SUNGKYUNKWAN UNIVERSITY
DEPARTMENT OF PHYSICS

Status Report

Donghun Jung

Andreev bound state and Majorana Fermion

This report summarizes my studies over the past month until Feb 12 and is submitted on February 20, 2023.

Summary

1. Andreev reflection makes the transfer of charge and momentum into S state, leading to the conversion of normal current in N into supercurrent in S, possible.

2. Multiple Andreev reflection leads to the formation of Andreev bound states under specific condition.

3. In Kitaev chain toy model, it is shown that there is a non-local Majorana fermionic state located on the opposite ends of the chain.

Table of Contents

Chapter 1

Andreev bound state

Abstract

At the boundary(or interface) between a normal metal(N) and a superconductor(S), there occurs a scattering process known as Andreev Reflection. Generally, when electrons in the N state have energy below the superconducting gap Δ, they are forbidden from propagating into the S state. However, this process makes the transfer of charge and momentum into S state, leading to the conversion of normal current in N into supercurrent in S, possible. Here, multiple scattering leads to the formation of Andreev bound states.

1.1 Quasi-particles

I was confused by the concept of quasi-particles, so I summarized them. Without any interactions, that is if we have just a bunch of electrons in the solid at zero temperature, electrons will form a sphere in the momentum space since they are fermion. The surface of this sphere is the Fermi surface whose momentum is the Fermi momentum. When electrons are allowed to interact a little bit, this description is still standing except that the electrons are no longer electrons.[1] They are quasi-particles defined with respect to the Fermi level. A quasi-particle is the elementary excitation above the ground state of a Fermi level or a Fermi surface. To make this, one electron is taken from under the Fermi sphere and brought above it, leaving behind an empty space

[1]This approximation theory may be Fermi liquid theory.

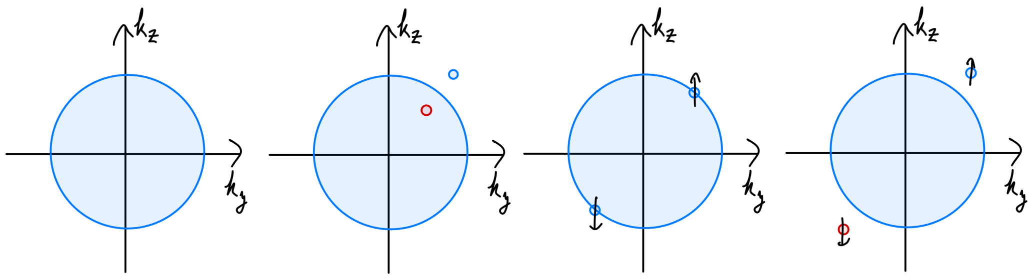

Figure 1.1: From left, The free electron gas, Fermi Liquid Quasiparticle, Cooper pair, Bogoliubov Quasiparticle.

called a hole. The taken electron is a quasi-electron. This is a quasiparticle excitation. Next one is Cooper pairs. In a BCS theory, Cooper pairs refer to the paired states of two electrons. First, they must have opposite momenta, which means that the center of mass of a Cooper pair is at rest on average. Cooper pairs exist in a Bose-Einstein condensate. This leads to the energy gap in the Fermionic excitation spectrum and the breaking of global U(1) symmetry related to the conservation of particle number. Second, they form a spin singlet, meaning that the two spins of the electrons are opposite. This equation describes the ground state of the superconductor.

$$|\psi_\phi\rangle = \Pi_{\vec{k}}(|u_k| + |v_k|e^{i\phi}c^*_{\vec{k}\uparrow}c^*_{-\vec{k}\downarrow})\,|\phi_0\rangle \tag{1.1}$$

It has a product of two creation operators for particles with opposite momentum and opposite spin acting on the vacuum. Here, the u_k term actually has nothing in front of it, so it represents a superposition of a vacuum and two particles. It seems strange but, noting that the number of particles is no longer conserved, the number of Cooper pairs is not a good number that can be counted.

