LECTURE - 23

CHEPTER-7

7.2 Transformations of Variables

Frequently in statistics, one encounters the need to derive the probability distribution of a function of one or more random variables. For example, suppose that X is a discrete random variable with probability distribution f(x), and suppose further that Y = u(X) defines a one-to-one transformation between the values of X and Y . We wish to find the probability distribution of Y . It is important to note that the one-to-one transformation implies that each value x is related to one, and only one, value $y =$ u(x) and that each value y is related to one, and only one, value $x = w(y)$, where w(y) is obtained by solving$y = u(x)$ for x in terms of y.

Theorem-7.1

Suppose that X is a discrete random variable with probability distribution f(x). Let Y = u(X) define a one-to-one transformation between the values of X and Y so that the equation $y = u(x)$ can be uniquely solved for x in terms of y, say $x = w(y)$. Then the probability distribution of Y is $g(y) = f[w(y)]$.

Example-7.1:

Let X be a geometric random variable with probability distribution
$f(x) = \frac{3}{4}(\frac{1}{4})^{x-1}, x = 1, 2, 3, ...$
Find the probability distribution of the random variable$Y = X^2$.
Solution:
Since the values of X are all positive, the transformation defines a one-to-one correspondence between the x and y values,$y = x^2$ and $x = \sqrt{y}$. Hence

$$g(y) = \begin{cases} f(\sqrt{y}) = \frac{3}{4}(\frac{1}{4})^{\sqrt{y}-1}, & y = 1, 4, 9, ... \\ 0 & elsewhere \end{cases}$$

Exercise-7.1:

Let X be a binomial random variable with probability distribution

$$f(x) = \begin{cases} \binom{3}{x}\left(\frac{2}{5}\right)^x\left(\frac{3}{5}\right)^{3-x}, & x = 0, 1, 2, , ... \\ 0 & elsewhere \end{cases}$$

Find the probability distribution of the random variable $Y = X^2$.
Solution:

Since the values of X are positive and 0, the transformation defines a one-to-one correspondence between the x and y values,$y = x^2$ and $x = \sqrt{y}$. Hence

$$g(y) = \begin{cases} f(\sqrt{y}) = \binom{3}{\sqrt{y}}(\frac{2}{5})^{\sqrt{y}}(\frac{3}{5})^{3-\sqrt{y}}, & x = 0, 1, 4, 9, ... \\ 0 & elsewhere \end{cases}$$

Theorem-7.2
Suppose thatX_1 andX_2 are discrete random variables with joint probability distribution$f(x_1, x_2)$. Let $Y_1 = u_1(X_1, X_2) and Y_2 = u_2(X_1, X_2)$ define a one-toone transformation between the points(x_1, x_2) and(y_1, y_2) so that the equations$y_1 = u_1(x_1, x_2)$ and$y_2 = u_2(x_1, x_2)$may be uniquely solved for $x_1 and x_2$ in terms of$y_1 and y_2$, say$x_1 = w_1(y_1, y_2)$ and$x_2 = w_2(y_1, y_2)$. Then the joint probability distribution ofY_1 andY_2 is $g(y_1, y_2) = f[w_1(y_1, y_2), w_2(y_1, y_2)]$.

Exercise-7.3
Let X_1 andX_2 be discrete random variables with the joint multinomial distribution
$f(x_1, x_2) = \binom{2}{x_1, x_2, 2-x_1-x_2}(\frac{1}{4})^{x_1}(\frac{1}{3})^{x_2}(\frac{5}{12})^{2-x_1-x_2}$

for $x_1 = 0, 1, 2; x_2 = 0, 1, 2; x_1 + x_2 \leq 2$; and zero elsewhere. Find the joint probability distribution of $Y_1 = X_1 + X_2 and Y_2 = X_1 - X_2$.
Solution:
Here $Y_1 = X_1 + X_2 and Y_2 = X_1 - X_2$.
So solving these we get $X_1 = \frac{Y_1+Y_2}{2}$ and $X_2 = \frac{Y_1-Y_2}{2}$
Given that the joint probability distribution of X_1 and X_2 is

