.$$

Base case: *When $n = 1$, the left side of $(*)$ is $f_1 = 1$, and the right side is $f_3 - 1 = 2 - 1 = 1$, so both sides are equal and $(*)$ is true for $n = 1$.*

Induction step: *Let $k \in \mathbb{Z}_+$ be given and suppose $(*)$ is true for $n = k$. Then*

$$\sum_{i=1}^{k+1} f_i = \sum_{i=1}^{k} f_i + f_{k+1}$$
$$= f_{k+2} - 1 + f_{k+1} \quad \text{(by ind. hyp. $(*)$ with $n = k$)}$$
$$= f_{k+3} - 1 \quad \text{(by recurrence for f_n)}$$

Thus, $(*)$ holds for $n = k + 1$, and the proof of the induction step is complete.

Conclusion: *By the principle of induction, it follows that $(*)$ is true for all $n \in \mathbb{Z}_+$.*

Remark: *Here standard induction was sufficient, since we were able to relate the $n = k+1$ case directly to the $n = k$ case, in the same way as in the induction proofs for summation formulas like $\sum_{i=1}^{n} i = n(n+1)/2$. Hence, a single base case was sufficient.*

10. Let the "Tribonacci sequence" be defined by $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$. Prove that $T_n < 2^n$ for all $n \in \mathbb{Z}_+$.

Proof: *We will prove by strong induction that, for all $n \in \mathbb{Z}_+$,*

$$(*) \qquad T_n < 2^n$$

Base case: *We will need to check $(*)$ directly for $n = 1, 2, 3$ since the induction step (below) is only valid when $k \geq 3$. For $n = 1, 2, 3$, T_n is equal to 1, whereas the right-hand side of $(*)$ is equal to $2^1 = 2$, $2^2 = 4$, and $2^3 = 8$, respectively. Thus, $(*)$ holds for $n = 1, 2, 3$.*

Induction step: *Let $k \geq 3$ be given and suppose $(*)$ is true for all $n = 1, 2, \ldots, k$. Then*

$$T_{k+1} = T_k + T_{k-1} + T_{k-2} \quad \text{(by recurrence for T_n)}$$
$$< 2^k + 2^{k-1} + 2^{k-2} \quad \text{(by strong ind. hyp. $(*)$ with $n = k$, $k - 1$, and $k - 2$)}$$
$$= 2^{k+1} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \right)$$
$$= 2^{k+1} \frac{7}{8}$$
$$< 2^{k+1}.$$

Thus, $(*)$ holds for $n = k + 1$, and the proof of the induction step is complete.

Conclusion: *By the principle of strong induction, it follows that $(*)$ is true for all $n \in \mathbb{Z}_+$.*

11. Let a_n be the sequence defined by $a_1 = 1, a_2 = 8$, $\quad a_n = a_{n-1} + 2a_{n-2}$ $\quad (n \geq 3)$. Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{Z}_+$.

Proof: *We will prove by strong induction that, for all $n \in \mathbb{Z}_+$,*

$$(*) \qquad a_n = 3 \cdot 2^{n-1} + 2(-1)^n.$$

Base case: *When $n = 1$, the left side of $(*)$ is $a_1 = 1$, and the right side is $3 \cdot 2^0 + 2 \cdot (-1)^1 = 1$, so both sides are equal and $(*)$ is true for $n = 1$.*

When $n = 2$, the left and right sides of $(*)$ are $a_2 = 8$ and $3 \cdot 2^1 + 2 \cdot (-1)^2 = 8$, so $(*)$ holds in this case as well.

Induction step: Let $k \in \mathbb{Z}_+$ with $k \geq 2$ be given and suppose $(*)$ is true for $n = 1, 2, \ldots, k$. Then

$$
\begin{aligned}
a_{k+1} &= a_k + 2a_{k-1} \quad \text{(by recurrence for } a_n) \\
&= 3 \cdot 2^{k-1} + 2 \cdot (-1)^k + 2\left(3 \cdot 2^{k-2} + 2 \cdot (-1)^{k-1}\right) \quad \text{(by } (*) \text{ for } n = k \text{ and } n = k-1) \\
&= 3 \cdot \left(2^{k-1} + 2^{k-1}\right) + 2\left((-1)^k + 2(-1)^{k-1}\right) \quad \text{(by algebra)} \\
&= 3 \cdot 2^k + 2(-1)^{k+1} \quad \text{(more algebra)}.
\end{aligned}
$$

Thus, $(*)$ holds for $n = k + 1$, and the proof of the induction step is complete.

Conclusion: By the strong induction principle, it follows that $(*)$ is true for all $n \in \mathbb{Z}_+$.

12. **Number of subsets:** Show that a set of n elements has 2^n subsets.

   Proof: We will prove by induction that, for all $n \in \mathbb{Z}_+$, the following holds:

   $P(n)$ Any set of n elements has 2^n subsets.

