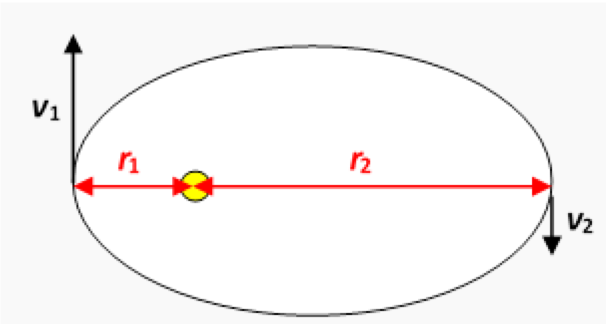
Kepler's law:

$$T^2 = \frac{4\pi^2 a^3}{GM}$$

$$E_{tot} = -\frac{GMm}{2a}$$

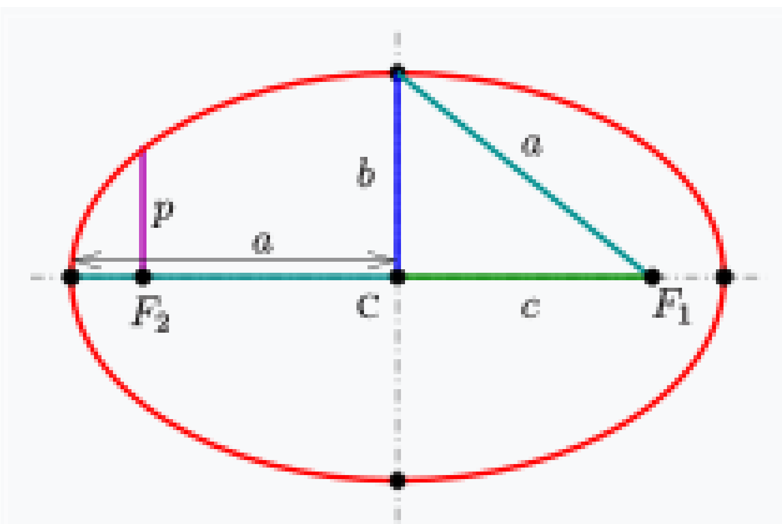
Angular Momentum Conservation:



$$mv_1r_1 = mv_2r_2 = L$$

$$\frac{L^2}{2m^2} = \frac{GM}{\left(\frac{1}{r_1} + \frac{1}{r_2}\right)} = \frac{GMb^2}{2a}$$

Shape parameters: *a*: semi-major axis, *b*: semi-minor axis, *c*: linear eccentricity, *p*: semi-latus rectum



Orbital and escape velocity. Different orbits:

The orbit shape will depend on the choices of r_0 and v_0. The escape speed of an object at a radius r from the center of Earth is

$$v_{esc} = \sqrt{\frac{2GM}{r}}.$$

If

$$v > \sqrt{\frac{2GM}{r}},$$

the object will have a hyperbolic escape trajectory.

If

$$v = \sqrt{\frac{GM}{r}},$$

the object will have a circular orbit.

If

$$\sqrt{\frac{2GM}{r}} > v > \sqrt{\frac{GM}{r}},$$

the object has an elliptical orbit starting closest to Earth.

If

$$\sqrt{\frac{GM}{r}} > v > 0,$$

the object has an elliptical orbit starting furthest from Earth.

Problem 2. Masses of two stars forming a binary star are m_1 and m_2. They rotate around each other (more accurately, around their center of mass), at distance l. Determine the period of their rotation.

Solution: The center of mass is at distances $l_1 = \frac{m_2 l}{m_1+m_2}$ and $l_2 = \frac{m_1 l}{m_1+m_2}$ from the two stars, respectively. This comes from $l_1 + l_2 = l$ and $m_1 l_1 = m_2 l_2$. The centripetal force is the gravity, so if their angular velocity is ω

$$m_1 l_1 \omega^2 = G\frac{m_1 m_2}{l^2}.$$

Therefore,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l^3}{G(m_1+m_2)}}.$$

Elastic collision: u is velocity before collision; v is after.

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 + \frac{2m_2}{m_1 + m_2} u_2$$

$$v_2 = \frac{2m_1}{m_1 + m_2} u_1 + \frac{m_2 - m_1}{m_1 + m_2} u_2$$

$$v_1 + u_1 = u_2 + v_2$$

Vertical projectile (+ is up)

$$\boxed{v_y = v_0 - gt} \quad \boxed{y = v_0 t - \frac{gt^2}{2}} \quad \boxed{v_y^2 = v_0^2 - 2gy}$$

General projectile motion with v_0 and θ (+ is up)

$$\boxed{v_x = \text{const} = v_0 \cos\theta} \quad \boxed{x = v_x t = v_0 t \cos\theta}$$

$$\boxed{v_y = v_0 \sin\theta - gt} \quad \boxed{y = v_0 t \sin\theta - \frac{gt^2}{2}} \quad \boxed{v_y^2 = v_0^2 \sin^2\theta - 2gy}$$

$$\boxed{y(x) = x \tan\theta - \frac{g}{2v_0^2 \cos^2\theta} x^2} \quad \boxed{v^2 = v_x^2 + v_y^2}$$

Time in air: $\boxed{T = \frac{2v_0 \sin\theta}{g}}$ Horiz.range: $\boxed{R_x = \frac{v_0^2 \sin 2\theta}{g}}$ Vert.range: $\boxed{R_y = \frac{v_0^2 \sin^2\theta}{2g}}$