You need to distinguish the Bogoliubov quasi-particle from the others introduced above. The Bogoliubov quasi-particle is an elementary excitation above the superconducting state whereas the other quasi-particles are excitation above non-interacting Fermi gas or liquid. Bogoliubov quasi-particle is described by these operators.

$$\gamma^*_{\vec{k}0} = u^*_{\vec{k}}c^*_{\vec{k}\uparrow} - v^*_{\vec{k}}c_{-\vec{k}\downarrow} \tag{1.2}$$

$$\gamma^*_{\vec{k}1} = u^*_{\vec{k}}c^*_{-\vec{k}\downarrow} + v^*_{\vec{k}}c_{\vec{k}\uparrow} \tag{1.3}$$

As seen above equation, the Bogoliubov quasi-particle differs from a simple quasi-particle as it consists of both creation and annihilation operators, representing both an electron and a hole in a single particle. Here, the Bogoliubov equation is a linear combination of creation and annihilation operators, but the (coherence) coefficients $u_{\vec{k}}$ and $v_{\vec{k}}$ come from the superconducting pairing. As a result, an elementary excitation above the superconducting state is not a simple quasi-particle, but rather a superposition of an electron and a hole with opposite momentum and opposite spin. A Cooper pair can be described as two Bogoliubov quasi-particles.

1.2 Andreev reflection

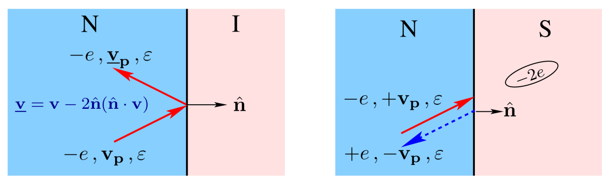

Figure 1.2: Left : Specular reflection at an N-I boundary, Right : Retro-reflection at an N-S boundary.

Generally, when electrons in the Normal state have energy below the superconducting gap Δ, they are forbidden from propagating into the Superconducting state. However, there is a tricky mechanism. An electron(hole) comes in, then makes a Cooper pair by Cooper pairing. Here, the second charge comes from sending back an outgoing hole(electron) which is retro-reflected into the normal metal. So the charge of $2e$ is transferred. There is no charge conservation.

In contrast, in the normal reflection that occurs in the boundary between the normal metal and insulator, the electron is just bounced off. An electron with momentum $\vec{p}$, group velocity $\vec{v}_{\vec{p}}$ and energy below the energy gap of the insulator, is reflected by the insulating gap into an outgoing electron with $\vec{p'} = \vec{p} - 2\hat{n}(\hat{n}\cdot\vec{p})$, $\vec{v'} = \vec{v} - 2\hat{n}(\hat{n}\cdot\vec{v})$. So it requires momentum change of the charge carriers about $2\hat{n}(\hat{n}\cdot\vec{p})$. However, the momentum change in the Andreev reflection is given by $\Delta\vec{p}_{\max} = \frac{dU}{dx}\Delta x$ with $dU \simeq \Delta$, $\Delta x = 2\xi_0$ where ξ_0 is penetration depth. Then, $\Delta\vec{p}_{\max} \simeq 2p_f\frac{\Delta}{E_F} = 2\frac{E_f}{v_f}$ which is small enough compared to the fermi momentum. So momentum is transferred but almost conserved. The exact conservation occurs at the fermi level. The momentum conservation is required since Copper pairs have zero momentum. And the energy is conserved. $E_e = \epsilon$, $E_h = -\epsilon$ and $E_{\text{pair}} = 0$ with respect to the fermi energy E_f.

Here is one more story. Whenever one undergoes a reflection or transmission, there is a phase shift. Andreev reflection is phase coherent which means that there is a relation between the phase of the electron and the retro-reflected hole.

$$\phi'_e = \phi_h - \phi_s - \arccos\frac{E}{\Delta} \tag{1.4}$$

$$\phi'_h = \phi_e + \phi_s - \arccos\frac{E}{\Delta} \tag{1.5}$$

Here, the superconducting phase ϕ_s does not play any roles and can be chosen to be zero by a gauge transformation.

