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Status Report

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Quantum State Tomography and Positive value measurement basis

This report summarizes my studies over two weeks until Apr 10 and is submitted on April 26, 2023.

Summary

1. Measurement process in QST extracts information from a quantum state by measuring the distance between the hyperplane of a measurement operator and the state vector of the system.

2. Although a PVM set with a rank of 15 can perform QST well, constructing a PVM set with 15 operators and minimum Hilbert-Schmidt norm is ideal for optimal QST.

3. In order to find the optimal PVM set for a 2-qubit system, one approach is to calculate ${\theta^{-1}}^k_{ij}$. However, this strategy may not be scalable for larger systems.

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Chapter 1

Quantum State Tomography

Abstract

Quantum State Tomography (QST) is a method used to reconstruct a quantum state of a given system by performing a series of measurements in different measurement bases. The measurement process extracts information from the quantum state by measuring the distance between the hyperplane and the state vector. To reconstruct the full quantum state, there are two main methods for conducting QST: Maximum Likelihood Estimation(MLE) and Linear Regression Estimation(LRE).

1.1 Quantum State Tomography

Quantum State Tomography(QST) is the process of reconstructing a quantum state of a given system through a series of measurements in different (measurement) basis. Similar to its classical counterpart, Tomography, which reconstructs a three-dimensional image via a series of two-dimensional projections, QST aims to reconstruct a full or the most probable density matrix or wave function through multiple measurements. [1]

There are various statistical methods available to reconstruct the original quantum state, even from incomplete information. (1) However, in general, without a priori information, informationally complete or overcomplete mea-

[1] Due to the quantum nature, measuring a state of the given state perturbs its state. Therefore, QST must be conducted on a number of identical copied states.

surements are required, which is referred to as full quantum state tomography (FQST).

1.1.1 Informational approach to QST

From an informational approach, determining only one density matrix of a d-dimensional quantum system requires $2d^2$ variables, since there are d^2 matrix elements and each element is complex. However, there are constraints on the density matrix. First, the density matrix must be Hermitian. Second, the density matrix must be positive semidefinite. Third, the trace of the density matrix must be 1. These constraints reduce the number of required variables to $d^2 - 1$, which means $d^2 - 1$ measurements are required to determine the state of a d-dimensional quantum system.

1.1.2 Mathematical approach to QST

From a mathematical perspective, the measurement process measures the distance between the hyperplane and the state vector. As discussed above, state vector is given by

$$\rho^i = \mathrm{tr}\left(\rho\Gamma^i\right) \tag{1.1}$$

where measurement basis is given by

$$\Omega^k = \theta_i^k \Gamma^i \tag{1.2}$$

Note that θ_i^k can be regarded as a normal vector of hyperplane of the measurement operator Ω^k in $\mathbf{R}^{d^2}$ space. Here, the result of the measurement can be written by

$$\left\langle \Omega^k \right\rangle = \mathrm{tr}\left(\rho\Omega^k\right) \tag{1.3}$$

$$= \theta_i^k \,\mathrm{tr}\left(\rho\Gamma^i\right) \tag{1.4}$$

$$= \theta_i^k \rho^i \tag{1.5}$$

That is, it is the inner product between the normal vector of hyperplane and the state vector. Although θ_i^k is not an unit vector, $\left\langle \Omega^k \right\rangle$ can be regarded as the distance between the hyperplane and the state vector. Therefore, noting that any density matrix can be taken as a form of linear combination $\rho = \xi_i \Gamma^i$,QST is to determine the value of ξ_i in the equation, given the normalization condition, $\xi_0 = 1$. Since there are $d^2 - 1$ undetermined variables, $d^2 - 1$ measurements are required. Then, ξ_i are determined uniquely since the space of density matrix is convex.

1.2 Methods for QST

The general idea reconstructing the given quantum state is to solve the following equation for Ξ, which represents the elements in the vector form of a given quantum state ρ.

$$X\Xi = P \tag{1.6}$$

where P is the measured frequency of the measurement operator Ω_i, and X is given by

$$X = \begin{pmatrix} \mathrm{Tr}(\Omega_1\Gamma^{00\cdots0}) & \mathrm{Tr}(\Omega_1\Gamma^{00\cdots1}) & \cdots & \mathrm{Tr}(\Omega_1\Gamma^{33\cdots3}) \\ \mathrm{Tr}(\Omega_2\Gamma^{00\cdots0}) & \mathrm{Tr}(\Omega_2\Gamma^{00\cdots1}) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Tr}(\Omega_M\Gamma^{00\cdots0}) & \cdots & \cdots & \mathrm{Tr}(\Omega_M\Gamma^{33\cdots3}) \end{pmatrix} \tag{1.7}$$