List of moment of inertia:

https://en.wikipedia.org/wiki/List_of_moments_of_inertia

Bernoulli’s principle

A common form of Bernoulli's equation is:

$$\frac{v^2}{2} + gz + \frac{p}{\rho} = \text{constant}$$

Torricelli's law

Article Talk

From Wikipedia, the free encyclopedia

Torricelli's law, also known as **Torricelli's theorem**, is a theorem in fluid dynamics relating the speed of fluid flowing from an orifice to the height of fluid above the opening. The law states that the speed v of efflux of a fluid through a sharp-edged hole at the bottom of the tank filled to a depth h is the same as the speed that a body would acquire in falling freely from a height h, i.e. $v = \sqrt{2gh}$, where g is the acceleration due to gravity. This expression comes from equating the kinetic energy gained, $\frac{1}{2}mv^2$, with the potential energy lost, mgh, and solving for v. The law was discovered (though not in this form) by the Italian scientist Evangelista Torricelli, in 1643. It was later shown to be a particular case of Bernoulli's principle.

Venturi Effect

The **Venturi effect** is the reduction in fluid pressure that results when a fluid flows through a constricted section (or choke) of a pipe. The Venturi effect is named after its discoverer, the 18th-century Italian physicist Giovanni Battista Venturi.

$$p_1 - p_2 = \frac{\rho}{2}(v_2^2 - v_1^2)$$

Rotation from ramp:

Problem 3. We are given an object that can roll down a slope and has a moment of inertia $I = cmr^2$. For example,

- a hollow cylinder with mass m and radius r has $c = 1$
- a solid cylinder with mass m and radius r has $c = 1/2$
- a hollow sphere with mass m and radius r has $c = 2/3$
- a solid ball with mass m and radius r has $c = 2/5$

The object rolls without slipping down a slope with angle θ to the horizon. Determine its acceleration. Note: If the object did not roll, i.e., if it was just sliding down the slope, the acceleration would be $a = g \sin\theta$.

Solution: One way to obtain this result is to write the equation for energies before the object starts rolling and when it traveled distance s along the slope,

$$E_p = E_k^{lin} + E_k^{rot} \quad \Rightarrow \quad mgh = \frac{mv^2}{2} + \frac{I\omega^2}{2}$$

There is no slipping so $v = \omega r$. Also, $h = s\sin\theta$ and we know $I = cmr^2$, so

$$2gs\sin\theta = v^2 + cv^2 \quad \Rightarrow \quad v^2 = \frac{2gs\sin\theta}{1+c}.$$

Combining this with $v^2 = 2as$ we get $a = \dfrac{v^2}{2s} = \dfrac{g\sin\theta}{1+c}$.

Problem 4. Using the result of the previous problem, discuss the factors that influence how fast an object will reach the bottom of a slope: radius, mass, density, shape, then order the following objects from fastest to slowest down the slope: (a) small hollow sphere made of light wood, (b) small hollow sphere made of lead, (c) large hollow sphere made of lead, (d) small solid ball made of copper, (e) large solid ball made of copper, (f) large hollow cylinder made of copper, (g) large solid cylinder made of copper.

Solution: From $a = \frac{g\sin\theta}{1+c}$ we see that only the shape and of the object determines how fast it will get to the bottom of the slope, nothing else. Mass, density, radius, none of that matters. Therefore, they will arrive in the order of increasing values of parameter c:

1. solid balls (small and large)
2. solid cylinder
3. hollow spheres (two small and one large)
4. hollow cylinder

Condition for rolling without slip from a ramp:

Because slipping does not occur, $f_S \le \mu_S N$. Solving for the friction force,

$$f_S = I_{CM}\frac{\alpha}{r} = I_{CM}\frac{(a_{CM})}{r^2} = \frac{I_{CM}}{r^2}\left(\frac{mg\sin\theta}{m+(I_{CM}/r^2)}\right) = \frac{mgI_{CM}\sin\theta}{mr^2+I_{CM}}.$$

Substituting this expression into the condition for no slipping, and noting that $N = mg\cos\theta$, we have

$$\frac{mgI_{CM}\sin\theta}{mr^2+I_{CM}} \le \mu_S mg\cos\theta$$

or

$$\mu_S \ge \frac{\tan\theta}{1+(mr^2/I_{CM})}.$$

Period of pendulum with large angle: grows with theta and converges to a finite number

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1}{16}\theta_0^2+\frac{11}{3072}\theta_0^4+\ldots\right]$$

Damped simple harmonic motion:

- The mechanical energy E in a real oscillating system decreases during the oscillations because external forces, such as a drag force, inhibit the oscillations and transfer mechanical energy to thermal energy. The real oscillator and its motion are then said to be damped.
- If the damping force is given by $\vec{F}_d = -b\vec{v}$, where $\vec{v}$ is the velocity of the oscillator and b is a damping constant, then the displacement of the oscillator is given by

$$x(t) = x_m e^{-bt/2m}\cos(\omega' t + \phi),$$

where ω', the angular frequency of the damped oscillator, is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$

- If the damping constant is small ($b \ll \sqrt{km}$), then $\omega' \approx \omega$, where ω is the angular frequency of the undamped oscillator. For small b, the mechanical energy E of the oscillator is given by

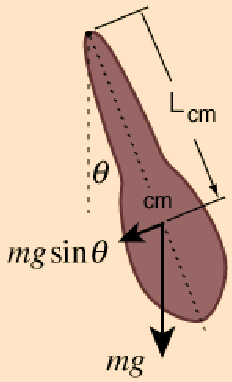
$$E(t) \approx \tfrac{1}{2}kx_m^2 e^{-bt/m}.$$

