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Status Report

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Quantum State Tomography and Measurement basis

This report summarizes my studies over the past two weeks until Mar 17 and is submitted on March 20, 2023.

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Abstract

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2.1 $SU(2) \otimes SU(2)$

Our system consists of two qubit, therefore, it possesses $SU(2) \otimes SU(2)$ symmetry. Let me choose $\{\Gamma^{ij}\}_{i,j=0}^{3}$ as an orthogonal representation basis set such that $\mathrm{tr}\left(\Gamma^{ij\dagger}\Gamma^{mn}\right) = \delta_{(ij)(mn)}$. We can choose $\Gamma^{ij} = \sigma^i \otimes \sigma^j$. It is convinient in that they are traceless excpet for Γ^{00}. See Appendix 2.2.2 how Γ^{ij} look like.

$\{\Gamma^{ij}\}$ possesses some interesting properties. First, the product between Γ^{ij} and Γ^{mn} is given by

$$\begin{aligned}
\Gamma^{ij}\Gamma^{mn} &= (\sigma^i \otimes \sigma^j) \otimes (\sigma^m \otimes \sigma^n) \\
&= (\sigma^i \sigma^m) \otimes (\sigma^j \otimes \sigma^n) \\
&= (\delta^{im} + i\epsilon_{imk}\sigma^k) \otimes (\delta_{jn} + i\epsilon_{jnl}\sigma^l) \\
&= \delta_{im}\delta_{jn} + i\delta_{im}\epsilon_{jnl}\sigma^l + i\delta^{jn}\epsilon_{imk}\sigma^k - \epsilon_{imk}\epsilon_{jnl}\sigma^k\sigma^l
\end{aligned} \tag{2.1}$$

Note that I supposed that $i, j, m, n \neq 0$ and didn't write $\otimes$ explicitly at the end of equation. And you don't have to pay attention to the upper/lower indecies. In a same way, we have

$$\Gamma^{jn}\Gamma^{ij} = \delta_{im}\delta_{jn} - i\delta_{im}\epsilon_{jnl}\sigma^l - i\delta^{jn}\epsilon_{imk}\sigma^k - \epsilon_{imk}\epsilon_{jnl}\sigma^k\sigma^l \tag{2.2}$$

in component form. Or more conviniently,

$$
\begin{aligned}
\Gamma^{jn}\Gamma^{ij} &= (\sigma^m \otimes \sigma^n) \otimes (\sigma^m \otimes \sigma^n) \\
&= -(\sigma^i \sigma^m - 2\delta_{im}) \otimes (\sigma^n \sigma^j) \\
&= -(\sigma^i \sigma^m) \otimes (\sigma^n \sigma^j) + 2\delta_{im} \otimes \sigma^n \sigma^j \qquad (2.3) \\
&= (\sigma^i \sigma^m - 2i\epsilon_{mik}\sigma^k) \otimes (\sigma^n \sigma^j) \\
&= (\sigma^i \sigma^m) \otimes (\sigma^n \sigma^j) - 2i\epsilon_{mik}\sigma^k \otimes \sigma^n \sigma^j. \qquad (2.4)
\end{aligned}
$$

Therefore, the commutator relation and the anticommutator relation are given by

$$
\begin{aligned}
\left[\Gamma^{ij}, \Gamma^{mn}\right] &= (\sigma^i \sigma^m) \otimes \left\{\sigma^j, \sigma^n\right\} - 2\delta_{im} \otimes \sigma^n \sigma^j \qquad (2.5) \\
&= (\sigma^i \sigma^m) \otimes \left[\sigma^j, \sigma^n\right] + 2i\epsilon_{mik}\sigma^k \otimes \sigma^n \sigma^j \qquad (2.6) \\
&= 2i\delta_{im}\epsilon_{jnl}\sigma^l + 2i\delta_{jn}\epsilon_{imk}\sigma^k \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
\left\{\Gamma^{ij}, \Gamma^{mn}\right\} &= (\sigma^i \sigma^m) \otimes \left[\sigma^j, \sigma^n\right] + 2\delta_{im} \otimes \sigma^n \sigma^j \qquad (2.8) \\
&= (\sigma^i \sigma^m) \otimes \left\{\sigma^j, \sigma^n\right\} - 2i\epsilon_{mik}\sigma^k \otimes \sigma^n \sigma^j \qquad (2.9) \\
&= 2\delta_{im}\delta_{jn} - 2\epsilon_{imk}\epsilon_{jnl}\sigma^k \sigma^l \qquad (2.10)
\end{aligned}
$$