$f(x_1, x_2) = \binom{2}{x_1, x_2, 2-x_1-x_2}(\frac{1}{4})^{x_1}(\frac{1}{3})^{x_2}(\frac{5}{12})^{2-x_1-x_2}$

for $x_1 = 0, 1, 2; x_2 = 0, 1, 2; x_1 + x_2 \leq 2$; and zero elsewhere
Now the joint probability distribution of $Y_1 = X_1 + X_2 and Y_2 = X_1 - X_2$. is

$g(y_1, y_2) = \binom{2}{\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2}, 2-y_1}(\frac{1}{4})^{\frac{y_1+y_2}{2}}(\frac{1}{3})^{\frac{y_1-y_2}{2}}(\frac{5}{12})^{2-y_1}$

for $y_1 = 0, 1, 2; y_2 = -2, -1, 0, 1, 2$ and since $y_1 - y_2 \geq 0 \implies y_2 \geq y_1$
x_1=0, 1, 2 and $x_1 = \frac{y_1+y_2}{2} \implies y_1 + y_2 = 0, 2, 4$.

Theorem-7.3:
Suppose that X is a continuous random variable with probability distribution f(x). Let Y = u(X) define a one-to-one correspondence between the values of X and Y so that the equation y = u(x) can be uniquely solved

for x in terms of y, say x = w(y). Then the probability distribution of Y is $g(y) = f[w(y)] \mid J \mid$, where $J = w'(y)$ and is called the Jacobian of the transformation.

Theorem-7.4:
Suppose thatX_1 and X_2 are continuous random variables with joint probability distribution $f(x_1, x_2)$. Let$Y_1 = u_1(X_1, X_2) and Y_2 = u_2(X_1, X_2)$ define a one-toone transformation between the points$(x_1, x_2) and (y_1, y_2)$ so that the equations$y_1 = u_1(x_1, x_2) and y_2 = u_2(x_1, x_2)$ may be uniquely solved forx_1 and x_2 in terms ofy_1 and y_2, say$x_1 = w_1(y_l, y_2) and x_2 = w_2(y_1, y_2)$. Then the joint probability distribution of Y_1 and Y_2 is $g(y_1, y_2) = f[w_1(y_1, y_2), w_2(y_1, y_2)] \mid J \mid$, where the Jacobian is the 2×2 determinant

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix}$$

$\frac{\partial x_1}{\partial y_1}$ is simply the derivative of$x_1 = w_1(y_1, y_2)$ with respect to y_1 with y_2 held constant, referred to in calculus as the partial derivative ofx_1 with respect to y_1.

Exercise-7.8
A dealer's profit, in units of \$5000, on a new automobile is given by $Y = X^2$, where X is a random variable having the density function

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1. \\ 0 & elsewhere \end{cases}$$

(a) Find the probability density function of the random variable Y .
(b) Using the density function of Y , find the probability that the profit on the next new automobile sold by this dealership will be less than \$500.
Solution:
(a) Since the values of X are all positive, the transformation defines a one-to-one correspondence between the x and y values,$y = x^2$ and $x = \sqrt{y}$ and $J = \frac{1}{2\sqrt{y}}$. Hence

$$g(y) = \begin{cases} 2(1-\sqrt{y})(\frac{1}{2\sqrt{y}}) = \frac{1-\sqrt{y}}{\sqrt{y}}, & 0 < y < 1. \\ 0 & elsewhere \end{cases}$$

(b) To find the probability that the profit on the next new automobile sold by this dealership will be less than \$500 is same as finding$P(0 < y < 0.1)$.

$$P(0 < y < 0.1) = \int_0^{0.1} g(y)dy = \int_0^{0.1} (y^{-\frac{1}{2}} - 1)dy$$
$$= (\sqrt{y} - y)\Big|_0^{0.1} = 0.316227 - 0.1 = 0.216227$$