Angular frequency for small oscillation around equilibrium point: $\boldsymbol{\omega = \sqrt{\frac{V''(x_0)}{m}}}$

Where V is the potential, x0 is the equilibrium point

Physical Pendulum

Hanging objects may be made to oscillate in a manner similar to a simple pendulum. The motion can be described by "Newton's 2nd law for rotation":



$$\tau = I_{\text{support}}\alpha$$

where the torque is

$$\tau = -mgL_{cm}\sin\theta$$

and the relevant moment of inertia is that about the point of suspension. The resulting equation of motion is:

$$\alpha \approx \frac{mgL_{cm}}{I_{\text{support}}}\theta$$

for small angles where $\sin\theta \approx \theta$ Show

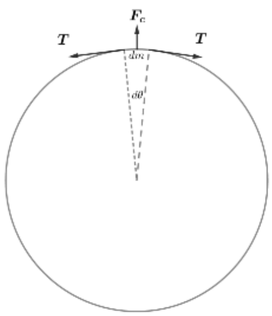
This is identical in form to the equation for the simple pendulum and yields a period:

$$T = 2\pi\sqrt{\frac{I_{\text{support}}}{mgL_{cm}}}$$

Speed of wave in a string:

$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{M/L}}$, where μ is mass per unit length.

Tension in a circular string: restoring force F = 2πT, where T is the tension.



Euler-Lagrange Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$, where L = T – V

If L does not depend on a certain coordinate q_k, then $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} = 0 \quad \Longrightarrow \quad \frac{\partial L}{\partial \dot{q}_k} = C,$

We say that is a cyclic coordinate, and $\partial L/\partial \dot{q}_k$ is a *conserved* quantity, if Cartesian coordinates are used then $\partial L/\partial \dot{x}_k$ is simply the momentum..

Energy: $$E = \left(\sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) - L$$

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}$$, so if L does not have depend on t explicitly (which is usually the case), the E is conserved.

25. Alice and Bob are working on a lab report. Alice measures the period of a pendulum to be 1.013 ± 0.008 s, while Bob independently measures the period to be 0.997 ± 0.016 s. Alice and Bob can combine their measurements in several ways.

1: Keep Alice's result and ignore Bob's

2: Average Alice's and Bob's results

3: Perform a weighted average of Alice's and Bob's results, with Alice's result weighted 4 times more than Bob's

How are the uncertainties of these results related?

(A) Method 1 has the lowest uncertainty, and method 2 has the highest

(B) Method 3 has the lowest uncertainty, and method 2 has the highest ← **CORRECT**

(C) Method 2 has the lowest uncertainty, and method 1 has the highest

(D) Method 3 has the lowest uncertainty, and method 1 has the highest

(E) Method 1 has the lowest uncertainty, and method 3 has the highest

Solution

Let Alice's measurement have uncertainty Δx, so Bob's has uncertainty $2\Delta x$. The uncertainty obeys the following rules, which can be derived by identifying the uncertainty with the standard deviation:

(A) Scaling: if x has uncertainty Δx, then cx has uncertainty $c\Delta x$.

(B) Addition in quadrature: if x has uncertainty Δx and an independent measurement y has uncertainty Δy, then $x + y$ has uncertainty $\sqrt{(\Delta x)^2 + (\Delta y)^2}$.

Then the uncertainties of the three possibilities are:

1: Δx

2: $\sqrt{(\Delta x)^2 + (2\Delta x)^2}/2 = (\sqrt{5}/2)\Delta x$

3: $\sqrt{(4\Delta x)^2 + (2\Delta x)^2}/5 = (2/\sqrt{5})\Delta x$

The lowest uncertainty is from method 3 (which is in fact optimal); the highest is from method 2.

Taylor expansion

Remark

If these questions seem complicated, rest assured that 90% of approximations on the USAPhO and IPhO boil down to using

$$\sin x \approx x, \quad \cos x \approx 1 - x^2/2, \quad (1+x)^n \approx 1 + xn, \quad e^x \approx 1 + x, \quad \log(1+x) \approx x.$$

Useful integral for Work W and Power P:

$$dW = mv\,dv$$

$$dW = P\,dt = P\frac{dt}{dx}\,dx = \frac{P}{v}\,dx$$

Spherical coordinate:

$$x = r\sin\theta\,\cos\varphi,$$
$$y = r\sin\theta\,\sin\varphi,$$
$$z = r\cos\theta.$$

Spherical coordinate integral: $dV = r^2 sin\theta\; drd\theta d\varphi$

The line element for an infinitesimal displacement from (r, θ, φ) to $(r + dr, \theta + d\theta, \varphi + d\varphi)$ is

Differentiation: $\mathrm{d}\mathbf{r} = \mathrm{d}r\,\hat{\mathbf{r}} + r\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\mathrm{d}\varphi\,\hat{\boldsymbol{\varphi}},$

E.g. velocity: $\mathbf{v} = \dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\,\dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}$

Coordinates transformation, gradient, divergence, curl, etc:

https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates

Table with the del operator in cartesian, cylindrical and spherical coordinates

Operation	Cartesian coordinates (x, y, z)	Cylindrical coordinates (ρ, φ, z)	Spherical coordinates (r, θ, φ), where θ is the polar angle and φ is the azimuthal angle[α]
Vector field $\mathbf{A}$	$A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$	$A_\rho\hat{\boldsymbol{\rho}} + A_\varphi\hat{\boldsymbol{\varphi}} + A_z\hat{\mathbf{z}}$	$A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\varphi\hat{\boldsymbol{\varphi}}$
Gradient ∇f[1]	$\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$
Divergence $\nabla\cdot\mathbf{A}$[1]	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla\times\mathbf{A}$[1]	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$	$\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{\mathbf{z}}$	$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\varphi\sin\theta) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$
Laplace operator $\nabla^2 f \equiv \Delta f$[1]	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$