If one of the elements is zero, i.e. $i = 0$, then,

$$
\begin{aligned}
\Gamma^{0j}\Gamma^{mn} &= (\sigma^0 \otimes \sigma^j) \otimes (\sigma^m \otimes \sigma^n) \\
&= (\sigma^0 \sigma^m) \otimes (\sigma^j \otimes \sigma^n) \\
&= \sigma^m \otimes \delta_{jn} + i\sigma^m \otimes \epsilon_{jnl}\sigma^l \qquad (2.11)
\end{aligned}
$$

and the commutator relation and the anticommutator relation are given by

$$
\left[\Gamma^{0j}, \Gamma^{mn}\right] = -2i\sigma^m \otimes \epsilon_{njk}\sigma^k \qquad (2.12)
$$

$$
\left\{\Gamma^{0j}, \Gamma^{mn}\right\} = 2\sigma^m \otimes \delta_{jn} \qquad (2.13)
$$

If two of the elements are zero, i.e. $i = j = 0$ or $i = m = 0$, then,

$$
\begin{aligned}
\Gamma^{00}\Gamma^{mn} &= (\sigma^m) \otimes (\sigma^n) \\
&= \Gamma^{mn} \\
&= \Gamma^{mn}\Gamma^{00}
\end{aligned}
$$

And obviously,

$$\left[\Gamma^{00}, \Gamma^{mn}\right] = 0 \tag{2.14}$$

$$\left\{\Gamma^{00}, \Gamma^{mn}\right\} = \Gamma^{mn} \tag{2.15}$$

On the other hand,

$$\Gamma^{0j}\Gamma^{0n} = \sigma^0 \otimes \left(\sigma^j \sigma^n\right)$$

and

$$\left[\Gamma^{0j}, \Gamma^{0n}\right] = \sigma^0 \otimes 2i\epsilon_{njk}\sigma^k \tag{2.16}$$

$$\left\{\Gamma^{0j}, \Gamma^{0n}\right\} = \sigma^0 \otimes 2\delta_{nj} \tag{2.17}$$

Therefore, $\{\Gamma^{ij}\}$ constructs an orthogonal representation basis. Considering the property of trace, $\operatorname{tr} A \otimes B = \operatorname{tr} A \times \operatorname{tr} B$, we have

$$\operatorname{tr}\left(\Gamma^{ij}\Gamma^{mn}\right) = 4\delta_{im}\delta_{jn}. \tag{2.18}$$

Note that here $\dagger$ is removed. The hermitivity of Γ^{ij} is guranteed by the hermitivity of pauli matrices.

It has been recently notice that $SU(2) \otimes SU(2)$ is actually $SO(1,3)$, which is familiar in the special relativity. It seems possible to use generaters in $SO(1,3)$ to represent the system. This study is undergoing.

2.2 Measurement basis

2.2.1 Bell basis measurement

2.2.2 POVM basis Ω^i

Given system is a two-qubit system with $\sigma^3 \otimes \sigma^3$ interaction and the projection measurement is performed with POVM basis. See Appendix B 2.2.2. There are well-calculated 29 POVM operators. Since the rank of the set of such basis is 16, it is assumed that the set of POVM operators can be reduced. There are some numerous possible combinations, but it is presumed that there are about 1,600 possible combinations to conduct QST.

Absolutly, POVM operators can be represented in a linear combination of the representation basis $\{\Gamma^{ij}\}$.

$$\Omega^k = \theta_{ij}^k \Gamma^{ij} \tag{2.19}$$

Using the property discussed above 2.18, θ_{ij}^k is given by

$$\theta_{ij}^k = \frac{1}{4} \operatorname{tr}\left(\Omega^k \Gamma^{ij}\right) \tag{2.20}$$

In this manner, we find θ_{ij}^k. See the table 2.2.2. For example, the first POVM basis Ω^1 is given by

$$\begin{aligned}
\Omega^1 &= \frac{1}{4}\Gamma^{00} + \frac{1}{4}\Gamma^{11} - \frac{1}{4}\Gamma^{22} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}
\end{aligned}$$

Interestingly, all the Ω^k consists of three Γ^{ij}. And one is always Γ^{00}.