Then, it is expected to find the given density matrix:

$$\rho = \xi_{ij\cdots k}\Gamma^{ij\cdots k} \tag{1.8}$$

However, there are potential problems with this method. Firstly, the exact measurement is impossible due to the limited number of possible copies of the quantum state. Secondly, M may not be equal to $d^2 - 1$, the dimensionality of the quantum state. This can lead to an unsolvable equation, or even if solvable, may result in a density matrix that violates important properties such as positive semidefiniteness and normalization. To overcome these problems, two alternative numerical methods exist for obtaining a physically valid density matrix.

1.2.1 Maximum Likelihood Estimation

Maximum Likelihood Estimation(MLE) method finds the density matrix that is most likely to have produced the observed measurement outcomes. Any positive semidefinite matrix $\mathcal{G}$ that satisfies $\langle\psi|\,\mathcal{G}\,|\psi\rangle \geq 0$ can be expressed as $\mathcal{G} = \hat{T}^\dagger\hat{T}$, where $\hat{T}$ is a tridiagonal matrix chosen for convenience. For a two-qubit system, $\hat{T}$ can be written as:

$$\hat{T} = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ t_5 + it_6 & t_2 & 0 & 0 \\ t_{11} + it_{12} & t_7 + it_8 & t_3 & 0 \\ t_{15} + it_{16} & t_{13} + it_{14} & t_9 + it_{10} & t_4 \end{pmatrix} \tag{1.9}$$

Furthermore, any manifestly physical density matrix can be written as:

$$\rho = \frac{\hat{T}^\dagger \hat{T}}{\mathrm{Tr}(\hat{T}^\dagger \hat{T})} \tag{1.10}$$

whether the density matrix is pure state or not. It is important to note that $\mathcal{G}$ must be Hermitian.

Let me assume that the noise on the coincidence measurements has a Gaussian distribution. Then, the probability of obtaining a measurement result n_{real} given a theoretical expectation value n_{exp} is given by:

$$p = \mathcal{N}_{\mathrm{norm}} \prod_i \exp\left[-\frac{(n^i_{\mathrm{real}} - n_{\mathrm{exp}})^2}{2\sigma_i^2}\right] \tag{1.11}$$

where the expectation value for ith measurement is given by $n^i_{\mathrm{ext}} = \mathrm{Tr}(\rho\Omega^i)$. Here, the standard deviation for the ith measurement can be approximated to $\sqrt{n^i_{\mathrm{exp}}}$. Then, the likelihood function is rewritten by:

$$p = \mathcal{N}_{\mathrm{norm}} \prod_i \exp\left[-\frac{(n^i_{\mathrm{real}} - N\,\mathrm{Tr}\,\rho\Omega^i)^2}{2N\,\mathrm{Tr}\,\rho\Omega^i}\right] \tag{1.12}$$

where N is the total number of observation for the measurement operator Ω^i. Here it is assumed that N is equal for every measurement. Finally, the parameters t are chosen to maximize the likelihood function or, equivalently, minimize the following function:

$$\mathcal{L} = \sum_i \frac{(n^i_{\mathrm{real}} - N\,\mathrm{Tr}\,\rho\Omega^i)^2}{2N\,\mathrm{Tr}\,\rho\Omega^i} \tag{1.13}$$

$$= \sum_i \frac{(p^i_{\mathrm{real}} - \mathrm{Tr}\,\rho\Omega^i)^2}{2\,\mathrm{Tr}\,\rho\Omega^i} \tag{1.14}$$

where $p^i = \frac{n^i_{real}}{N}$.

MLE method can be implemented with scipy.minimize or cvxopt in python.

1.2.2 Linear Regression

As seen, FQST requires a minimum of $d^2 - 1$ measurement operators, leading to a large number of measurements and a high computational time

complexity for data processing due to the curse of dimensionality. On the other hand, the MLE method requires d^2 variables, but a well-known method exists to solve the (possibly) unsolvable equation 1.6. This method, known as least-squares estimation, gives a direct estimate of Θ_{LS} as:

$$\Theta_{\mathrm{LS}} = (X^T X)^{-1} X^T P \tag{1.15}$$

Then, the density matrix can be directly reconstructed as:

$$\mu_{\mathrm{LS}} = \theta^{\mathrm{LS}}_{ij\cdots k} \Gamma^{ij\cdots k} \tag{1.16}$$

However, the recovered density matrix μ_{LS} may not satisfy the physical properties of a density matrix. To resolve this issue, a fast algorithm introduced in this letter (2) can be used. This algorithm, like the MLE method, uses the likelihood function given in equation 1.12 and finds the nearest physical density matrix under the 2-norm.