1.3 Bogouliubov equation

To describe electrons and holes coming into the N-S interface, Bogoliubov equation is introduced.

$$\begin{pmatrix} \hat{H}_0 & \Delta e^{i\phi} \\ \Delta e^{-i\phi} & -\hat{H}_0 \end{pmatrix} \begin{pmatrix} \psi_e \\ \psi_h \end{pmatrix} = E \begin{pmatrix} \psi_e \\ \psi_h \end{pmatrix} \tag{1.6}$$

where Δ is the superconductor paring. It becomes zero in the normal metal. And if the superconductor is non-homogeneous, Δ becomes a function that depends on $\vec{r}$ and $\vec{p}$. The off-diagonal components describe the correlations between the electrons and holes.

1.4 Andreev bound state

Consider an S-N-S system. At one side of the N-S junction, an electron can enter the superconductor via Andreev reflection, where it is reflected as a hole with opposite energy, creating a Cooper pair. An electron is exchanged for a hole, resulting in a charge transfer of $2e$. The energy is conserved since the electron energy is $+\epsilon$, the hole energy is $-\epsilon$, and the Cooper pair energy is zero with respect to the Fermi energy. Until here, it is the same story with Andreev reflection.

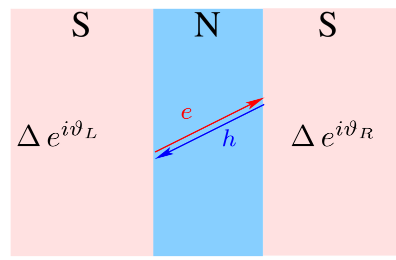

Figure 1.3: Andreev bound state confined within an S-N-S sandwich.

On the other hand, when there is another superconductor on the other side, a hole, created via Andreev reflection above, it can undergo another Andreev reflection in the second superconductor, producing another Cooper pair. The electron can then return to the first superconductor, creating an infinite loop where the electron and the hole go back and forth between two superconductors. The particle is bound between the two superconductors, but it's not like an electron in a box. It is because, at every reflection, a Cooper pair is created and goes into the superconductor.

The phase accumulated through the entire process that the electron goes and comes back is 2π. Then,

$$\phi_{s2} - \phi_{s1} + (k_e - k_h)L - 2\arccos\frac{E}{\Delta} = 2\pi n \tag{1.7}$$

where n is an integer. For a short junction, that L is small enough and n is taken to be zero, we have

$$E_{\pm} = \pm\Delta\cos\frac{\phi}{2} \tag{1.8}$$

where ϕ is a phase difference between the superconductors. For a long junction,

$$E_n = \frac{\hbar v_f}{2L}\left(2\pi\left(n + \frac{1}{2}\right) \pm \phi\right) \tag{1.9}$$

Upon the sign introduced above equation, there are hole-like Andreev bound states and electron-like bound states. The + sign represents the electron right-moving or the hole left-moving, while the - sign represents the electron left-moving or the hole right-moving. But they are always symmetric with respect to Fermi energy.

Chapter 2

Majorana Fermion

Abstract

The Kitaev chain is a one-dimensional tight-binding (toy) model with a *p*-wave superconductor. In this model, the Hamiltonian can be diagonalized using Majorana fermion operators and conventional fermionic operators. Then, it is shown that there is a non-local state located on the opposite ends of the chain.

2.1 Majorana fermion

Majorana fermions(MFs) are particles that are their own antiparticles. MF operators can be rewritten, in mathematical terms, by conventional operators for the creation and annihilation of conventional fermions like electrons and vice versa.

$$\gamma_1 = c + c^\dagger \qquad c = \frac{1}{2}(\gamma_1 + i\gamma_2) \tag{2.1}$$

$$\gamma_2 = i(c^\dagger - c) \qquad c^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2) \tag{2.2}$$

where γ represents MF.[1] It is clear from this equation that $\gamma = \gamma^\dagger$.

[1]Note that anti-commutator relation holds.

$$\{\gamma_{i,\alpha}\gamma_{j,\beta}\} = 2\delta_{ij}\delta_{\alpha\beta} \tag{2.3}$$

Also, MFs are basically obtained by splitting a fermion into its real and imaginary part.