It is presumed that the sets consisting of the selected 16 POVM operations, which conduct QST well, may have almost the same amount of Γ^{ij}. That is, the sum of square of coefficient would be same or similar for all Γ^{ij} excpet for Γ^{00}. However, it does not. For example, in a possible POVM basis $\{\Omega^1, \Omega^2, \Omega^3, \Omega^7, \Omega^9, \Omega^{11}, \Omega^{13}, \Omega^{14}, \Omega^{16}, \Omega^{18}, \Omega^{19}, \Omega^{20}, \Omega^{21}, \Omega^{22}, \Omega^{26}, \Omega^{29}\}$, Ω^1, Ω^2 ,Ω^{18}, Ω^{19}, Ω^{20}, Ω^{26} consists of Γ^{11}, while only Ω^{14} consists of Γ^{01}. The sum of square of coefficient for Γ^{11} is $\frac{3}{8}$, while the sum of square of coefficient for Γ^{01} is $\frac{1}{16}$.

Nevertheless, all the Γ^{ij} are included in the set of POVM basis. All the selected POVM sets include all the elements of $\{\Gamma^{ij}\}$.

	Γ^{00}	Γ^{01}	Γ^{02}	Γ^{03}	Γ^{10}	Γ^{11}	Γ^{12}	Γ^{13}	Γ^{20}	Γ^{21}	Γ^{22}	Γ^{23}	Γ^{30}	Γ^{31}	Γ^{32}	Γ^{33}
Ω^{1}	$\frac{1}{2}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0
Ω^{2}	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0
Ω^{3}	$\frac{1}{2}$	0	0	0	0	0	$-\frac{1}{4}$	0	0	$-\frac{1}{4}$	0	0	0	0	0	0
Ω^{4}	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	0	0	0	0
Ω^{5}	$\frac{1}{2}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0
Ω^{6}	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0
Ω^{7}	$\frac{1}{2}$	0	0	0	0	0	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	0	0	0	0
Ω^{8}	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{4}$	0	0	$-\frac{1}{4}$	0	0	0	0	0	0
Ω^{9}	$\frac{1}{2}$	0	0	0	0	0	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	0	0	0	0	0
Ω^{10}	$\frac{1}{2}$	0	0	0	0	0	0	$-\frac{1}{4}$	0	$-\frac{1}{4}$	0	0	0	0	0	0
Ω^{11}	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	$\frac{1}{4}$	0	0
Ω^{12}	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	$-\frac{1}{4}$	0	0
Ω^{13}	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	0	0
Ω^{14}	$\frac{1}{2}$	$\frac{1}{4}$	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0
Ω^{15}	$\frac{1}{2}$	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0
Ω^{16}	$\frac{1}{2}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	0	0
Ω^{17}	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	$\frac{1}{4}$	0	0	0	0
Ω^{18}	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	$-\frac{1}{4}$	0	0	0	0
Ω^{19}	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0
Ω^{20}	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0	0	$-\frac{1}{4}$	0
Ω^{21}	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0
Ω^{22}	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0
Ω^{23}	$\frac{1}{2}$	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0
Ω^{24}	$\frac{1}{2}$	0	$-\frac{1}{4}$	0	0	0	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0
Ω^{25}	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0	0	0	$\frac{1}{4}$
Ω^{26}	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0	0	0	$-\frac{1}{4}$
Ω^{27}	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0	0	0
Ω^{28}	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0
Ω^{29}	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	0	0	0	0	0	0	0	0	$-\frac{1}{4}$	0	0	0