Chapter 2

Measurement basis

Abstract

In order to conduct QST, a minimum of $d^2 - 1$ measurements are required. In the case of a 2-qubit system, there are 29 measurement operators generated using a relatively small number of operators. Although a PVM set with a rank of 15 can perform QST well without considering any noises, constructing a PVM set with 15 operators is possible. However, to ensure optimal measurement that uniformly extracts information in terms of Γ^{ij}, the PVM set needs to be constructed with a minimum Hilbert-Schmidt norm.

2.1 Positive value measurement in 2 qubit system

The system under consideration is a two-qubit system with $\sigma^3 \otimes \sigma^3$ interaction, and projection measurements are performed using a PVM basis consisting of 29 operators. Some of PVM operators are single qubit operators, and the others are entangling(?) gates which include one CNOT gate and four gates. Our PVM operators are generated by the following formula.

$$\Omega^i = U_i^\dagger \frac{1}{2} \left(|0\rangle\langle 0| \otimes I_2 + I_2 \otimes |0\rangle\langle 0| \right) U_i \tag{2.1}$$

Considering the rank of the set of basis operators is 16, it is expected that the set of PVM operators can be reduced. For more information on the PVM basis, see Appendix 2.4

2.1.1 $SU(2) \otimes SU(2)$ and Pauli representation basis

Our system consists of two qubits, which exhibit $SU(2) \otimes SU(2)$ symmetry. To represent the POVM operators, we choose an orthogonal basis set $\{\Gamma^{ij}\}_{i,j=0}^{3}$ that satisfies $\mathrm{tr}\left(\Gamma^{ij\dagger}\Gamma^{mn}\right) = \delta_{im}\delta_{jn}$. We use the representation basis $\Gamma^{ij} = \sigma^i \otimes \sigma^j$, which is convenient because it is traceless except for Γ^{00}. You can see what the Γ^{ij} look like in Appendix 2.3.

$\{\Gamma^{ij}\}$ possesses some interesting properties. First, the product between Γ^{ij} and Γ^{mn} is given by

$$\begin{aligned}
\Gamma^{ij}\Gamma^{mn} &= (\sigma^i \otimes \sigma^j) \otimes (\sigma^m \otimes \sigma^n) \\
&= (\sigma^i \otimes \sigma^m) \otimes (\sigma^j \otimes \sigma^n) \\
&= (\delta^{im} + i\epsilon_{imk}\sigma^k) \otimes (\delta^{jn} + i\epsilon_{jnl}\sigma^l) \\
&= \delta_{im}\delta_{jn} + i\delta_{im}\epsilon_{jnl}\sigma^l + i\delta^{jn}\epsilon_{imk}\sigma^k - \epsilon_{imk}\epsilon_{jnl}\sigma^k\sigma^l
\end{aligned} \tag{2.2}$$

Note that I supposed that $i, j, m, n \neq 0$ and didn't write $\otimes$ explicitly at the end of equation. And you don't have to pay attention to the upper/lower indecies. In a same way, we have

$$\Gamma^{jn}\Gamma^{ij} = \delta_{im}\delta_{jn} - i\delta_{im}\epsilon_{jnl}\sigma^l - i\delta^{jn}\epsilon_{imk}\sigma^k - \epsilon_{imk}\epsilon_{jnl}\sigma^k\sigma^l \tag{2.3}$$

in component form. Or more conviniently,

$$\begin{aligned}
\Gamma^{jn}\Gamma^{ij} &= (\sigma^m \otimes \sigma^n) \otimes (\sigma^m \otimes \sigma^n) \\
&= -(\sigma^i\sigma^m - 2\delta_{im}) \otimes (\sigma^n\sigma^j) \\
&= -(\sigma^i\sigma^m) \otimes (\sigma^n\sigma^j) + 2\delta_{im} \otimes \sigma^n\sigma^j \qquad (2.4) \\
&= (\sigma^i\sigma^m - 2i\epsilon_{mik}\sigma^k) \otimes (\sigma^n\sigma^j) \\
&= (\sigma^i\sigma^m) \otimes (\sigma^n\sigma^j) - 2i\epsilon_{mik}\sigma^k \otimes \sigma^n\sigma^j. \qquad (2.5)
\end{aligned}$$