As seen in the equation, it is a superposition of an electron and a hole. In a sense, a MF is half of a normal fermion and a fermionic state is obtained by a superposition of two MFs.

In practice, it is natural to observe MFs in superconductors. In the superconductor, with the existence of the Cooper pair, it is possible to convert a hole into an electron by adding a Cooper pair. As shown in the figure, the single particle experiences the superconducting gap, but superconductivity can be locally suppressed somehow, such as a quantum dot.[2] In that area, a single particle can exist. But by the particle-hole symmetry of superconductivity, $\gamma^\dagger(E) = \gamma(-E)$ and the energy spectrum is symmetric around the fermi level. If $E = 0$, we may have $\gamma = \gamma^\dagger$. Generally, it is impossible so there can never be a bound state at zero energy in such a trap. However, using p-wave superconductor, then the zero point motion is canceled by the Barry phase in the p-wave superconductor and there will be a stable energy level at zero energy.

2.2 Kitaev chain

Consider the one-dimensional tight-binding chain with p-wave superconducting pairing. The Hamiltonian is given by

$$\mathcal{H}_{\text{chain}} = -\mu \sum_{i=1}^{N} n_i - \sum_{i=1}^{N-1} (tc_i^\dagger c_{i+1} + tc_{i+1}^\dagger c_i + \Delta c_i c_{i+1} + \Delta c_{c+1}^\dagger c_i^\dagger) \tag{2.4}$$

To see the Majorana physics let $\mu = 0$and $t = \Delta$. Then, this Hamiltonian is reduced by substituting the creation and annihilation operators with Majorana operators and using the anti-commutator relation.

$$\mathcal{H}_{\text{chain}} = -it \sum_{i=1}^{N-1} \gamma_{i,2}\gamma_{i+1,1} \tag{2.5}$$

It is just an alternative way of writing the diagonalized Hamiltonian. To see this, using new fermionic operator[3] $\bar{c}_i$ which is constructed by combining MF

[2] This is a superconducting vortex.

[3] It is a fermionic operator in that

$$\{\bar{c}_i, \bar{c}_j^\dagger\} = \delta_{ij} \tag{2.6}$$

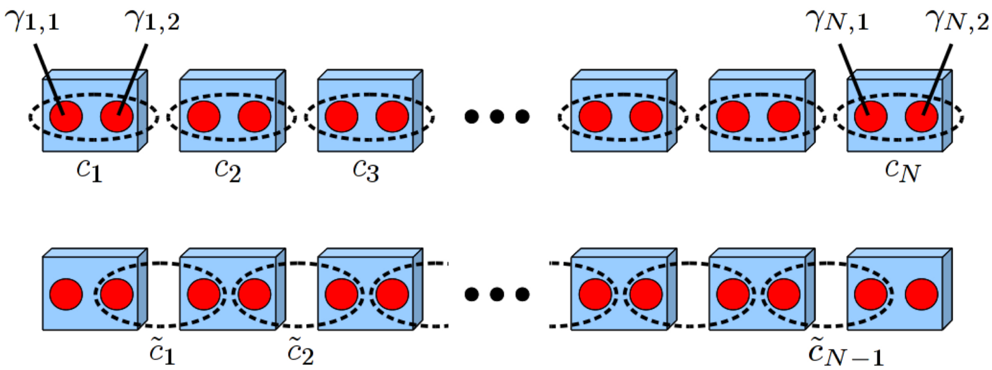

Figure 2.1: Kitaev 1D p-wave superconducting tight-binding chain. Upper: The fermion operator can be split into two Majorana operators, $\gamma_{i,1}$ and $\gamma_{i,2}$. Lower : The Hamiltonian is diagonalized in the fermion operator obtained by combining Majorana operators can be re-split into two Majorana operators, $\gamma_{i,2}$ and $\gamma_{i+1,1}$. This leaves two unparied Majorana operators, $\gamma_{1,1}$ and $\gamma_{N,2}$, which can be combined to form one zero energy, highly non-local fermionic state.