Gauss's law is written in integral form as

$$\oint \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\epsilon_0}.$$

In practice, you will only apply this form to situations with high symmetry, where

$$E = \begin{cases} Q/4\pi\epsilon_0 r^2 & \text{spherical symmetry,} \\ \lambda/2\pi\epsilon_0 r & \text{cylindrical symmetry,} \\ \sigma/2\epsilon_0 & \text{infinite plane.} \end{cases}$$

Electric field for electric dipole:

Problem 18. Consider an electric charge q placed at $x = 0$ and a charge $-q$ placed at $x = d$. The electric field along the x axis is then

$$E(x) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{x^2} - \frac{1}{(x-d)^2}\right).$$

For large x, use the binomial theorem to approximate the field.

$E(x) = -\dfrac{2qd}{4\pi\epsilon_0 x^3} = -\dfrac{qd}{2\pi\epsilon_0 x^3}$**, for x >> d.**

The dipole moment of two charges q and $-q$ separated by $\mathbf{d}$ is $\mathbf{p} = q\mathbf{d}$. More generally, the dipole moment of a charge configuration is defined as

$$\mathbf{p} = \int \rho(\mathbf{r})\mathbf{r}\, d^3\mathbf{r}.$$

For an overall neutral charge configuration, the leading contribution to its electric potential far away is the dipole potential,

$$\phi(r,\theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

where θ is the angle of $\mathbf{r}$ to $\mathbf{p}$.

Electric field of general dipole: $\mathbf{E}(\mathbf{r}) = \dfrac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \dfrac{1}{4\pi\epsilon_0 r^3}(3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p})$

Total Potential energy of a uniformly charged solid sphere: $\boldsymbol{U = \frac{3}{5}\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R}}$**,** which can be calculated as bringing small charge continuously from infinite distance to build the sphere, or integrating the field intensity $\boldsymbol{\frac{E^2}{2\varepsilon_0}}$ over the whole space inside and outside the sphere.

Image charge of point charge q which is distance b from the center of a grounded conducting sphere. The sphere radius is r:

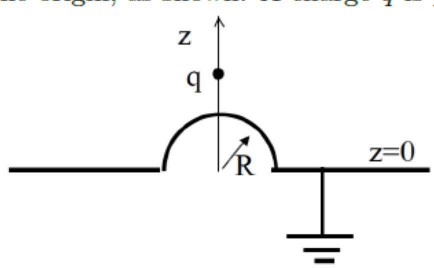
Image charge distance: $b' = \frac{r^2}{b}, for\ both\ b < r\ or\ b > r.$

Image charge: $q' = -q\sqrt{\frac{b'}{b}} = -q\frac{r}{b}$

The force on the charge is: $$F = \frac{qq'}{4\pi\epsilon_0(b-b')^2} = \frac{q^2 rb}{4\pi\epsilon_0(b^2-r^2)^2}$$

Image charges for composite grounded sphere + plane:

Problem 3. An infinite grounded conducting plane at $z = 0$ is deformed with a hemispherical bump of radius R centered at the origin, as shown. A charge q is placed at $z = a$ as shown.



Can the method of images be used to find the potential in the region with the charge? If so, specify the image charges; if not, explain why not.

Solution. It can be done with three image charges, all on the z-axis:

- A charge $-q$ at $z = -a$.
- A charge $-qR/a$ at $z = R^2/a$.
- A charge qR/a at $z = -R^2/a$.

These ensure that the voltage vanishes on both the whole plane $z = 0$ and on the sphere $r = R$.

Image charge of a non-grounded sphere with net charge q:

Problem 4 (Purcell 3.50). A point charge q is located a distance $b > r$ from the center of a *nongrounded* conducting spherical shell of radius r, which also has charge q. When b is close to r, the charge is attracted to the shell because it induces negative charge; when b is large the charge is clearly repelled. Find the value of b so that the point charge is in equilibrium. (Hint: you should have to solve a difficult polynomial equation. You can either use a computer or calculator, or use the fact that it contains a factor of $1 - x - x^2$.)

Solution. The image charge has value $-q' = -qr/b$, and is at position r^2/b. As of now, the spherical shell has total charge $-q'$ (by Gauss's law), so to compensate, we add a charge of value $q + q'$ at the center. Thus, to balance forces we have

$$\frac{q'}{(b - r^2/b)^2} = \frac{q + q'}{b^2}.$$

Capacitance of two concentric spherical metal shells with radii a<b:

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \qquad C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{ab}{b - a}$$

DC circuits / resistors:

Idea 7

If any two points in a resistor network are at the same potential, nothing will change if the two points are connected together and treated as one. More generally, the resistance of any resistor directly connecting the two points may be changed freely.

21.6 DC Circuits Containing Resistors and Capacitors

- An RC circuit is one that has both a resistor and a capacitor.
- The time constant τ for an RC circuit is $\tau = RC$.
- When an initially uncharged ($V_0 = 0$ at $t = 0$) capacitor in series with a resistor is charged by a DC voltage source, the voltage rises, asymptotically approaching the emf of the voltage source; as a function of time,

$$V = \text{emf}(1 - e^{-t/RC})(\text{charging}).$$

- Within the span of each time constant τ, the voltage rises by 0.632 of the remaining value, approaching the final voltage asymptotically.
- If a capacitor with an initial voltage V_0 is discharged through a resistor starting at $t = 0$, then its voltage decreases exponentially as given by

$$V = V_0 e^{-t/RC}\,(\text{discharging}).$$

- In each time constant τ, the voltage falls by 0.368 of its remaining initial value, approaching zero asymptotically.