Therefore, the commutator relation and the anticommutator relation are given by

$$\left[\Gamma^{ij}, \Gamma^{mn}\right] = (\sigma^i\sigma^m) \otimes \left\{\sigma^j, \sigma^n\right\} - 2\delta_{im} \otimes \sigma^n\sigma^j \tag{2.6}$$

$$= (\sigma^i\sigma^m) \otimes \left[\sigma^j, \sigma^n\right] + 2i\epsilon_{mik}\sigma^k \otimes \sigma^n\sigma^j \tag{2.7}$$

$$= 2i\delta_{im}\epsilon_{jnl}\sigma^l + 2i\delta_{jn}\epsilon_{imk}\sigma^k \tag{2.8}$$

$$\left\{\Gamma^{ij},\Gamma^{mn}\right\} = (\sigma^i\sigma^m)\otimes\left[\sigma^j,\sigma^n\right] + 2\delta_{im}\otimes\sigma^n\sigma^j \tag{2.9}$$
$$= (\sigma^i\sigma^m)\otimes\left\{\sigma^j,\sigma^n\right\} - 2i\epsilon_{mik}\sigma^k\otimes\sigma^n\sigma^j \tag{2.10}$$
$$= 2\delta_{im}\delta_{jn} - 2\epsilon_{imk}\epsilon_{jnl}\sigma^k\sigma^l \tag{2.11}$$

If one of the elements is zero, i.e. $i = 0$, then,

$$\begin{aligned}\Gamma^{0j}\Gamma^{mn} &= (\sigma^0\otimes\sigma^j)\otimes(\sigma^m\otimes\sigma^n)\\ &= (\sigma^0\sigma^m)\otimes(\sigma^j\otimes\sigma^n)\\ &= \sigma^m\otimes\delta_{jn} + i\sigma^m\otimes\epsilon_{jnl}\sigma^l\end{aligned} \tag{2.12}$$

and the commutator relation and the anticommutator relation are given by

$$\left[\Gamma^{0j},\Gamma^{mn}\right] = -2i\sigma^m\otimes\epsilon_{njk}\sigma^k \tag{2.13}$$

$$\left\{\Gamma^{0j},\Gamma^{mn}\right\} = 2\sigma^m\otimes\delta_{jn} \tag{2.14}$$

If two of the elements are zero, i.e. $i = j = 0$ or $i = m = 0$, then,

$$\begin{aligned}\Gamma^{00}\Gamma^{mn} &= (\sigma^m)\otimes(\sigma^n)\\ &= \Gamma^{mn}\\ &= \Gamma^{mn}\Gamma^{00}\end{aligned}$$

And obviously,

$$\left[\Gamma^{00},\Gamma^{mn}\right] = 0 \tag{2.15}$$

$$\left\{\Gamma^{00},\Gamma^{mn}\right\} = \Gamma^{mn} \tag{2.16}$$

On the other hand,

$$\Gamma^{0j}\Gamma^{0n} = \sigma^0\otimes(\sigma^j\sigma^n)$$

and

$$\left[\Gamma^{0j},\Gamma^{0n}\right] = \sigma^0\otimes 2i\epsilon_{njk}\sigma^k \tag{2.17}$$

$$\left\{\Gamma^{0j},\Gamma^{0n}\right\} = \sigma^0\otimes 2\delta_{nj} \tag{2.18}$$

Therefore, $\{\Gamma^{ij}\}$ constructs an orthogonal representation basis. Considering the property of trace, $\mathrm{tr}\, A \otimes B = \mathrm{tr}\, A \times \mathrm{tr}\, B$, we have

$$\mathrm{tr}\left(\Gamma^{ij}\Gamma^{mn}\right) = 4\delta_{im}\delta_{jn}. \tag{2.19}$$

Note that here † is removed. The hermitivity of Γ^{ij} is guranteed by the hermitivity of pauli matrices. To represent the PVM operators, we choose this orthogonal basis set $\{\Gamma^{ij}\}_{i,j=0}^{3}$.

On the other hand, It is expected to use an invariance, which comes from the symmetry of $SU(2) \otimes SU(2)$, since it is 2 qubit system. Here note that $SU(2) \otimes SU(2)$ is just $SO(1,3)$ which is well-known group in special relativity. Unfortunatly, it is thought to be there is no useful property related to this group representation.