   Base case: Since any 1-element set has 2 subsets, namely the empty set and the set itself, and $2^1 = 2$, the statement $P(n)$ is true for $n = 1$.

   Induction step:
   - Let $k \in \mathbb{Z}_+$ be given and suppose $P(k)$ is true, i.e., that any k-element set has 2^k subsets.
   - We seek to show that $P(k + 1)$ is true as well, i.e., that any $(k + 1)$-element set has 2^{k+1} subsets.
   - Let A be a set with $(k + 1)$ elements.
   - Let a be an element of A, and let $A' = A - \{a\}$ (so that A' is a set with k elements).
   - We classify the subsets of A into two types: (I) subsets that do not contain a, and (II) subsets that do contain a.
   - The subsets of type (I) are exactly the subsets of the set A'. Since A' has k elements, the induction hypothesis can be applied to this set and we get that there are 2^k subsets of type (I).
   - The subsets of type (II) are exactly the sets of the form $B = B' \cup \{a\}$, where B' is a subset of A'. By the induction hypothesis there are 2^k such sets B', and hence 2^k subsets of type (II).
   - Since there are 2^k subsets of each of the two types, the total number of subsets of A is $2^k + 2^k = 2^{k+1}$.
   - Since A was an arbitrary $(k + 1)$-element set, we have proved that any $(k + 1)$-element set has 2^{k+1} subsets. Thus $P(k + 1)$ is true, completing the induction step.

   Conclusion: By the principle of induction, $P(n)$ is true for all $n \in \mathbb{Z}_+$.

13. **Number of 2-element subsets:** Show that a set of n elements has $n(n - 1)/2$ subsets with 2 elements. (Take $n = 2$ as the base case.)

   Proof: (Sketch) This can be proved in the same way as the formula for the number of all subsets. For the number of Type I subsets (i.e., those not containing the element a), the induction hypothesis can be used as before. The Type II subsets of A can be counted directly: the latter subsets are those of the form $\{a, b\}$, where a is the selected element and b is an arbitrary element in $A - \{a\}$. There are k choices for b, and hence k subsets of Type II.

14. **Generalization of De Morgan's Law to unions of n sets.** Show that if $A_1, \ldots, A_n$ are sets, then

$$
\overline{(A_1 \cup \cdots \cup A_n)} = \overline{A_1} \cap \cdots \cap \overline{A_n}.
$$

Proof: *We seek to prove by induction on n the following statement:*

$P(n)$:   *For all sets $A_1, \ldots, A_n$ we have*

$$(*) \qquad\qquad \overline{(A_1 \cup \cdots \cup A_n)} = \overline{A_1} \cap \cdots \cap \overline{A_n}.$$

The key to the argument is two set version of De Morgan's Law:

$$(**) \qquad\qquad \overline{(A \cup B)} = \overline{A} \cap \overline{B},$$

which holds for any sets A and B.

Base case: *For $n = 1$, the left and right sides of $(*)$ are both equal to $\overline{A_1}$, so $(*)$ holds trivially in this case. Hence $P(1)$ is true.*

Though not absolutely necessary, we can also easily verify the next case, $n = 2$: In this case, the left and right sides of $(*)$ are $\overline{(A_1 \cup A_2)}$ and $\overline{A_1} \cap \overline{A_2}$, respectively, so the identity is just the two set version of De Morgan's Law, i.e., $(**)$ with $A = A_1$ and $B = A_2$.

Induction step: *Let $k \geq 1$, and suppose $P(k)$ is true, i.e., suppose that $(*)$ holds for $n = k$ and any sets $A_1, \ldots, A_k$. We seek to show that $P(k+1)$ is true, i.e., that for any sets $A_1, \ldots, A_{k+1}$, $(*)$ holds.*

Let $A_1, \ldots, A_{k+1}$ be given sets. Then

$$
\begin{aligned}
\overline{(A_1 \cup \cdots \cup A_{k+1})} &= \overline{((A_1 \cup \cdots \cup A_k) \cup A_{k+1})} \\
&= \overline{(A_1 \cup \cdots \cup A_k)} \cap \overline{A_{k+1}} \quad \text{(by $(**)$ with $A = (A_1 \cup \cdots \cup A_k)$ and $B = A_{k+1}$)} \\
&= \left(\overline{A_1} \cap \cdots \cap \overline{A_k} \right) \cap \overline{A_{k+1}} \quad \text{(by induction hypothesis applied to $A_1, \ldots, A_k$)} \\
&= \overline{A_1} \cap \cdots \cap \overline{A_k} \cap \overline{A_{k+1}}.
\end{aligned}
$$

Thus, $(*)$ holds for $n = k+1$, and since the $A_1, \ldots, A_{k+1}$ were arbitrary sets, we have obtained statement $P(k+1)$. Hence, the proof of the induction step is complete.

Conclusion: *By the principle of induction, it follows that $P(n)$ is true for all $n \in \mathbb{Z}_+$.*