Table 2.1: The elements of θ_{ij}^{k}

	Γ^{00}	Γ^{01}	Γ^{02}	Γ^{03}	Γ^{10}	Γ^{11}	Γ^{12}	Γ^{13}	Γ^{20}	Γ^{21}	Γ^{22}	Γ^{23}	Γ^{30}	Γ^{31}	Γ^{32}	Γ^{33}
Ω_1^1	$\frac{1}{2}$	0	0	0	0	-$\frac{1}{4}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0
Ω_1^2	$\frac{1}{2}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	-$\frac{1}{4}$	0	0	0	0	0
Ω_1^3	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	0	0	0	0
Ω_1^4	$\frac{1}{2}$	0	0	0	0	0	-$\frac{1}{4}$	0	0	-$\frac{1}{4}$	0	0	0	0	0	0
Ω_1^5	$\frac{1}{2}$	0	0	0	0	-$\frac{1}{4}$	0	0	0	0	-$\frac{1}{4}$	0	0	0	0	0
Ω_1^6	$\frac{1}{2}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0
Ω_1^7	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{4}$	0	0	-$\frac{1}{4}$	0	0	0	0	0	0
Ω_1^8	$\frac{1}{2}$	0	0	0	0	0	-$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	0	0	0	0
Ω_1^9	$\frac{1}{2}$	0	0	0	0	0	0	-$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	0	0	0
Ω_1^{10}	$\frac{1}{2}$	0	0	0	0	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	0	0	0
Ω_1^{11}	$\frac{1}{2}$	0	0	0	0	0	-$\frac{1}{4}$	0	0	0	0	0	0	-$\frac{1}{4}$	0	0
Ω_1^{12}	$\frac{1}{2}$	0	0	0	0	0	-$\frac{1}{4}$	0	0	0	0	0	0	$\frac{1}{4}$	0	0
Ω_1^{13}	$\frac{1}{2}$	0	0	$\frac{1}{4}$	-$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	0	0
Ω_1^{14}	$\frac{1}{2}$	-$\frac{1}{4}$	0	0	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0
Ω_1^{15}	$\frac{1}{2}$	-$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0	0	0
Ω_1^{16}	$\frac{1}{2}$	0	-$\frac{1}{4}$	0	-$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	0	0
Ω_1^{17}	$\frac{1}{2}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	-$\frac{1}{4}$	0	0	0	0
Ω_1^{18}	$\frac{1}{2}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	$\frac{1}{4}$	0	0	0	0
Ω_1^{19}	$\frac{1}{2}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	-$\frac{1}{4}$	0
Ω_1^{20}	$\frac{1}{2}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0
Ω_1^{21}	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0	0	0	0	-$\frac{1}{4}$	0	0	0	0	0	0	0
Ω_1^{22}	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0
Ω_1^{23}	$\frac{1}{2}$	0	-$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0	0	0
Ω_1^{24}	$\frac{1}{2}$	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0	0	0
Ω_1^{25}	$\frac{1}{2}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	-$\frac{1}{4}$
Ω_1^{26}	$\frac{1}{2}$	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	$\frac{1}{4}$
Ω_1^{27}	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	-$\frac{1}{4}$	0	0	0
Ω_1^{28}	$\frac{1}{2}$	0	0	-$\frac{1}{4}$	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0	0	0
Ω_1^{29}	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0	0	0

Table 2.2: The elements of θ_{ij}^k

Appendix A : Representation basis Γ^{ij}

$$\Gamma^{00} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Gamma^{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Gamma^{02} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \qquad \Gamma^{03} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Gamma^{10} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \Gamma^{11} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^{12} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\Gamma^{20} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \qquad \Gamma^{21} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^{22} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^{23} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$\Gamma^{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \Gamma^{31} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\Gamma^{32} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \qquad \Gamma^{33} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Appendix B : POVM basis Ω^{ij}

$$\Omega_0^1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2}i \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2}i & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^4 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2}i \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2}i & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^5 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^6 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^7 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2}i & 0 \\ 0 & \frac{1}{2}i & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^8 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2}i & 0 \\ 0 & -\frac{1}{2}i & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^9 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4}i \\ 0 & \frac{1}{2} & \frac{1}{4}i & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4}i & \frac{1}{2} & 0 \\ -\frac{1}{4}i & -\frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^{10} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{4} & \frac{1}{4}i \\ 0 & \frac{1}{2} & \frac{1}{4}i & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4}i & \frac{1}{2} & 0 \\ -\frac{1}{4}i & \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^{11} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & -\frac{1}{4}i \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}i & 0 \\ 0 & -\frac{1}{4}i & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4}i & 0 & -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^{12} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 & -\frac{1}{4}i \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4}i & 0 \\ 0 & -\frac{1}{4}i & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4}i & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^{13} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix} \qquad \Omega_0^{14} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4}i & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4}i \\ -\frac{1}{4}i & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & -\frac{1}{4}i & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^{15} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix} \qquad \Omega_0^{16} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4}i & \frac{1}{4} & 0 \\ \frac{1}{4}i & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} & -\frac{1}{4}i \\ 0 & \frac{1}{4} & \frac{1}{4}i & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^{17} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{4}i & -\frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4}i \\ \frac{1}{4}i & -\frac{1}{4} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{4}i & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^{18} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4}i & -\frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4}i \\ -\frac{1}{4}i & -\frac{1}{4} & \frac{1}{2} & 0 \\ -\frac{1}{4} & \frac{1}{4}i & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^{19} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4}i & 0 & -\frac{1}{4} \\ \frac{1}{4}i & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4}i \\ -\frac{1}{4} & 0 & -\frac{1}{4}i & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^{20} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4}i & 0 & -\frac{1}{4} \\ -\frac{1}{4}i & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4}i \\ -\frac{1}{4} & 0 & \frac{1}{4}i & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^{21} = \begin{pmatrix} \frac{1}{4} & 0 & -\frac{1}{4}i & 0 \\ 0 & \frac{3}{4} & 0 & -\frac{1}{4}i \\ \frac{1}{4}i & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4}i & 0 & \frac{3}{4} \end{pmatrix} \qquad \Omega_0^{22} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4}i & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4}i \\ -\frac{1}{4}i & 0 & \frac{1}{4} & 0 \\ 0 & -\frac{1}{4}i & 0 & \frac{3}{4} \end{pmatrix}$$