2.1.2 PVM operators

Our PVM operators are generated by the following formula.

$$\Omega^i = U_i^\dagger \frac{1}{2}\left(|0\rangle\langle 0| \otimes I_2 + I_2 \otimes |0\rangle\langle 0|\right) U_i \tag{2.20}$$

$$= U_i^\dagger (S_1, S_1) U_i \tag{2.21}$$

where U_i is an unitary operator, which corresponds to single or two qubit operation.

$$U_i^{\text{single}} = e^{\pm\frac{i\pi}{4}\sigma^j} \otimes e^{\pm\frac{i\pi}{4}\sigma^k} \tag{2.22}$$

$$U_i^{\text{entangled}} = \left(e^{\pm\frac{i\pi}{4}\sigma^j} \otimes e^{\pm\frac{i\pi}{4}\sigma^k}\right) e^{\pm\frac{i\pi}{4}\sigma^3\otimes\sigma^3} \left(e^{\pm\frac{i\pi}{4}\sigma^m} \otimes e^{\pm\frac{i\pi}{4}\sigma^n}\right) \tag{2.23}$$

Absolutely, the PVM operators can be represented as linear combinations of the representation basis Γ^{ij}.

$$\Omega^k = \theta_{ij}^k \Gamma^{ij} \tag{2.24}$$

Using the property discussed above 2.19, θ_{ij}^k is given by

$$\theta_{ij}^k = \frac{1}{4}\,\mathrm{tr}\left(\Omega^k \Gamma_{ij}\right) \tag{2.25}$$

In this manner, we find θ_{ij}^k. See the table 2.1.

The PVM set generated by $(S1, S1)$ has been found to perform QST well. There are numerous possible combinations of these operators as long as the rank of PVM set is 15. There are approximately 13 milion possible sets.

	Γ^{00}	Γ^{01}	Γ^{02}	Γ^{03}	Γ^{10}	Γ^{11}	Γ^{12}	Γ^{13}	Γ^{20}	Γ^{21}	Γ^{22}	Γ^{23}	Γ^{30}	Γ^{31}	Γ^{32}	Γ^{33}
Ω_0^1	$\frac{1}{2}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0
Ω_0^2	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0
Ω_0^3	$\frac{1}{2}$	0	0	0	0	0	$-\frac{1}{4}$	0	0	$-\frac{1}{4}$	0	0	0	0	0	0
Ω_0^4	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	0	0	0	0
Ω_0^5	$\frac{1}{2}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0
Ω_0^6	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0
Ω_0^7	$\frac{1}{2}$	0	0	0	0	0	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	0	0	0	0
Ω_0^8	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{4}$	0	0	$-\frac{1}{4}$	0	0	0	0	0	0
Ω_0^9	$\frac{1}{2}$	0	0	0	0	0	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	0	0	0	0	0
Ω_0^{10}	$\frac{1}{2}$	0	0	0	0	0	0	$-\frac{1}{4}$	0	$-\frac{1}{4}$	0	0	0	0	0	0
Ω_0^{11}	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	$\frac{1}{4}$	0	0
Ω_0^{12}	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	$-\frac{1}{4}$	0	0
Ω_0^{13}	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	0	0
Ω_0^{14}	$\frac{1}{2}$	$\frac{1}{4}$	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0
Ω_0^{15}	$\frac{1}{2}$	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0
Ω_0^{16}	$\frac{1}{2}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	0	0
Ω_0^{17}	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	$\frac{1}{4}$	0	0	0	0
Ω_0^{18}	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	$-\frac{1}{4}$	0	0	0	0
Ω_0^{19}	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0
Ω_0^{20}	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0	0	$-\frac{1}{4}$	0
Ω_0^{21}	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0
Ω_0^{22}	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0
Ω_0^{23}	$\frac{1}{2}$	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0
Ω_0^{24}	$\frac{1}{2}$	0	$-\frac{1}{4}$	0	0	0	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0
Ω_0^{25}	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0	0	0	$\frac{1}{4}$
Ω_0^{26}	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0	0	0	$-\frac{1}{4}$
Ω_0^{27}	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0	0	0
Ω_0^{28}	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0
Ω_0^{29}	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0

Table 2.1: The elements of θ_{ij}^k, $(S1, S1) = \frac{1}{2}\left(|0\rangle\langle 0| \otimes \sigma^0 + \sigma^0 \otimes |0\rangle\langle 0|\right)$

2.2 Optimal QST in 2 qubit system

2.2.1 Optimal QST

In order for FQST to be possible, it is crucial to ensure that the rank of the PVM set is 15, which is in line with the minimum number of measurements discussed earlier. However, optimizing the QST process is a separate challenge altogether. Since each measurement simultaneously extracts two pieces of information from the quantum system, conducting a uniform measurement operation on all elements in the PVM set may lead to non-uniformity in terms of the Pauli basis representation. For instance, consider a PVM set that includes Ω_0^1, Ω_0^2, Ω_0^5, Ω_0^6, Ω_0^{17}, Ω_0^{18}, Ω_0^{19}, Ω_0^{20}, Ω_0^{25}, and Ω_0^{26}. In this case, the measurement for Γ^{03} is over-performed for the given PVM set. Therefore, careful selection of PVM sets is necessary to achieve an "optimal" QST that ensures uniform extraction of information.