22.3 Magnetic Fields and Magnetic Field Lines

- Magnetic fields can be pictorially represented by magnetic field lines, the properties of which are as follows:

1. The field is tangent to the magnetic field line.
2. Field strength is proportional to the line density.
3. Field lines cannot cross.
4. Field lines are continuous loops.

22.4 Magnetic Field Strength: Force on a Moving Charge in a Magnetic Field

- Magnetic fields exert a force on a moving charge *q*, the magnitude of which is

$$F = qvB \sin \theta,$$

where θ is the angle between the directions of v and B.

- The SI unit for magnetic field strength B is the tesla (T), which is related to other units by

$$1\text{ T} = \frac{1\text{ N}}{\text{C} \cdot \text{m/s}} = \frac{1\text{ N}}{\text{A} \cdot \text{m}}.$$

- The *direction* of the force on a moving charge is given by right hand rule 1 (RHR-1): Point the thumb of the right hand in the direction of v, the fingers in the direction of B, and a perpendicular to the palm points in the direction of F.
- The force is perpendicular to the plane formed by $\mathbf{v}$ and $\mathbf{B}$. Since the force is zero if $\mathbf{v}$ is parallel to $\mathbf{B}$, charged particles often follow magnetic field lines rather than cross them.

22.5 Force on a Moving Charge in a Magnetic Field: Examples and Applications

- Magnetic force can supply centripetal force and cause a charged particle to move in a circular path of radius

$$r = \frac{mv}{qB},$$

where v is the component of the velocity perpendicular to B for a charged particle with mass m and charge q.

22.6 The Hall Effect

- The Hall effect is the creation of voltage ε, known as the Hall emf, across a current-carrying conductor by a magnetic field.
- The Hall emf is given by

$$\varepsilon = Blv \text{ } (B, v, \text{ and } l, \text{ mutually perpendicular})$$

for a conductor of width l through which charges move at a speed v.

22.7 Magnetic Force on a Current-Carrying Conductor

- The magnetic force on current-carrying conductors is given by

$$F = IlB \sin \theta,$$

where I is the current, l is the length of a straight conductor in a uniform magnetic field B, and θ is the angle between I and B. The force follows RHR-1 with the thumb in the direction of I.

22.8 Torque on a Current Loop: Motors and Meters

- The torque τ on a current-carrying loop of any shape in a uniform magnetic field. is

$$\tau = NIAB \sin \theta,$$

where N is the number of turns, I is the current, A is the area of the loop, B is the magnetic field strength, and θ is the angle between the perpendicular to the loop and the magnetic field.

22.9 Magnetic Fields Produced by Currents: Ampere's Law

- The strength of the magnetic field created by current in a long straight wire is given by

$$B = \frac{\mu_0 I}{2\pi r} \text{ (long straight wire)},$$

I is the current, r is the shortest distance to the wire, and the constant $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$ is the permeability of free space.

- The direction of the magnetic field created by a long straight wire is given by right hand rule 2 (RHR-2): *Point the thumb of the right hand in the direction of current, and the fingers curl in the direction of the magnetic field loops* created by it.
- The magnetic field created by current following any path is the sum (or integral) of the fields due to segments along the path (magnitude and direction as for a straight wire), resulting in a general relationship between current and field known as Ampere's law.
- The magnetic field strength at the center of a circular loop is given by

$$B = \frac{\mu_0 I}{2R} \text{ (at center of loop)},$$

R is the radius of the loop. This equation becomes $B = \mu_0 nI/(2R)$ for a flat coil of N loops. RHR-2 gives the direction of the field about the loop. A long coil is called a solenoid.

- The magnetic field strength inside a solenoid is

$$B = \mu_0 nI \text{ (inside a solenoid)},$$

where n is the number of loops per unit length of the solenoid. The field inside is very uniform in magnitude and direction.

22.10 Magnetic Force between Two Parallel Conductors

- The force between two parallel currents I_1 and I_2, separated by a distance r, has a magnitude per unit length given by

$$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r}.$$

- The force is attractive if the currents are in the same direction, repulsive if they are in opposite directions.

22.11 More Applications of Magnetism

- Crossed (perpendicular) electric and magnetic fields act as a velocity filter, giving equal and opposite forces on any charge with velocity perpendicular to the fields and of magnitude

$$v = \frac{E}{B}.$$

23.1 Induced Emf and Magnetic Flux

- The crucial quantity in induction is magnetic flux Φ, defined to be $\Phi = BA\cos\theta$, where B is the magnetic field strength over an area A at an angle θ with the perpendicular to the area.
- Units of magnetic flux Φ are $\mathrm{T}\cdot\mathrm{m}^2$.
- Any change in magnetic flux Φ induces an emf—the process is defined to be electromagnetic induction.

23.2 Faraday's Law of Induction: Lenz's Law

- Faraday's law of induction states that the emfinduced by a change in magnetic flux is

  $$\mathrm{emf} = -N\frac{\Delta\Phi}{\Delta t}$$

  when flux changes by $\Delta\Phi$ in a time Δt.
- If emf is induced in a coil, N is its number of turns.
- The minus sign means that the emf creates a current I and magnetic field B that *oppose the change in flux* $\Delta\Phi$ —this opposition is known as Lenz's law.

23.3 Motional Emf

- An emf induced by motion relative to a magnetic field B is called a *motional emf* and is given by

  $$\mathrm{emf} = B\ell v \qquad (B, \ell, \text{and } v \text{ perpendicular}),$$

  where ℓ is the length of the object moving at speed v relative to the field.