operators,

$$\bar{c}_i = \frac{1}{2}(\gamma_{i+1,1} + i\gamma_{i,2}) \tag{2.7}$$

the Hamiltonian can be rewritten in fermionic representation.

$$\mathcal{H}_{\text{chain}} = 2t \sum_{i=1}^{N-1} \left(\bar{c}_i^\dagger \bar{c}_i - \frac{1}{2} \right) \tag{2.8}$$

$$= 2t \sum_{i=1}^{N-1} \left(\bar{n}_i - \frac{1}{2} \right) \tag{2.9}$$

Then, $2t$ corresponds to the energy cost of creating a $\bar{c}_i$ fermion.[4] However, there are some missing terms, $\gamma_{1,1}$ and $\gamma_{N,2}$. These can be described by a

[4]My result differs from that presented in the review paper. It is told that

$$\mathcal{H}_{\text{chain}} = 2t \sum_{i=1}^{N-1} \bar{c}_i^\dagger \bar{c}_i \tag{2.10}$$

$$= 2t \sum_{i=1}^{N-1} \bar{n}_i \tag{2.11}$$

single fermionic state with the fermionic operator $\bar{c}_M$.

$$\bar{c_M} = \frac{1}{2}(\gamma_{N,2} + i\gamma_{1,1}) \tag{2.12}$$

This state constructs a highly non-local state since two MF operators, $\gamma_{1,1}$ and $\gamma_{N,2}$, are localized on the opposite ends of the chain. And considering the absence of that operator in the Hamiltonian, it occupies a zero energy state. [5]

without the constant terms. I will try it later.

[5] Here, the paper states the two-fold degeneracy of the ground state, which is related to even or oddness of the number of electrons. I don't understand it yet, but I think we need more knowledge about superconductors. In this paper, it is called parity. This corresponds to the eigenvalue of the number operator $\bar{n}_M$, 0(1) for even(odd) parity.

Chapter 3

What to do?

What I've done and haven't

For a month, I had the opportunity to study Andreev bound states and Majorana fermions based on Dr. Park Tae Ho's recommendation. I have a good grasp of the fundamental physics underlying Andreev bound states and have reviewed relevant experiments on Andreev reflection and bound states, although these details are not included in my report. However, I have yet to tackle the Bogoliubov equation. During the two weeks, I dedicated to studying Majorana fermions, I focused on the detailed calculations presented in the review paper, specifically concerning the Kitaev chain. While I have not yet delved into the two-dimensional case and the properties of Majorana fermions or their non-Abelian statistics, I plan to explore these topics in the future.

What I'm going to do

From now on, I have to do KIST research internship, graduation thesis preparation, and coursework at the same time. Therefore, the status report will be uploaded every two to three weeks. The next status report about this topic will be uploaded on Mar 05.

The followings are what I'm planning to do with reference.

- Try to solve the Boboliubov equation.
  - JA Sauls. Andreev bound states and their signatures, 2018

  - GE Blonder, m M Tinkham, and k TM Klapwijk. Transition from metallic to tunneling regimes in superconducting microconstrictions: Excess current, charge imbalance, and supercurrent conversion. *Physical Review B*, 25(7):4515, 1982

- Study superconductivity(what is p-wave superconductor?)

  - Michael Tinkham. *Introduction to superconductivity*. Courier Corporation, 2004

- Study properties of Majorana fermion and Non-abelian statistics

  - Martin Leijnse and Karsten Flensberg. Introduction to topological superconductivity and majorana fermions. *Semiconductor Science and Technology*, 27(12):124003, 2012

Bibliography

[1] GE Blonder, m M Tinkham, and k TM Klapwijk. Transition from metallic to tunneling regimes in superconducting microconstrictions: Excess current, charge imbalance, and supercurrent conversion. *Physical Review B*, 25(7):4515, 1982.

[2] Martin Leijnse and Karsten Flensberg. Introduction to topological superconductivity and majorana fermions. *Semiconductor Science and Technology*, 27(12):124003, 2012.

[3] JA Sauls. Andreev bound states and their signatures, 2018.

[4] Michael Tinkham. *Introduction to superconductivity*. Courier Corporation, 2004.