2.2.2 Choosing PVM basis

As discussed in the chaper 1.1.2, the measurement process is nothing but measuring the distance between the hyperplane and the state vector. Hence, the more orthogonal the elements of the PVM set are to each other, the more optimal the set becomes. Except for the component of Γ^{00}, which is the identity matrix, the coefficient of each Pauli representation basis represents the normal vector of the hyperplane. For example, the PVM operator $\Omega_0^1 = \frac{1}{4}\Gamma^{00} + \frac{1}{4}\Gamma^{11} - \frac{1}{4}\Gamma^{22}$ represents a hyperplane whose normal vector is given as:

$$\theta^1 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \tag{2.26}$$

In a similar manner, the normal vector of Ω_0^5 and Ω_0^6 represent hyperplanes that are fully orthogonal to Ω_0^1, then combining Ω_0^1 with either Ω_0^5 or Ω_0^6 is a good combination for optimal QST. Thus, we can choose the PVM basis by selecting normal vectors that are as orthogonal as possible to each other. To evaluate how orthogonal the set of normal vectors is, we can use the Hilbert-Schmidt norm. Even for a reduced PVM set, we can write:

$$\Omega^k_{\text{Reduced}} = \theta_{\text{Reduced}}{}^k_{ij}\Gamma^{ij}, \tag{2.27}$$

and the Hilbert-Schmidt norm of ${\theta_{\text{Reduced}}}_{ij}^{k}$ is given by:

$$|\Theta_{Reduced}| = \sqrt{\sum_{i,j=0}^{15} {\theta_{\text{Reduced}}}_{ij}^{k\,2}} = \sqrt{\text{Tr}\,\Theta^{\dagger}\Theta} \tag{2.28}$$

where the Hilbert-Schmidt norm of ${\theta_{\text{Reduced}}}_{ij}^{k}$ represents how optimal the PVM set is. Thus, we can choose the PVM set whose Hilbert-Schmidt norm is minimum.

However, there are millions of possible PVM sets with a rank of 15, making it necessary to reduce the number of cases. Here, we can use (Θ^{-1}), which can be calculated using the equation:

$$\left(\Theta^{-1}\right) = (\Theta^{T}\Theta)^{-1}\Theta^{T} \tag{2.29}$$

as a guide for selecting PVM operators. It is shown in the equation 2.30. By using (Θ^{-1}), we can directly see which operators are required to obtain information about specific Pauli matrices.

For example, a set of PVM operators including Ω_0^{13}, Ω_0^{14}, Ω_0^{15}, Ω_0^{16}, Ω_0^{21}, Ω_0^{22}, Ω_0^{23}, Ω_0^{24}, Ω_0^{27}, and Ω_0^{28} can provide information about Γ^{01}, Γ^{02}, Γ^{03}, Γ^{10}, Γ^{20}, and Γ^{30}. However, each operator may not necessarily provide information about all of the referenced Γ matrices, but it can help eliminate mixed information about other matrices. For instance, while Ω_0^{13} provides mixed information about Γ^{03} and Γ^{10}, other PVM operators can be used to eliminate information about Γ^{03} in order to obtain information about Γ^{10} for a given quantum system.