$$\Omega_0^{23} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4}i & 0 & 0 \\ \frac{1}{4}i & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{3}{4} & -\frac{1}{4}i \\ 0 & 0 & \frac{1}{4}i & \frac{3}{4} \end{pmatrix} \qquad \Omega_0^{24} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4}i & 0 & 0 \\ -\frac{1}{4}i & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4}i \\ 0 & 0 & -\frac{1}{4}i & \frac{3}{4} \end{pmatrix}$$

$$\Omega_0^{25} = \begin{pmatrix} \frac{3}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix} \qquad \Omega_0^{26} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ -\frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$\Omega_0^{27} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_0^{28} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_0^{29} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Omega_1^1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_1^2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_1^3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2}i \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2}i & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_1^4 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2}i \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2}i & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_1^5 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_1^6 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_1^7 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2}i & 0 \\ 0 & -\frac{1}{2}i & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_1^8 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2}i & 0 \\ 0 & \frac{1}{2}i & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_1^9 = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{4} & -\frac{1}{4}i \\ 0 & \frac{1}{2} & -\frac{1}{4}i & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4}i & \frac{1}{2} & 0 \\ \frac{1}{4}i & \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_1^{10} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & -\frac{1}{4}i \\ 0 & \frac{1}{2} & -\frac{1}{4}i & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4}i & \frac{1}{2} & 0 \\ \frac{1}{4}i & -\frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_1^{11} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 & \frac{1}{4}i \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4}i & 0 \\ 0 & \frac{1}{4}i & \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4}i & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \qquad \Omega_1^{12} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4}i \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4}i & 0 \\ 0 & \frac{1}{4}i & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4}i & 0 & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$\Omega_1^{13} = \begin{pmatrix} \frac{3}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & -\frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} \qquad \Omega_1^{14} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4}i & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 & -\frac{1}{4}i \\ \frac{1}{4}i & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{4}i & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$\Omega_1^{15} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} \end{pmatrix} \qquad \Omega_1^{16} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4}i & -\frac{1}{4} & 0 \\ -\frac{1}{4}i & \frac{1}{2} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4}i \\ 0 & -\frac{1}{4} & -\frac{1}{4}i & \frac{1}{2} \end{pmatrix}$$

$$\Omega_1^{17} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4}i & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & -\frac{1}{4}i \\ -\frac{1}{4}i & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4}i & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_1^{18} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{4}i & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4}i \\ \frac{1}{4}i & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{4}i & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_1^{19} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4}i & 0 & \frac{1}{4} \\ -\frac{1}{4}i & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4}i \\ \frac{1}{4} & 0 & \frac{1}{4}i & \frac{1}{2} \end{pmatrix} \qquad \Omega_1^{20} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4}i & 0 & \frac{1}{4} \\ \frac{1}{4}i & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4}i \\ \frac{1}{4} & 0 & -\frac{1}{4}i & \frac{1}{2} \end{pmatrix}$$

$$\Omega_1^{21} = \begin{pmatrix} \frac{3}{4} & 0 & \frac{1}{4}i & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4}i \\ -\frac{1}{4}i & 0 & \frac{3}{4} & 0 \\ 0 & -\frac{1}{4}i & 0 & \frac{1}{4} \end{pmatrix} \qquad \Omega_1^{22} = \begin{pmatrix} \frac{3}{4} & 0 & -\frac{1}{4}i & 0 \\ 0 & \frac{1}{4} & 0 & -\frac{1}{4}i \\ \frac{1}{4}i & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{4}i & 0 & \frac{1}{4} \end{pmatrix}$$

$$\Omega_1^{23} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4}i & 0 & 0 \\ -\frac{1}{4}i & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4}i \\ 0 & 0 & -\frac{1}{4}i & \frac{1}{4} \end{pmatrix} \qquad \Omega_1^{24} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4}i & 0 & 0 \\ \frac{1}{4}i & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{1}{4}i \\ 0 & 0 & \frac{1}{4}i & \frac{1}{4} \end{pmatrix}$$

$$\Omega_1^{25} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix} \qquad \Omega_1^{26} = \begin{pmatrix} \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix}$$

$$\Omega_1^{27} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \Omega_1^{28} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Omega_1^{29} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Chapter 3

What to do?

What I've done and haven't

What I'm going to do

example. example title, 4

Bibliography

[1] example. example title, 4.