23.4 Eddy Currents and Magnetic Damping

- Current loops induced in moving conductors are called eddy currents.
- They can create significant drag, called magnetic damping.

23.5 Electric Generators

- An electric generator rotates a coil in a magnetic field, inducing an emfgiven as a function of time by

  $$\mathrm{emf} = NAB\omega\sin\omega t,$$

  where A is the area of an N-turn coil rotated at a constant angular velocity ω in a uniform magnetic field B.
- The peak emf emf_0 of a generator is

  $$\mathrm{emf}_0 = NAB\omega.$$

23.6 Back Emf

- Any rotating coil will have an induced emf—in motors, this is called back emf, since it opposes the emf input to the motor.

23.7 Transformers

- Transformers use induction to transform voltages from one value to another.
- For a transformer, the voltages across the primary and secondary coils are related by

  $$\frac{V_s}{V_p} = \frac{N_s}{N_p},$$

  where V_p and V_s are the voltages across primary and secondary coils having N_p and N_s turns.
- The currents I_p and I_s in the primary and secondary coils are related by $\frac{I_s}{I_p} = \frac{N_p}{N_s}$.
- A step-up transformer increases voltage and decreases current, whereas a step-down transformer decreases voltage and increases current.

23.9 Inductance

- Inductance is the property of a device that tells how effectively it induces an emf in another device.
- Mutual inductance is the effect of two devices in inducing emfs in each other.
- A change in current $\Delta I_1 / \Delta t$ in one induces an emf emf_2 in the second:

$$\text{emf}_2 = -M\frac{\Delta I_1}{\Delta t},$$

  where M is defined to be the mutual inductance between the two devices, and the minus sign is due to Lenz's law.
- Symmetrically, a change in current $\Delta I_2 / \Delta t$ through the second device induces an emf emf_1 in the first:

$$\text{emf}_1 = -M\frac{\Delta I_2}{\Delta t},$$

  where M is the same mutual inductance as in the reverse process.
- Current changes in a device induce an emf in the device itself.
- Self-inductance is the effect of the device inducing emf in itself.
- The device is called an inductor, and the emf induced in it by a change in current through it is

$$\text{emf} = -L\frac{\Delta I}{\Delta t},$$

  where L is the self-inductance of the inductor, and $\Delta I / \Delta t$ is the rate of change of current through it. The minus sign indicates that emf opposes the change in current, as required by Lenz's law.
- The unit of self- and mutual inductance is the henry (H), where $1\ \text{H} = 1\ \Omega \cdot \text{s}$.
- The self-inductance L of an inductor is proportional to how much flux changes with current. For an N-turn inductor,

$$L = N\frac{\Delta \Phi}{\Delta I}.$$

- The self-inductance of a solenoid is

$$L = \frac{\mu_0 N^2 A}{\ell} (\text{solenoid}),$$

  where N is its number of turns in the solenoid, A is its cross-sectional area, ℓ is its length, and $\mu_0 = 4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A}$ is the permeability of free space.
- The energy stored in an inductor E_{ind} is

$$E_{\text{ind}} = \frac{1}{2}LI^2.$$

23.10 RL Circuits

- When a series connection of a resistor and an inductor—an *RL* circuit—is connected to a voltage source, the time variation of the current is

$$I = I_0(1 - e^{-t/\tau}) \quad (\text{turning on}).$$

  where $I_0 = V / R$ is the final current.
- The characteristic time constant τ is $\tau = \frac{L}{R}$, where L is the inductance and R is the resistance.
- In the first time constant τ, the current rises from zero to $0.632 I_0$, and 0.632 of the remainder in every subsequent time interval τ.
- When the inductor is shorted through a resistor, current decreases as

$$I = I_0 e^{-t/\tau} \quad (\text{turning off})$$

  Here I_0 is the initial current.

- Current falls to $0.368 I_0$ in the first time interval τ, and 0.368 of the remainder toward zero in each subsequent time τ.

23.11 Reactance, Inductive and Capacitive

- For inductors in AC circuits, we find that when a sinusoidal voltage is applied to an inductor, the voltage leads the current by one-fourth of a cycle, or by a 90° phase angle.
- The opposition of an inductor to a change in current is expressed as a type of AC resistance.
- Ohm's law for an inductor is

$$I = \frac{V}{X_L},$$

where V is the rms voltage across the inductor.

- X_L is defined to be the inductive reactance, given by

$$X_L = 2\pi fL,$$

with f the frequency of the AC voltage source in hertz.

- Inductive reactance X_L has units of ohms and is greatest at high frequencies.
- For capacitors, we find that when a sinusoidal voltage is applied to a capacitor, the voltage follows the current by one-fourth of a cycle, or by a 90° phase angle.
- Since a capacitor can stop current when fully charged, it limits current and offers another form of AC resistance; Ohm's law for a capacitor is

$$I = \frac{V}{X_C},$$

where V is the rms voltage across the capacitor.

- X_C is defined to be the capacitive reactance, given by

$$X_C = \frac{1}{2\pi fC}.$$

- X_C has units of ohms and is greatest at low frequencies.

23.12 RLC Series AC Circuits

- The AC analogy to resistance is impedance Z, the combined effect of resistors, inductors, and capacitors, defined by the AC version of Ohm's law:

$$I_0 = \frac{V_0}{Z} \text{ or } I_{\text{rms}} = \frac{V_{\text{rms}}}{Z},$$

where I_0 is the peak current and V_0 is the peak source voltage.