$$\left(\theta^{-1}\right)^{k}_{ij} = \left(\begin{array}{cccccccccccc|cccccccccccc|cccccccccccc|cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{176} & -\frac{157}{176} & -\frac{195}{176} & -\frac{3}{176} & 0 & 0 & 0 & 0 & -\frac{71}{176} & \frac{43}{88} & \frac{21}{88} & \frac{39}{176} & 0 & 0 & -\frac{3}{16} & \frac{3}{16} & \frac{3}{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{65}{176} & \frac{1}{176} & -\frac{1}{176} & -\frac{65}{176} & 0 & 0 & 0 & 0 & -\frac{13}{176} & -\frac{7}{88} & -\frac{73}{88} & \frac{141}{176} & 0 & 0 & -\frac{1}{16} & \frac{1}{16} & -\frac{1}{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{27}{176} & -\frac{5}{176} & \frac{5}{176} & -\frac{27}{176} & 0 & 0 & 0 & 0 & \frac{65}{176} & \frac{35}{88} & \frac{13}{88} & -\frac{1}{176} & 0 & 0 & \frac{5}{16} & -\frac{5}{16} & \frac{5}{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{195}{176} & -\frac{3}{176} & \frac{3}{176} & -\frac{157}{1760} & 0 & 0 & 0 & 0 & \frac{39}{176} & \frac{21}{88} & \frac{43}{88} & -\frac{71}{176} & 0 & 0 & \frac{3}{16} & -\frac{3}{16} & \frac{3}{11} \\
\hline
-\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{176} & \frac{65}{176} & -\frac{65}{176} & -\frac{1}{176} & 0 & 0 & 0 & 0 & -\frac{141}{176} & \frac{73}{88} & \frac{7}{88} & \frac{13}{176} & 0 & 0 & -\frac{1}{16} & \frac{1}{16} & \frac{1}{11} \\
0 & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{176} & -\frac{27}{176} & \frac{27}{176} & -\frac{5}{176} & 0 & 0 & 0 & 0 & -\frac{1}{176} & \frac{13}{88} & \frac{35}{88} & \frac{65}{176} & 0 & 0 & -\frac{5}{16} & \frac{5}{16} & \frac{5}{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0
\end{array}\right) \tag{2.30}$$

To choose the appropriate PVM operators, we have devised a strategy that reduces the number of cases. First, we identified three sets of PVM operators that provide information about different sets of Γ parameters. The first set includes Ω_0^{13}, Ω_0^{14}, Ω_0^{15}, Ω_0^{16}, Ω_0^{21}, Ω_0^{22}, Ω_0^{23}, Ω_0^{24}, Ω_0^{27}, and Ω_0^{28}, and provides information about Γ^{01}, Γ^{02}, Γ^{03}, Γ^{10}, Γ^{20}, and Γ^{30}. The second set includes Ω_0^{1}, Ω_0^{2}, Ω_0^{5}, Ω_0^{6}, Ω_0^{17}, Ω_0^{18}, Ω_0^{19}, Ω_0^{20}, Ω_0^{25}, and Ω_0^{26}, and provides information about Γ^{11}, Γ^{22}, Γ^{23}, Γ^{32}, and Γ^{33}. The third set includes Ω_0^{3}, Ω_0^{4}, Ω_0^{7}, Ω_0^{8}, Ω_0^{9}, Ω_0^{10}, Ω_0^{11}, and Ω_0^{12}, and provides information about Γ^{12}, Γ^{13}, Γ^{21}, and Γ^{31}.

We also noted that some measurement operators are identical, such as Ω_0^1 and Ω_0^2, which have oppositely directed normal vectors, and Ω_0^3 and Ω_0^4, Ω_0^5 and Ω_0^6, Ω_0^7 and Ω_0^8, and Ω_0^{27} and Ω_0^{28}, which have the same normal vectors but different origins. Therefore, we can ignore one of the identical operators and reduce the number of measurement operators to 6, 5, and 4 for each set while maintaining the rank of each set.

By following this strategy, we can significantly reduce the number of possible combinations to less than 80,000, which is much smaller than the 13 million possible PVM sets with rank 15. With this reduced set, we can efficiently select the appropriate PVM operators to obtain uniform information about Γ^{ij}.

Once we find the optimal set for the first set with the smallest Hilbert-Schmidt norm, we can find the PVM set for the optimal QST. This is because it is relatively easy to find the optimal set by hand for the second and third sets.

However, this strategy may not be scalable. If the system becomes bigger than now, i.e. increasing the number of qubits or using qudits, this approach may not applicable. For here, $\Theta(\Theta^{-1})$ was $29 \times 16(16 \times 29)$ matrix. But, for example, in 3 qubit system, Θ becomes at least 64×64 matrix. Therefore, a general strategy for selecting PVM operators to minimize the Hilbert-Schmidt norm is required.