- Impedance has units of ohms and is given by $Z = \sqrt{R^2 + (X_L - X_C)^2}$.
- The resonant frequency f_0, at which $X_L = X_C$, is

$$f_0 = \frac{1}{2\pi\sqrt{LC}}.$$

- In an AC circuit, there is a phase angle ϕ between source voltage V and the current I, which can be found from

$$\cos\phi = \frac{R}{Z},$$

- $\phi = 0^{\circ}$ for a purely resistive circuit or an *RLC* circuit at resonance.
- The average power delivered to an *RLC* circuit is affected by the phase angle and is given by

$$P_{\text{ave}} = I_{\text{rms}} V_{\text{rms}} \cos\phi,$$

$\cos\phi$ is called the power factor, which ranges from 0 to 1.

Electromagnetic wave:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}, \qquad \frac{E}{B} = c, \quad c = f\lambda,$$

24.4 Energy in Electromagnetic Waves

- The energy carried by any wave is proportional to its amplitude squared. For electromagnetic waves, this means intensity can be expressed as

$$I_{\text{ave}} = \frac{c\varepsilon_0 E_0^2}{2},$$

where I_{ave} is the average intensity in W/m^2, and E_0 is the maximum electric field strength of a continuous sinusoidal wave.

- This can also be expressed in terms of the maximum magnetic field strength B_0 as

$$I_{\text{ave}} = \frac{cB_0^2}{2\mu_0}$$

and in terms of both electric and magnetic fields as

$$I_{\text{ave}} = \frac{E_0 B_0}{2\mu_0}.$$

- The three expressions for I_{ave} are all equivalent.

Peak energy intensity: ***$I_0 = 2\, I_{ave}$***

Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Integral form:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0} \quad \text{(Gauss' law for electric fields)}$$

$$\Phi_B = \oint \vec{B} \cdot d\vec{A} = 0 \quad \text{(Gauss' law for magnetic fields).}$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law of induction).}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} + \mu_0 i_{\text{enc}} \quad \text{(Ampere–Maxwell law).}$$

Heat by radiation:

14.7 Radiation

- Radiation is the rate of heat transfer through the emission or absorption of electromagnetic waves.
- The rate of heat transfer depends on the surface area and the fourth power of the absolute temperature:

$$\frac{Q}{t} = \sigma e A T^4,$$

where $\sigma = 5.67 \times 10^{-8}\ \text{J/s} \cdot \text{m}^2 \cdot \text{K}^4$ is the Stefan-Boltzmann constant and e is the emissivity of the body. For a black body, $e = 1$ whereas a shiny white or perfect reflector has $e = 0$, with real objects having values of e between 1 and 0. The net rate of heat transfer by radiation is

$$\frac{Q_{\text{net}}}{t} = \sigma e A\left(T_2^4 - T_1^4\right)$$

where T_1 is the temperature of an object surrounded by an environment with uniform temperature T_2 and e is the emissivity of the *object*.

Change of internal energy equals heat transferred into the system, minus work done by the system to outside:

$$\Delta U = Q - W.$$

Table 15.1 Summary of Terms for the First Law of Thermodynamics, *ΔU=Q−W*

Term	Definition
U	Internal energy—the sum of the kinetic and potential energies of a system's atoms and molecules. Can be divided into many subcategories, such as thermal and chemical energy. Depends only on the state of a system (such as its P, V, and T), not on how the energy entered the system. Change in internal energy is path independent.
Q	Heat—energy transferred because of a temperature difference. Characterized by random molecular motion. Highly dependent on path. Q entering a system is positive.
W	Work—energy transferred by a force moving through a distance. An organized, orderly process. Path dependent. W done by a system (either against an external force or to increase the volume of the system) is positive.

Table 15.2 Summary of Simple Thermodynamic Processes

Isobaric	Constant pressure $W = P\Delta V$
Isochoric	Constant volume $W = 0$
Isothermal	Constant temperature $Q = W$
Adiabatic	No heat transfer $Q = 0$

Kinetic energy of the an atom of monatomic ideal gas is: $\frac{1}{2}m\bar{v}^2 = \frac{3}{2}kT$

The kinetic energy is the only form of internal energy for monatomic ideal gas, so it's total internal energy is: $U = N\frac{1}{2}m\bar{v}^2 = \frac{3}{2}NkT,\ \text{(monatomic ideal gas)}$

And for monatomic gas, the molar (1 mole) heat capacity at constant volume is $\frac{3}{2}R$, where *R=N*$_A$*k* is the gas constant.

The molar (1 mole) heat capacity at constant pressure is $\frac{5}{2}R$. (Extra heat is needed to do work to the environment as volume expands)

- In a cyclical process, such as a heat engine, the net work done by the system equals the net heat transfer into the system, or $W = Q_h - Q_c$, where Q_h is the heat transfer from the hot object (hot reservoir), and Q_c is the heat transfer into the cold object (cold reservoir).
- Efficiency can be expressed as $Eff = \frac{W}{Q_h}$, the ratio of work output divided by the amount of energy input.

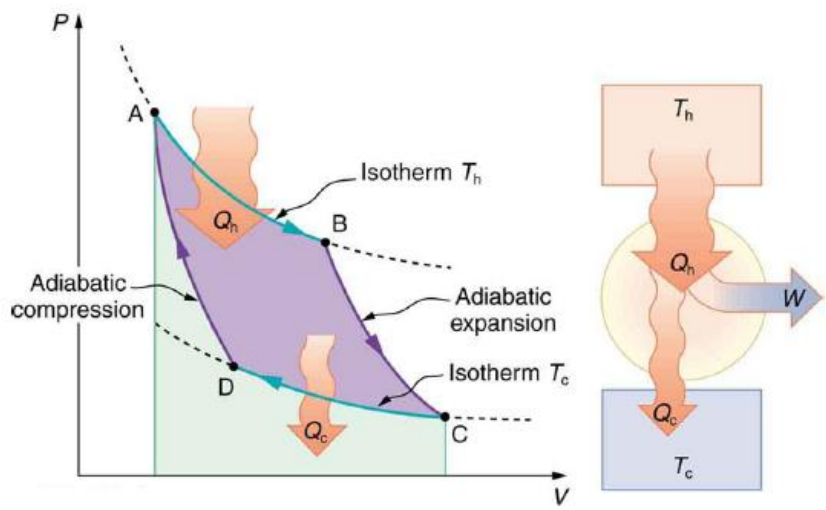


Figure 15.23 PV diagram for a Carnot cycle, employing only reversible isothermal and adiabatic processes. Heat transfer Q_h occurs into the working substance during the isothermal path AB, which takes place at constant temperature T_h. Heat transfer Q_c occurs out of the working substance during the isothermal path CD, which takes place at constant temperature T_c. The net work output W equals the area inside the path ABCDA. Also shown is a schematic of a Carnot engine operating between hot and cold reservoirs at temperatures T_h and T_c. Any heat engine using reversible processes and operating between these two temperatures will have the same maximum efficiency as the Carnot engine.

Carnot Efficiency: $Eff_C = 1 - \frac{T_c}{T_h}.$

Heat pump coefficient of performance: $COP_{hp} = \frac{Q_h}{W} = 1 / Eff$

Air conditioner and refrigerator performance: $COP_{ref} = \frac{Q_c}{W} = COP_{hp} - 1$

Carnot cycle: $\frac{Q_c}{T_c} = \frac{Q_h}{T_h}$

Change in entropy: $\Delta S = \left(\frac{Q}{T}\right)_{\text{rev}}$

Entropy is constant for a reversible process; and it increases for an irreversible process.

Entropy is also the loss of energy available to do work.

The entropy of a system in a given state (a macrostate) can be written as

$$S = k \ln W,$$

where $k = 1.38 \times 10^{-23}$ J/K is Boltzmann's constant, and $\ln W$ is the natural logarithm of the number of microstates W corresponding to the given macrostate.

Wave:

16.10 Superposition and Interference

- Superposition is the combination of two waves at the same location.
- Constructive interference occurs when two identical waves are superimposed in phase.
- Destructive interference occurs when two identical waves are superimposed exactly out of phase.
- A standing wave is one in which two waves superimpose to produce a wave that varies in amplitude but does not propagate.
- Nodes are points of no motion in standing waves.
- An antinode is the location of maximum amplitude of a standing wave.
- Waves on a string are resonant standing waves with a fundamental frequency and can occur at higher multiples of the fundamental, called overtones or harmonics.
- Beats occur when waves of similar frequencies f_1 and f_2 are superimposed. The resulting amplitude oscillates with a beat frequency given by

$$f_B = |f_1 - f_2|.$$

$$x = X\cos(2\pi f_1 t) + X\cos(2\pi f_2 t) = 2X\cos(\pi f_B t)\cos(2\pi f_{ave} t)$$

Standing wave: $\lambda = 2L/n$, where n=1,2,3,…. The longest wavelength for standing wave is 2L.

Relativity:

Time dilation:

Observers moving at a relative velocity v do not measure the same elapsed time for an event. Proper time Δt_0 is the time measured by an observer at rest relative to the event being observed. Proper time is related to the time Δt measured by an Earth-bound observer by the equation

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \Delta t_0,$$

Length contraction:

Length contraction L is the shortening of the measured length of an object moving relative to the observer's frame:

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} = \frac{L_0}{\gamma}.$$

Addition of velocities: $u = \dfrac{v + u'}{1 + \frac{vu'}{c^2}}$

Relativistic Doppler effects: $\lambda_{obs} = \lambda_s \sqrt{\dfrac{1 + \frac{u}{c}}{1 - \frac{u}{c}}}$

Where λ_{obs} is the observed wavelength, λ_s is the source wavelength, and u is the relative velocity of the source to the observer.

Momentum: $p = \gamma m u$

Energy: $E = \gamma mc^2$

Relativistic kinetic energy is $\mathrm{KE}_{\mathrm{rel}} = (\gamma - 1)mc^2$

$$E^2 = (pc)^2 + (mc^2)^2$$

Lorentz transformation

$$\begin{aligned} x' &= \gamma(x - vt), \\ y' &= y, \\ z' &= z, \\ t' &= \gamma(t - vx/c^2) \end{aligned}$$

(Lorentz transformation equations; valid at all physically possible speeds).

Matrix form:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -r\beta & 0 & 0 \\ -r\beta & -r\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$, Where $\beta = v/c$

Similarly for energy and momentum:

$$\begin{pmatrix} E'/c \\ p_x{}' \\ p_y{}' \\ p_z{}' \end{pmatrix} = \begin{pmatrix} \gamma & -r\beta & 0 & 0 \\ -r\beta & -r\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

Relativistic momentum and force and effective mass:

$$F = \frac{dp}{dt} = m\frac{d}{dt}\left(\frac{v}{\sqrt{1 - v^2/c^2}}\right) = \frac{m}{(1 - v^2/c^2)^{3/2}}\frac{dv}{dt}$$

So effective mass: $m_{\mathrm{eff}} = \dfrac{m}{(1 - v^2/c^2)^{3/2}}$, which can be used to solve for oscillation frequency, etc.

Electric charge is Lorentz invariant.