2.3 Appendix A : Representation basis Γ^{ij}

$$\Gamma^{00} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Gamma^{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Gamma^{02} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \qquad \Gamma^{03} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Gamma^{10} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \Gamma^{11} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^{12} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\Gamma^{20} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \qquad \Gamma^{21} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^{22} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^{23} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$\Gamma^{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \Gamma^{31} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\Gamma^{32} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \qquad \Gamma^{33} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2.4 Appendix B : POVM basis Ω_0^{ij}

$$\Omega_0^1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2}i \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2}i & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^4 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2}i \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2}i & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^5 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^6 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^7 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2}i & 0 \\ 0 & \frac{1}{2}i & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^8 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2}i & 0 \\ 0 & -\frac{1}{2}i & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^9 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4}i \\ 0 & \frac{1}{2} & \frac{1}{4}i & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4}i & \frac{1}{2} & 0 \\ -\frac{1}{4}i & -\frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^{10} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{4} & \frac{1}{4}i \\ 0 & \frac{1}{2} & \frac{1}{4}i & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4}i & \frac{1}{2} & 0 \\ -\frac{1}{4}i & \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^{11} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & -\frac{1}{4}i \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}i & 0 \\ 0 & -\frac{1}{4}i & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4}i & 0 & -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^{12} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 & -\frac{1}{4}i \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4}i & 0 \\ 0 & -\frac{1}{4}i & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4}i & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^{13} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix} \qquad \Omega_0^{14} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4}i & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4}i \\ -\frac{1}{4}i & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & -\frac{1}{4}i & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^{15} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix} \qquad \Omega_0^{16} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4}i & \frac{1}{4} & 0 \\ \frac{1}{4}i & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} & -\frac{1}{4}i \\ 0 & \frac{1}{4} & \frac{1}{4}i & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^{17} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{4}i & -\frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4}i \\ \frac{1}{4}i & -\frac{1}{4} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{4}i & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^{18} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4}i & -\frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4}i \\ -\frac{1}{4}i & -\frac{1}{4} & \frac{1}{2} & 0 \\ -\frac{1}{4} & \frac{1}{4}i & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^{19} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4}i & 0 & -\frac{1}{4} \\ \frac{1}{4}i & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4}i \\ -\frac{1}{4} & 0 & -\frac{1}{4}i & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^{20} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4}i & 0 & -\frac{1}{4} \\ -\frac{1}{4}i & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4}i \\ -\frac{1}{4} & 0 & \frac{1}{4}i & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^{21} = \begin{pmatrix} \frac{1}{4} & 0 & -\frac{1}{4}i & 0 \\ 0 & \frac{3}{4} & 0 & -\frac{1}{4}i \\ \frac{1}{4}i & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4}i & 0 & \frac{3}{4} \end{pmatrix} \qquad \Omega_0^{22} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4}i & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4}i \\ -\frac{1}{4}i & 0 & \frac{1}{4} & 0 \\ 0 & -\frac{1}{4}i & 0 & \frac{3}{4} \end{pmatrix}$$

$$\Omega_0^{23} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4}i & 0 & 0 \\ \frac{1}{4}i & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{3}{4} & -\frac{1}{4}i \\ 0 & 0 & \frac{1}{4}i & \frac{3}{4} \end{pmatrix} \qquad \Omega_0^{24} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4}i & 0 & 0 \\ -\frac{1}{4}i & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4}i \\ 0 & 0 & -\frac{1}{4}i & \frac{3}{4} \end{pmatrix}$$

$$\Omega_0^{25} = \begin{pmatrix} \frac{3}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix} \qquad \Omega_0^{26} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ -\frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$\Omega_0^{27} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^{28} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^{29} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Chapter 3

What to do?

What I've done and haven't

Over the past few weeks, I have focused on studying the construction of a PVM set for conducting QST. I have also made improvements to the Python code used for conducting the actual process of QST. Through this work, I have discovered that it is possible to construct a PVM set with 15 operators, as long as the rank of the PVM set is also 15. Furthermore, I have explored different possible PVM basis sets in order to find the PVM set for conducting optimal QST that extracts information uniformly. Based on my analysis, I believe that the PVM set for optimal QST should have a minimum Hilbert-Schmidt norm, so that the hyperplanes are as orthogonal to each other as possible. However, my attempts to find an useful invariance quantity have been unsuccessful. Additionally, I have found that operators constructed from (S_z, S_z) are unnecessary since they are not PVM.

What I'm going to do

I have proposed a potential solution to minimize Hilbert-Schmidt norm by calculating ${\theta^{-1}}^k_{ij}$, but it may not be practical for larger quantum systems. Therefore, a more general approach is necessary to identify a PVM set with a minimum Hilbert-Schmidt norm. Once I have completed the work on finding the general strategy for constructing PVM sets with a minimum Hilbert-Schmidt norm, my timeline for this month will be completed. My next task will be focused on the Variational Quantum Eigensolver(VQE).

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